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**Design and Calculating Bounded Area**

**Stage 2 Mathematics Methods**

**SACE No. 216771E**

**Introduction**

The problem presented in this task is to design a mathematically defined visual design and to calculate the exact area of the regions enclosed within its boundaries. Unlike a purely artistic drawing, the logo must be expressed through explicit mathematical functions and models so that it can be rigorously analysed, reproduced, and quantified. The challenge therefore lies in combining creativity with precise mathematical reasoning to transform abstract curves into a measurable and well-defined design.

This task is significant both for learning and for real-world applications. From a learning perspective, it requires the comprehensive application of advanced mathematical concepts, including transformations, interpolation, and integration. It also develops problem-solving skills by requiring efficient and accurate solutions to both routine and non-routine problems, such as finding areas bounded by complex curves. From a real-world perspective, calculating the area of a design has practical uses: for example, determining the material required to produce it as a decoration or sign, estimating costs for embroidery or printing, or scaling it appropriately for digital and physical contexts. In this way, the task demonstrates how mathematical models directly support design, production, and communication in professional settings.

To solve this problem, a wide range of mathematical techniques and technologies will be employed. Polynomial functions, logarithmic and sigmoid functions, Bézier curves, Hermite interpolation, and Fourier series will be used to model different sections of the logo. Each method will be clearly defined and applied with appropriate mathematical notation and terminology, ensuring accurate modelling, effective problem-solving, and logical interpretation of results. Through this approach, the task not only highlights the depth and versatility of mathematical modelling but also demonstrates the ability to communicate reasoning effectively, draw logical conclusions, and recognise both the strengths and limitations of the methods used.

Integration is a fundamental mathematical technique used to calculate the accumulation of quantities. In the context of this task, integration allows us to determine the area enclosed between curves by summing infinitely many infinitesimal slices. This provides an exact and reliable method for quantifying regions bounded by complex functions. Thus, integration acts as the bridge between abstract function definitions and practical measures such as the total area of the visual design.

Python with NumPy and Matplotlib will be used to visualise curves, confirm accuracy, and efficiently compute results. Geogebra also helped in the design of specific regions that require manual adjustments.

1. **Part A: Working with a Given Function and Designing with Functions**
   1. **Overview**

Part A of the task requires a graphic design that is clearly described mathematically. It also asks to integrate two given functions and into the design. A good mathematical expression given in this section will be important in Part B when evaluating the area. To achieve this, a series of methods will be introduced, and based on them, a rather complex but not overly elaborate pattern is expected.

* 1. **Calculate the area of the given region**

Given a function . The second function It is created by vertically translating down by 2 units. The region bounded by the two curves is

(see Figure 1)

To calculate its area, there are 2 methods.

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| This area can be evaluated by the formula:    This is a general formula to calculate the region bounded by a pair of functions in a given domain. To use this formula, the equation of is required. By the definition of translation, . |  | original  Figure 1. Region *D0* |
| An easier way: the region between 2 curves is formed by translation. Hence, when evaluating the integral in Eq. 1, every infinitesimal rectangle of the Riemann summation has the same length , so the total area is equal to the area of a 4.5 by 2 rectangle (see the orange rectangles in Figure 2). |  | Figure 2. Put all infinitesimal rectangles together |

* 1. **Purpose of the design and a rough sketch of final design**

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| This is a design of a logo for the loved subject ­ Mathematics Methods. By modelling the design by mathematical expressions, it can be better used when it needs to be resized, moved and rotated in contexts such as printing it on uniforms or putting it on websites. Besides, a clear expression provides a way to evaluate its area analytically by integration.  Specifically for all regions of the final design, they will be arranged as Figure 3. The blue part is a fancy frame called transformed by . The red part is a wing of the bird shown in green called . is relatively easy, so it will be designed to use some basic functions like logs and exponentials. is more complex and will be designed using Bezier curves. Purple and grey regions and , designed based on the image of a handwritten letter M, stand for “Math” and “Methods”. Since their boundaries are expected to have the property of functions, an polynomial interpolation will be used to design it. Finally, the yellow flower vine Regionis most complex. To reflect all the details, a Fourier series will be suitable for translating a image to a region bounded by a enclosed curve. These rationales of methods used to design each region will be further discussed in following sections. | Figure 3.Rough sketch |

