**Design and Calculating Bounded Area**

**Introduction**

**1 Part A - Working with a Given Function and Designing with Functions**

**1.1 Calculate the area of the given region**

Given a function . The second function It is created by vertically translating down by 2 units. The region bounded by the two curves is

To calculate its area, there are 2 methods.

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| An intuitive way: the region between 2 curves is formed by translation, hence its area is equal to the area of a 4.5 by 2 rectangle (you can see that as you put all the infinitesimal rectangles of the Riemann summation together). |  |
| This area can also be evaluated by the formula:  This is a general formula to calculate the region bounded by a pair of functions in a given domain. To use this formula, the equation of is required. By the definition of translation, . |  |

**1.2 Affine transformation and parametric equation**

The task asks to integrate the region between the given function and into the final design, and it is encouraged to use transformations including reflection, translation, or dilations, which are **affine transformations**.

A simple definition of an affine transformation is to perform a linear transformation on a vector space, and additionally, a translation. Generally, an affine transformation on vector in can be written as

where is a 2 by 2 matrix representing a linear transformation, and represents a translation. The transformation of a function is more complicated. For this specific purpose, since what is transformed is the whole shape of a function , which means the actual subject of the transformation is every point on the plane, namely

This equation naturally satisfies the restriction on the **domain.** But note that after the transformation, the relation between and will probably no longer obey the restriction of a function (one-to-one correspondence). To better describe this kind of curve, **parametric equations** are introduced. Hence, a curve can be written as

where vector P records the position of a point on the curve at time t. Consequently, the whole curve is the trajectory of the point. According to Eq. 1.2, a general affine transformation on a parametric equation is

Here are some examples on .

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| **Operation and comment** | **Graph** | **Working out** |
| **Parameterise**  To parameterise a function is all about picking the most accessible parameter to make life easier. Here for , the best choice is to let . | N/A | Let , |
| **Translate to get**  Since it is only a translation, the linear transformation matrix should be identical, namely  Translating down by 2 units gives the translation vector |  |  |
| **Scale transform**  To compress vertically to 1/5 of the original size and stretch horizontally to four times the original size requires the matrix  By the way, if the graph is sclaed by a negative factor, the graph will be flipped.  Since there is no translation, b is the zero vector. |  |  |
| **Shear transform**  A shear transformation is a type of linear transformation that slants the shape of an object. To shear the region in the y-direction by 1/2 requires the matrix  Since there is no translation, b is the zero vector. | C:\Users\Administrator\AppData\Local\Microsoft\Windows\INetCache\Content.Word\transformation_shear.png |  |
| **Rotation transform**  A general rotation transformation can be denoted as  where positive represents a counter-clockwise rotation. Here put , then | C:\Users\Administrator\AppData\Local\Microsoft\Windows\INetCache\Content.Word\transformation_rot.png |  |

There are two very good things about affine transformations. First, they are composable. For example, the transformation to first scale the given curves and , copy a pair of curves and , flip this pair of curves around the y-axis, translate them to align with the original curves, rotate them all by 180 degrees, and finally translate them to 10 units above the origin, can be written as

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Another advantage of affine transformations is that the amount of the area changed in the transform can be easily calculated by the **determinant** of the transformation matrix. This will be further discussed in Part B.

**1.3 Non-linear transformation and Jacobian**

Here, I want the shape to be more “enclosed”, **by curling** it to a circular shape. This requires a non-linear transformation. A non-linear transformation does not guarantee that the paralleled lines in the original graph remain paralleled, and hence do not have a constant matrix to describe the transformation for every point on the cartesian plane. A general way is to establish a map. For this purpose, the map is:

since the new graph’s x-coordinates have a range from -18 to 18. This **curl** transform gives a shape shown in Fig. 1.As a matter of convenience, after the transformation, the curves shown in Fig. 1 are named .

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| **Fig.1.** |  |

Though this transform do not have a matrix to describe it, it do have a local linear behaviour when one look it closely enough. This sort of linear behaviour can be measured by Jacobian.For transformation, the Jacobian is

The method to evaluate the transformed area based on Jacobian will be further discussed in Part B.

**1.4 Smoothness Consideration**

Generally, the smoothness can be measured by continuity , namely the nth derivatives at the two ends are equal. However, when dealing with a parametric curve, this approach will probably fail since the rate of change in position of a point with respect to t is not only related to the smoothness of the shape but also the smoothness of the motion (change in velocity). Thereby, the geometric continuity is introduced.

A curve or surface can be described as having continuity, with being the increasing measure of smoothness. Consider there are two curves, their continuity is defined as:

: The curves touch at the join point.

: The curves also share a common tangent direction at the join point, which means the derivatives are collinear.

: The curves also share a common centre of curvature at the join point (Wikipedia, 2023).

Besides the conclusion , are smoother than , if n>m, it is also obvious that is stronger than ( ). For graph designs, is sufficiently smooth. Another property of continuity is that an affine transformation will not affect the original continuity, and actually, every continuous mapping can keep the original continuity.

To construct continuous joins, I copy one set of curves shown in Fig. 1 and rotate them 180 degrees to get curves .

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