Design and Calculating Bounded Area

Stage 2 Mathematics Methods

SACE No. 216771E

Introduction

The problem presented in this task is to design a mathematically defined visual design and to calculate the exact area of the regions enclosed within its boundaries. Unlike a purely artistic drawing, the logo must be expressed through explicit mathematical functions and models so that it can be rigorously analysed, reproduced, and quantified. The challenge therefore lies in combining creativity with precise mathematical reasoning to transform abstract curves into a measurable and well-defined design.

This task is significant both for learning and for real-world applications. From a learning perspective, it requires the comprehensive application of advanced mathematical concepts, including transformations, interpolation, and integration. It also develops problem-solving skills by requiring efficient and accurate solutions to both routine and non-routine problems, such as finding areas bounded by complex curves. From a real-world perspective, calculating the area of a design has practical uses: for example, determining the material required to produce it as a decoration or sign, estimating costs for embroidery or printing, or scaling it appropriately for digital and physical contexts. In this way, the task demonstrates how mathematical models directly support design, production, and communication in professional settings.

To solve this problem, a wide range of mathematical techniques and technologies will be employed. Polynomial functions, logarithmic and sigmoid functions, Bézier curves, Hermite interpolation, and Fourier series will be used to model different sections of the logo. Each method will be clearly defined and applied with appropriate mathematical notation and terminology, ensuring accurate modelling, effective problem-solving, and logical interpretation of results. Through this approach, the task not only highlights the depth and versatility of mathematical modelling but also demonstrates the ability to communicate reasoning effectively, draw logical conclusions, and recognise both the strengths and limitations of the methods used.

Integration is a fundamental mathematical technique used to calculate the accumulation of quantities. In the context of this task, integration allows us to determine the area enclosed between curves by summing infinitely many infinitesimal slices. This provides an exact and reliable method for quantifying regions bounded by complex functions. Thus, integration acts as the bridge between abstract function definitions and practical measures such as the total area of the visual design.

Python with NumPy and Matplotlib will be used to visualise curves, confirm accuracy, and efficiently compute results. Geogebra also helped in the design of specific regions that require manual adjustments.

1. Part A: Working with a Given Function and Designing with Functions

1.1. Overview

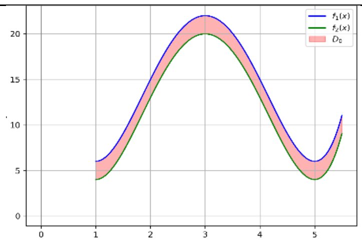
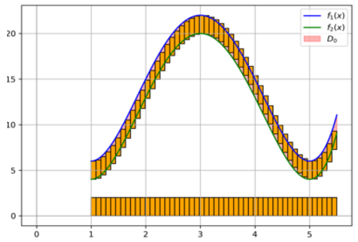
Part A of the task requires a graphic design that is clearly described mathematically. It also asks to integrate two given functions $f_1(x)$ and $f_2(x)$ into the design. A good mathematical expression given in this section will be important in Part B when evaluating the area. To achieve this, a series of methods will be introduced, and based on them, a rather complex but not overly elaborate pattern is expected.

1.2. Calculate the area of the given region

Given a function $f_1(x) = (x-1)^2(x-5)^2 + 6, 1 \le x \le 5.5$. The second function $f_2(x)$ It is created by vertically translating down $f_1(x)$ by 2 units. The region bounded by the two curves is

$$D_0 = \{(x, y) \in \mathbb{R}^2 \mid 1.5 \le x \le 5,\ f_2(x) \le y \le f_1(x)\} \text{ (see Figure 1)}$$

To calculate its area, there are 2 methods.

This area can be evaluated by the formula: $$\int_a^b [f_{upper}(x) - f_{lower}(x)] \quad (1)$$ This is a general formula to calculate the region bounded by a pair of functions in a given domain. To use this formula, the equation of $f_2(x)$ is required. By the definition of translation, $f_2(x) = f_1(x) - 2$.	$A_0 = \int_1^{5.5} [f_1(x) - f_2(x)]dx$ $= \int_1^{5.5} [f_1(x) - (f_1(x) - 2)]dx$ $= \int_1^{5.5} 2\,dx = [2x]_1^{5.5}$ $= 9 \text{ units}^2$	 Figure 1. Region D_0
An easier way: the region between 2 curves is formed by translation. Hence, when evaluating the integral in Eq. 1, every infinitesimal rectangle of the Riemann summation has the same length $f_1(x) - f_2(x) = 2$, so the total area is equal to the area of a 4.5 by 2 rectangle (see the orange rectangles in Figure 2).	$A_0 = A_{\text{yellow}} = (5.5 - 1) \times 2$ $= 9 \text{ units}^2$	 Figure 2. Put all infinitesimal rectangles together

1.3. Purpose of the design and a rough sketch of final design

This is a design of a logo for the loved subject Mathematics Methods. By modelling the design by mathematical expressions, it can be better used when it needs to be resized, moved and rotated in contexts such as printing it on uniforms or putting it on websites. Besides, a clear expression provides a way to evaluate its area analytically by integration.

Figure 3.Rough sketch

Specifically for all regions of the final design, they will be arranged as Figure 3. The blue part is a fancy frame called D_1 transformed by D_0. The red part D_2 is a wing of the bird shown in green called D_3. D_2 is relatively easy, so it will be designed to use some basic functions like logs and exponentials. D_3 is more complex and will be designed using Bezier curves. Purple and grey regions D_4 and D_4', designed based on the image of a handwritten letter M, stand for “Math” and “Methods”. Since their boundaries are expected to have the property of functions, an polynomial interpolation will be used to design it. Finally, the yellow flower vine Region D_5 is most complex. To reflect all the details, a Fourier series will be suitable for translating a image to a region bounded by a enclosed curve. These rationales of methods used to design each region will be further discussed in following sections.

1.4. Design of region D_1

1.4.1. Graph transformation of a function

The task asks to integrate the given functions $f_1(x)$ and $f_2(x)$ into the final design, and it is encouraged to use transformations including reflection, translation, and scaling. Generally, for a function $f(x)$, the transformation above can be done by following operations:

a) Translate right by a: $f(x-a)$; Translate left by a: $f(x+a)$; Translate upward by b: $f(x)+b$; Translate downward by b: $f(x)-b$.

b) Horizontal scaling: $f(kx)$ ($k>1$: compressed, $0<k<1$: stretched). Vertical scaling: $kf(x)$.

c) Reflection across the x-axis: $-f(x)$. Reflection across the y-axis: $f(-x)$

Moreover, these transformations can be combined. For example, $y=-2f(x-3)+4$ means shift right by 3, stretch vertically by a factor of 2, reflect in the x-axis, and shift up by 4.

1.4.2. Non-single-valuedness and affine transformation of a parametric curve

The definition of the function $f: X \to Y$ is that

For every $x \in X$, there exists a unique $y \in Y$ such that $f(x)=y$.

It is also known as the **single-valuedness** of functions. However, the design is expected to be **non-single-valued**. Hence, affine transformations and parametric curves are introduced. A simple definition of an **affine transformation** is to perform a linear transformation on a vector space, and additionally, a translation. Generally, an affine transformation $T(\mathbf{x})$ on vector $\mathbf{x}$ in $\mathbb{R}^2$ can be written as

$$T(\mathbf{x}) = A\mathbf{x} + \mathbf{b} \tag{2}$$

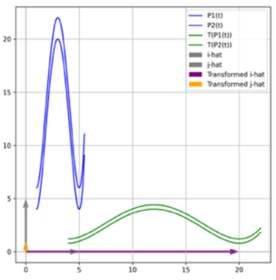
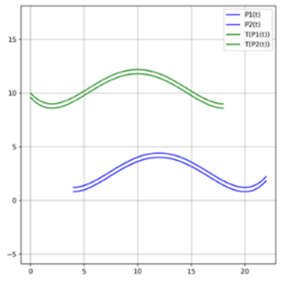
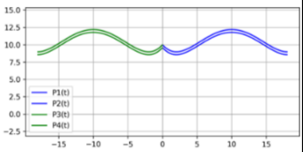
where A is a 2 by 2 matrix representing a linear transformation, and $\mathbf{b}$ represents a translation. The transformation of a function is more complicated. For this specific purpose, since what is transformed is the whole shape of a function $f(x)$, which means the actual subject of the transformation is every point $(x, f(x))$ on the plane, namely

$$\begin{pmatrix} x_T \\ y_T \end{pmatrix} = T(x, f(x)) = A\begin{pmatrix} x \\ f(x) \end{pmatrix} + \mathbf{b} \tag{3}$$

This equation naturally satisfies the restriction on the **domain.** But note that after the transformation, the relation between x_T and y_T will probably no longer obey the single-valuedness. To better describe this kind of curve, **parametric equations** are introduced. According to Eq. 2, an affine transformation on a parametric equation $\mathbf{P}(t)$ is

$$T(\mathbf{P}(t)) = A\mathbf{P}(t) + \mathbf{b} \tag{4}$$

Parameterise $f_1(x)$ is simply letting $t=x$, so $\mathbf{P_1}(t) = \begin{pmatrix} x \\ f_1(x) \end{pmatrix} = \begin{pmatrix} t \\ f_1(t) \end{pmatrix}, 1 \le t \le 5.5$. And $\mathbf{P_2}(t)$, the parametric form of $f_2(x)$, is given by transformation $\mathbf{P_2}(t) = T(\mathbf{P_1}(t)) = \mathbf{P_1}(t) + \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} t \\ f_1(t)+2 \end{pmatrix}$. Now all kinds of transformation can be applied to. To adjust the given function to be a fancy frame of the design, the following transformations are applied:

The region D_0 looks too long in the vertical direction. To compress it vertically to 1/5 of the original size and stretch horizontally to four times the original size requires the matrix $A = \begin{pmatrix} 4 & 0 \\ 0 & \frac{1}{5} \end{pmatrix}$ Since there is no translation, b is the zero vector.	 Figure 4. Transformation 1	$T(\mathbf{P_1}(t)) = A\mathbf{P_1}(t) + \mathbf{b}$ $= \begin{pmatrix} 4 & 0 \\ 0 & \frac{1}{5} \end{pmatrix}\begin{pmatrix} t \\ f_1(t) \end{pmatrix}$ $= \begin{pmatrix} 4t \\ \frac{1}{5}f_1(t) \end{pmatrix}$	$T(\mathbf{P_2}(t)) = A\mathbf{P_2}(t) + \mathbf{b}$ $= \begin{pmatrix} 4t \\ \frac{1}{5}f_2(t) \end{pmatrix}$
Then the wiggling end of the curves is expected to be centered and aligned with the y-axis. Hence a flip transformation and a translation are needed. The flip matrix is $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. The translating vector is $\mathbf{b} = \begin{pmatrix} 22 \\ \frac{623}{80} \end{pmatrix}$	 Figure 5. Transformation 2	$T(\mathbf{P_1}(t))$ $= A\mathbf{P_1}(t) + \mathbf{b}$ $= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4t \\ \frac{1}{5}f_1(t) \end{pmatrix} + \begin{pmatrix} 22 \\ -\frac{623}{80} \end{pmatrix}$ $= \begin{pmatrix} -4t+22 \\ \frac{1}{5}f_1(t) + \frac{623}{80} \end{pmatrix}$	$T(\mathbf{P_1}(t)) = A\mathbf{P_2}(t) + \mathbf{b}$ $= \begin{pmatrix} -4t+22 \\ \frac{1}{5}f_2(t) + \frac{623}{80} \end{pmatrix}$
Copy a pair of curve $\mathbf{P_3}$ and $\mathbf{P_4}$ to make a symmetrical shape, where the same flip matrix is needed.	 Figure 6. Transformation 3	$\mathbf{P_3}(t) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{P_1}(t)$ $= \begin{pmatrix} 4t-22 \\ \frac{1}{5}f_1(t) + \frac{623}{80} \end{pmatrix}$	$\mathbf{P_4}(t) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\mathbf{P_2}(t)$ $= \begin{pmatrix} 4t-22 \\ \frac{1}{5}f_2(t) + \frac{623}{80} \end{pmatrix}$

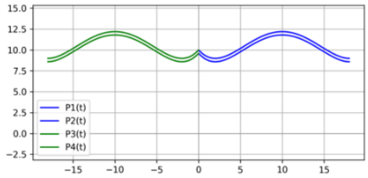
An advantage of affine transformations is that the amount of the scaling of the area can be easily calculated by the **determinant** of the transformation matrix. This will be further discussed in Part B.

1.4.3. Non-linear transformation and Jacobian

The shape of a frame is expected to be more "enclosed", by curling it into a circular shape. This requires a **non-linear transformation**. A non-linear transformation does not guarantee that the parallel lines in the original graph remain parallel, and hence do not have a constant matrix to describe the transformation for every point on the Cartesian plane. A general way is to use a **map**. For this purpose, the map is:

$$T_{curl}(x,y): (x,y) \to \left(y\cos\left(-\frac{x\pi}{18}+\frac{\pi}{2}\right), y\sin\left(-\frac{x\pi}{18}+\frac{\pi}{2}\right)\right) \tag{5}$$

since the new graph's x-coordinates have a range from -18 to 18, and the curl starts at $\frac{\pi}{2}$ in the polar frame. What this **curl** transformation actually does is to wrap the Cartesian grid into a polar shape (see Figure 7). For convenience, the transformed curves are named $\mathbf{P_1} \sim \mathbf{P_4}$.



↓ curl

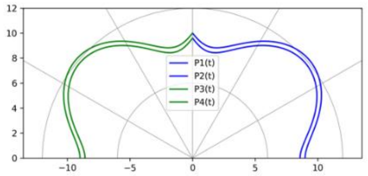


Figure 7. Curl transformation

$$\mathbf{P_1}(t) = T_{curl}\left(\begin{pmatrix} -4t+22 \\ \frac{1}{5}f_1(t)+\frac{623}{80}\end{pmatrix}\right) = \begin{pmatrix} \left(\frac{1}{5}f_1(t)+\frac{623}{80}\right)\cos\left(-\frac{\pi(11-2t)}{9}+\frac{\pi}{2}\right) \\ \left(\frac{1}{5}f_1(t)+\frac{623}{80}\right)\sin\left(-\frac{\pi(11-2t)}{9}+\frac{\pi}{2}\right)\end{pmatrix}$$

$$\mathbf{P_2}(t) = T_{curl}\left(\begin{pmatrix} -4t+22 \\ \frac{1}{5}f_2(t)+\frac{623}{80}\end{pmatrix}\right) = \begin{pmatrix} \left(\frac{1}{5}f_2(t)+\frac{623}{80}\right)\cos\left(-\frac{\pi(11-2t)}{9}+\frac{\pi}{2}\right) \\ \left(\frac{1}{5}f_2(t)+\frac{623}{80}\right)\sin\left(-\frac{\pi(11-2t)}{9}+\frac{\pi}{2}\right)\end{pmatrix}$$

$$\mathbf{P_3}(t) = T_{curl}\left(\begin{pmatrix} 4t-22 \\ \frac{1}{5}f_1(t)+\frac{623}{80}\end{pmatrix}\right) = \begin{pmatrix} \left(\frac{1}{5}f_1(t)+\frac{623}{80}\right)\cos\left(-\frac{\pi(2t-11)}{9}+\frac{\pi}{2}\right) \\ \left(\frac{1}{5}f_1(t)+\frac{623}{80}\right)\sin\left(-\frac{\pi(2t-11)}{9}+\frac{\pi}{2}\right)\end{pmatrix}$$

$$\mathbf{P_4}(t) = T_{curl}\left(\begin{pmatrix} 4t-22 \\ \frac{1}{5}f_2(t)+\frac{623}{80}\end{pmatrix}\right) = \begin{pmatrix} \left(\frac{1}{5}f_2(t)+\frac{623}{80}\right)\cos\left(-\frac{\pi(2t-11)}{9}+\frac{\pi}{2}\right) \\ \left(\frac{1}{5}f_2(t)+\frac{623}{80}\right)\sin\left(\frac{\pi(2t-11)}{9}-\frac{\pi}{2}\right)\end{pmatrix}$$

Though this transform do not have a matrix to describe it, it do behave locally as a linear transformation. This sort of linear behaviour can be measured by the Jacobian. For transformation T_{curl}, the Jacobian is

$$J = \frac{\partial(X,Y)}{\partial(x,y)} = \begin{bmatrix} \frac{\partial}{\partial x} y\cos\left(-\frac{x\pi}{18}+\frac{\pi}{2}\right) & \frac{\partial}{\partial y} y\cos\left(-\frac{x\pi}{18}+\frac{\pi}{2}\right) \\ \frac{\partial}{\partial x} y\sin\left(-\frac{x\pi}{18}+\frac{\pi}{2}\right) & \frac{\partial}{\partial y} y\sin\left(-\frac{x\pi}{18}+\frac{\pi}{2}\right)\end{bmatrix} = \begin{bmatrix} \frac{\pi y}{18}\sin\left(-\frac{x\pi}{18}+\frac{\pi}{2}\right) & \cos\left(-\frac{x\pi}{18}+\frac{\pi}{2}\right) \\ -\frac{\pi y}{18}\cos\left(-\frac{x\pi}{18}+\frac{\pi}{2}\right) & \sin\left(-\frac{x\pi}{18}+\frac{\pi}{2}\right)\end{bmatrix} \tag{6}$$

The method to evaluate the transformed area based on the Jacobian will be further discussed in Part B.