* 1. **Design of region *D1***
     1. **Graph transformation of a function**

The task asks to integrate the given functions and into the final design, and it is encouraged to use transformations including reflection, translation, and scaling. Generally, for a function , the transformation above can be done by following operations:

1. Translate right by : ; Translate left by : ; Translate upward by : ; Translate downward by : .
2. Horizontal scaling: . Vertical scaling: .
3. Reflection across the x-axis: . Reflection across the y-axis:

Moreover, these transformations can be combined. For example, means shift right by 3, stretch vertically by a factor of 2, reflect in the x-axis, and shift up by 4.

* + 1. **Non-single-valuedness and affine transformation of a parametric curve**

The definition of the function is that

It is also known as the **single-valuedness** of functions. However, the design is expected to be **non-single-valued**. Hence, affine transformations and parametric curves are introduced. A simple definition of an **affine transformation** is to perform a linear transformation on a vector space, and additionally, a translation. Generally, an affine transformation on vector in can be written as

where is a 2 by 2 matrix representing a linear transformation, and represents a translation. The transformation of a function is more complicated. For this specific purpose, since what is transformed is the whole shape of a function , which means the actual subject of the transformation is every point on the plane, namely

This equation naturally satisfies the restriction on the **domain.** But note that after the transformation, the relation between and will probably no longer obey the single-valuedness. To better describe this kind of curve, **parametric equations** are introduced. According to Eq. 2, an affine transformation on a parametric equation is

Parameterise is simply letting , so . And , the parametric form of , is given by transformation . Now all kinds of transformation can be applied to. To adjust the given function to be a fancy frame of the design, the following transformations are applied:

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| The region looks too long in the vertical direction. To compress it vertically to 1/5 of the original size and stretch horizontally to four times the original size requires the matrix  Since there is no translation, b is the zero vector. | Figure 4. Transformation 1 |  |  |
| Then the wiggling end of the curves is expected to be centered and aligned with the y-axis. Hence a flip transformation and a translation are needed. The flip matrix is .  The translating vector is | Figure 5. Transformation 2 |  |  |
| Copy a pair of curve and to make a symmetrical shape, where the same flip matrix is needed. | Figure 6. Transformation 3 |  |  |

An advantage of affine transformations is that the amount of the scaling of the area can be easily calculated by the **determinant** of the transformation matrix. This will be further discussed in Part B.

* + 1. **Non-linear transformation and Jacobian**

The shape of a frame is expected to be more “enclosed”, by curlingit into a circular shape. This requires a **non-linear transformation**. A non-linear transformation does not guarantee that the parallel lines in the original graph remain parallel, and hence do not have a constant matrix to describe the transformation for every point on the Cartesian plane. A general way is to use a **map**. For this purpose, the map is:

since the new graph’s x-coordinates have a range from -18 to 18, and the curl starts at in the polar frame. What this **curl** transformation actually does is to wrap the Cartesian grid into a polar shape (see Figure 6). For convenience, the transformed curves are named .

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| ↓ curl    Figure 7. Curl transformation |  |

Though this transform do not have a matrix to describe it, it do behave locally as a linear transformation. This sort of linear behaviour can be measured by the Jacobian. For transformation , the Jacobian is

The method to evaluate the transformed area based on the Jacobian will be further discussed in Part B.

* + 1. **Smoothness Consideration**

Generally, the smoothness can be measured by continuity , namely the nth derivatives at the two ends are equal. However, when dealing with a parametric curve, this approach will probably fail since the rate of change in position of a point with respect to t is not only related to the smoothness of the shape but also the smoothness of the motion (change in velocity). Thereby, the geometric continuity is introduced. A curve or surface can be described as having continuity, with being the increasing measure of smoothness. Consider there are two curves, and their continuity is defined as: (1) , the curves touch at the join point; (2) , the curves also share a common tangent direction at the join point, which means the derivatives are collinear; (3) , the curves also share a common centre of curvature at the join point (Wikipedia, 2023). For graph designs, is sufficiently smooth. Also, an affine transformation will not affect the original continuity

To construct continuous joins, copy one set of curves shown in Figure 6 and rotate them 180 degrees to get curves **,** therefore,

(for full equations, see Appendix 1).

