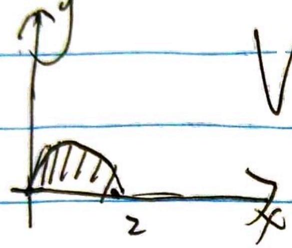
PS6 29th Dec 2024

Lecture 22 3rd Jan ~~2024~~ 2025

4B-2 e)



$$V=\int_0^2 2\pi x\cdot(2x-x^2)\,dx$$

$$=2\pi\int_0^2(2x^2-x^3)\,dx$$

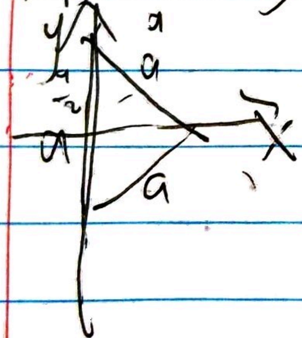
$$=2\pi\left(\frac{2}{3}x^3-\frac{1}{4}x^4\right)\Big|_0^2=2\pi\left(\frac{16}{3}-4\right)=\frac{8}{3}\pi$$

g)

$$V=\int_0^a 2\pi x(\sqrt{ax})\,dx=2\pi\sqrt{a}\int_0^a x^{\frac{3}{2}}\,dx$$

$$=2\pi\sqrt{a}\left(\frac{2}{5}x^{\frac{5}{2}}\right)\Big|_0^a=\frac{4\pi}{5}a^3$$

5



$$V=\int_0^{\frac{\sqrt{3}a}{2}}2\pi x\left(\frac{a}{2}-\frac{x}{\sqrt{3}}-\left(-\frac{a}{2}+\frac{x}{\sqrt{3}}\right)\right)dx$$

$$=2\pi\int_0^{\frac{\sqrt{3}a}{2}}x\left(a-\frac{2\sqrt{3}x}{3}\right)dx$$

$$=2\pi\int_0^{\frac{\sqrt{3}a}{2}}\left(ax-\frac{2\sqrt{3}x^2}{3}\right)dx$$

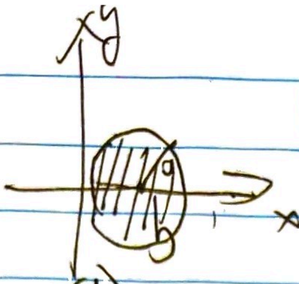
$$=2\pi\left(\frac{a}{2}x^2-\frac{2\sqrt{3}}{9}x^3\right)\Big|_0^{\frac{\sqrt{3}a}{2}}$$

$$=2\pi\left(\frac{3a^3}{8}-\frac{2\sqrt{3}}{9}\cdot\frac{3\sqrt{3}}{8}a^3\right)=\frac{\pi a^3}{4}$$

$\frac{1}{2\sqrt{2}\sqrt{x}}dx$

$\hat{V}(h)$

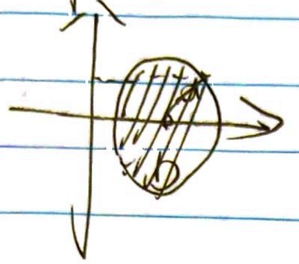
4C-1 a)



$$V = 2\int_{b-a}^{b+a} \sqrt{a^2-(x-b)^2}\cdot 2\pi x \cdot dx$$

$$\cancel{= 4\pi \int_{b-a}^{b+a} \sqrt{a^2x^2 - x^4 + 2bx^3 - b^2x^2}\, dx}$$

b)



$$x = \sqrt{a^2-y^2}+b$$

$$V = 2\pi\int_0^a \left[\left(\sqrt{a^2-y^2}+b\right)^2 - \left(-\sqrt{a^2-y^2}+b\right)^2\right] dy$$

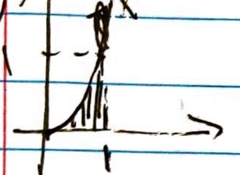
$$= 2\pi\int_0^a (2b)\left(2\sqrt{a^2-y^2}\right) dy$$

$$= 8\pi b\int_0^a \sqrt{a^2-y^2}\, dy$$

c)

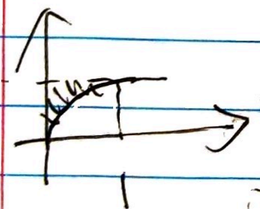
$$V = 8\pi b \cdot \frac{\pi a^2}{4} = 2\pi^2 a^2 b.$$