1.4.4. Smoothness Consideration

Generally, the smoothness can be measured by continuity C^n, namely the n[th] derivatives at the two ends are equal. However, when dealing with a parametric curve, this approach will probably fail since the rate of change in position of a point with respect to t is not only related to the smoothness of the shape but also the smoothness of the motion (change in velocity). Thereby, the geometric continuity is introduced. A curve or surface can be described as having G^n continuity, with n being the increasing measure of smoothness. Consider there are two curves, and their continuity is defined as: (1) G^0, the curves touch at the join point; (2) G^1, the curves also share a common tangent direction at the join point, which means the derivatives are collinear; (3) G^2, the curves also share a common centre of curvature at the join point (Wikipedia, 2023). For graph designs, G^1 is sufficiently smooth. Also, an affine transformation will not affect the original continuity

To construct continuous joins, copy one set of curves shown in Figure 7 and rotate them 180 degrees to get curves $\mathbf{P_5} \sim \mathbf{P_8}$, therefore,

$$\mathbf{P_{n+4}}(t) = \begin{pmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{pmatrix}\mathbf{P_n}(t) = \begin{pmatrix} -1 & 0 \\ 0 & -1\end{pmatrix}\mathbf{P_n}(t), n = 1,2,3,4 \text{ (for full equations, see Appendix 1).}$$

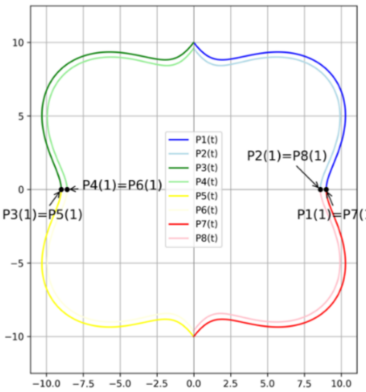


Figure 8. Region D_I

Curves $\mathbf{P_1} \sim \mathbf{P_8}$ give us Region D_I as shown in Fig. 2, where there are four joins satisfying C^0 and C^1 continuity: $\mathbf{P_1}(1) = \mathbf{P_7}(1)$, $\mathbf{P_2}(1) = \mathbf{P_8}(1)$, $\mathbf{P_3}(1) = \mathbf{P_5}(1)$, and $\mathbf{P_4}(1) = \mathbf{P_6}(1)$. Take $\mathbf{P_1}(1) = \mathbf{P_7}(1)$ as an example.

$$\frac{d}{dt}\mathbf{P_1}(t) = \begin{pmatrix} \frac{1}{5}f_1'(t)\cos\left(-\frac{\pi(11-2t)}{9}+\frac{\pi}{2}\right) - \left(\frac{1}{5}f_1(t)+\frac{623}{80}\right)\left(-\frac{2\pi}{9}\right)\sin\left(-\frac{\pi(11-2t)}{9}+\frac{\pi}{2}\right) \\ \frac{1}{5}f_1'(t)\sin\left(-\frac{\pi(11-2t)}{9}+\frac{\pi}{2}\right) + \left(\frac{1}{5}f_1(t)+\frac{623}{80}\right)\left(-\frac{2\pi}{9}\right)\cos\left(-\frac{\pi(11-2t)}{9}+\frac{\pi}{2}\right)\end{pmatrix}$$

$$\frac{d}{dt}\mathbf{P_7}(t) = \begin{pmatrix} \frac{1}{5}f_1'(t)\cos\left(-\frac{\pi(2t-11)}{9}+\frac{\pi}{2}\right) - \left(\frac{1}{5}f_1(t)+\frac{623}{80}\right)\left(\frac{2\pi}{9}\right)\sin\left(-\frac{\pi(2t-11)}{9}+\frac{\pi}{2}\right) \\ -\frac{1}{5}f_1'(t)\sin\left(-\frac{\pi(2t-11)}{9}+\frac{\pi}{2}\right) - \left(\frac{1}{5}f_1(t)+\frac{623}{80}\right)\left(\frac{2\pi}{9}\right)\cos\left(-\frac{\pi(2t-11)}{9}+\frac{\pi}{2}\right)\end{pmatrix}$$

$$\frac{d}{dt}\mathbf{P_1}(1) = \begin{pmatrix} \left(\frac{1}{5}f_1(t)+\frac{623}{80}\right)\left(\frac{2\pi}{9}\right) \\ \frac{1}{5}f_1'(t)\end{pmatrix} = -\frac{d}{dt}\mathbf{P_7}(1) \tag{7}$$

Since the direction of the motion is reversible in a parametric equation, Eq. 7 actually implies there is a C^1 continuity. By applying similar approaches to the other three joins, it can be found that all joins are C^1 continuous. Now, these curves can be denoted as

$$\Gamma_1^{\text{out}} = \bigcup_{i=1}^{4}\mathbf{P_{2i-1}}\,([1.5,5]), \qquad \Gamma_1^{\text{in}} = \bigcup_{i=1}^{4}\mathbf{P_{2i}}\,([1.5,5])$$

And let Region $D_1 \subset \mathbb{R}^2$ be the region bounded by the piecewise smooth closed curve $\Gamma_1^{\text{out},}$ but not by Γ_1^{in}, namely

$$D_1 = \text{int}\,(\Gamma_1^{\text{out}}) \setminus \text{int}\,(\Gamma_1^{\text{in}})$$

1.5. Design of region D_2

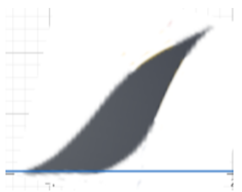


Figure 9. $g_1(x)$

This part of the design is the wing of a bird, which would be formed by some basic functions. Its shape is roughly like Figure 10, constituted by functions $g_1 \sim g_4$.

$g_1(x)$ is simply a straight line. For convenience, it is put horizontally and assigned to be

$$g_1(x) = 0.05$$

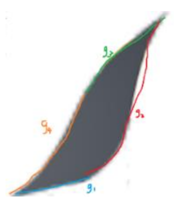


Figure 10. D_2 Rough shape

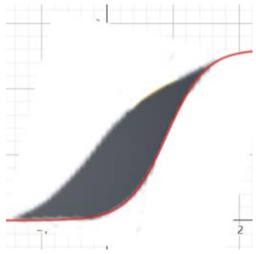


Figure 11. $g_2(x)$

$g_2(x)$ is winding and has an "S" shape. A very suitable function is Sigmoid, which is defined as

$$\sigma(x) = \frac{1}{1+e^{-x}}$$

To fit this curve to the shape, use the graph transformation in 1.4.1, as it is easier to apply to functions:

$$g_2(x) = \frac{a}{1+e^{-b(x-c)}}$$

Where a, b, and c control the dilations in y and x directions and horizontal translation. By adjusting them, it is found that the curve in Figure 11 is given when a, b, and c are 2.6, 3.4, and 0.9.

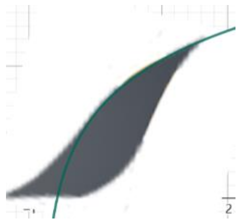


Figure 12. $g_3(x)$

$g_3(x)$ is the upper half of the left side of the wing. It is like a log function and can be modelled as

$$g_3(x) = d\ln k(x-l) + m$$

Where d, k, l, and m control the dilations in y and x directions and translation in y and x directions. The curve in Figure 11 is given when d, k, l, and m are 1, 0.1, -0.8, and 3.8, respectively. It is expected that the g_4 and g_3 connect at $\mathrm{C}\big(0, g_3(0)\big)$ with C^1 continuity. The derivative of g_3 at $x=0$ is

$$g_3'(x) = (\ln(x+0.8) + \ln 0.1 + 3.8)' = \frac{1}{x+0.8}, g_3'(0) = \frac{1}{0.8} = \frac{5}{4}$$

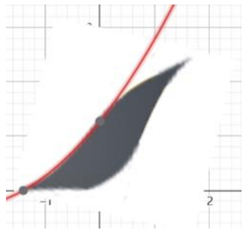


Figure 13. $g_4(x)$

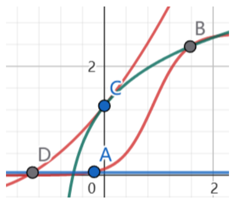


Figure 14. D_2 no domain

$g_4(x)$ is modelled by a quadratic function:

$$g_4(x) = p(x-q)^2 + s$$

Whose derivative is $g_4'(0) = 2p(-q) = -2pq$. To make g_4 and g_3 connect at $\mathrm{C}\big(0, g_3(0)\big)$ with C^1 continuity, the following 2 equations need to be satisfied.

$$\begin{cases} g_4'(0) = g_3'(0) \\ g_4(0) = g_3(0) \end{cases} \Rightarrow \begin{cases} -2pq = \dfrac{5}{4} \\ pq^2 + s = \ln 0.08 + 3.8 \end{cases}$$

Note that the equations are now hard to solve. To simplify it, let $u = pq, v = pq^2$. Thus, $q = \frac{v}{u}, p = \frac{u^2}{v}$.

$$\begin{cases} -2u = \dfrac{5}{4} \\ v + s = \ln 0.08 + 3.8 \end{cases} \Rightarrow \begin{cases} u = -\dfrac{5}{8} \\ v + s = \ln 0.08 + 3.8 \end{cases}$$

3 variables and 2 equations leave 1 degree of freedom to adjust. When set $v = 1.6$, Figure 13 is given, and $s = \ln 0.08 + 3.8 - v = \ln 0.08 + 2.2, p = \frac{u^2}{v} = \frac{125}{512}, q = -2.56$.

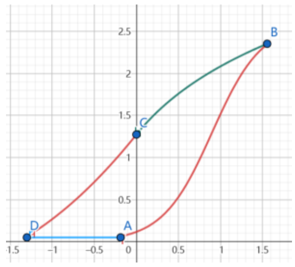


Figure 15. D_2

To get the joints A, B, D, to get the **domains**, it is necessary to solve the following equations:

$$\frac{2.6}{1+e^{-3.4\,(x_A-0.9)}} = 0.05$$

$$\ln 0.1(x_B + 0.8) + 3.8 = \frac{2.6}{1+e^{-3.4\,(x_B-0.9)}} \text{ (the greater solution)}$$

$$\frac{125}{512}(x_D - 2.56)^2 + \ln 0.08 + 2.2 = 0.05 \text{ (greater solution)}$$

The solution is $x_A = -0.188, x_B = 1.576, x_D = -1.319$. Hence, the domains are $g_1\!: x \in [-1.319, 0.188]$, $g_2\!: x \in [-0.188, 1.576]$, $g_3\!: x \in [0, 1.576]$, and $g_4\!: x \in [-1.319, 0]$.

Let the upper and the lower functions be respectively

$$g_{upper}(x) = \begin{cases} \frac{125}{512}(x+2.56)^2 + \ln 0.08 + 2.2, & -1.319 \le x \le 0 \\ \ln 0.1(x+0.8) + 3.8, & 0 < x \le 1.576 \end{cases}, \quad g_{lower}(x) = \begin{cases} 0.05, & -1.319 \le x \le -0.188 \\ \frac{2.6}{1+e^{-3.4(x-0.9)}}, & -0.188 < x \le 1.576 \end{cases}$$

And Region $D_2 = \big\{(x,y) \in \mathbb{R}^2 \,\big|\, -1.319 \le x \le 1.576,\ g_{lower}(x) \le y \le g_{upper}(x)\big\}$ (see Figure 15).

1.6. Design of region D_3

1.6.1. Introduction to Bézier Curves

Bézier Curves are a type of interpolation method intended to approximate a real-world shape that otherwise has no mathematical representation. The general equation of a Bezier curve of order n is

$$\mathbf{Bezier_n}(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} \mathbf{C_{i+1}}, t \in [0,1] \tag{8}$$

Where $\boldsymbol{C_i}$ are the control points**.** The curves are expected to be rather wavy when designing the pattern using Bezier curves. It can be achieved by using a high-order Bezier curve, but there are problems. First, when there are many control points, it is hard to imagine the shape of the curve. Secondly, one cannot modify only a certain part of the curve, because if one control point is moved, the whole curve will change. In addition, applying a higher-order curve can cause a problem of oscillation at the edges of an interval. It is called **Runge's phenomenon**. A viable solution is a composite Bezier curve, namely, composing a curve of multiple shorter Bezier curves. To do that, the smoothness of their joints must be guaranteed. The derivative of Eq. 8 is

$$\mathbf{Bezier_n'}(t) = n \sum_{i=0}^{n-1} \binom{n-1}{i} t^i (1-t)^{n-i-1} (\mathbf{C_{i+2}} - \mathbf{C_{i+1}}) \tag{9}$$

(for the full derivation, see Appendix 2) Put $t = 0$ and $t = 1$, the conclusion can be obtained that: the tangent vector at the start point is $\overrightarrow{C_1C_2}$ and at the terminal point is $\overrightarrow{C_nC_{n+1}}$. Thus, to ensure that the two curves have G^1 continuity at the junction is to put the four control points near the junction collinearly, namely

$$\overrightarrow{C_1C_2} = k\overrightarrow{C_nC_{n+1}} \Rightarrow G^1 \text{ continuity} \tag{10}$$

1.6.2. Design based on Bézier Curves

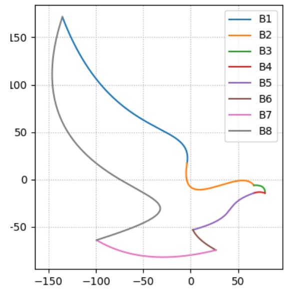


Figure 16. D3

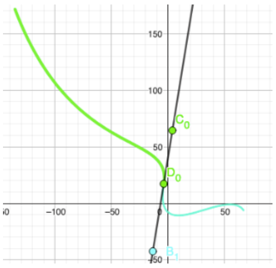


Figure 17. Continuous joint between B1 and B2

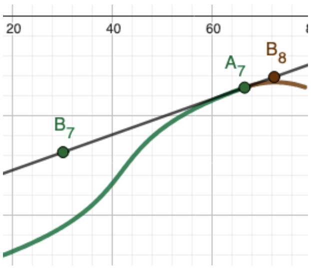


Figure 18. Continuous joint between B6 and B7

Region D_3 is enclosed by 8 Bezier curves $\mathbf{B_1}$~$\mathbf{B_8}$ with control points $C_n^{(1)}$~$C_n^{(8)}$. The curves are connected end to end and form an enclosed shape (see Figure 16). Note that Bezier curves naturally have a domain of $t \in [0,1]$. As shown in Figure 17 & Figure 18, there are 2 pairs of curves that have continuous joints, which are taken as samples to show calculations (for full equations see Appendix 3).