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| Figure 8. Region *D1* | Curves give us Region *D1* as shown in Fig. 2, where there are four joins satisfying and continuity: , , , and . Take as an example. |

Since the direction of the motion is reversible in a parametric equation, Eq. 7 actually implies there is a continuity. By applying similar approaches to the other three joins, it can be found that all joins are continuous. Now, these curves can be denoted as

And let Region be the region bounded by the piecewise smooth closed curve but not by , namely

* 1. **Design of region *D2***

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| Figure 9. g1(x) | This part of the design is the wing of a bird, which would be formed by some basic functions. Its shape is roughly like Figure 9, constituted by functions .  is simply a straight line. For convenience, it is put horizontally and assigned to be | C:\Users\Administrator\AppData\Local\Microsoft\Windows\INetCache\Content.Word\wing.png  Figure 10. D2 Rough shape |

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| --- | --- |
| Figure 11. g2(x) | is winding and has an “S” shape. A very suitable function is Sigmoid, which is defined as  To fit this curve to the shape, use the graph transformation in 1.4.1, as it is easier to apply to functions:  Where , , and control the dilations in and directions and horizontal translation. By adjusting them, it is found that the curve in Figure 10 is given when , , and are 2.6, 3.4, and 0.9. |
| Figure 12. g3(x) | is the upper half of the left side of the wing. It is like a log function and can be modelled as  Where , , , and control the dilations in and directions and translation in and directions. The curve in Figure 10 is given when , , , and are 1, 0.1, -0.8, and 3.8, respectively. It is expected that the and connect at with continuity. The derivative of at is |
| Figure 13. g4(x)    Figure 14. D2 no domain | is modelled by a quadratic function:  Whose derivative is . To make and connect at with continuity, the following 2 equations need to be satisfied.  Note that the equations are now hard to solve. To simplify it, let . Thus, .  3 variables and 2 equations leave 1 degree of freedom to adjust. When set , Figure 12 is given, and . |
| Figure 15. D2 | To get the joints A, B, D, to get the **domains**, it is necessary to solve the following equations:  The solution is . Hence, the domains are , , , and . |

Let the upper and the lower functions be respectively

And Region (see Figure 14).

* 1. **Design of region *D3***
     1. **Introduction to Bézier Curves**

Bézier Curves are a type of interpolation method intended to approximate a real-world shape that otherwise has no mathematical representation. The general equation of a Bezier curve of order n is

Where are the control points**.** The curves are expected to be rather wavy when designing the pattern using Bezier curves. It can be achieved by using a high-order Bezier curve, but there are problems. First, when there are many control points, it is hard to imagine the shape of the curve. Secondly, one cannot modify only a certain part of the curve, because if one control point is moved, the whole curve will change. In addition, applying a higher-order curve can cause a problem of oscillation at the edges of an interval. It is called **Runge's phenomenon**. A viable solution is a composite Bezier curve, namely, composing a curve of multiple shorter Bezier curves. To do that, the smoothness of their joints must be guaranteed. The derivative of Eq. 8 is

(for the full derivation, see Appendix 2) Put and , the conclusion can be obtained that: the tangent vector at the start point is and at the terminal point is . Thus, to ensure that the two curves have continuity at the junction is to put the four control points near the junction collinearly, namely

* + 1. **Design based on Bézier Curves**

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| Figur*e* 16. D3 | Figure 17. Continuous joint between B1 and B2 | Figure 18*.* Continuous joint between B*6*and B*7* | Region D3 is enclosed by 8 Bezier curves with control points *.* The curves are connected end to end and form an enclosed shape (see Figure 15). Note that Bezier curves naturally have a domain of . As shown in Figure 16 & Figure 17, there are 2 pairs of curves that have continuous joints, which are taken as samples to show calculations (for full equations see Appendix 3). |