2 a)



$$V = \int_0^1 x^2 \cdot 2\pi x \cdot dx = 2\pi\left(\frac{1}{4}x^4\right)\Big|_0^1 = \frac{\pi}{2}$$

3



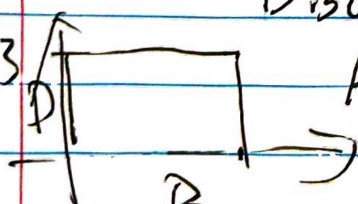
shells: $V = \int_0^1 (1-\sqrt{x})\, 2\pi x\, dx = 2\pi\int_0^1 \left(x - x^{\frac{3}{2}}\right) dx$

$$= 2\pi\left(\frac{1}{2}x^2 - \frac{2}{5}x^{\frac{5}{2}}\right)\Big|_0^1$$

$$= 2\pi\left(\frac{5}{10} - \frac{4}{10}\right) = \frac{\pi}{5}$$

Discs: $V = \int_0^1 (y^2)^2 \pi\, dy = \pi\left(\frac{1}{5}y^5\right)\Big|_0^1 = \frac{\pi}{5}$

4J-3



$$A = \int_0^R \frac{k^0}{1+r^2} D 2\pi r\, dr$$

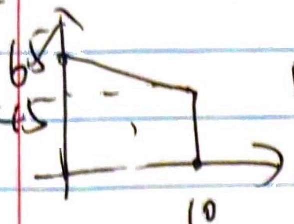
$$= 2\pi k D\int_0^R \frac{r}{1+r^2}\, dr$$

$$= 2\pi k D\int_1^{1+R^2} \frac{\sqrt{u-1}}{u}\, d\sqrt{u-1}$$

$$= \pi k D\int_1^{1+R^2} \frac{du}{u} = \pi k D \ln u\Big|_1^{1+R^2} = \pi k D \ln(1+R^2)$$

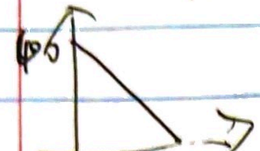
Lecture 23 3rd Jan 2025

49/5



$W = (65+45)10/2 = 110\times 5 = 550$ lb·ft

6

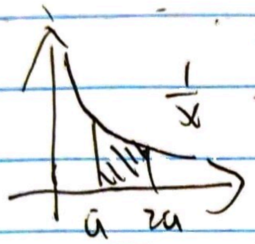


$W = \frac{400\times 10}{2} = 2000$ ~~lb/ft~~ lb·ft

15 M — a — m

$W = \int_a^{2a} G\frac{Mm}{r^2}\,dr$
$= GMm\int_a^{2a}\frac{dr}{r^2} = GMm\left(-\frac{1}{r}\right)\Big|_a^{2a} = GMm\left(-\frac{1}{2a}+\frac{2}{2a}\right)$
$= \frac{GMm}{2a}$

f)-2



$a>0$.
$\frac{1}{a}\int_a^{2a}\frac{dx}{x} = \frac{1}{a}(\ln x)\Big|_a^{2a} = \frac{\ln 2}{a}$

3 $\bar{v}(t) = \frac{1}{b-a}\int_a^b v(t)\,dt = \frac{1}{b-a}(s(b)-s(a)) = \bar{v}$

5 $g(x) = \frac{1}{x}\int_0^x$ ~~$f(x)dx$~~ $f(t)\,dt$
$xg(x) = \int_0^x f(t)\,dt$
$g(x) + xg'(x) = f(x)$

Lecture 24 4th Jan 2025

3(a-1a) Riemann Sum: $\sum_{k=0}^{4}\sqrt{k}\cdot\frac{1}{4} = 0+\frac{1}{4}+\frac{\sqrt{2}}{4}+\frac{\sqrt{3}}{4} = \frac{1+\sqrt{2}+\sqrt{3}}{4} \approx 1.031$

Trapezoidal rule: $\left(\frac{\sqrt{0}}{2}+1+\sqrt{2}+\sqrt{3}+\frac{2}{2}\right)\frac{1}{4} = \frac{2+\sqrt{2}+\sqrt{3}}{4} \approx 1.287$

Simpson's rule: $\frac{1}{3}\cdot\frac{1}{4}(0+4+2\sqrt{2}+4\sqrt{3}+2) = \frac{6+2\sqrt{2}+4\sqrt{3}}{12} = \frac{3+\sqrt{2}+2\sqrt{3}}{6} \approx 1.313$

d) Riemann Sum: $\sum_{k=1}^{4}\frac{1}{k}\cdot\frac{1}{4} = \left(1+\frac{4}{5}+\frac{4}{6}+\frac{4}{7}\right)\cdot\frac{1}{4} = \frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7} \approx 0.7595$