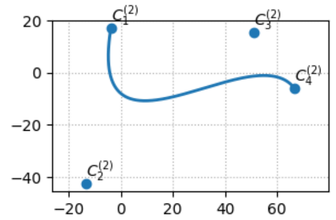


Figure 19. B2

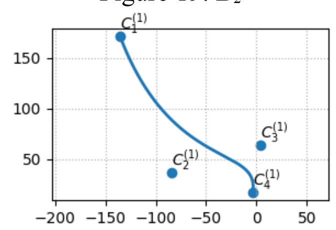


Figure 20. B1

$\mathbf{B_2}(t)$ (see Figure 19) has control points

$$C_1^{(2)} = (-3.74, 17.19), C_2^{(2)} = (-13.4, -42.48), C_3^{(2)} = (51.12, 15.53), C_4^{(2)} = (66.82, -6.08)$$

Substitute into Eq. 8, there is

$$\begin{aligned} \mathrm{B}_2(t) &= C_1^{(2)}(1-t)^3 + 3C_2^{(2)}t(1-t)^2 + 3C_3^{(2)}t^2(1-t) + C_4^{(2)}t^3 \\ &= (-123.0t^3 + 222.54t^2 - 28.98t - 3.74,\ -197.3t^3 + 353.04t^2 - 179.01t + 17.19), t \in [0,1] \end{aligned}$$

To ensure G^1 continuity, it must be

$$C_1^{(2)} = C_4^{(1)}, \overrightarrow{C_3^{(1)}C_4^{(1)}} = k\overrightarrow{C_1^{(2)}C_2^{(2)}} = k\begin{pmatrix} -9.66 \\ -59.67 \end{pmatrix}$$

That is to say,

$$C_3^{(1)} = C_1^{(2)} - k(-9.66, -59.67) = (-3.74 + 9.66k, 17.19 + 59.67k)$$

Under this constraint, the control points of B_1 (see Figure 20) are designed to be

$$C_1^{(1)} = (-135.39, 171.46), C_2^{(1)} = (-83.86, 36.14), C_3^{(1)} = (3.90, 64.40), C_4^{(1)} = (-3.74, 17.19)$$

where the k is set to be 3.90. Hence the equation is

$$\begin{aligned} \mathbf{B_1}(t) &= C_1^{(1)}(1-t)^3 + 3C_2^{(1)}t(1-t)^2 + 3C_3^{(1)}t^2(1-t) + C_4^{(1)}t^3 \\ &= (-131.66t^3 + 108.72t^2 + 154.59t - 135.39,\ -239.05t^3 + 490.74t^2 - 405.96t + 171.46) \end{aligned}$$

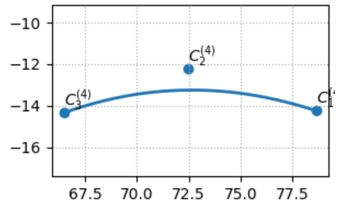


Figure 21. B4

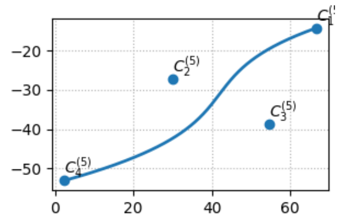


Figure 22. B5

$\mathbf{B_4}(t)$ (see Figure 21) has control points

$$C_1^{(4)} = (78.62, -14.24), C_2^{(4)} = (72.46, -12.21), C_3^{(4)} = (66.52, -14.33)$$

Note there is only one turning point is required for this curve. So it only needs 3 control points, namely it is a quadratic Bezier curve. Substitute into Eq. 8, there is

$$\begin{aligned} B_2(t) &= C_1^{(4)}(1-t)^2 + 2C_2^{(4)}t(1-t) + C_3^{(4)}t^2 \\ &= (0.22t^2 - 12.32t + 78.62,\ -4.15t^2 + 4.06t - 14.24), t \in [0,1] \end{aligned}$$

To ensure G^1 continuity, it must be

$$C_1^{(5)} = C_4^{(4)}, \overrightarrow{C_1^{(5)}C_2^{(5)}} = k\,\overrightarrow{C_2^{(4)}C_3^{(4)}} = k\begin{pmatrix} -5.49 \\ -2.12 \end{pmatrix}$$

That is to say,

$$C_2^{(5)} = k(-5.49, -2.12) + C_3^{(4)} = (66.52 - 5.49k, -14.33 - 2.12k)$$

Under this constraint, B_5 (see Figure 22) are designed to be

$$C_1^{(5)} = (66.52, -14.33), C_2^{(5)} = (30.15, -27.30), C_3^{(5)} = (54.53, -38.65), C_4^{(5)} = (2.45, -53.06)$$

$$\begin{aligned} \mathbf{B_5}(t) &= C_1^{(5)}(1-t)^3 + 3C_2^{(5)}t(1-t)^2 + 3C_3^{(5)}t^2(1-t) + C_4^{(5)}t^3 \\ &= (-137.21t^3 + 182.25t^2 - 109.11t + 66.52,\ -4.68t^3 + 4.86t^2 - 38.91t - 14.33), t \in [0,1] \end{aligned}$$

All these 8 curves and the Region D_3 can be denoted as

$$\Gamma_3 = \bigcup_{i=1}^{8} \mathbf{B_i}([0,1]), \qquad D_3 = \mathrm{int}(\Gamma_3)$$

1.7. Design of region D_4

1.7.1. Curve fitting and interpolation

Note that what is done in the design of D_2 and D_3 is pretty much fitting a shape with some curve. In numerical analysis, it is called interpolation, which means the process of predicting the continuous value between discrete data points with some function or curve. Bezier curves are a classical method of interpolation, which is essentially a parameterised polynomial. Hence, why not just use a polynomial to fit some given pattern and get a design?


Figure 23. Handwritten letter M

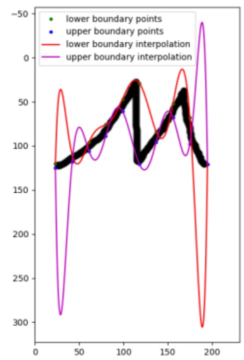


Figure 24. Runge phenomenon of 10-point Lagrange interpolation

Given a handwritten letter M shown in Figure 23, a polynomial is used for interpolation for 20 sample points of upper and lower boundaries. Take the upper boundary as an example. Take 10 sample points $(x_0, y_0), (x_1, y_1), \dots, (x_9, y_9)$, The Lagrange interpolating polynomial through them is

$$L(x) = \sum_{j=0}^{9} y_j l_j(x) \tag{11}$$

where $l_j(x)$ is called the Lagrange basis for this linear combination. Each $l_j(x_i) = 0$ if $m \neq j$ and $l_j(x_j) = 1$, which guarantees that the polynomial must go through all points. This interpolation yields the curve in Figure 25. It is not well-fitted and has severe oscillations at the edges of the domain, which is the Runge phenomenon investigated in the context of Bezier curves. The solution is the same – consider composing multiple polynomials together.

1.7.2. Piecewise interpolation polynomial and Hermite basis functions

To construct a polynomial $p(x)$ between each interval $[x_i, x_{i+1}]$ and ensure that adjacent ones connect with C^1 continuity, it requires:

$$p(x_i) = y_i, \qquad p'(x_i) = d_i, \qquad p(x_{i+1}) = y_{i+1}, \qquad p'(x_{i+1}) = d_{i+1}$$

where d_i is a parameter we assign to the slope shared by the ends of the two adjacent polynomials. Notice that there are 4 constraints, so consider using cubic polynomials with 4 free parameters. For the convenience of the derivation, use a factor $h = x_{i+1} - x_i$ to scale the interval to $[0,1]$ by letting $t = \frac{x - x_i}{h}$. Hence, the intended $p(x)$ satisfies

$$p(0) = y_i, \qquad p(1) = y_{i+1}, \qquad p'(0) = hd_i, \qquad p'(1) = hd_{i+1}$$

If assuming $p(x) = at^3 + bt^2 + ct + d$, there are

$$\begin{cases} d = y_i \\ c = hd_i \\ a + b + c + d = y_{i+1} \\ 3a + 2b + c = hm_{i+1} \end{cases} \Rightarrow \begin{cases} a = 2y_i - 2y_{i+1} + h(d_i + d_{i+1}) \\ b = -3y_i + 3y_{i+1} - h(2d_i + d_{i+1}) \\ c = hm_i \\ d = y_i \end{cases} \tag{12}$$

For convenience, write $p(x)$ as a linear combination of $y_i, y_{i+1}, hd_i, hd_{i+1}$, such that

$$p(x) = h_{00}(t)y_i + h_{10}(t)hd_i + h_{01}(t)y_{i+1} + h_{11}(t)hd_{i+1} \tag{13}$$

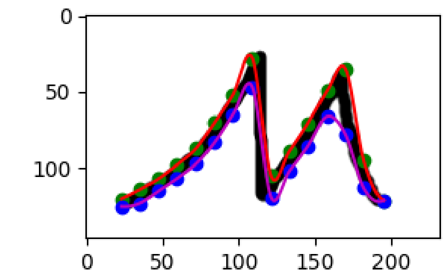
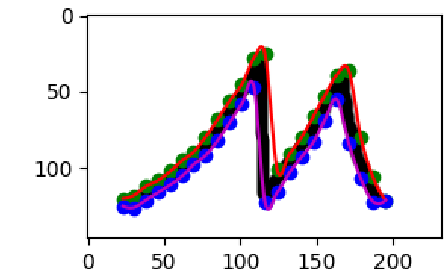
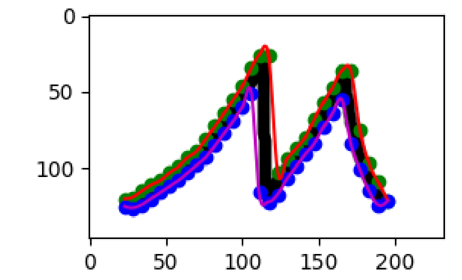
with the result in Eq. 12, it can be found that the bases (called Hermite basis functions) are

$$h_{00}(t) = 2t^3 - 3t^2 + 1, \qquad h_{10}(t) = t^3 - 2t^2 + t, \qquad h_{01}(t) = -2t^3 + 3t^2, \qquad h_{11}(t) = t^3 - t^2$$

1.7.3. Design with the Piecewise Cubic Hermite Interpolating Polynomial (PCHIP)

There are 2 parameters to adjust: the number of sample points and joint slope d_i. For the sample, OpenCV-Python is used to extract sample pointa from the boundaries. Different numbers of sample points can influence how many details of the original graph can be caught by the PCHIP. After trials, it is found that 23 is a suitable value (see Table 1)

Table 1. Comparison between different numbers of sample points

15 sample points	23 sample points	30 sample points
		

Here d_i is set to be $\left(\frac{y_i - y_{i-1}}{x_i - x_{i-1}} + \frac{y_{i+1} - y_i}{x_{i+1} - x_i}\right)/2$, which is the average of the secant slopes of the adjacent points with the edge conditions $d_0 = d_{22} = 0$. But in figures in Table 1, some unnatural wiggles are observed at sharp turns. It is because when a sample point is a stationary point, the corresponding d might not be 0, making the PCHIP faking a new stationary point (see Figure 26). The solution is, when dealing with a local maximum or minimum, namely

If $y_i - y_{i-1}$ and $y_{i+1} - y_i$ have different signs, $d_i = 0$.

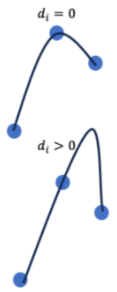


Figure 26. Fake peak

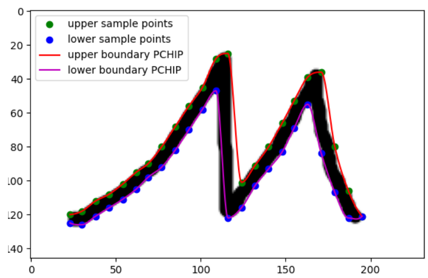


Figure 27. D_4

Which yields the curves in Figure 27, which contains two sets of 22 PCHIPs p_i^{up}, p_i^{low}, namely,

$$h_{upper}(x) = \begin{cases} p_1^{up}(x), & x_0 \le x \le x_1 \\ \vdots & \\ p_{22}^{up}(x), & x_{21} \le x \le x_{22} \end{cases}, \qquad h_{lower}(x) = \begin{cases} p_1^{low}(x), & x_0 \le x \le x_1 \\ \vdots & \\ p_{22}^{low}(x), & x_{21} \le x \le x_{22} \end{cases}$$

And Region $D_4 = \{(x, y) \in \mathbb{R}^2 \mid x_0 \le x \le x_{22},\ h_{lower}(x) \le y \le h_{upper}(x)\}$. For expanded equations and Python code see Apendix 4 and Code Appendix.

1.8. Design of region D_5

1.8.1. Generate an interpolation curve from the given points based on the Discrete Fourier Transform

The PCHIP can only be applied to points that have a function relation between ys and xs, namely, for every y, there is only one corresponding x. That is to say, it cannot be applied to parametric curves. That is why the **Fourier Transform** is introduced. Consider an arbitrary closed (end-to-end) parametric curve $\mathbf{r}(t) = (x(t), y(t)), t \in [0,1]$. Denote it as a complex function

$$z(t) = x(t) + iy(t)$$

Note that this complex function can be expressed as a sum of infinite simple orbiting complex numbers $ce^{i2\pi nt}$ with different frequencies n and complex coefficients $c \in \mathbb{C}$ to describe the phases and amplitude. Hence, there is

$$z(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi int} \tag{14}$$

This series, known as the **Fourier series**, was proposed by Joseph Fourier in 1807. The reason why the frequency is $n \in \mathbb{Z}$ is that for any closed curve, $z(0) = z(1) \Rightarrow e^{2\pi n} = 1$ is true if and only if $n \in \mathbb{Z}$; you can see this as a sort of period ($T = 1,\ z(t) = z(t + T)$). The **Fourier Transform** is used to transform a function from the original domain to the frequency domain, namely to get corresponding c_ns for different frequencies ns, which is:

$$c_n = \int_0^1 z(t)e^{-2\pi int}\,dt \tag{15}$$

Proof: According to Eqs. 17 and 15,

$$c_n = \int_0^1 e^{-2\pi int} \sum_{k=-\infty}^{\infty} c_k e^{2\pi ikt}\,dt = \sum_{k=-\infty}^{\infty} \int_0^1 c_k e^{2\pi it(k-n)}\,dt \tag{16}$$

According to **Euler's formula** $e^{i\theta} = \cos\theta + i\sin\theta$, the complex number $e^{2\pi ikt}, k \in \mathbb{Z}, t \in [0,1]$ really describes a "revolution". Therefore, if integrate it through t, it will give zero as:

$$\int_0^1 e^{2\pi ikt}\,dt = \left[-\frac{1}{2\pi k}e^{2\pi ikt}\right]_0^1 = \frac{1}{2\pi k}(1 - e^{2\pi ik}) = \frac{1}{2\pi k}(1 - \cos 2k\pi) = 0 \tag{17}$$

Hence, all terms in Eq. 15 vanish except $\int_0^1 c_n e^{2\pi it(0)}\,dt = c_n$. In the real-world problem, it is unlikely to get a continuous curve, and most of the time, we have to deal with discrete sample points. Suppose there are N discrete points on a curve in complex form: $z_0, z_1, \ldots, z_{N-1}$. The **Discrete Fourier Transform** and the Fourier series fetched are:

$$c_n = \sum_{k=0}^{N-1} z_k e^{-\frac{2\pi ink}{N}}, \qquad z(t) = \sum_{n=-M}^{M} c_n e^{2\pi int} \tag{18; 19}$$

A higher M can give a more accurate fitting of the original curve, and a relatively small M can filter the high-frequency details and give a gentler (no steep wiggle) shape.

1.8.2. Design with the Fourier series

Given an original graph like Figure 28, first take sample points with OpenCV-Python on the contour of it. Note that the contour must be enclosed for the periodic property of the Fourier series. Since this is a rather complex shape, to keep more details, 2500 sample points are taken. Next, the only parameter that needs to be adjusted is the number of terms retained from the series (the value of M). Different Ms give different curves (see Figure 29).