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| Figure 19. B*2*    Figure 20. B*1* | (see Figure 18) has control points  Substitute into Eq. 8, there is  **To ensure G1 continuity, it must be**  That is to say,  Under this constraint, the control points of B1 (see Figure 19) are designed to be  where the k is set to be 3.90. Hence the equation is |
| Figure 21. B*4*    Figure 22. B*5* | (see Figure 20) has control points  Note there is only one turning point is required for this curve. So it only needs 3 control points, namely it is a quadratic Bezier curve. Substitute into Eq. 8, there is  **To ensure G1 continuity, it must be**  **That is to say,**  Under this constraint, B5 (see Figure 21) are designed to be |

All these 8 curves and the Regioncan be denoted as

* 1. **Design of region *D4***
     1. **Curve fitting and interpolation**

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| Note that what is done in the design of *D2* and *D3* is pretty much fitting a shape with some curve. In numerical analysis, it is called interpolation, which means the process of predicting the continuous value between discrete data points with some function or curve. Bezier curves are a classical method of interpolation, which is essentially a parameterised polynomial. Hence, why not just use a polynomial to fit some given pattern and get a design? | Figure 23. Handwritten letter M | Figure 24. Runge phenomenon of 10-point Lagrange interpolation |
| Given a handwritten letter M shown in Figure 22, a polynomial is used for interpolation for 20 sample points of upper and lower boundaries. Take the upper boundary as an example. Take 10 sample points , The Lagrange interpolating polynomial through them is | |

where is called the Lagrange basis for this linear combination. Each if and , which guarantees that the polynomial must go through all points. This interpolation yields the curve in Figure 25. It is not well-fitted and has severe oscillations at the edges of the domain, which is the Runge phenomenon investigated in the context of Bezier curves. The solution is the same – consider composing multiple polynomials together.

* + 1. **Piecewise interpolation polynomial and Hermite basis functions**

To construct a polynomial between each interval and ensure that adjacent ones connect with continuity, it requires:

where is a parameter we assign to the slope shared by the ends of the two adjacent polynomials. Notice that there are 4 constraints, so consider using cubic polynomials with 4 free parameters.For the convenience of the derivation, use a factor to scale the interval to by letting . Hence, the intended satisfies

If assuming , there are

For convenience, write as a linear combination of , such that

with the result in Eq. 12, it can be found that the bases (called Hermite basis functions) are

* + 1. **Design with the Piecewise Cubic Hermite Interpolating Polynomial (PCHIP)**

There are 2 parameters to adjust: the number of sample points and joint slope . For the sample, OpenCV-Python is used to extract sample pointa from the boundaries. Different numbers of sample points can influence how many details of the original graph can be caught by the PCHIP. After trials, it is found that 23 is a suitable value (see Table 1)

Table 1. Comparison between different numbers of sample points

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| --- | --- | --- |
| 15 sample points | 23 sample points | 30 sample points |
|  |  |  |

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| --- | --- | --- |
| Here is set to be , which is the average of the secant slopes of the adjacent points with the edge conditions . But in figures in Table 1, some unnatural wiggles are observed at sharp turns. It is because when a sample point is a stationary point, the corresponding might not be 0, making the PCHIP faking a new stationary point (see Figure 25). The solution is, when dealing with a local maximum or minimum, namely  If and have different signs, . | Figure 26. Fake peak | Figure 27. D4 |

Which yields the curves in Figure 26, which contains two sets of 22 PCHIPs , namely,

And Region . For expanded equations and Python code see Apendix 4 and Code Appendix.

* 1. **Design of region *D5***
     1. **Generate an interpolation curve from the given points based on the Discrete Fourier Transform**

The PCHIP can only be applied to points that have a function relation between ys and xs, namely, for every y, there is only one corresponding x. That is to say, it cannot be applied to parametric curves. That is why the **Fourier Transform** is introduced. Consider an arbitrary closed (end-to-end) parametric curve . Denote it as a complex function

Note that this complex function can be expressed as a sum of infinite simple orbiting complex numbers with different frequencies and complex coefficients to describe the phases and amplitude. Hence, there is

This series, known as the **Fourier series**, was proposed by Joseph Fourier in 1807. The reason why the frequency is is that for any closed curve, is true if and only if ; you can see this as a sort of period ( ). The **Fourier Transform** is used to transform a function from the original domain to the frequency domain, namely to get corresponding s for different frequencies s, which is:

Proof: According to Eqs. 17 and 15,

According to **Euler’s formula** , the complex number really describes a “revolution”. Therefore, if integrate it through t, it will give zero as:

Hence, all terms in Eq. 15 vanish except . In the real-world problem, it is unlikely to get a continuous curve, and most of the time, we have to deal with discrete sample points. Suppose there are discrete points on a curve in complex form: . The **Discrete Fourier Transform** and the Fourier series fetched are:

A higher can give a more accurate fitting of the original curve, and a relatively small can filter the high-frequency details and give a gentler (no steep wiggle) shape.