Trapezoidal rule: $\left(\frac{1}{2}+\frac{4}{5}+\frac{4}{6}+\frac{4}{7}+\frac{2}{8}\right)\frac{1}{4} = \frac{1}{8}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{16} \approx 0.6970$

Simpson's rule: $\frac{1}{3}\cdot\frac{1}{4}\left(1+\frac{16}{5}+\frac{8}{6}+\frac{16}{7}+\frac{4}{8}\right) = \left(1+\frac{1}{2}+\frac{4}{3}+\frac{16}{5}+\frac{16}{7}\right)/12 \approx$ ~~0.6933~~ 0.5144

$\ln 2 \approx 0.6931$

34-4 $\left(\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{n}+\sum_{k=1}^{n-1}\frac{1}{k}\right)\frac{1}{n} \approx \int_1^n \frac{dx}{x} = \ln x\Big|_1^n = \ln n$

$\text{Sum} \approx \ln n + \frac{1}{2} + \frac{1}{2n}$

$\text{Sum} > \ln n + \frac{1}{2} + \frac{1}{2n}$

Lec 15 4th Jan 2025

5B-9 $\int e^x(1+e^x)^{-3}dx$ let $u=e^x+1$ $du=e^x dx = (u-1)dx$

$= \int \cancel{(u-1)/u^3 \cdot u-1}\, du = \cancel{\ldots} -\frac{1}{2}u^{-2}+C = -\frac{1}{2u^2}+C$

11 $\int \sec^2 9x\, dx$ $\cancel{= \int \frac{\ldots}{\cos^2 x} = \ldots}$

$\int \sec^2 u \cdot \frac{1}{9}du = \frac{1}{9}\tan u + C = \cancel{\tan 9x}\ \frac{\tan 9x}{9}+C$

13 $\int \frac{x^2 dx}{1+x^6}$ let $u=x^3$ $du=3x^2dx$ $\cancel{\text{let } v = \ldots\ dv = \ldots}$

$= \int \frac{\frac{du}{3}}{1+u^2} = \int \frac{du}{3+3u^2} = \frac{1}{3}\int \frac{du}{1+u^2} = \frac{1}{3}\int \frac{du}{1+u^2} = \frac{\arctan u}{3}+C = \frac{\arctan x^3}{3}+C$

16 $\int_{-1}^{1} \frac{\tan^{-1}x\,dx}{1+x^2}$ let $u=\tan^{-1}x$ $du=\frac{dx}{1+x^2}$

$1 \to \frac{\pi}{4}$

$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} u\,du = \frac{u^2}{2}\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \left(\frac{\frac{\pi^2}{16}}{2} - \frac{\frac{\pi^2}{16}}{2}\right) = 0$

5C-5 $\int \sin^3 x \cos^2 x\, dx = \int (1-\cos^2 x)(\cos^2 x)\, d(-\cos x)$ let $\cos x = u$

$= \int (1-u^2)(u^2)\,du = \int -u^4+u^2 = -\frac{1}{5}u^5+\frac{1}{3}u^3+C$

7 $\int \sin^2(4x)\cos^2(4x)\,dx = \int \sin^2(4x)\cdot\frac{\cos(8x)+1}{2}dx$ $\cancel{\int \ldots}$

$= \int \frac{1-\cos 8x}{2}\cdot\frac{1+\cos 8x}{2}dx = \int \frac{1-\cos^2 8x}{4}dx = \int \frac{\sin^2 8x}{4}dx$

$= \int \frac{1-\cos 16x}{8}dx = \frac{1}{8}x - \frac{1}{128}\sin 16x + C$

9 $\int \sin^3 x \sec^2 x\, dx = \int \frac{(1-\cos^2 x)\, d(-\cos x)}{\cos^2 x} = \int \frac{(1-u^2)du}{u^2} = \int \left(\frac{1}{u^2}-1\right)du$

$= -\frac{1}{u} - u + C = -\frac{1}{\cos x} + \cos x + C$

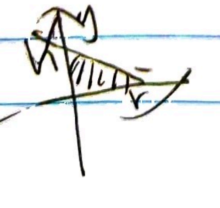
11 $\int \sin x \cos(2x)\,dx = \int (\sin x - \sin^3 x)\,dx = \int (\cos^2 x)\,d(-\cos x)$

$= \int u^2 du = \frac{1}{3}u^3 + C = -\frac{1}{3}\cos^3 x + C$

PART II 5th Jan 2025 0:12.