Figure 28. D_5 original graph

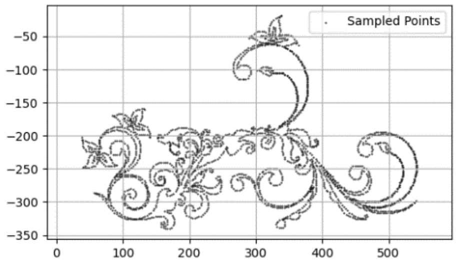
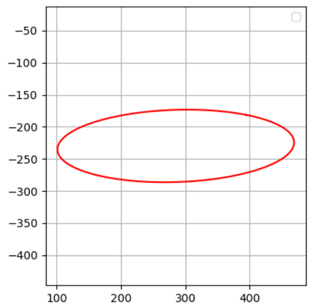
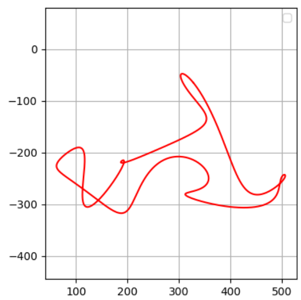
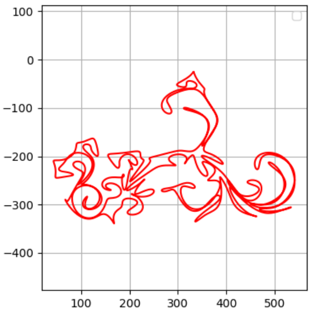
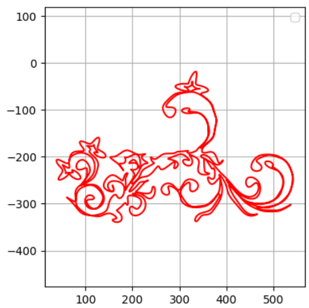
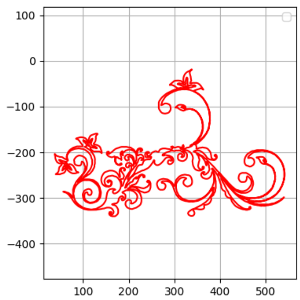
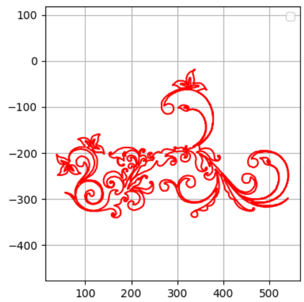


Figure 29. Sampled graph

Table 2. Different *values of* M *and corresponding* curves

M=2	M=10	M=100
		
M=200	**M=1000**	**M=2500**
		

The process to get the Discrete Fourier Transform from 2500 sample points requires massive calculation. Therefore, a Python program is used (see Code Appendix). Notice that when M=2500, the curve start to have sawteeth from the pixels of the original image, which is not expected. Hence, M=1000 is considered more suitable. The features of the graphs with different numbers of terms of the Fourier series kept can be better observe from this video (https://github.com/Yanttttt/SACE_Math_Method_Investigation/raw/refs/heads/main/D5/epicycles.mp4). Now, an enclosed curve is obtained. It only consists of one curve. Thus, it has C^∞ continuity everywhere. Translate it from complex form to vector form:

$$\mathbf{r}(t) = (Re[z(t)], Im[z(t)]) = \left(\sum_{n=-1000}^{1000} c_n \cos 2\pi nt\,,\ \sum_{n=-1000}^{1000} c_n \sin 2\pi nt\right), \qquad t \in [0,1]$$

Let $\Gamma_5 = \mathbf{r}([0,1])$, the Region $D_5 = \operatorname{int}(\Gamma_5)$ (see the figure where M=1000 in Table 2).

1.9. Assemble and arrange parts

Now all parts are settled and they are going to be assembled. However, at this stage, are they not at the same scale, specifically, some are too big while some are too small (see Figure 30). Moreover, some final adjustments of the patterns are required. Hence, the arrangement is based on affine transformations.

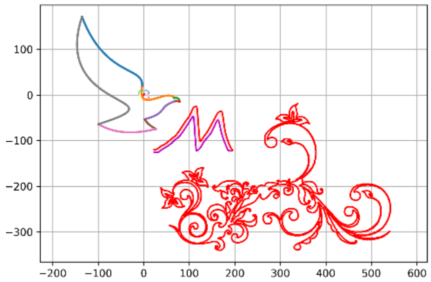


Figure 30. Parts before adjustments

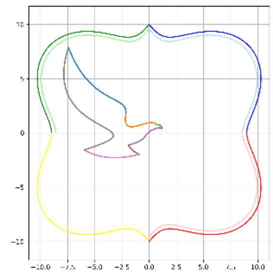


Figure 31. Assemble D_3

First, assemble D_3 since it determines where the wing and the vine lie. D_3 is too big, so before the translation $\mathbf{b_3} = (-2,1.3)$, compress it first by linear transformation $A_3 = \begin{pmatrix}0.04 & 0\\ 0 & 0.04\end{pmatrix}$. Thus, the affine transformation is

$$T_3(\Gamma_3) = \begin{pmatrix}0.04 & 0\\ 0 & 0.04\end{pmatrix}\Gamma_3 + \begin{pmatrix}-2\\ 1.3\end{pmatrix}$$
$$= \bigcup_{i=1}^{8}\left[\begin{pmatrix}0.04 & 0\\ 0 & 0.04\end{pmatrix}\mathbf{B_i}([0,1]) + \begin{pmatrix}-2\\ 1.3\end{pmatrix}\right]$$

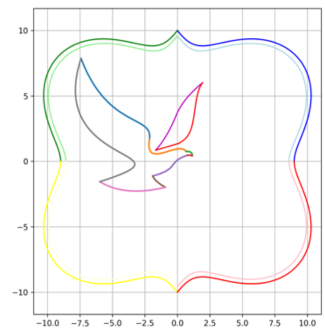


Figure 32. Assemble D_2

To assemble the wing to the bird, a rotation and a scaling are required. The composed transformation is

$$A_2 = \begin{pmatrix}\cos\frac{\pi}{11} & -\sin\frac{\pi}{11}\\ \sin\frac{\pi}{11} & \cos\frac{\pi}{11}\end{pmatrix}\begin{pmatrix}1.7 & 0\\ 0 & 1.7\end{pmatrix} = 1.7\begin{pmatrix}\cos\frac{\pi}{11} & -\sin\frac{\pi}{11}\\ \sin\frac{\pi}{11} & \cos\frac{\pi}{11}\end{pmatrix}$$

And the translation is $\mathbf{b_2} = (0.5,1.7)$. The affine transformations on the parameterised functions are

$$T_2\left(\begin{pmatrix}x\\ g_{upper}(x)\end{pmatrix}\right) = 1.7\begin{pmatrix}\cos\frac{\pi}{11} & -\sin\frac{\pi}{11}\\ \sin\frac{\pi}{11} & \cos\frac{\pi}{11}\end{pmatrix}\begin{pmatrix}x\\ g_{upper}(x)\end{pmatrix} + \begin{pmatrix}0.5\\ 1.7\end{pmatrix} = 1.7\begin{pmatrix}x\cos\frac{\pi}{11} - g_{upper}(x)\sin\frac{\pi}{11} + 0.5\\ x\sin\frac{\pi}{11} + g_{upper}(x)\cos\frac{\pi}{11} + 1.7\end{pmatrix}$$

$$T_2\left(\begin{pmatrix}x\\ g_{lower}(x)\end{pmatrix}\right) = 1.7\begin{pmatrix}x\cos\frac{\pi}{11} - g_{lower}(x)\sin\frac{\pi}{11} + 0.5\\ x\sin\frac{\pi}{11} + g_{lower}(x)\cos\frac{\pi}{11} + 1.7\end{pmatrix}$$

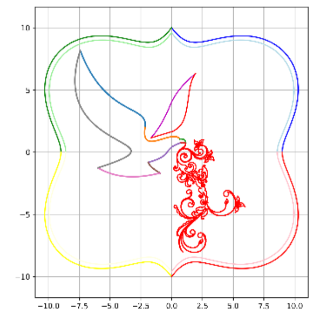


Figure 33. Assemble D_5

The vine needs a rotation of 90° and a scaling. The composed matrix is

$$A_5 = \begin{pmatrix}\cos\frac{\pi}{2} & -\sin\frac{\pi}{2}\\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2}\end{pmatrix}\begin{pmatrix}0.018 & 0\\ 0 & 0.018\end{pmatrix} = 0.018\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

And the translation is $\mathbf{b_5} = (6.3,1.7)$. The affine transformation is

$$T_5(\mathbf{r}(t)) = 0.018\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\sum_{n=-1000}^{1000}\begin{pmatrix}c_n\cos 2\pi nt\\ c_n\sin 2\pi nt\end{pmatrix} + \begin{pmatrix}6.3\\ 1.7\end{pmatrix} = 0.018\sum_{n=-1000}^{1000}\begin{pmatrix}-c_n\sin 2\pi nt\\ c_n\cos 2\pi nt\end{pmatrix} + \begin{pmatrix}6.3\\ 1.7\end{pmatrix}$$

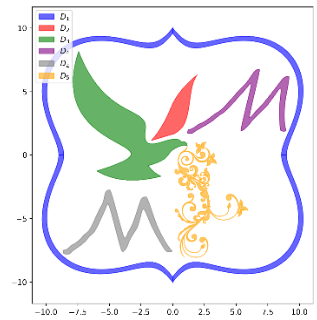


Figure 34. Final design

For the handwritten letter M in D_4, copy an identical region D_4'. Different scalings and shearings are applied:

$$A_4 = \begin{pmatrix}1 & 0.3\\ 0 & 1\end{pmatrix}\begin{pmatrix}0.045 & 0\\ 0 & 0.05\end{pmatrix} = \begin{pmatrix}0.045 & 1.015\\ 0 & 0.05\end{pmatrix}, \quad A_4' = \begin{pmatrix}1 & -0.2\\ 0 & 1\end{pmatrix}\begin{pmatrix}0.05 & 0\\ 0 & 0.05\end{pmatrix} = 0.05\begin{pmatrix}1 & -0.2\\ 0 & 1\end{pmatrix}$$

And the translations are $\mathbf{b_4} = (2,8)$ and $\mathbf{b_4'} = (-11,-1.5)$. The affine transformations are.

$$T_4\left(\begin{pmatrix}x\\ h_{upper}(x)\end{pmatrix}\right) = \begin{pmatrix}0.045 & 1.015\\ 0 & 0.05\end{pmatrix}\begin{pmatrix}x\\ h_{upper}(x)\end{pmatrix} + \begin{pmatrix}2\\ 8\end{pmatrix} = \begin{pmatrix}0.045x + 1.015\,h_{upper}(x)\\ 0.05\,h_{upper}(x)\end{pmatrix}$$

$$T_4'\left(\begin{pmatrix}x\\ h_{lower}(x)\end{pmatrix}\right) = 0.05\begin{pmatrix}1 & -0.2\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ h_{lower}(x)\end{pmatrix} + \begin{pmatrix}-11\\ -1.5\end{pmatrix} = 0.05\begin{pmatrix}x - 2\,h_{lower}(x)\\ h_{lower}(x)\end{pmatrix}$$

Here, all designs are done. Their areas are ready to be evaluated.

Evaluation: The final design in Figure 34 design demonstrates high quality. Its construction integrates polynomial functions, logarithmic and sigmoid (exponential) functions, 8 Bézier curves, 22 pieces of Hermite interpolation functions, a curve formed by Fourier transforms, and various transformations. This diverse combination ensures both local smoothness and global accuracy, while allowing flexible control of the shape. The design is composed of 6 distinct regions, each with clear analytic or constructive definitions, which provides a solid foundation for precise area and geometric calculations and highlights the effective integration of various mathematical tools and curve design techniques.

2. Part B: Calculations of the Area

2.1. Overview

The task asks to apply calculus and mathematical reasoning to analyse full designs. Specifically, calculate the area of the design using algebraic integration, justify assumptions and limitations in modelling processes, and discuss how the model simplifies real-world structures.

2.2. Selection and rationale: the area calculation methods based on the algebraic integration

For regions formed by different types of curves, a suitable selection of area calculation methods is crucial. Generally, in $\mathbb{R}^2$, the area of a Region D is defined by a double integral as

$$\text{Area}(D) = \iint_D 1\,dS \tag{20}$$

Regions bounded by 2 functions and bounded by an enclosed parametric curve have different applications of this equation. For regions like $D = \{(x,y)|a \le x \le b, g(x) \le y \le f(x)\}$, its area is

$$\text{Area}(D) = \iint_D 1\,dS = \int_a^b\int_{g(x)}^{f(x)} 1\,dy\,dx = \int_a^b y|_{g(x)}^{f(x)}dx = \int_a^b [f(x) - g(x)]dx \tag{21}$$

which is exactly the Eq. 1 used in Part A when evaluating the area of D_0. For parametric curves, **Green's theorem** is introduced:

$$\iint_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right) dS = \oint_{\partial D} P\,dx + Q\,dy$$

where $\oint$ means integrate on an enclosed curve, and ∂D means the boundary of D. By letting $Q = x, P = 0$, there is

$$\text{Area}(D) = \iint_D 1\,dS = \iint_D \left(\frac{\partial x}{\partial x}-\frac{\partial 0}{\partial y}\right) dS = \oint_{\partial D} x\,dy = \int x(t)y'(t)\,dt \tag{22}$$

Note that this area has direction. If integrate along the curve clockwise, it gives a negative value.

Additionally, for an affine transformation $T(\mathbf{x}) = A\mathbf{x}+\mathbf{b}$, there is

$$\text{Area}\big(T(D)\big) = |\det A|\cdot \text{Area}(D) \tag{23}$$

where $\det A$ is the determinant of a matrix A, indicating the area scaling factor. For a non-linear transformation $T(x,y)$, the Jacobian J reflects the linear transformation behaviour in each infinitesimal element in the region. Summing these behaviours yields the scaling of the area:

$$\text{Area}\big(T(D)\big) = \iint_{T(D)} 1\,dS_T = \iint_D |\det J|\,dS \tag{24}$$

Summary: Select the area calculation method in different situations			
Region bounded by functions	Region enclosed by parametric curves	Affine transformation	Non-linear transformation
Eq. 21	Eq. 22	Eq. 23	Eq. 24

2.3. Calculate the area of Region D_1

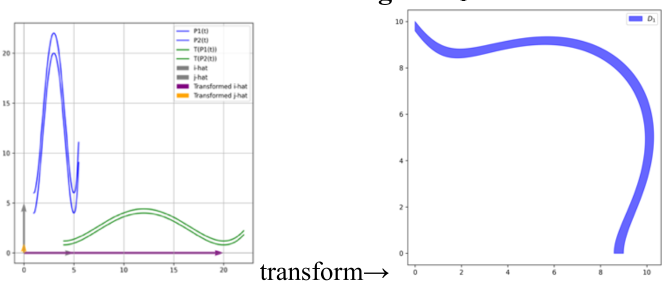


Figure 35. 1/4 of D₁

Region D_1 can be divided into 4 chunks with the same area, as shown in Figure 35. This region is obtained by transforming the given region D_0, whose area is 9 unit². According to 1.4.2, it is first transformed to

$$T(\mathbf{P_1}(t)) = \begin{pmatrix} -4t+22 \\ \frac{1}{5}f_1(t)+\frac{623}{80}\end{pmatrix},\ T(\mathbf{P_2}(t))\begin{pmatrix} -4t+22 \\ \frac{1}{5}f_2(t)+\frac{623}{80}\end{pmatrix}$$

which corresponds to functions

$$f_{1,T}(x) = \frac{1}{5}f_1\left(\frac{22-x}{4}\right)+\frac{623}{80}$$

$$f_{2,T}(x) = \frac{1}{5}f_2\left(\frac{22-x}{4}\right)+\frac{623}{80} = \frac{1}{5}\left[f_1\left(\frac{22-x}{4}\right)-2\right]+\frac{623}{80}$$

Note that the domain is very important here. Since both $f_1(x)$ and $f_2(x)$ have the domain $1 \le x \le 5.5$, there must be

$$1 \le \frac{22-x}{4} \le 5.5 \Rightarrow 0 \le x \le 18$$

Thus, the region $T(D_0) = \{(x,y)|0 \le x \le 18, f_{2,T}(x) \le y \le f_{1,T}(x)\}$. Then, the curl transformation was applied on $T(D_0)$ with the Jacobian:

$$\det J = \begin{vmatrix} \frac{\pi y}{18}\sin\left(-\frac{x\pi}{18}+\frac{\pi}{2}\right) & \cos\left(-\frac{x\pi}{18}+\frac{\pi}{2}\right) \\ -\frac{\pi y}{18}\cos\left(-\frac{x\pi}{18}+\frac{\pi}{2}\right) & \sin\left(-\frac{x\pi}{18}+\frac{\pi}{2}\right)\end{vmatrix} = \frac{\pi y}{18}\sin^2\left(-\frac{x\pi}{18}+\frac{\pi}{2}\right)+\frac{\pi y}{18}\cos^2\left(-\frac{x\pi}{18}+\frac{\pi}{2}\right) = \frac{\pi y}{18}$$

Hence, according to Eq. 24,

$$\frac{1}{4}\text{Area}(D_1) = \iint_{T(D_0)} |\det J|\,dS = \int_0^{18}\int_{f_{2,T}(x)}^{f_{1,T}(x)} \frac{\pi y}{18}\,dy\,dx = \frac{\pi}{18}\int_0^{18}\left[\frac{y^2}{2}\right]_{f_{2,T}(x)}^{f_{1,T}(x)} dx = \frac{\pi}{36}\int_0^{18}\left[f_{1,T}^2(x)-f_{2,T}^2(x)\right]dx$$

$$f_{1,T}^2(x)-f_{2,T}^2(x) = \left(\frac{1}{5}f_1\left(\frac{22-x}{4}\right)+\frac{623}{80}\right)^2-\left(\frac{1}{5}\left[f_1\left(\frac{22-x}{4}\right)-2\right]+\frac{623}{80}\right)^2 = \frac{16f_1\left(\frac{22-x}{4}\right)+607}{100} = \frac{x^4-40x^3+472x^2-1440x+12544}{1600}$$

$$\text{Area}(D_1) = \frac{\pi}{9}\int_0^{18}\frac{x^4-40x^3+472x^2-1440x+12544}{1600}dx = \frac{\pi}{9}\cdot\frac{1}{1600}\left[\frac{x^5}{5}-10x^4+\frac{472}{3}x^3-720x^2+12544x\right]_0^{18} = \frac{\pi}{9}\cdot\frac{18612}{125} = \frac{2068\pi}{125} \approx 51.97 \text{ (2 d.p.)}$$