* + 1. **Design with the Fourier series**

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| Given an original graph like Figure 27, first take sample points with OpenCV-Python on the contour of it. Note that the contour must be enclosed for the periodic property of the Fourier series. Since this is a rather complex shape, to keep more details, 2500 sample points are taken. Next, the only parameter that needs to be adjusted is the number of terms retained from the series (the value of *M*). Different *M*s give different curves (see Figure 28). | Figure 28. D5 original graph | Figure 29. Sampled graph |

*Table 2.* Different *values of* M *and corresponding* curves

|  |  |  |
| --- | --- | --- |
| M=2 | M=10 | M=100 |
|  |  |  |
| M=200 | M=1000 | M=2500 |
|  |  |  |

The process to get the Discrete Fourier Transform from 2500 sample points requires massive calculation. Therefore, a Python program is used (see Code Appendix). Notice that when M=2500, the curve start to have sawteeth from the pixels of the original image, which is not expected. Hence, M=1000 is considered more suitable. The features of the graphs with different numbers of terms of the Fourier series kept can be better observe from this video (<https://github.com/Yanttttt/SACE_Math_Method_Investigation/raw/refs/heads/main/D5/epicycles.mp4> ). Now, an enclosed curve is obtained. It only consists of one curve. Thus, it has continuity everywhere. Translate it from complex form to vector form:

Let , the Region (see the figure where M=1000 in Table 2).

* 1. **Assemble and arrange parts**

Now all parts are settled and they are going to be assembled. However, at this stage, are they not at the same scale, specifically, some are too big while some are too small (see Figure 29). Moreover, some final adjustments of the patterns are required. Hence, the arrangement is based on affine transformations.

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| C:\Users\Administrator\AppData\Local\Microsoft\Windows\INetCache\Content.Word\assemble.png  Figure 30. Parts before adjustments | | C:\Users\Administrator\AppData\Local\Microsoft\Windows\INetCache\Content.Word\assemble_D3.png  Figure 31. Assemble *D3* | First, assemble *D3* since it determines where the wing and the vine lie. *D3* is too big, so before the translation , compress it first by linear transformation . Thus, the affine transformation is |
| Figure 32. Assemble *D2* | To assemble the wing to the bird, a rotation and a scaling are required. The composed transformation is    And the translation is . The affine transformations on the parameterised functions are | | |
| C:\Users\Administrator\AppData\Local\Microsoft\Windows\INetCache\Content.Word\assemble_D5.png  Figure 33. Assemble *D5* | The vine needs a rotation of 90° and a scaling. The composed matrix is  And the translation is . The affine transformation is | | |
| C:\Users\Administrator\AppData\Local\Microsoft\Windows\INetCache\Content.Word\assemble_label.png  Figure 34. Final design | For the handwritten letter M in *,* copy an identical region . Different scalings and shearings are applied:  And the translations are and . The affine transformations are.  Here, all designs are done. Their areas are ready to be evaluated. | | |

**Evaluation:** The final design in Figure 33 design demonstrates high quality. Its construction integrates polynomial functions, logarithmic and sigmoid (exponential) functions, 8 Bézier curves, 22 pieces of Hermite interpolation functions, a curve formed by Fourier transforms, and various transformations. This diverse combination ensures both local smoothness and global accuracy, while allowing flexible control of the shape. The design is composed of 6 distinct regions, each with clear analytic or constructive definitions, which provides a solid foundation for precise area and geometric calculations and highlights the effective integration of various mathematical tools and curve design techniques.

1. **Part B: Calculations of the Area** 
   1. **Overview**

The task asks to apply calculus and mathematical reasoning to analyse full designs. Specifically, calculate the area of the design using algebraic integration, justify assumptions and limitations in modelling processes, and discuss how the model simplifies real-world structures.