1. 7.4.

12.



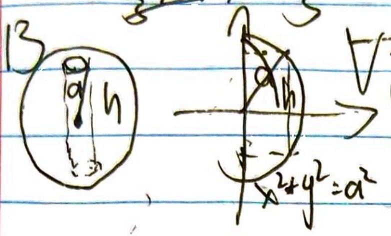
$V = \int_0^r \left(-\frac{h}{r}\cdot x + h\right)\cdot 2\pi x \cdot dx$

$= 2\pi h \int_0^r \left(-\frac{1}{r}x^2 + x\right)dx = 2\pi h\left(-\frac{1}{3r}x^3 + \frac{1}{2}x^2\right)\Big|_0^r$

$$= \left(\frac{\pi h}{3r}x^3 + \pi h x^2\right)\Big|_0^r$$

$$= \cancel{\tfrac{1}{3}\pi h x^2}\ \frac{1}{3}\pi h r^2$$

13



$$V = 2\int_{\sqrt{a^2-\frac{h^2}{4}}}^{a} \sqrt{a^2-x^2}\cdot 2\pi x\,dx$$

$$\cancel{= 4\pi\int_{\sqrt{a^2-\frac{h^2}{4}}}^{a} x\sqrt{a^2-x^2}\,dx}$$

$$V = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(-\left(\sqrt{a^2-h^2}\right)^2\pi + \left(\sqrt{a^2-y^2}\right)^2\pi\right)$$

$$= \pi\int_{-\frac{h}{2}}^{\frac{h}{2}} (a^2-y^2-a^2+h^2)\,dy = \pi\int_{-\frac{h}{2}}^{\frac{h}{2}} (h^2-y^2)\,dy$$

$$= \pi\left(h^2y - \frac{1}{3}y^3\right)\Big|_{-\frac{h}{2}}^{\frac{h}{2}} = \pi\left(\frac{h^3}{2} - \frac{h^3}{24} - \left(-\frac{h^3}{2} - \frac{h^3}{24}\right)\right)$$

$$= \pi\left(\frac{h^3}{4}\right) = \frac{\pi h^3}{4}$$

2 $V^2(t) = C^2\sin^2(120\pi t)$

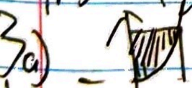
~~RMS~~ $\overline{V^2(t)} = \frac{1}{\frac{1}{60}}\cancel{\int_0^t V^2(t) = \frac{1}{t}\int_0^t}\ \frac{1}{\frac{1}{60}}\int_0^{\frac{1}{60}} V^2(t)\,dt$

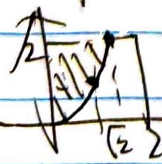
$$= 60C^2\int_0^{\frac{1}{60}} \sin^2(120\pi t)\,dt \quad \frac{C^2}{60}\int_0^{\frac{1}{60}} \frac{1-\cos(240\pi t)}{2}\,dt$$

$$= 30C^2\left(t - \frac{1}{240\pi}\sin(240\pi t)\right)\Big|_0^{\frac{1}{60}}$$

$$= 30C^2\left(\frac{1}{60} - \frac{1}{240\pi}\sin 4\pi\right) = \frac{C^2}{2}$$

$$RMS = \sqrt{\frac{C^2}{2}} = \frac{C}{\sqrt{2}}$$

3 a)  $\rightarrow$ $P = \int_0^1 (1-x^2)\,dx = x - \frac{1}{3}x^3\Big|_0^1 = \frac{2}{3}$

b)  $\rightarrow$ $P = \frac{1}{4}\int_0^{\sqrt{2}} (1-x^2)\,dx = \frac{1}{4}\left(x - \frac{1}{3}x^3\right)\Big|_0^{\sqrt{2}} = \frac{1}{4}\left(\sqrt{2} - \frac{2}{3}\sqrt{2}\right) = \frac{\sqrt{2}}{12}$

c) $W = \int_0^{\infty} e^{-at}\,dt = \lim_{N\to\infty} \int_0^N e^{-at}\,dt$

$= \lim_{N\to\infty} \left(\left(-\frac{1}{a} e^{-at}\right)\Big|_0^N \right) = \lim_{N\to\infty} \left(-\frac{1}{a} e^{-aN} + \frac{1}{a}\right) = \frac{1}{a}$

d) $P([0,T]) = a\int_0^T e^{-at}\,dt = \frac{1}{2}$

$= a\left(-\frac{1}{a} e^{-aT} + \frac{1}{a}\right) = -e^{-aT} + 1 = \frac{1}{2}$