2.4. Calculate the area of Region D_2

Region $D_2 = \{(x,y)\in\mathbb{R}^2\,|-1.319 \le x \le 1.576,\ g_{lower}(x) \le y \le g_{upper}(x)\}$ (see Figure 15), where

$$g_{upper}(x) = \begin{cases}\frac{125}{512}(x+2.56)^2+\ln 0.08+2.2, & -1.319 \le x \le 0 \\ \ln 0.1(x+0.8)+3.8, & 0 < x \le 1.576\end{cases},\ g_{lower}(x) = \begin{cases} 0.05, & -1.319 \le x \le -0.188 \\ \frac{2.6}{1+e^{-3.4(x-0.9)}}, & -0.188 < x \le 1.576\end{cases}$$

It is clear that Eq. 21 should be used here:

$$\text{Area}(D_2) = \int_{-1.319}^{1.576}\left[g_{upper}(x)-g_{lower}(x)\right]dx = \int_{-1.319}^{1.576} g_{upper}(x)dx - \int_{-1.319}^{1.576} g_{lower}(x)dx$$

$$= \int_{-1.319}^{0}\left(\frac{125}{512}(x+2.56)^2+\ln 0.08+2.2\right)dx + \int_0^{1.576}(\ln 0.1(x+0.8)+3.8)\,dx - \int_{-1.319}^{-0.188} 0.05\,dx - \int_{-0.188}^{1.576}\frac{2.6\,dx}{1+e^{-3.4(x-0.9)}}$$

Calculate the 4 integrals separately:

a) $\int_{-1.319}^{0}\left(\frac{125}{512}(x+2.56)^2+\ln 0.08+2.2\right)dx = \left[\frac{125}{512}\frac{(x+2.56)^3}{3}\right]_{-1.319}^{0} + [(\ln 0.08+2.2)x]_{-1.319}^{0} = \frac{125}{1536}(2.56^3-(2.56-1.319)^3) + 1.319(\ln 0.08+2.2) \approx 0.78 \text{ unit}^2 \text{ (2 d.p.)}$

b) $\int_0^{1.576}(\ln 0.1(x+0.8)+3.8)\,dx = \int_0^{1.576}\ln(x+0.8)\,dx + \int_0^{1.576}(\ln 0.1+3.8)\,dx$

The rear part is easy as $\int_0^{1.576}(\ln 0.1+3.8)\,dx = 1.576(\ln 0.1+3.8)$. The front part with ln cannot be integrated directly. Thus, consider **integrating by parts**. Let $u = \ln(x+0.8), u' = \frac{1}{x+0.8}$ and $v' = 1, v = x$, then

$$\int_0^{1.576} \ln(x+0.8)\ dx = \int_b^a uv' dx = [uv]_b^a - \int_b^a u'v\ dx = [x\ln(x+0.8)]_0^{1.576} - \int_0^{1.576} \frac{x\ dx}{x+0.8}$$
$$= 1.576\ln 2.376 + \int_0^{1.576} dx - 0.8\int_0^{1.576} \frac{dx}{x+0.8} = 1.576\ln 2.376 + 1.576 - 0.8(\ln 2.376 - \ln 0.8)$$

Add the constant part, there is $\int_0^{1.576}(\ln 0.1(x+0.8)+3.8)\ dx = 1.576(\ln 0.1 + 3.8) + 1.576\ln 2.376 + 1.576 - 0.8(\ln 2.376 - \ln 0.8) \approx 3.02\ \text{unit}^2$ (2 d.p.)

c) $-\int_{-1.319}^{-0.188} 0.05\ dx = 0.05 \times (-0.188 + 1.319) = -0.06\ \text{unit}^2$ (2 d.p.)

d) $-\int_{-0.188}^{1.576} \frac{2.6\ dx}{1+e^{-3.4(x-0.9)}} = -2.6\int_{-0.188}^{1.576} \frac{e^{3.4(x-0.9)}\ dx}{e^{3.4(x-0.9)}+1}$. Notice that **integration by substitution** is available as

$$e^{3.4(x-0.9)}\ \mathrm{d}x = \frac{1}{3.4}\mathrm{d}\left(e^{3.4(x-0.9)}\right)$$

$$-\int_{-0.188}^{1.576} \frac{2.6\ dx}{1+e^{-3.4(x-0.9)}} = -\frac{2.6}{3.4}\int_{-0.188}^{1.576} \frac{\mathrm{d}\left(e^{3.4(x-0.9)}\right)}{1+e^{-3.4(x-0.9)}} = -\frac{2.6}{3.4}\left[\ln\left(1+e^{3.4(x-0.9)}\right)\right]_{0.188}^{1.578} \approx -1.81\ \text{unit}^2\ (2\text{ d.p.})$$

Hence, the total area of D_2 is

$$\text{Area}(D_2) \approx 0.78 + 3.02 - 0.06 - 1.81 = 1.93\ \text{unit}^2$$

Note that a transformation $T_2(\mathbf{x}) = A_3\mathbf{x} + \mathbf{b}$ is used to adjust it finally, where

$$\det A_3 = \left|1.7\begin{pmatrix} \cos\frac{\pi}{11} & -\sin\frac{\pi}{11} \\ \sin\frac{\pi}{11} & \cos\frac{\pi}{11} \end{pmatrix}\right| = 1.7^2\left(\cos^2\frac{\pi}{11} + \sin^2\frac{\pi}{11}\right) = 2.89$$

Therefore, according to Eq. 23, the area after the transformation is $\text{Area}\left(T_2(D_2)\right) = 2.89 \times 1.93 \approx 5.58\ \text{unit}^2$ (2 d.p.).

2.5. Calculate the area of Region *D₃*

This region is bounded by the curve

$$\Gamma_3 = \bigcup_{i=1}^{8} \mathbf{B_i}([0,1])$$

where $\mathbf{B_i}(t) = \left(x_i(t), y_i(t)\right)$ is a parametric equation. Therefore, Eq. 22 should be used:

$$\text{Area}(D_3) = \left|\sum_{i=1}^{8}\int_0^1 x_i y_i'\ dt\right|$$

Take $\mathrm{B}_1(t) = (-131.66t^3 + 108.72t^2 + 154.59t - 135.39,\ -239.05t^3 + 490.74t^2 - 405.96t + 171.46)$ for a sample calculation. First,

$$y_1'(t) = -3 \times 239.05t^2 + 2 \times 490.74t - 405.96 = -717.15t^2 + 981.48t - 405.96$$
$$x_1(t)\cdot y_1'(t) = (-131.66t^3 + 108.72t^2 + 154.59t - 135.39)(-717.15t^2 + 981.48t - 405.96)$$
$$= 94419.969t^5 - 207190.2048t^4 + 49290.9807t^3 + 204685.9605t^2 - 195639.9336 + 54962.9244$$

The curve integral is

$$\int_0^1 x_1(t)\cdot y_1'(t)\ dt = \int_0^1 (94419.969t^5 - 207190.2048t^4 + 49290.9807t^3 + 204685.9605t^2 - 195639.9336 + 54962.9244)dt$$
$$= \frac{94419.969}{6} - \frac{207190.2048}{5} + \frac{49290.9807}{4} + \frac{204685.9605}{3} - \frac{195639.9336}{2} + 54962.9244 \approx 11992.98$$

Similarly, $\int_0^1 x_i y_i'\ dt$ for all $i = 2\sim 8$ can be found. Sum them up, the total area of Region D_3 is

$$\sum_{i=1}^{8}\int_0^1 x_i y_i'\ dt \approx -17691.28\ (2\text{ d.p.})$$

The negative result makes sense since the Bezier curves designed go clockwise. Take the absolute value, $\text{Area}(D_3) = 17691.28\ \text{unit}^2$

The transformation on D_3 is a scaling with a factor

$$\det A_3 = \begin{vmatrix} 0.04 & 0 \\ 0 & 0.04 \end{vmatrix} = 0.0016$$

Therefore the final area of D_3 is $\text{Area}(T_3(D_3)) = 0.0016 \times 17691.28 \approx 28.31\ \text{unit}^2$ (2 d.p.).

2.6. Calculate the area of Region *D₄*

Region $D_4 = \left\{(x,y) \in \mathbb{R}^2\ \middle|\ x_0 \le x \le x_{22},\ h_{lower}(x) \le y \le h_{upper}(x)\right\}$, where

$$h_{upper}(x) = \begin{cases} p_1^{up}(x), & x_0 \le x \le x_1 \\ \vdots & \vdots \\ p_{22}^{up}(x), & x_{21} \le x \le x_{22} \end{cases}, \quad h_{lower}(x) = \begin{cases} p_1^{low}(x), & x_0 \le x \le x_1 \\ \vdots & \vdots \\ p_{22}^{low}(x), & x_{21} \le x \le x_{22} \end{cases}$$

For functions, consider using Eq. 21:

$$\text{Area}(D_4) = \int_{x_0}^{x_{22}} \left[h_{upper}(x) - h_{lower}(x)\right]dx = \sum_{i=1}^{22}\int_{x_{i-1}}^{x_i} \left[p_i^{up}(x) - p_i^{low}(x)\right]dx$$

That is to say, the values of $\int_{x_{i-1}}^{x_i}[p_i^{up}(x)-p_i^{low}(x)]dx$ of 22 segments need to be evaluated. Take segment 1 with interval [23.0, 30.0] for sample calculation. The upper boundary is $p_1(x) = 0.004397\,x^3 - 0.3854\,x^2 + 11.2507x + 16.6050$, while the lower boundary is $p_l(x) = -0.0003158\,x^3 - 0.006948\,x^2 + 0.7518\,x + 110.23$. The difference is

$$p_1^{up}(x) - p_1^{low}(x) = 0.004713\,x^3 - 0.3784\,x^2 + 10.50\,x - 93.62$$

The integral is

$$\int_{23.0}^{30.0} 0.004713\,x^3 - 0.3784\,x^2 + 10.50\,x - 93.62\,dx = [0.001178\,x^4 - 0.1261\,x^3 + 5.249\,x^2 - 93.62\,x]_{23}^{30} \approx 85.19\ (2\ \mathrm{d.p.})$$

For full calculation, see Appendix 5. These calculations are long and repeated, so it is helped by Python. The total area is

$$\mathrm{Area}(D_4) = \sum_{i=1}^{22}\int_{x_{i-1}}^{x_i}[p_i^{up}(x)-p_i^{low}(x)]dx \approx 3135.64\ \mathrm{unit}^2\ (2\ d.p.)$$

Regions $T_4(D_4)$ and $T_4'(D_4')$ is sclaed by factor

$$\det A_4 = \begin{vmatrix}0.045 & 0\\ 0 & 0.05\end{vmatrix} = 0.00225, \qquad \det A_4' = \begin{vmatrix}0.05 & 0\\ 0 & 0.05\end{vmatrix} = 0.0025$$

The areas are

$$\mathrm{Area}(T_4(D_4)) = 0.00225 \times 3135.64 \approx 7.06\ \mathrm{unit}^2, \qquad \mathrm{Area}(T_4'(D_4')) = 0.0025 \times 3135.64 \approx 7.84\ \mathrm{unit}^2$$

2.7. Calculate the area of Region D_5

This region is enclosed by the curve

$$\mathbf{r}(t) = \left(\sum_{n=-1000}^{1000} c_n \cos 2\pi nt\,, \sum_{n=-1000}^{1000} c_n \sin 2\pi nt\right), t \in [0,1]$$

For a parametric curve, consider using Eq. 22:

$$\mathrm{Area}(D_5) = \int_0^1 xy'\,dt = \int_0^1\left(\sum_{n=-1000}^{1000} c_n\cos 2\pi nt\right)\left(\sum_{n=-1000}^{1000} 2\pi n c_n \cos 2\pi nt\right)dt = 2\pi\int_0^1 \sum_{n=-1000}^{1000}\sum_{m=-1000}^{1000} mc_nc_m \cos 2\pi nt \cos 2\pi mt\,dt$$

Swap the integration and summations:

$$= 2\pi \sum_{n=-1000}^{1000}\sum_{m=-1000}^{1000} mc_nc_m \int_0^1 \cos 2\pi nt \cos 2\pi mt\,dt$$

Note that

$$\int_0^1 \cos 2\pi nt \cos 2\pi mt\,dt = \begin{cases} 1/2, & n = m \neq 0\\ 1, & n = m = 0\\ 0, & n \neq m\end{cases}$$

Additionally, the contribution of the $n = 0$ term is also 0. Hence, only terms with $n = m \neq 0$ remain. Substituting c_ns, the area is

$$\mathrm{Area}(D_5) = \pi \sum_{n=-1000}^{1000} nc_n^2 = 18957.90\ \mathrm{unit}^2\ (2\ \mathrm{d.p.})$$

The transformed area is

$$\mathrm{Area}\big(T_5(D_5)\big) = \det A_5\ 18957.90 = \begin{vmatrix}0.018 & 0\\ 0 & 0.018\end{vmatrix} \times 18957.90 = 0.000324 \times 18957.90 \approx 6.14\ \mathrm{unit}^2\ (2\ \mathrm{d.p.})$$

2.8. Sum the areas of parts & the meanings in real-world contexts

The overall area of the design is

$$A \approx 51.97 + 5.58 + 28.31 + 7.06 + 7.84 + 6.14 = 106.90\ \mathrm{unit}^2\ (2\ \mathrm{d.p.})$$

Meanings in real-world contexts: This design is for a logo in the real-world context. Hence, the area of a logo can be very useful in certain situations. For example, when printing it on uniforms to determine the space it occupies, or when resizing the logo for websites or apps, the area can be used to preserve proportions and balance visual weight. Moreover, in embroidery, screen printing, or laser cutting, when producing the logo as a decoration or sign, the area is directly related to production cost and material use.

2.9. Evaluation of the strengths and limitations of the mathematics methods of design used

2.9.1. A comparison of the transformation methods (Green: Strength. Red: Limitation)

	Graph transformation	Affine transformation	Non-linear transformation
Object	Function	Vector space (hence any vector, parametric equation, and parameterised function)	Every point in the plane.
Non-single-valuedness	Not supported	Supported before and after the transformation	Supported before and after the transformation
Computational complexity	Low	Moderate. Requires matrix operations.	High, involves nonlinear equations and singularity.
Type of transformation supported	Translation, scaling, reflection. Very limited transformational capability	Moderate, supports translation, scaling, reflection, rotation, and shearing.	Most, including complex distortions, bending, and even topological changes
Preserved properties	Straightness, parallelism, ratio, continuity, monotonicity and function property (single-valuedness)	Straightness, parallelism, continuity, and ratio.	Generally does not preserve geometric properties.

Intuitiveness	High, easy to understand and visualise	Moderate. Requires intuitive understandings of matrices and vector spaces.	Low
Computational complexity of the area scaling factor	Low	Low, just need to evaluate the determinant.	High, involves partial derivatives (Jacobians) and double integrals.

2.9.2. A comparison of the methods of curve design by adjusting

	Design with functions	**Design with Bezier curves**
Outcome	Arbitrary functions	Quadratic or cubic parametric curves
Non-single-valuedness	Not supported for a single function. Can be achieved by multiple single-valued functions.	Supported
Computational complexity	Relatively low	Moderate, needs to expand equations.
Intuitiveness	Low, needs to construct and imagine the shape in mind	High
Ensure continuity	Hard, needs to take derivatives of both at joint.	Easy

2.9.3. A comparison of the methods of curve design by fitting

	PCHIP	**Fourier transform**
Outcome	Piecewise cubic functions	Cartesian form of a complex Fourier series
Non-single-valuedness	Not supported for a single function. Can be achieved by multiple single-valued functions.	Supported
Computational complexity	Moderate	Very high when doing summations of 1000+ terms
Degree of process automation	Relatively low, parameters such as number of sample points and the secant slope need to be assigned.	Very high, only needs to set the number of terms of the series kept.
Ensure continuity	Ensures C^1 continuity	Naturally has C^∞ continuity

2.9.4. A comparison of the design by adjusting & the design by fitting

	Design by adjusting	**Design by fitting**
Strength	Highly customised.	Highly automatic process. Can be customised by inputting hand-drawn images.
Limitation	Slow and complicated process.	Usually requires lots of calculations that have to be done using computers

3. Conclusion

This project examined the analysis of geometric regions bounded by curves, incorporating a variety of functions, including polynomial, trigonometric, logarithmic and exponential functions, to model boundaries with differing growth and saturation behaviours. Observations indicate that these functional choices allowed for flexible and accurate modelling of curve shapes.

