* 1. **Design of region *D4***
     1. **Curve fitting and interpolation**

|  |  |  |
| --- | --- | --- |
| Note that what is done in the design of *D2* and *D3* is pretty much fitting a shape with some curve. In numerical analysis, it is called interpolation, which means the process of predicting the continuous value between discrete data points with some function or curve. Bezier curves are a classical method of interpolation, which is essentially a parameterised polynomial. Hence, why not just use a polynomial to fit some given pattern and get a design? | Figure 1. Handwritten letter M | Figure 2. Runge phenomenon of 10 points Lagrange interpolation |
| Given a handwritten letter M shown in Figure 1, a polynomial is used for interpolation for 10 sample points of upper and lower boundaries respectively. Take the upper boundary as an example. Take 10 sample points , The Lagrange interpolating polynomial through them is | |

where is called Lagrange basis for this linear combination. Each if and , which guarantees that the polynomial must go through all points. This interpolation yields the curve in Figure 3. It is not well-fitted and has severe oscillations at the edges of the domain, which is the Runge phenomenon investigated in the context of Bezier curves. The solution is the same – consider composing multiple polynomials together.

* + 1. **Piecewise interpolation and Hermite basis functions**

To construct a polynomial between each interval and ensure that adjacent ones connect with continuity, it requires:

where is a parameter we assign to the slope shared by the ends of the two adjacent polynomials. Notice that there are 4 constraints, so consider using cubic polynomials with 4 free parameters.For the convenience of the derivation, use a factor to scale the interval to by letting . Hence, the intended satisfies

If assuming , there are

For convenience, write as a linear combination of , such that

with the result in Eq. 3, it can be found that the bases (called Hermite basis functions) are

* + 1. **Design with the Piecewise Cubic Hermite Interpolating Polynomial (PCHIP)**

There are 2 parameter to adjust: number of sample points and joint slope . For the sample, different numbers of sample points can influence how many details of the original graph can be caught by the PCHIP. After trials, it is found that 23 is a suitable value (see Table 1)

Table 1. Comparison between different numbers of sample points

|  |  |  |  |
| --- | --- | --- | --- |
| Sample number | 15 | 23 | 30 |
| Figure |  |  |  |

|  |  |  |
| --- | --- | --- |
| Here is set to be which is the average of the secant slopes of the adjacent points with the edge conditions . But in figure in Table 1, some unnatural wiggles are observed at sharp turns. It is because when a sample point is a stationary point, the corresponding might not be 0, making the PCHIP faking a new stationary point (see Figure 4). The solution is, when dealing with a local maximum or minimum, namely  If and have different signs, .  Which yields the curves Figure 5, which contains two sets of 22 PCHIPs . | Figure 4. slope fails | Figure 5. D4 |