$e^{-aT} = \frac{1}{2}$

$-aT = \ln\frac{1}{2}$

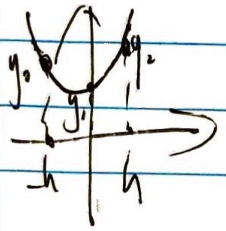
$aT = \ln 2$

$W = \frac{1}{a} = \frac{1}{\ln 2}$

$P = \frac{1}{2} \cdot a \int_1^2 e^{-at}\,dt$

$= \frac{a}{2} \cdot \left(-\frac{1}{a} e^{-2a} + \frac{1}{a} e^{-a}\right) = -\frac{1}{2} e^{-2a} + \frac{1}{2} e^{-a} = -\frac{1}{8} + \frac{1}{4} = \frac{1}{8}$

4.



$C = y_1$

$\int_{-h}^{h} y\,dx = \int_{-h}^{h} \left(\frac{A}{3}x^3 + \frac{B}{2}x^2 + Cx\right) dx$

$= \left(\frac{Ah^3}{3} + \frac{Bh^2}{2} + Ch + \frac{Ah^3}{3} - \frac{B}{2}h^2 + Ch\right)$

$= \frac{2Ah^3}{3} + 2Ch = \frac{h}{3}(2Ah^2 + 6y_1)$

$y_0 = Ah^2 - Bh + C$

$y_1 = C$

$y_2 = Ah^2 + Bh + C$

$y_0 + y_1 + y_2 = 2Ah^2 + 3C$

$\therefore \int_{-h}^{h} y\,dx = \frac{h}{3}(y_0 + y_1 + y_2 + 3y_1)$

$= \frac{h}{3}(y_0 + 4y_1 + y_2)$

5.

y	x	$\cos t^2$
y_0	0	$\cos 0 = 1$
y_1	$\frac{\sqrt{\pi}}{4\sqrt{2}}$	$\cos\frac{\pi}{32}$
y_2	$\frac{\sqrt{\pi}}{2\sqrt{2}}$	$\cos\frac{\pi}{8}$
y_3	$\frac{3\sqrt{\pi}}{4\sqrt{2}}$	$\cos\frac{9\pi}{32}$
y_4	$\frac{\sqrt{\pi}}{\sqrt{2}}$	$\cos\frac{\pi}{2} = 0$

$\int_0^a \cos(t^2)\,dt \approx (y_0 + 4y_1 + 2y_2 + 4y_3 + y_4)\frac{\frac{a}{2}}{3}$

≈ 0.9782019

y	x	$\cos t^2$
y_0	0	$\cos 0 = 1$
y_1	$\frac{\sqrt{\pi}}{8\sqrt{2}}$	$\cos\frac{\pi}{128}$
y_2	$\frac{\sqrt{\pi}}{4\sqrt{2}}$	$\cos\frac{\pi}{32}$
y_3	$\frac{3\sqrt{\pi}}{8\sqrt{2}}$	$\cos\frac{9\pi}{128}$
y_4	$\frac{\sqrt{\pi}}{2\sqrt{2}}$	$\cos\frac{\pi}{8}$
y_5	$\frac{5\sqrt{\pi}}{8\sqrt{2}}$	$\cos\frac{25\pi}{128}$
y_6	$\frac{3\sqrt{\pi}}{4\sqrt{2}}$	$\cos\frac{9\pi}{32}$
y_7	$\frac{7\sqrt{\pi}}{8\sqrt{2}}$	$\cos\frac{49\pi}{128}$
y_8	$\sqrt{\frac{\pi}{2}}$	$\cos\frac{\pi}{2} = 0$

$$\int_0^{a} \cos t^2 \, dt$$

$$\approx \left(1 + 4\cos\frac{\pi}{128} + 2\cos\frac{\pi}{32} + 4\cos\frac{9\pi}{128} + 2\cos\frac{\pi}{8} + 4\cos\frac{25\pi}{128} + 2\cos\frac{9\pi}{32} + 4\cos\frac{49\pi}{128}\right)\frac{\sqrt{\frac{\pi}{2}}}{24}$$

$$\approx 0.977503$$