* 1. **Selection and rationale: the area calculation methods based on the algebraic integration**

For regions formed by different types of curves, a suitable selection of area calculation methods is crucial. Generally, in , the area of a Region is defined by a double integral as

Regions bounded by 2 functions and bounded by an enclosed parametric curve have different applications of this equation. For regions like , its area is

which is exactly the Eq. 1 used in Part A when evaluating the area of . For parametric curves, **Green’s theorem** is introduced:

where means integrate on an enclosed curve, and means the boundary of . By letting , there is

Note that this area has direction. If integrate along the curve clockwise, it gives a negative value.

Additionally, for an affine transformation , there is

where is the determinant of a matrix , indicating the area scaling factor. For a non-linear transformation , the Jacobian reflects the linear transformation behaviour in each infinitesimal element in the region.Summing these behaviours yields the scaling of the area:

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| **Summary: Select the area calculation method in different situations** | | | |
| Region bounded by functions | Region enclosed by parametric curves | Affine transformation | Non-linear transformation |
| Eq. 21 | Eq. 22 | Eq. 23 | Eq. 24 |

* 1. **Calculate the area of Region *D1***

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| transform→  Figure 35. 1/4 of D1 | Region can be divided into 4 chunks with the same area, as shown in Figure 34. This region is obtained by transforming the given region , whose area is 9 unit². According to 1.4.2, it is first transformed to    which corresponds to functions |

Note that the domain is very important here. Since both and have the domain , there must be

Thus, the region . Then, the curl transformation was applied on with the Jacobian:

Hence, according to Eq. 24,

(2 d.p.)

* 1. **Calculate the area of Region *D2***

Region (see Figure 14), where

It is clear that Eq. 21 should be used here:

Calculate the 4 integrals separately:

1. (2 d.p.)

The rear part is easy as . The front part with cannot be integrated directly. Thus, consider **integrating by parts**. Let and , then

Add the constant part, there is (2 d.p.)

1. (2 d.p.)
2. . Notice that **integration by substitution** is available as

Hence, the total area of is

Note that a transformation is used to adjust it finally, where

Therefore, according to Eq. 23, the area after the transformation is (2 d.p.).

* 1. **Calculate the area of Region *D3***

This region is bounded by the curve

where is a parametric equation. Therefore, Eq. 22 should be used:

Take  **for a sample calculation. First,**

**The curve integral is**

Similarly, for all can be found. Sum them up, the total area of Region is

The negative result makes sense since the Bezier curves designed go clockwise. Take the absolute value,

The transformation on is a scaling with a factor

Therefore the final area of is (2 d.p.).

* 1. **Calculate the area of Region *D4***

Region , where

For functions, consider using Eq. 21:

That is to say, the values of of 22 segments need to be evaluated. Take segment 1 with interval for sample calculation. The upper boundary is , while the lower boundary is . The difference is

The integral is

For full calculation, see Appendix 5. These calculations are long and repeated, so it is helped by Python. The total area is

Regions and is sclaed by factor

The areas are

* 1. **Calculate the area of Region *D5***

This region is enclosed by the curve

For a parametric curve, consider using Eq. 22:

Swap the integration and summations:

Note that

Additionally, the contribution of the term is also 0. Hence, only terms with remain. Substituting s, the area is

The transformed area is

* 1. **Sum the areas of parts & the meanings in real-world contexts**

The overall area of the design is

**Meanings in real-world contexts:** This design is for a logo in the real-world context. Hence, the area of a logo can be very useful in certain situations. For example, when printing it on uniforms to determine the space it occupies, or when resizing the logo for websites or apps, the area can be used to preserve proportions and balance visual weight. Moreover, in embroidery, screen printing, or laser cutting, when producing the logo as a decoration or sign, the area is directly related to production cost and material use.