Mathematical methods, including Bezier curves, piecewise Hermite interpolation and the discrete Fourier transformation, proved effective in simplifying design tasks and ensuring continuity and smoothness where necessary. A pattern is observed that the design methods based on adjustment, like constructing functions and Bezier curves, are more flexible in design, but sometimes hard to control and imagine the shape, while methods based on interpolation have a more automatic process but require more calculation. Patterns on methods using sample points like Bezier curves and polynomial interpolation indicate are also observed that with more sample points, the output design is more accurate, but at the same time tends to have Runge's phenomenon, which can be solved by using piecewise and composite curves.

Differentiation techniques are applied in this investigation mainly to ensure the smooth connection between curves. Observations indicate that curves and functions like (Bezier curves and Piecewise Cubic Hermite Interpolating Polynomials (PCHIP)) that have pre-defined slope properties at joints can better ensure smooth connections and easy to make adjustments without interfering original shape.

The use of definite integrals was central to quantifying areas between the curves. Analytical integration of piecewise-defined functions, combined with the curve integration of parametric equations, provided reliable results. The effects of affine transformations and non-linear transformations on the area are also investigated. It is observed that for some special regions, calculating and applying the scaling factor of a transformation on the original area is easier than doing the integration directly. By using various integration techniques, the overall area of the final design is calculated to be 106.90 unit^2 (2 d.p.).

Potential improvements include using Monte Carlo methods to numerically validate the area, increasing the density of sample points for designs formed by interpolations in regions where the curves change rapidly, or experimenting with alternative functional forms to better capture extreme variations. These adjustments would further reduce approximation errors and enhance the precision of integral-based area calculations.

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Code Appendix

All source code is available at: https://github.com/Yanttttt/SACE_Math_Method_Investigation

Appendix 1. D_1 curves

$$\mathbf{P_1}(t) = T_{curl}\left(\begin{pmatrix} -4t+22 \\ \frac{1}{5}f_1(t)+\frac{623}{80}\end{pmatrix}\right) = \begin{pmatrix} \left(\frac{1}{5}f_1(t)+\frac{623}{80}\right)\cos\left(-\frac{\pi(11-2t)}{9}+\frac{\pi}{2}\right) \\ \left(\frac{1}{5}f_1(t)+\frac{623}{80}\right)\sin\left(-\frac{\pi(11-2t)}{9}+\frac{\pi}{2}\right)\end{pmatrix}$$

$$\mathbf{P_2}(t) = T_{curl}\left(\begin{pmatrix} -4t+22 \\ \frac{1}{5}f_2(t)+\frac{623}{80}\end{pmatrix}\right) = \begin{pmatrix} \left(\frac{1}{5}f_2(t)+\frac{623}{80}\right)\cos\left(-\frac{\pi(11-2t)}{9}+\frac{\pi}{2}\right) \\ \left(\frac{1}{5}f_2(t)+\frac{623}{80}\right)\sin\left(-\frac{\pi(11-2t)}{9}+\frac{\pi}{2}\right)\end{pmatrix}$$

$$\mathbf{P_3}(t) = T_{curl}\left(\begin{pmatrix} 4t-22 \\ \frac{1}{5}f_1(t)+\frac{623}{80}\end{pmatrix}\right) = \begin{pmatrix} \left(\frac{1}{5}f_1(t)+\frac{623}{80}\right)\cos\left(-\frac{\pi(2t-11)}{9}+\frac{\pi}{2}\right) \\ \left(\frac{1}{5}f_1(t)+\frac{623}{80}\right)\sin\left(-\frac{\pi(2t-11)}{9}+\frac{\pi}{2}\right)\end{pmatrix}$$

$$\mathbf{P_4}(t) = T_{curl}\left(\begin{pmatrix} 4t-22 \\ \frac{1}{5}f_2(t)+\frac{623}{80}\end{pmatrix}\right) = \begin{pmatrix} \left(\frac{1}{5}f_2(t)+\frac{623}{80}\right)\cos\left(-\frac{\pi(2t-11)}{9}+\frac{\pi}{2}\right) \\ \left(\frac{1}{5}f_2(t)+\frac{623}{80}\right)\sin\left(\frac{\pi(2t-11)}{9}-\frac{\pi}{2}\right)\end{pmatrix}$$

$$\mathbf{P_5}(t) = \begin{pmatrix} -1 & 0 \\ 0 & -1\end{pmatrix}\mathbf{P_1}(t) = \begin{pmatrix} -\left(\frac{1}{5}f_1(t)+\frac{623}{80}\right)\cos\left(-\frac{\pi(11-2t)}{9}+\frac{\pi}{2}\right) \\ -\left(\frac{1}{5}f_1(t)+\frac{623}{80}\right)\sin\left(-\frac{\pi(11-2t)}{9}+\frac{\pi}{2}\right)\end{pmatrix}$$

$$\mathbf{P_6}(t) = \begin{pmatrix} -1 & 0 \\ 0 & -1\end{pmatrix}\mathbf{P_2}(t) = \begin{pmatrix} -\left(\frac{1}{5}f_2(t)+\frac{623}{80}\right)\cos\left(-\frac{\pi(11-2t)}{9}+\frac{\pi}{2}\right) \\ -\left(\frac{1}{5}f_2(t)+\frac{623}{80}\right)\sin\left(-\frac{\pi(11-2t)}{9}+\frac{\pi}{2}\right)\end{pmatrix}$$

$$\mathbf{P_7}(t) = \begin{pmatrix} -1 & 0 \\ 0 & -1\end{pmatrix}\mathbf{P_3}(t) = \begin{pmatrix} -\left(\frac{1}{5}f_1(t)+\frac{623}{80}\right)\cos\left(-\frac{\pi(2t-11)}{9}+\frac{\pi}{2}\right) \\ -\left(\frac{1}{5}f_1(t)+\frac{623}{80}\right)\sin\left(-\frac{\pi(2t-11)}{9}+\frac{\pi}{2}\right)\end{pmatrix}$$

$$\mathbf{P_8}(t) = \begin{pmatrix} -1 & 0 \\ 0 & -1\end{pmatrix}\mathbf{P_4}(t) = \begin{pmatrix} -\left(\frac{1}{5}f_2(t)+\frac{623}{80}\right)\cos\left(-\frac{\pi(2t-11)}{9}+\frac{\pi}{2}\right) \\ -\left(\frac{1}{5}f_2(t)+\frac{623}{80}\right)\sin\left(-\frac{\pi(2t-11)}{9}+\frac{\pi}{2}\right)\end{pmatrix}$$

Appendix 2. Derivative of the Bezier curves.

We have discussed the general form of Bezier curves. But to make the index of control point sstart from 1, we didn't use the most general equation, which is supposed to be

$$\boldsymbol{Bezier_n}(t) = \sum_{i=0}^{n}\binom{n}{i}t^i(1-t)^{n-i}\boldsymbol{P_i}, t \in [0,1]$$

First, take the derivative of the k^{th} term of a Bezier curve without control points as coefficients:

$$\begin{aligned} B'_{n,k}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\binom{n}{k}t^k(1-t)^{n-k}\right) \\ &= \binom{n}{k}\left(k\cdot t^{k-1}(1-t)^{n-k} - t^k(n-k)(1-t)^{n-k-1}\right) \end{aligned}$$

Expand the binomial coefficient:

$$\begin{aligned} \ldots &= \frac{kn!}{k!\,(n-k)!}t^{k-1}(1-t)^{n-k} - \frac{(n-k)n!}{k!\,(n-k)!}t^k(1-t)^{n-k-1} \\ &= \frac{n!}{(k-1)!\,(n-k)!}t^{k-1}(1-t)^{n-k} - \frac{n!}{k!\,(n-1-k)!}t^k(1-t)^{n-1-k} \\ &= n\left(\frac{(n-1)!}{(k-1)!\,((n-1)-(k-1))!}t^{k-1}(1-t)^{(n-1)-(k-1)} - \frac{(n-1)!}{k!\,((n-1)-k)!}t^k(1-t)^{n-1-k}\right) \end{aligned}$$

Notice that the two terms are typical Bezier curves of lower orders. Hence,

$$B'_{n,k}(t) = n\left(B_{n-1,k-1}(t) - B_{n-1,k}(t)\right)$$

Now, apply the control points.

$$\begin{aligned} Bezier_n(t) &= B_{n,0}(t)\cdot P_0 + B_{n,1}(t)\cdot P_1 + B_{n,2}(t)\cdot P_2 + B_{n,3}(t)\cdot P_3 + \cdots \\ Bezier'_n(t) &= n\cdot\left(B_{n-1,-1}(t) - B_{n-1,0}(t)\right)\cdot P_0 + \\ &\quad n\cdot\left(B_{n-1,0}(t) - B_{n-1,1}(t)\right)\cdot P_1 + \\ &\quad n\cdot\left(B_{n-1,1}(t) - B_{n-1,2}(t)\right)\cdot P_2 + \\ &\quad n\cdot\left(B_{n-1,2}(t) - B_{n-1,3}(t)\right)\cdot P_3 + \\ &\quad \ldots \end{aligned}$$

Expand this and reorder the terms by k:

$$\begin{aligned} n\cdot B_{n-1,0}(t)\cdot P_1 - n\cdot B_{n-1,0}(t)\cdot P_0 &+ \\ n\cdot B_{n-1,1}(t)\cdot P_2 - n\cdot B_{n-1,1}(t)\cdot P_1 &+ \\ n\cdot B_{n-1,2}(t)\cdot P_3 - n\cdot B_{n-1,2}(t)\cdot P_2 &+ \\ \ldots - n\cdot B_{n-1,3}(t)\cdot P_3 &+ \end{aligned}$$

In this transformation, there are two eliminations, namely $nB_{n-1,-1}(t)P_0$ as the $n-1^{\text{th}}$ term does not exist and the very last term with the binomial coefficient $\binom{n-1}{n}$, which is usually considered to be zero. Write it in the form of a sum:

$$\boldsymbol{Bezier'_n}(t) = n\sum_{i=0}^{n-1}\binom{n-1}{i}t^i(1-t)^{n-i-1}(\boldsymbol{P_{i+1}} - \boldsymbol{P_i})$$

This form can also be translated to our convetion by letting $C_{i+1} = P_i$.

Appendix 3. $B_1 \sim B_8$ Equations.

The control points are,

$C_1^{(1)} = (-135.39,171.46), C_2^{(1)} = (-83.86,36.14), C_3^{(1)} = (3.90,64.40), C_4^{(1)} = (-3.74,17.19)$

$C_1^{(2)} = (-3.74,17.19), C_2^{(2)} = (-13.4,-42.48), C_3^{(2)} = (51.12,15.53), C_4^{(2)} = (66.82,-6.08)$

$C_1^{(3)} = (66.82,-6.08), C_2^{(3)} = (79.02,-5.29), C_3^{(3)} = (78.62,-14.24)$

$C_1^{(4)} = (78.62,-14.24), C_2^{(4)} = (72.46,-12.21), C_3^{(4)} = (66.52,-14.33)$

$C_1^{(5)} = (66.52,-14.33), C_2^{(5)} = (30.15,-27.30), C_3^{(5)} = (54.53,-38.65), C_4^{(5)} = (2.45,-53.06)$

$C_1^{(6)} = (2.45,-53.06), C_2^{(6)} = (7.57,-63.97), C_3^{(6)} = (26.78,-74.38)$

$C_1^{(7)} = (26.78,-74.38), C_2^{(7)} = (-49.98,-93.38), C_3^{(7)} = (-99.30,-63.83)$

$C_1^{(8)} = (-99.30,-63.83), C_2^{(8)} = (96.95,-1.58), C_3^{(8)} = (-205.88,-40.99), C_4^{(8)} = (-135.39,171.46)$

According to Eq. 8,

$B_1(t) = (-131.66t^3 + 108.72t^2 + 154.59t - 135.39,\ -239.05t^3 + 490.74t^2 - 405.96t + 171.46)$

$B_2(t) = (-123.0t^3 + 222.54t^2 - 28.98t - 3.74,\ -197.3t^3 + 353.04t^2 - 179.01t + 17.19)$

$B_3(t) = (-12.6t^2 + 24.4t + 66.82,\ -9.74t^2 + 1.58t - 6.08)$

$B_4(t) = (0.22t^2 - 12.32t + 78.62,\ -4.15t^2 + 4.06t - 14.24)$

$B_5(t) = (-137.21t^3 + 182.25t^2 - 109.11t + 66.52,\ -4.68t^3 + 4.86t^2 - 38.91t - 14.33)$

$B_6(t) = (14.09t^2 + 10.24t + 2.45,\ 0.5t^2 - 21.82t - 53.06)$

$B_7(t) = (27.44t^2 - 153.52t + 26.78,\ 48.55t^2 - 38.0t - 74.38)$

$B_8(t) = (872.4t^3 - 1497.24t^2 + 588.75t - 99.3,\ 353.52t^3 - 304.98t^2 + 186.75t - 63.83)$

Appendix 4. PCHIP Equations

For the upper boundary, the sample points are:

(23.0000,125.0000),(30.0000,126.0000),(38.0000,121.0000),(46.0000,116.0000),(54.0000,111.0000),(62.0000,105.0000),(69.0000,98.0000),(77.0000,92.0000),(85.0000,82.0000),(93.0000,70.0000),(101.0000,58.0000),(109.0000,47.0000),(116.0000,122.0000),(124.0000,116.0000),(132.0000,103.0000),(140.0000,93.0000),(148.0000,83.0000),(155.0000,69.0000),(163.0000,55.0000),(171.0000,84.0000),(179.0000,107.0000),(187.0000,122.0000),(195.0000,121.0000)

Hence, according to Eq. 12 and the Hermite basis functions,

$$\begin{aligned} p_1(x) &= h_{00}(t)y_1 + h_{10}(t)h\,d_1 + h_{01}(t)y_2 + h_{11}(t)h\,d_2, h = 7.0000, t = \frac{x-23.0000}{h} \\ &= 0.004397473275024266x^3 - 0.3853984450923349x^2 + 11.250728862973801x + 16.60495626822194 \end{aligned}$$

$$\begin{aligned} p_2(x) &= h_{00}(t)y_2 + h_{10}(t)h\,d_2 + h_{01}(t)y_3 + h_{11}(t)h\,d_3, h = 8.0000, t = \frac{x-30.0000}{h} \\ &= 0.009765624999999944x^3 - 1.03515625x^2 + 35.7421875x - 278.296875 \end{aligned}$$

$$p_3(x) = h_{00}(t)y_3 + h_{10}(t)h\,d_3 + h_{01}(t)y_4 + h_{11}(t)h\,d_4, h = 8.0000, t = \frac{x-38.0000}{h} = 0.0x^3 + 0.0x^2 - 0.625x + 144.75$$

$$\begin{aligned} p_4(x) &= h_{00}(t)y_4 + h_{10}(t)h\,d_4 + h_{01}(t)y_5 + h_{11}(t)h\,d_5, h = 8.0000, t = \frac{x-46.0000}{h} \\ &= -0.0009765625x^3 + 0.142578125x^2 - 7.54296875x + 256.3359375 \end{aligned}$$

$$\begin{aligned} p_5(x) &= h_{00}(t)y_5 + h_{10}(t)h\,d_5 + h_{01}(t)y_6 + h_{11}(t)h\,d_6, h = 8.0000, t = \frac{x-54.0000}{h} \\ &= -0.0009765625x^3 + 0.158203125x^2 - 9.23046875x + 301.8984375 \end{aligned}$$

$$\begin{aligned} p_6(x) &= h_{00}(t)y_6 + h_{10}(t)h\,d_6 + h_{01}(t)y_7 + h_{11}(t)h\,d_7, h = 7.0000, t = \frac{x-62.0000}{h} \\ &= 0.005102040816326529x^3 - 1.002551020408161x^2 + 64.60459183673464x - 1262.6377551020305 \end{aligned}$$

$$\begin{aligned} p_7(x) &= h_{00}(t)y_7 + h_{10}(t)h\,d_7 + h_{01}(t)y_8 + h_{11}(t)h\,d_8, h = 8.0000, t = \frac{x-69.0000}{h} \\ &= -0.005859374999999445x^3 + 1.2753906249999858x^2 - 93.189453125x + 2380.794921875 \end{aligned}$$

