12. The Blackman-Tukey Method

Prior to the advent of the FFT and fast computers, power density spectral estimation was almost never done as described in the last section. Rather the onerous computational load led scientists, as far as possible, to reduce the number of calculations required. The so-called Blackman-Tukey method, which became the de facto standard, begins with a purely theoretical idea. Let $< x_n >= 0$. Define the "sample autocovariance",

$$\tilde{R}(\tau) = \frac{1}{N}\sum_{n=0}^{N-1-|\tau|} x_n x_{n+\tau}, \tau = 0, \pm 1, \pm 2, ..., \pm N-1, \tag{12.1}$$

where as τ grows, the number of terms in the sum necessarily diminishes. From the discrete convolution theorem, it follows that,

$$\mathcal{F}\left(\tilde{R}(\tau)\right) = \sum_{\tau=-(N-1)}^{N-1} \tilde{R}(\tau)\exp(-2\pi i s\tau) = \frac{1}{N}|\hat{x}(s)|^2 = N|\alpha_n|^2 \tag{12.2}$$

Then the desired power density is,

$$< N|\alpha_n|^2 >= \Phi(s) = \sum_{\tau=-(N-1)}^{N-1} < \tilde{R}(\tau) > \exp(-2\pi i s\tau). \tag{12.3}$$

Consider

$$\begin{aligned} < \tilde{R}(\tau) >&= \frac{1}{N}\sum_{m=0}^{N-1-|\tau|} < x_m x_{m+\tau} >, \tau = 0, \pm 1, \pm 2, ..., \pm N-1 \\ &= \frac{N-|\tau|}{N} R(\tau), \end{aligned} \tag{12.4}$$

by definition of $R(\tau)$. First letting $N \to \infty$, and then $\tau \to \infty$, we have the Wiener-Khinchin theorem:

$$\Phi(s) = \sum_{\tau=-\infty}^{\infty} R(\tau)\exp(-2\pi i s\tau) = \sum_{\tau=-\infty}^{\infty} R(\tau)\cos(2\pi s\tau) \tag{12.5}$$

the power density spectrum of a stochastic process is the Fourier transform of the autocovariance. This relationship is an extremely important *theoretical* one. (One of the main mathematical issues of time series analysis is that the limit as $T = N\Delta t \to \infty$ of the Fourier transform or series,

$$\alpha_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\exp\left(\frac{2\pi i n t}{T}\right) dt, \tag{12.6}$$

whether an integral or sum does not exist (does not converge) because of the stochastic behavior of $x(t)$, but the Fourier transform of the autocovariance (which is *not random*) does exist.). It is important to recognize that unlike the definition of "power spectrum" used above for non-random (deterministic) functions, an expected value operation is an essential ingredient when discussing stochastic processes.

It is very tempting (and many people succumbed) to assert that $\tilde{R}(\tau) \to R(\tau)$ as N becomes very large. The idea is plausible because (12.1) looks just like an average. The problem is that no matter how large N becomes, (12.2) requires the Fourier transform of the *entire* sample autocovariance. As the

lag $\tau \to N$, the number of terms in the average (12.1) diminishes until the last lag has only a single value in it—a very poor average. While the lag 0 term may have thousands of terms in the average, the last term has only one. The Fourier transform of the sample autocovariance includes these very poorly determined sample covariances; indeed we know from (12.2) that the statistical behavior of the result must be exactly that of the periodogram—it is unstable (inconsistent) as an estimator because its variance does not diminish with N.

The origin of this instability is directly derived from the poorly estimated large-lag sample covariances.[4] The Blackman-Tukey method does two things at once: it reduces the variance of the periodogram, and minimizes the number of elements which must be Fourier transformed. This is a bit confusing because the two goals are quite different. Once one identifies the large lag τ values of $\tilde{R}(\tau)$ as the source of the statistical instability, the remedy is clear: get rid of them. One multiplies $\tilde{R}(\tau)$ by a "window" w_τ and Fourier transforms the result

$$\tilde{\Phi}^{\nu}(s) = \sum_{\tau=-(N-1)}^{\tau=N-1} \tilde{R}(\tau)\, w_\tau \exp(-2\pi i s\tau) \tag{12.8}$$

By the convolution theorem, this is just

$$\tilde{\Phi}^{\nu}(s) = \mathcal{F}\left(\tilde{R}(\tau)\right) * \mathcal{F}(w_\tau) \tag{12.9}$$

If w_τ is such that its Fourier transform is a local averaging operator, then (12.9) is exactly what we seek, a local average of the periodogram. If we can select w_τ so that it simultaneously has this property, and so that it actually vanishes for $|\tau| > M$, then the Fourier transform in (12.8) is reduced from being taken over N-terms to over $M << N$, that is,

$$\tilde{\Phi}^{\nu}(s) = \sum_{\tau=-(M-1)}^{\tau=M-1} \tilde{R}(\tau)\, w_\tau \exp(-2\pi i s\tau). \tag{12.10}$$

The Blackman-Tukey estimate is based upon (12.9, and 12.10) and the choice of suitable window weights w_τ. A large literature grew up devoted to the window choice. Again, one trades bias against variance through the value M, which one prefers greatly to minimize. The method is now obsolete because the ability to generate the Fourier coefficients directly permits much greater control over the result. The bias discussion of the Blackman-Tukey method is particularly tricky, as is the determination of ν. Use of the method should be avoided except under those exceptional circumstances when for some reason only

[4] Some investigators made the situation much worse by the following plausible argument. For finite τ, the number of terms in (12.1) is actually not N, but $N - |\tau|$; .they argued therefore, that the proper way to calculate $R(\tau)$ was actually

$$\tilde{R}_1(\tau) = \frac{1}{N-|\tau|} \sum_{n=0}^{N-1-|\tau|} x_t x_{t+\tau} \tag{12.7}$$

which would be (correctly) an unbiassed estimator of $R(\tau)$. They then Fourier transformed $\tilde{R}_1(\tau)$ instead of $R(\tau)$. But this makes the situation much worse: by using (12.7) one gives greatest weight to the least well-determined components in the Fourier analysis. One has traded a reduction in bias for a vastly increased variance, in this case, a very poor choice indeed. (Bias does not enter if x_t is white noise, as all terms of both $< \tilde{R} >, < \tilde{R}_1 >$ vanish except for $\tau = 0$.)

$\tilde{R}(\tau)$ is known. (For large data sets, it is actually computationally more efficient to compute $\tilde{R}(\tau)$, should one wish it, by first forming $\tilde{\Phi}^{\nu}(s)$ using FFT methods, and then obtaining $\tilde{R}(\tau)$ as its inverse Fourier transform.)