* 1. **Evaluation of the strengths and limitations of the mathematics methods of design used**
     1. **A comparison of the transformation methods (Green: Strength. Red: Limitation)**

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| --- | --- | --- | --- |
|  | **Graph transformation** | **Affine transformation** | **Non-linear transformation** |
| **Object** | Function | Vector space (hence any vector, parametric equation, and parameterised function) | Every point in the plane. |
| **Non-single-valuedness** | Not supported | Supported before and after the transformation | Supported before and after the transformation |
| **Computational complexity** | Low | Moderate. Requires matrix operations. | High, involves nonlinear equations and singularity. |
| **Type of transformation supported** | Translation, scaling, reflection. Very limited transformational capability | Moderate, supports translation, scaling, reflection, rotation, and shearing. | Most, including complex distortions, bending, and even topological changes |
| **Preserved properties** | Straightness, parallelism, ratio, continuity, monotonicity and function property (single-valuedness) | Straightness, parallelism, continuity, and ratio. | Generally does not preserve geometric properties. |
| **Intuitiveness** | High, easy to understand and visualise | Moderate. Requires intuitive understandings of matrices and vector spaces. | Low |
| **Computational complexity of the area scaling factor** | Low | Low, just need to evaluate the determinant. | High, involves partial derivatives (Jacobians) and double integrals. |

* + 1. **A comparison of the methods of curve design by adjusting**

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|  | **Design with functions** | **Design with Bezier curves** |
| **Outcome** | Arbitrary functions | Quadratic or cubic parametric curves |
| **Non-single-valuedness** | Not supported for a single function. Can be achieved by multiple single-valued functions. | Supported |
| **Computational complexity** | Relatively low | Moderate, needs to expand equations. |
| **Intuitiveness** | Low, needs to construct and imagine the shape in mind | High |
| **Ensure continuity** | Hard, needs to take derivatives of both at joint. | Easy |

* + 1. **A comparison of the methods of curve design by fitting**

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| --- | --- | --- |
|  | **PCHIP** | **Fourier transform** |
| **Outcome** | Piecewise cubic functions | Cartesian form of a complex Fourier series |
| **Non-single-valuedness** | Not supported for a single function. Can be achieved by multiple single-valued functions. | Supported |
| **Computational complexity** | Moderate | Very high when doing summations of 1000+ terms |
| **Degree of process automation** | Relatively low, parameters such as number of sample points and the secant slope need to be assigned. | Very high, only needs to set the number of terms of the series kept. |
| **Ensure continuity** | Ensures continuity | Naturally has continuity |

* + 1. **A comparison of the design by adjusting & the design by fitting**

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| --- | --- | --- |
|  | **Design by adjusting** | **Design by fitting** |
| **Strength** | Highly customised. | Highly automatic process. Can be customised by inputting hand-drawn images. |
| **Limitation** | Slow and complicated process. | Usually requires lots of calculations that have to be done using computers |

1. **Conclusion**

This project examined the analysis of geometric regions bounded by curves, incorporating a variety of functions, including polynomial, trigonometric, logarithmic and exponential functions, to model boundaries with differing growth and saturation behaviours. Observations indicate that these functional choices allowed for flexible and accurate modelling of curve shapes.

Mathematical methods, including Bezier curves, piecewise Hermite interpolation and the discrete Fourier transformation, proved effective in simplifying design tasks and ensuring continuity and smoothness where necessary. A pattern is observed that the design methods based on adjustment, like constructing functions and Bezier curves, are more flexible in design, but sometimes hard to control and imagine the shape, while methods based on interpolation have a more automatic process but require more calculation. Patterns on methods using sample points like Bezier curves and polynomial interpolation indicate are also observed that with more sample points, the output design is more accurate, but at the same time tends to have Runge’s phenomenon, which can be solved by using piecewise and composite curves.

Differentiation techniques are applied in this investigation mainly to ensure the smooth connection between curves. Observations indicate that curves and functions like (Bezier curves and Piecewise Cubic Hermite Interpolating Polynomials (PCHIP)) that have pre-defined slope properties at joints can better ensure smooth connections and easy to make adjustments without interfering original shape.

The use of definite integrals was central to quantifying areas between the curves. Analytical integration of piecewise-defined functions, combined with the curve integration of parametric equations, provided reliable results. The effects of affine transformations and non-linear transformations on the area are also investigated. It is observed that for some special regions, calculating and applying the scaling factor of a transformation on the original area is easier than doing the integration directly. By using various integration techniques, the overall area of the final design is calculated to be 106.90 unit2 (2 d.p.).