$$\begin{aligned} p_8(x) &= h_{00}(t)y_8 + h_{10}(t)h\,d_8 + h_{01}(t)y_9 + h_{11}(t)h\,d_9, h = 8.0000, t = \frac{x-77.0000}{h} \\ &= 0.001953124999999889x^3 - 0.498046875x^2 + 40.958984375x - 1000.587890625 \end{aligned}$$

$$\begin{aligned} p_9(x) &= h_{00}(t)y_9 + h_{10}(t)h\,d_9 + h_{01}(t)y_{10} + h_{11}(t)h\,d_{10}, h = 8.0000, t = \frac{x-85.0000}{h} \\ &= 0.001953125000000111x^3 - 0.5292968750000142x^2 + 46.271484375x - 1226.369140625 \end{aligned}$$

$$p_{10}(x) = h_{00}(t)y_{10} + h_{10}(t)h\, d_{10} + h_{01}(t)y_{11} + h_{11}(t)h\, d_{11}, h = 8.0000, t = \frac{x - 93.0000}{h}$$
$$= 0.0009765624999999445x^3 - 0.2802734375x^2 + 25.2919921875x - 643.5751953125$$
$$p_{11}(x) = h_{00}(t)y_{11} + h_{10}(t)h\, d_{11} + h_{01}(t)y_{12} + h_{11}(t)h\, d_{12}, h = 8.0000, t = \frac{x - 101.0000}{h}$$
$$= 0.0205078125x^3 - 6.3701171875x^2 + 657.7255859375x - 22519.9384765625$$
$$p_{12}(x) = h_{00}(t)y_{12} + h_{10}(t)h\, d_{12} + h_{01}(t)y_{13} + h_{11}(t)h\, d_{13}, h = 7.0000, t = \frac{x - 109.0000}{h}$$
$$= -0.4373177842565595x^3 + 147.59475218658883x^2 - 16588.33819241982x + 620941.8250728862$$
$$p_{13}(x) = h_{00}(t)y_{13} + h_{10}(t)h\, d_{13} + h_{01}(t)y_{14} + h_{11}(t)h\, d_{14}, h = 8.0000, t = \frac{x - 116.0000}{h}$$
$$= 0.0048828125x^3 - 1.83203125x^2 + 227.921875x - 9286.6875$$
$$p_{14}(x) = h_{00}(t)y_{14} + h_{10}(t)h\, d_{14} + h_{01}(t)y_{15} + h_{11}(t)h\, d_{15}, h = 8.0000, t = \frac{x - 124.0000}{h}$$
$$= 0.009765625000000111x^3 - 3.765625x^2 + 482.21875x - 20398.25$$
$$p_{15}(x) = h_{00}(t)y_{15} + h_{10}(t)h\, d_{15} + h_{01}(t)y_{16} + h_{11}(t)h\, d_{16}, h = 8.0000, t = \frac{x - 132.0000}{h}$$
$$= -0.002929687500001076x^3 + 1.20703125x^2 - 166.953125x + 7847.6875$$
$$p_{16}(x) = h_{00}(t)y_{16} + h_{10}(t)h\, d_{16} + h_{01}(t)y_{17} + h_{11}(t)h\, d_{17}, h = 8.0000, t = \frac{x - 140.0000}{h}$$
$$= -0.005859374999999889x^3 + 2.5078125x^2 - 358.90625x + 17264.875$$
$$p_{17}(x) = h_{00}(t)y_{17} + h_{10}(t)h\, d_{17} + h_{01}(t)y_{18} + h_{11}(t)h\, d_{18}, h = 7.0000, t = \frac{x - 148.0000}{h}$$
$$= 0.010204081632653107x^3 - 4.655612244897968x^2 + 705.9056122448987x - 35494.010204081525$$
$$p_{18}(x) = h_{00}(t)y_{18} + h_{10}(t)h\, d_{18} + h_{01}(t)y_{19} + h_{11}(t)h\, d_{19}, h = 8.0000, t = \frac{x - 155.0000}{h}$$
$$= 0.025390625000000056x^3 - 11.994140625x^2 + 1886.279296875x - 98696.576171875$$
$$p_{19}(x) = h_{00}(t)y_{19} + h_{10}(t)h\, d_{19} + h_{01}(t)y_{20} + h_{11}(t)h\, d_{20}, h = 8.0000, t = \frac{x - 163.0000}{h}$$
$$= -0.06250000000000006x^3 + 31.515625x^2 - 5292.40625x + 296050.265625$$
$$p_{20}(x) = h_{00}(t)y_{20} + h_{10}(t)h\, d_{20} + h_{01}(t)y_{21} + h_{11}(t)h\, d_{21}, h = 8.0000, t = \frac{x - 171.0000}{h}$$
$$= -0.0019531249999999445x^3 + 0.970703125x^2 - 157.396484375x + 8380.505859375$$
$$p_{21}(x) = h_{00}(t)y_{21} + h_{10}(t)h\, d_{21} + h_{01}(t)y_{22} + h_{11}(t)h\, d_{22}, h = 8.0000, t = \frac{x - 179.0000}{h}$$
$$= -0.021484375x^3 + 11.646484375x^2 - 2101.923828125x + 126406.533203125$$
$$p_{22}(x) = h_{00}(t)y_{22} + h_{10}(t)h\, d_{22} + h_{01}(t)y_{23} + h_{11}(t)h\, d_{23}, h = 8.0000, t = \frac{x - 187.0000}{h}$$
$$= -0.013671874999999944x^3 + 7.763671875x^2 - 1469.337890625x + 92803.509765625$$

Similarly, for the lower boundary, the sample ponits are (23.0000,120.0000), (30.0000,118.0000), (38.0000,112.0000), (46.0000,108.0000), (54.0000,102.0000), (62.0000,95.0000), (69.0000,90.0000), (77.0000,80.0000), (85.0000,68.0000), (93.0000,56.0000), (101.0000,45.0000), (109.0000,28.0000), (116.0000,25.0000), (124.0000,101.0000), (132.0000,91.0000), (140.0000,80.0000), (148.0000,66.0000), (155.0000,53.0000), (163.0000,39.0000), (171.0000,36.0000), (179.0000,80.0000), (187.0000,106.0000), (195.0000,121.0000).
Hence, according to Eq. 12 and the Hermite basis functions,

$$p_1(x) = h_{00}(t)y_1 + h_{10}(t)h\, d_1 + h_{01}(t)y_2 + h_{11}(t)h\, d_2, h = 7.0000, t = \frac{x - 23.0000}{h}$$
$$= -0.00031584062196292253x^3 - 0.006948493683190893x^2 + 0.751822157434602x + 110.22667638483988$$
$$p_2(x) = h_{00}(t)y_2 + h_{10}(t)h\, d_2 + h_{01}(t)y_3 + h_{11}(t)h\, d_3, h = 8.0000, t = \frac{x - 30.0000}{h}$$
$$= 0.00558035714285709x^3 - 0.5758928571428572x^2 + 18.968750000000004x - 83.42857142857144$$
$$p_3(x) = h_{00}(t)y_3 + h_{10}(t)h\, d_3 + h_{01}(t)y_4 + h_{11}(t)h\, d_4, h = 8.0000, t = \frac{x - 38.0000}{h}$$
$$= -0.00390625x^3 + 0.4921875x^2 - 21.109375x + 417.78125$$
$$p_4(x) = h_{00}(t)y_4 + h_{10}(t)h\, d_4 + h_{01}(t)y_5 + h_{11}(t)h\, d_5, h = 8.0000, t = \frac{x - 46.0000}{h}$$
$$= 0.0009765625x^3 - 0.158203125x^2 + 7.73046875x - 7.8984375$$
$$p_5(x) = h_{00}(t)y_5 + h_{10}(t)h\, d_5 + h_{01}(t)y_6 + h_{11}(t)h\, d_6, h = 8.0000, t = \frac{x - 54.0000}{h}$$
$$= 0.0022321428571428024x^3 - 0.3872767857142856x^2 + 21.48660714285714x - 280.4598214285711$$
$$p_6(x) = h_{00}(t)y_6 + h_{10}(t)h\, d_6 + h_{01}(t)y_7 + h_{11}(t)h\, d_7, h = 7.0000, t = \frac{x - 62.0000}{h}$$
$$= -0.0071064139941691105x^3 + 1.3830174927113772x^2 - 90.33764577259439x + 2073.27223032068$$
$$p_7(x) = h_{00}(t)y_7 + h_{10}(t)h\, d_7 + h_{01}(t)y_8 + h_{11}(t)h\, d_8, h = 8.0000, t = \frac{x - 69.0000}{h}$$
$$= 0.002232142857142917x^3 - 0.513392857142871x^2 + 37.984375x - 819.9375$$
$$p_8(x) = h_{00}(t)y_8 + h_{10}(t)h\, d_8 + h_{01}(t)y_9 + h_{11}(t)h\, d_9, h = 8.0000, t = \frac{x - 77.0000}{h}$$
$$= 0.001953124999999889x^3 - 0.482421875x^2 + 38.177734375x - 891.072265625$$
$$p_9(x) = h_{00}(t)y_9 + h_{10}(t)h\, d_9 + h_{01}(t)y_{10} + h_{11}(t)h\, d_{10}, h = 8.0000, t = \frac{x - 85.0000}{h}$$
$$= 0.0009765625000000867x^3 - 0.2568359375000142x^2 + 20.9951171875x - 460.6767578125$$

$$p_{10}(x) = h_{00}(t)y_{10} + h_{10}(t)h\, d_{10} + h_{01}(t)y_{11} + h_{11}(t)h\, d_{11}, h = 8.0000, t = \frac{x - 93.0000}{h}$$
$$= -0.0068359375x^3 + 1.9697265625x^2 - 190.4345703125x + 6228.7841796875$$
$$p_{11}(x) = h_{00}(t)y_{11} + h_{10}(t)h\, d_{11} + h_{01}(t)y_{12} + h_{11}(t)h\, d_{12}, h = 8.0000, t = \frac{x - 101.0000}{h}$$
$$= 0.019112723214285702x^3 - 5.990931919642858x^2 + 623.5115792410714x - 21508.030831473217$$
$$p_{12}(x) = h_{00}(t)y_{12} + h_{10}(t)h\, d_{12} + h_{01}(t)y_{13} + h_{11}(t)h\, d_{13}, h = 7.0000, t = \frac{x - 109.0000}{h}$$
$$= -0.00856413994169097 4x^3 + 2.981596209912542x^2 - 346.01311953352786x + 13409.895043731798$$
$$p_{13}(x) = h_{00}(t)y_{13} + h_{10}(t)h\, d_{13} + h_{01}(t)y_{14} + h_{11}(t)h\, d_{14}, h = 8.0000, t = \frac{x - 116.0000}{h}$$
$$= -0.29687499999999994x^3 + 106.875x^2 - 12810.75x + 511353.0$$
$$p_{14}(x) = h_{00}(t)y_{14} + h_{10}(t)h\, d_{14} + h_{01}(t)y_{15} + h_{11}(t)h\, d_{15}, h = 8.0000, t = \frac{x - 124.0000}{h}$$
$$= 0.01855468750000011x^3 - 7.20703125x^2 + 931.453125x - 39960.6875$$
$$p_{15}(x) = h_{00}(t)y_{15} + h_{10}(t)h\, d_{15} + h_{01}(t)y_{16} + h_{11}(t)h\, d_{16}, h = 8.0000, t = \frac{x - 132.0000}{h}$$
$$= -0.001953125000000111x^3 + 0.78125x^2 - 105.46875x + 4892.5$$
$$p_{16}(x) = h_{00}(t)y_{16} + h_{10}(t)h\, d_{16} + h_{01}(t)y_{17} + h_{11}(t)h\, d_{17}, h = 8.0000, t = \frac{x - 140.0000}{h}$$
$$= 0.002092633928571504 4x^3 - 0.9190848214285694x^2 + 132.734375x - 6230.9375$$
$$p_{17}(x) = h_{00}(t)y_{17} + h_{10}(t)h\, d_{17} + h_{01}(t)y_{18} + h_{11}(t)h\, d_{18}, h = 7.0000, t = \frac{x - 148.0000}{h}$$
$$= 0.002186588921282817x^3 - 0.9938046647230401x^2 + 148.67747813410915x - 7258.4358600583655$$
$$p_{18}(x) = h_{00}(t)y_{18} + h_{10}(t)h\, d_{18} + h_{01}(t)y_{19} + h_{11}(t)h\, d_{19}, h = 8.0000, t = \frac{x - 155.0000}{h}$$
$$= 0.009905133928571449x^3 - 4.678431919642854x^2 + 734.5977957589284x - 38295.81208147321$$
$$p_{19}(x) = h_{00}(t)y_{19} + h_{10}(t)h\, d_{19} + h_{01}(t)y_{20} + h_{11}(t)h\, d_{20}, h = 8.0000, t = \frac{x - 163.0000}{h}$$
$$= -0.004882812500000052x^3 + 2.5126953125x^2 - 431.0068359375x + 24679.5380859375$$
$$p_{20}(x) = h_{00}(t)y_{20} + h_{10}(t)h\, d_{20} + h_{01}(t)y_{21} + h_{11}(t)h\, d_{21}, h = 8.0000, t = \frac{x - 171.0000}{h}$$
$$= -0.10351562499999994x^3 + 54.619140625x^2 - 9599.044921875x + 561954.357421875$$
$$p_{21}(x) = h_{00}(t)y_{21} + h_{10}(t)h\, d_{21} + h_{01}(t)y_{22} + h_{11}(t)h\, d_{22}, h = 8.0000, t = \frac{x - 179.0000}{h}$$
$$= 0.0068359375x^3 - 3.8662109375x^2 + 731.3876953125x - 46167.5517578125$$

Appendix 5. D_4 area full calculation

$Segment\ 1: interval\ [23.0, 30.0]$

$Upper\ p_u(x) = 0.00439747327502427 * x ** 3 - 0.385398445092335 * x ** 2 + 11.2507288629738 * x + 16.6049562682219$

$Lower\ p_l(x) = -0.000315840621962923 * x ** 3 - 0.00694849368319089 * x ** 2 + 0.751822157434602 * x + 110.22667638484$

$Difference\ p_d(x) = p_u - p_l$
$= 0.00471331389698719 * x ** 3 - 0.378449951409144 * x ** 2 + 10.4989067055392 * x - 93.6217201166179$

$Antiderivative\ F\ (x)$
$= 0.0011783284742468 * x ** 4 - 0.126149983803048 * x ** 3 + 5.2494533527696 * x ** 2 - 93.6217201166179 * x$

$$\int_{23.0}^{30.0} p_d(x)\, dx = 45.713888888793$$

$Segment\ 2: interval\ [30.0, 38.0]$

$Upper\ p_u(x) = 0.00976562499999994 * x ** 3 - 1.03515625 * x ** 2 + 35.7421875 * x - 278.296875$

$Lower\ p_l(x) = 0.00558035714285709 * x ** 3 - 0.575892857142857 * x ** 2 + 18.96875 * x - 83.4285714285714$

$Difference\ p_d(x) = p_u - p_l$
$= 0.00418526785714285 * x ** 3 - 0.459263392857143 * x ** 2 + 16.7734375 * x - 194.868303571429$

$Antiderivative\ F\ (x)$
$= 0.00104631696428571 * x ** 4 - 0.153087797619048 * x ** 3 + 8.38671875 * x ** 2 - 194.868303571429 * x$

$$\int_{30.0}^{38.0} p_d(x)\, dx = 70.761904761905$$

$Segment\ 3: interval\ [38.0, 46.0]$

$Upper\ p_u(x) = 144.75 - 0.625 * x$

$Lower\ p_l(x) = -0.00390625 * x ** 3 + 0.4921875 * x ** 2 - 21.109375 * x + 417.78125$

$Difference\ p_d(x) = p_u - p_l = 0.00390625 * x ** 3 - 0.4921875 * x ** 2 + 20.484375 * x - 273.03125$

