**1.5 General models to translate designs to mathematical expressions**

**1.5.1 Generate an interpolation curve from the given points based on the Discrete Fourier Transform**

Consider an arbitrary closed (end-to-end) parametric curve

Denote it as a complex function

Note that this complex function can be expressed as a sum of infinite simple orbiting complex numbers with different frequencies and complex coefficients to describe the phases and amplitude. Hence, there is

This series, known as the **Fourier series**, was proposed by Joseph Fourier in 1807. The reason why the frequency is is that for any closed curve, is true if and only if ; you can see this as a sort of period ( ).The **Fourier Transform** is used to transform a function from the original domain to the frequency domain, namely to get corresponding s for different frequencies s. For Eq. 1.13, the transform is

There are two perspectives to understand this equation.

1. Understand the Fourier Transform from the nature of Euler‘s formula

According to Eqs. 1.13 and 1.14.

According to **Euler’s formula** , the complex number really describes a “revolution”. Therefore, if integrate it through t, it will give zero. Proof:

Hence, all terms in Eq. 15 are killed except .

1. Understand the Fourier Transform from Hilbert space

**Hilbert space** is a vector space which have infinite dimensions. In this space, the vectors are functions, and there is an operation on them called the inner product, which is analogous to the dot product in the regular vector spaces. For example, in Hilbert space defined on , the inner product of two functions and is

where the bar stands for the complex conjugate. Thus, the Fourier transform is an inner product

Here is the interesting part. Taking any two different integers , there is

which means are orthogonal. Therefore, Eq. 1.18 is actually projecting onto infinite **orthogonal bases** , which gives the component of function in the direction of frequency , i.e. .

However, in the real-world problem, it is unlikely to get a continuous curve, and most of the time, we have to deal with discrete sample points. Suppose there are discrete points on a curve in complex form:

Apply the **Discrete Fourier Transform** on it,

And finally, the curve is formed by the sum of the first terms of the Fourier series:

A higher can give a more accurate fitting of the original curve, and a relatively small can filter the high-frequency details and give a gentler (no steep wiggle) shape.