Potential improvements include using Monte Carlo methods to numerically validate the area, increasing the density of sample points for designs formed by interpolations in regions where the curves change rapidly, or experimenting with alternative functional forms to better capture extreme variations. These adjustments would further reduce approximation errors and enhance the precision of integral-based area calculations.

**Bibliography**

3Blue1Brown (2019). *But what is a Fourier series? From heat flow to drawing with circles*. [online] YouTube. Available at: <https://www.youtube.com/watch?v=r6sGWTCMz2k&list=PL4VT47y1w7A1-T_VIcufa7mCM3XrSA5DD&index=4>.

David Jerison. (2006). *Single Variable Calculus.* MIT OpenCourseWare. [online] Available at: <https://ocw.mit.edu/courses/18-01-single-variable-calculus-fall-2006/>.

Denis Auroux. (2007). *Multivariable Calculus*. MIT OpenCourseWare. [online] Available at: <https://ocw.mit.edu/courses/18-02-multivariable-calculus-fall-2007/>.

Gilbert Strang. (2010). *Linear Algebra.* MIT OpenCourseWare. [online] Available at: <https://ocw.mit.edu/courses/18-06-linear-algebra-spring-2010/>

Laurent Demanet. (2012). *Introduction to Numerical Analysis.* MIT OpenCourseWare. [online] Available at: <https://ocw.mit.edu/courses/18-330-introduction-to-numerical-analysis-spring-2012/>

Wikipedia. (2022). Cubic Hermite spline. [online] Available at: <https://en.wikipedia.org/wiki/Cubic_Hermite_spline>.

Wikipedia. (2023). *Smoothness*. [online] Available at: <https://en.wikipedia.org/wiki/Smoothness>.

**Code Appendix**

All source code is available at: <https://github.com/Yanttttt/SACE_Math_Method_Investigation>

**Appendix 1. *D1* curves**

**Appendix 2. Derivative of the Bezier curves.**

We have discussed the general form of Bezier curves. But to make the index of control point sstart from 1, we didn’t use the most general equation, which is supposed to be

First, take the derivative of the th term of a Bezier curve without control points as coefficients:

Expand the binomial coefficient:

Notice that the two terms are typical Bezier curves of lower orders. Hence,

Now, apply the control points.

Expand this and reorder the terms by :

In this transformation, there are two eliminations, namely as the th term does not exist and the very last term with the binomial coefficient , which is usually considered to be zero. Write it in the form of a sum:

This form can also be translated to our convetion by letting .

**Appendix 3. Equations.**

The control points are,

According to Eq. 8,

**Appendix 4. PCHIP Equations**

For the upper boundary, the sample points are:

(23.0000,125.0000),(30.0000,126.0000),(38.0000,121.0000),(46.0000,116.0000),(54.0000,111.0000),(62.0000,105.0000),(69.0000,98.0000),(77.0000,92.0000),(85.0000,82.0000),(93.0000,70.0000),(101.0000,58.0000),(109.0000,47.0000),(116.0000,122.0000),(124.0000,116.0000),(132.0000,103.0000),(140.0000,93.0000),(148.0000,83.0000),(155.0000,69.0000),(163.0000,55.0000),(171.0000,84.0000),(179.0000,107.0000),(187.0000,122.0000),(195.0000,121.0000)

Hence, according to Eq. 12 and the Hermite basis functions,

Similarly, for the lower boundary, the sample ponits are (23.0000,120.0000), (30.0000,118.0000), (38.0000,112.0000), (46.0000,108.0000), (54.0000,102.0000), (62.0000,95.0000), (69.0000,90.0000), (77.0000,80.0000), (85.0000,68.0000), (93.0000,56.0000), (101.0000,45.0000), (109.0000,28.0000), (116.0000,25.0000), (124.0000,101.0000), (132.0000,91.0000), (140.0000,80.0000), (148.0000,66.0000), (155.0000,53.0000), (163.0000,39.0000), (171.0000,36.0000), (179.0000,80.0000), (187.0000,106.0000), (195.0000,121.0000).

Hence, according to Eq. 12 and the Hermite basis functions,

**Appendix 5. *D4* area full calculation**