$Antiderivative\ F\ (x) = 0.0009765625 * x ** 4 - 0.1640625 * x ** 3 + 10.2421875 * x ** 2 - 273.03125 * x$

$$\int_{38.0}^{46.0} p_d(x)\,dx \;=\; 68.000000000000$$

$Segment\ 4: interval\ [46.0, 54.0]$
$Upper\ p_u(x) = -0.0009765625 * x ** 3 + 0.142578125 * x ** 2 - 7.54296875 * x + 256.3359375$
$Lower\ p_l(x) = 0.0009765625 * x ** 3 - 0.158203125 * x ** 2 + 7.73046875 * x - 7.8984375$
$Difference\ p_d(x) = p_u - p_l = -0.001953125 * x ** 3 + 0.30078125 * x ** 2 - 15.2734375 * x + 264.234375$
$Antiderivative\ F\ (x) = -0.00048828125 * x ** 4 + 0.100260416666667 * x ** 3 - 7.63671875 * x ** 2 + 264.234375 * x$

$$\int_{46.0}^{54.0} p_d(x)\,dx \;=\; 67.333333333332$$

$Segment\ 5: interval\ [54.0, 62.0]$
$Upper\ p_u(x) = -0.0009765625 * x ** 3 + 0.158203125 * x ** 2 - 9.23046875 * x + 301.8984375$
$Lower\ p_l(x) = 0.0022321428571428 * x ** 3 - 0.387276785714286 * x ** 2 + 21.4866071428571 * x - 280.459821428571$
$Difference\ p_d(x) = p_u - p_l$
$= -0.0032087053571428 * x ** 3 + 0.545479910714286 * x ** 2 - 30.7170758928571 * x$
$+ 582.358258928571$
$Antiderivative\ F\ (x)$
$= -0.000802176339285701 * x ** 4 + 0.181826636904762 * x ** 3 - 15.3585379464286 * x ** 2$
$+ 582.358258928571 * x$

$$\int_{54.0}^{62.0} p_d(x)\,dx \;=\; 77.095238095326$$

$Segment\ 6: interval\ [62.0, 69.0]$
$Upper\ p_u(x) = 0.00510204081632653 * x ** 3 - 1.00255102040816 * x ** 2 + 64.6045918367346 * x - 1262.63775510203$
$Lower\ p_l(x) = -0.00710641399416911 * x ** 3 + 1.38301749271138 * x ** 2 - 90.3376457725944 * x + 2073.27223032068$
$Difference\ p_d(x) = p_u - p_l = 0.0122084548104956 * x ** 3 - 2.38556851311954 * x ** 2 + 154.942237609329 * x$
$- 3335.90998542271$
$Antiderivative\ F\ (x)$
$= 0.00305211370262391 * x ** 4 - 0.795189504373179 * x ** 3 + 77.4711188046645 * x ** 2$
$- 3335.90998542271 * x$

$$\int_{62.0}^{69.0} p_d(x)\,dx \;=\; 62.234374999854$$

$Segment\ 7: interval\ [69.0, 77.0]$
$Upper\ p_u(x) = -0.00585937499999994 * x ** 3 + 1.27539062499999 * x ** 2 - 93.189453125 * x + 2380.794921875$
$Lower\ p_l(x) = 0.00223214285714291 * x ** 3 - 0.513392857142871 * x ** 2 + 37.984375 * x - 819.9375$
$Difference\ p_d(x) = p_u - p_l$
$= -0.00809151785714286 * x ** 3 + 1.78878348214286 * x ** 2 - 131.173828125 * x + 3200.732421875$
$Antiderivative\ F\ (x)$
$= -0.00202287946428571 * x ** 4 + 0.596261160714286 * x ** 3 - 65.5869140625 * x ** 2$
$+ 3200.732421875 * x$

$$\int_{69.0}^{77.0} p_d(x)\,dx \;=\; 78.571428571464$$

$Segment\ 8: interval\ [77.0, 85.0]$
$Upper\ p_u(x) = 0.00195312499999989 * x ** 3 - 0.498046875 * x ** 2 + 40.958984375 * x - 1000.587890625$
$Lower\ p_l(x) = 0.00195312499999989 * x ** 3 - 0.482421875 * x ** 2 + 38.177734375 * x - 891.072265625$
$Difference\ p_d(x) = p_u - p_l = -0.015625 * x ** 2 + 2.78125 * x - 109.515625$
$Antiderivative\ F\ (x) = -0.00520833333333333 * x ** 3 + 1.390625 * x ** 2 - 109.515625 * x$

$$\int_{77.0}^{85.0} p_d(x)\,dx \;=\; 105.333333333333$$

$Segment\ 9: interval\ [85.0, 93.0]$
$Upper\ p_u(x) = 0.00195312500000011 * x ** 3 - 0.529296875000014 * x ** 2 + 46.271484375 * x - 1226.369140625$
$Lower\ p_l(x) = 0.000976562500000087 * x ** 3 - 0.256835937500014 * x ** 2 + 20.9951171875 * x - 460.6767578125$
$Difference\ p_d(x) = p_u - p_l$
$= 0.000976562500000024 * x ** 3 - 0.2724609375 * x ** 2 + 25.2763671875 * x - 765.6923828125$
$Antiderivative\ F\ (x)$
$= 0.000244140625000006 * x ** 4 - 0.0908203125 * x ** 3 + 12.63818359375 * x ** 2 - 765.6923828125 * x$

$$\int_{85.0}^{93.0} p_d(x)\,dx \;=\; 113.000000000131$$

$Segment\ 10{:}\ interval\ [93.0, 101.0]$
$Upper\ p_u(x) = \ 0.000976562499999944 * x ** 3 \ - \ 0.2802734375 * x ** 2 \ + \ 25.2919921875 * x \ - \ 643.5751953125$
$Lower\ p_l(x) = \ -0.0068359375 * x ** 3 \ + \ 1.9697265625 * x ** 2 \ - \ 190.4345703125 * x \ + \ 6228.7841796875$
$Difference\ p_d(x) = \ p_u - \ p_l = \ 0.00781249999999994 * x ** 3 \ - \ 2.25 * x ** 2 \ + \ 215.7265625 * x \ - \ 6872.359375$
$Antiderivative\ F\ (x) = \ 0.00195312499999999 * x ** 4 \ - \ 0.75 * x ** 3 \ + \ 107.86328125 * x ** 2 \ - \ 6872.359375 * x$
$$\int_{93.0}^{101.0} p_d(x)\,dx \;=\; 105.999999999534$$

$Segment\ 11{:}\ interval\ [101.0, 109.0]$
$Upper\ p_u(x) = \ 0.0205078125 * x ** 3 \ - \ 6.3701171875 * x ** 2 \ + \ 657.7255859375 * x \ - \ 22519.9384765625$
$Lower\ p_l(x) = \ 0.0191127232142857 * x ** 3 \ - \ 5.99093191964286 * x ** 2 \ + \ 623.511579241071 * x \ - \ 21508.0308314732$
$Difference\ p_d(x) = \ p_u - \ p_l$
$= \ 0.0013950892857143 * x ** 3 \ - \ 0.379185267857142 * x ** 2 \ + \ 34.2140066964286 * x$
$- \ 1011.90764508928$
$Antiderivative\ F\ (x)$
$= \ 0.000348772321428575 * x ** 4 \ - \ 0.126395089285714 * x ** 3 \ + \ 17.1070033482143 * x ** 2$
$- \ 1011.90764508928 * x$
$$\int_{101.0}^{109.0} p_d(x)\,dx \;=\; 122.857142857334$$

$Segment\ 12{:}\ interval\ [109.0, 116.0]$
$Upper\ p_u(x) = \ -0.437317784256559 * x ** 3 \ + \ 147.594752186589 * x ** 2 \ - \ 16588.3381924198 * x \ + \ 620941.825072886$
$Lower\ p_l(x) = \ -0.00856413994169097 * x ** 3 \ + \ 2.98159620991254 * x ** 2 \ - \ 346.013119533528 * x \ + \ 13409.8950437318$
$Difference\ p_d(x) = \ p_u - \ p_l = \ -0.428753644314869 * x ** 3 \ + \ 144.613155976676 * x ** 2 \ - \ 16242.3250728863 * x$
$+ \ 607531.930029154$
$Antiderivative\ F\ (x)$
$= \ -0.107188411078717 * x ** 4 \ + \ 48.2043853255588 * x ** 3 \ - \ 8121.16253644315 * x ** 2$
$+ \ 607531.930029154 * x$
$$\int_{109.0}^{116.0} p_d(x)\,dx \;=\; 411.213541671634$$

$Segment\ 13{:}\ interval\ [116.0, 124.0]$
$Upper\ p_u(x) = \ 0.0048828125 * x ** 3 \ - \ 1.83203125 * x ** 2 \ + \ 227.921875 * x \ - \ 9286.6875$
$Lower\ p_l(x) = \ -0.296875 * x ** 3 \ + \ 106.875 * x ** 2 \ - \ 12810.75 * x \ + \ 511353.0$
$Difference\ p_d(x) = \ p_u - \ p_l = \ 0.3017578125 * x ** 3 \ - \ 108.70703125 * x ** 2 \ + \ 13038.671875 * x \ - \ 520639.6875$
$Antiderivative\ F\ (x)$
$= \ 0.075439453125 * x ** 4 \ - \ 36.2356770833333 * x ** 3 \ + \ 6519.3359375 * x ** 2 \ - \ 520639.6875$
$$* x \int_{116.0}^{124.0} p_d(x)dx \;=\; 454.333333328366$$

$Segment\ 14{:}\ interval\ [124.0, 132.0]$
$Upper\ p_u(x) = \ 0.00976562500000011 * x ** 3 \ - \ 3.765625 * x ** 2 \ + \ 482.21875 * x \ - \ 20398.25$
$Lower\ p_l(x) = \ 0.0185546875000001 * x ** 3 \ - \ 7.20703125 * x ** 2 \ + \ 931.453125 * x \ - \ 39960.6875$
$Difference\ p_d(x) = \ p_u - \ p_l = \ -0.0087890625 * x ** 3 \ + \ 3.44140625 * x ** 2 \ - \ 449.234375 * x \ + \ 19562.4375$
$Antiderivative\ F\ (x) = \ -0.002197265625 * x ** 4 \ + \ 1.14713541666667 * x ** 3 \ - \ 224.6171875 * x ** 2 \ + \ 19562.4375 * x$
$$\int_{124.0}^{132.0} p_d(x)\,dx \;=\; 102.333333333023$$

$Segment\ 15{:}\ interval\ [132.0, 140.0]$
$Upper\ p_u(x) = \ -0.00292968750000011 * x ** 3 \ + \ 1.20703125 * x ** 2 \ - \ 166.953125 * x \ + \ 7847.6875$
$Lower\ p_l(x) = \ -0.00195312500000011 * x ** 3 \ + \ 0.78125 * x ** 2 \ - \ 105.46875 * x \ + \ 4892.5$
$Difference\ p_d(x) = \ p_u - \ p_l = \ -0.000976562499999997 * x ** 3 \ + \ 0.42578125 * x ** 2 \ - \ 61.484375 * x \ + \ 2955.1875$
$Antiderivative\ F\ (x)$
$= \ -0.000244140624999999 * x ** 4 \ + \ 0.141927083333333 * x ** 3 \ - \ 30.7421875 * x ** 2 \ + \ 2955.1875 * x$
$$\int_{132.0}^{140.0} p_d(x)\,dx \;=\; 97.666666666817$$

$Segment\ 16$: $interval\ [140.0, 148.0]$
$Upper\ p_u(x) = -0.00585937499999989 * x ** 3 + 2.5078125 * x ** 2 - 358.90625 * x + 17264.875$
$Lower\ p_l(x) = 0.0020926339285715 * x ** 3 - 0.919084821428569 * x ** 2 + 132.734375 * x - 6230.9375$
$Difference\ p_d(x) = p_u - p_l = -0.00795200892857139 * x ** 3 + 3.42689732142857 * x ** 2 - 491.640625 * x + 23495.8125$
$Antiderivative\ F(x)$
$= -0.00198800223214285 * x ** 4 + 1.14229910714286 * x ** 3 - 245.8203125 * x ** 2 + 23495.8125 * x$
$$\int_{140.0}^{148.0} p_d(x)\,dx = 120.714285715250$$

$Segment\ 17$: $interval\ [148.0, 155.0]$
$Upper\ p_u(x) = 0.0102040816326531 * x ** 3 - 4.65561224489797 * x ** 2 + 705.905612244899 * x - 35494.0102040815$
$Lower\ p_l(x) = 0.00218658892128282 * x ** 3 - 0.99380466472304 * x ** 2 + 148.677478134109 * x - 7258.43586005837$
$Difference\ p_d(x) = p_u - p_l = 0.00801749271137029 * x ** 3 - 3.66180758017493 * x ** 2 + 557.22813411079 * x - 28235.5743440232$
$Antiderivative\ F(x)$
$= 0.00200437317784257 * x ** 4 - 1.22060252672498 * x ** 3 + 278.614067055395 * x ** 2 - 28235.5743440232 * x$
$$\int_{148.0}^{155.0} p_d(x)\,dx = 116.520833337214$$

$Segment\ 18$: $interval\ [155.0, 163.0]$
$Upper\ p_u(x) = 0.0253906250000001 * x ** 3 - 11.994140625 * x ** 2 + 1886.279296875 * x - 98696.576171875$
$Lower\ p_l(x) = 0.00990513392857145 * x ** 3 - 4.67843191964285 * x ** 2 + 734.597795758928 * x - 38295.8120814732$
$Difference\ p_d(x) = p_u - p_l = 0.0154854910714286 * x ** 3 - 7.31570870535715 * x ** 2 + 1151.68150111607 * x - 60400.7640904018$
$Antiderivative\ F(x)$
$= 0.00387137276785715 * x ** 4 - 2.43856956845238 * x ** 3 + 575.840750558036 * x ** 2 - 60400.7640904018 * x$
$$\int_{155.0}^{163.0} p_d(x)\,dx = 121.952380951494$$

$Segment\ 19$: $interval\ [163.0, 171.0]$
$Upper\ p_u(x) = -0.0625000000000001 * x ** 3 + 31.515625 * x ** 2 - 5292.40625 * x + 296050.265625$
$Lower\ p_l(x) = -0.00488281250000005 * x ** 3 + 2.5126953125 * x ** 2 - 431.0068359375 * x + 24679.5380859375$
$Difference\ p_d(x) = p_u - p_l$
$= -0.0576171875 * x ** 3 + 29.0029296875 * x ** 2 - 4861.3994140625 * x + 271370.727539063$
$Antiderivative\ F(x)$
$= -0.014404296875 * x ** 4 + 9.66764322916667 * x ** 3 - 2430.69970703125 * x ** 2 + 271370.727539063 * x$
$$\int_{163.0}^{171.0} p_d(x)\,dx = 244.333333343267$$

$Segment\ 20$: $interval\ [171.0, 179.0]$
$Upper\ p_u(x) = -0.00195312499999994 * x ** 3 + 0.970703125 * x ** 2 - 157.396484375 * x + 8380.505859375$
$Lower\ p_l(x) = -0.103515625 * x ** 3 + 54.619140625 * x ** 2 - 9599.044921875 * x + 561954.357421875$
$Difference\ p_d(x) = p_u - p_l = 0.1015625 * x ** 3 - 53.6484375 * x ** 2 + 9441.6484375 * x - 553573.8515625$
$Antiderivative\ F(x) = 0.025390625 * x ** 4 - 17.8828125 * x ** 3 + 4720.82421875 * x ** 2 - 553573.8515625 * x$
$$\int_{171.0}^{179.0} p_d(x)\,dx = 328.000000000000$$

$Segment\ 21$: $interval\ [179.0, 187.0]$
$Upper\ p_u(x) = -0.021484375 * x ** 3 + 11.646484375 * x ** 2 - 2101.923828125 * x + 126406.533203125$
$Lower\ p_l(x) = 0.0068359375 * x ** 3 - 3.8662109375 * x ** 2 + 731.3876953125 * x - 46167.5517578125$
$Difference\ p_d(x) = p_u - p_l$
$= -0.0283203125 * x ** 3 + 15.5126953125 * x ** 2 - 2833.3115234375 * x + 172574.084960938$
$Antiderivative\ F(x)$
$= -0.007080078125 * x ** 4 + 5.1708984375 * x ** 3 - 1416.65576171875 * x ** 2 + 172574.084960938 * x$
$$\int_{179.0}^{187.0} p_d(x)\,dx = 175.000000000000$$

$Segment\ 22$: $interval\ [187.0, 195.0]$
$Upper\ p_u(x) = -0.0136718749999999 * x ** 3 + 7.763671875 * x ** 2 - 1469.337890625 * x + 92803.509765625$

$Lower\ p_l(x) =\ 5.55111512312578e - 17 * x ** 3\ -\ 0.0859375 * x ** 2\ +\ 34.703125 * x\ -\ 3378.3359375$
$Difference\ p_d(x) =\ p_u -\ p_l =\ -0.013671875 * x ** 3\ +\ 7.849609375 * x ** 2\ -\ 1504.041015625 * x\ +\ 96181.845703125$
$Antiderivative\ F\ (x)$
$=\ -0.00341796875 * x ** 4\ +\ 2.61653645833333 * x ** 3\ -\ 752.0205078125 * x ** 2\ +\ 96181.845703125 * x$

$$\int_{187.0}^{195.0} p_d(x)\, dx\ =\ 62.666666664183$$