**Design and Calculating Bounded Area**

**Introduction**

**1 Part A - Working with a Given Function and Designing with Functions**

**1.1 Calculate the area of the given region**

Given a function . The second function It is created by vertically translating down by 2 units. The region bounded by the two curves is

To calculate its area, there are 2 methods.

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| An intuitive way: the region between 2 curves is formed by translation, hence its area is equal to the area of a 4.5 by 2 rectangle (you can see that as you put all the infinitesimal rectangles of the Riemann summation together). |  |
| This area can also be evaluated by the formula:  This is a general formula to calculate the region bounded by a pair of functions in a given domain. To use this formula, the equation of is required. By the definition of translation, . |  |

**1.2 Affine transformation and parametric equation**

The task asks to integrate the region between the given function and into the final design, and it is encouraged to use transformations including reflection, translation, or dilations, which are **affine transformations**.

A simple definition of an affine transformation is to perform a linear transformation on a vector space, and additionally, a translation. Generally, an affine transformation on vector in can be written as

where is a 2 by 2 matrix representing a linear transformation, and represents a translation. The transformation of a function is more complicated. For this specific purpose, since what is transformed is the whole shape of a function , which means the actual subject of the transformation is every point on the plane, namely

This equation naturally satisfies the restriction on the **domain.** But note that after the transformation, the relation between and will probably no longer obey the restriction of a function (one-to-one correspondence). To better describe this kind of curve, **parametric equations** are introduced. Hence, a curve can be written as

where vector P records the position of a point on the curve at time t. Consequently, the whole curve is the trajectory of the point. According to Eq. 1.2, a general affine transformation on a parametric equation is

Here are some examples on .

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| **Operation and comment** | **Graph** | **Working out** |
| **Parameterise**  To parameterise a function is all about picking the most accessible parameter to make life easier. Here for , the best choice is to let . | N/A | Let , |
| **Translate to get**  Since it is only a translation, the linear transformation matrix should be identical, namely  Translating down by 2 units gives the translation vector |  |  |
| **Scale transform**  To compress vertically to 1/5 of the original size and stretch horizontally to four times the original size requires the matrix  By the way, if the graph is sclaed by a negative factor, the graph will be flipped.  Since there is no translation, b is the zero vector. |  |  |
| **Shear transform**  A shear transformation is a type of linear transformation that slants the shape of an object. To shear the region in the y-direction by 1/2 requires the matrix  Since there is no translation, b is the zero vector. | C:\Users\Administrator\AppData\Local\Microsoft\Windows\INetCache\Content.Word\transformation_shear.png |  |
| **Rotation transform**  A general rotation transformation can be denoted as  where positive represents a counter-clockwise rotation. Here put , then | C:\Users\Administrator\AppData\Local\Microsoft\Windows\INetCache\Content.Word\transformation_rot.png |  |

There are two very good things about affine transformations. First, they are composable. For example, the transformation to first scale the given curves and , copy a pair of curves and , flip this pair of curves around the y-axis, translate them to align with the original curves, rotate them all by 180 degrees, and finally translate them to 10 units above the origin, can be written as

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Another advantage of affine transformations is that the amount of the area changed in the transform can be easily calculated by the **determinant** of the transformation matrix. This will be further discussed in Part B.

**1.3 Non-linear transformation and Jacobian**

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| Here, I want the shape to be more “enclosed”, by curling it to a circular shape. This requires a non-linear transformation. A non-linear transformation does not guarantee that the paralleled lines in the original graph remain paralleled, and hence do not have a constant matrix to describe the transformation for every point on the cartesian plane. A general way is to establish a map. For this purpose, the map is:  since the new graph’s x-coordinates have a range from -18 to 18. This roll transform gives a shape shown in Figure. Though this transform do not have a matrix to describe it, it do have a local linear behaviour when one look it closely enough. This sort of linear behaviour can be measured by Jacobian. For transformation , the Jacobian is |  |

The method to evaluate the transformed area based on Jacobian will be further discussed in Part B.

**1.4 Consideration about smoothness**

Generally, the smoothness can be measured by continuity , namely the nth derivatives at the two ends are equal. However, when dealing with a parametric curve, this approach will probably fail since the rate of change in position of a point with respect to t is not only relate to the smoothness of the shape but also the smoothness of the motion (velocity). A curve or surface can be described as having continuity, with being the increasing measure of smoothness. Consider there are two curves, their continuity is defined as:

: The curves touch at the join point.

: The curves also share a common tangent direction at the join point, which means the derivatives are collinear.

: The curves also share a common centre of curvature at the join point. (Wikipedia, 2023)