Problem Set 0

Name: Connor Yan

Problem 0-1.

(a) $\binom{5}{0} = 1$, $\binom{5}{1} = 5$, $\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{120}{12} = 10$, $\binom{5}{3} = \binom{5}{2} = 10$, $\binom{5}{4} = \binom{5}{1} = 5$
$\therefore A = \{1, 6, 9, 12, 13\}$
$B = \{3, 6, 12, 15\}$, $A \bigcap B = \{6, 12\}$

(b) $A \bigcup B = \{1, 3, 6, 9, 12, 13, 15\} \therefore |A \bigcup B| = 7$

(c) $|A - B| = 3$

Problem 0-2.

(a) $\mathrm{E}[X] = \frac{1}{2^3} \times 3 + 2 \times \frac{1}{2^3} \times 3 + 3 \times \frac{1}{2^3} = \frac{12}{8} = \frac{3}{2}$

(b) $\mathrm{E}[Y] = \frac{1}{6^2} + 2 \times \frac{2}{6^2} + 3 \times \frac{2}{6^2} + 4 \times (\frac{2}{6^2} + \frac{1}{6^2}) + 5 \times \frac{2}{6^2} + 6 \times \frac{2}{6^2} \times 2 + 8 \times \frac{2}{6^2} + 9 \times \frac{1}{6^2} + 10 \times \frac{2}{6^2} + 12 \times \frac{4}{6^2} + 15 \times \frac{2}{6^2} + 16 \times \frac{1}{6^2} + 18 \times \frac{2}{6^2} + 20 \times \frac{2}{6^2} + 24 \times \frac{2}{6^2} + 25 \times \frac{1}{6^2} + 30 \times \frac{2}{6^2} + 36 \times \frac{1}{6^2} = 49/4$

(c) $\mathrm{E}[X + Y] = 55/4$

Problem 0-3.

(a) $A = 100, B = 18$
$A \mod 2 = 0, B \mod 2 = 0$
$\therefore$ True.

(b) $A \mod 3 = 1, B \mod 3 = 0$
$\therefore$ False.

(c) $A \mod 4 = 0, B \mod 4 = 2$
$\therefore$ False.

Problem 0-4.

Prove $\sum_{i=1}^{n} i^3 = (\frac{n(n+1)}{2})^2$ where $n \geqslant 1$

As $n = 1$,

$$\sum_{i=1}^{1} i^3 = 1 = \left(\frac{1 \times (1+1)}{2}\right)^2$$

Assume that the equition is true as $n = m$. Therefore,

$$\sum_{i=1}^{m} i^3 = \left(\frac{m(m+1)}{2}\right)^2$$

As $n = m + 1$,

$$\begin{aligned}\sum_{i=1}^{m+1} i^3 &= \sum_{i=1}^{m} i^3 + (m+1)^3 \\ &= \left(\frac{m(m+1)}{2}\right)^2 + (m+1)^3 \\ &= \frac{m^4 + 6m^3 + 13m^2 + 12m + 4}{4} \\ &= \left(\frac{(m+1)(m+1+1)}{2}\right)^2\end{aligned}$$

Hence, for any positive integer n, $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$ is true.

Problem 0-5.

Prove:

Let $G_1 = (V_1, E_1)$, where $|V_1 = 1|$ and $|E_1 = 0|$. Since there are only a vertex in the graph, it is apparently acyclic.

Assume that a graph $G_k = (V_k, E_k)$ that has m vertices and $m - 1$ edges is acyclic. If we now add a edge and a vertex, constructing a graph with $k + 1$ vertices and k edges, since the new vertex cannot form a cycle with those original vertices the new graph is still acyclic.

Hence, any connected undirected graph $G = (V, E)$, for wich$|V| - 1 = |E|$ is acyclic.

Solution: Induct on the number of vertices k. Base Case: a graph containing one vertex and zero edges is trivially acyclic. Now assume for induction the claim is true for any connected graph having exactly k vertices and k − 1 edges, and consider any connected graph G containing exactly k + 1 vertices and k edges. G is connected so every vertex connects to at least one edge. Since each of the k edges connects to two vertices, the average degree of vertices in G is 2k/(k + 1) < 2, so there exists at least one vertex v with degree 1, connected to exactly one vertex u. Removing v and the edge connecting v to u yields a graph G0 on k vertices and k − 1 edges that is also connected. Vertex v cannot be in any cycle of G since a vertex in a cycle has degree at least 2, so G contains a cycle only if G0 contains a cycle. By the inductive hypothesis G0 is acyclic, so G is also.

Problem 0-6. Submit your implementation to alg.mit.edu.

```
def count_long_subarray(A):
    '''
    Input:  A     | Python Tuple of positive integers
    Output: count | number of longest increasing subarrays of A
    '''
    count = 0
    ##################
    # YOUR CODE HERE #
    ##################
    return count
```