PS10 1/4/2025

Lec 28

5A-1 a) $\int_0^2 \int_{-1}^1 \int_0^1 (x+y+z)\,dx\,dy\,dz$

$= \int_0^2 \int_{-1}^1 \left[\frac{x^2}{2} + (y+z)x\right]_0^1 dy\,dz$

$= \int_0^2 \int_{-1}^1 \left(\frac{1}{2} + y + z\right) dy\,dz$

$= \int_0^2 \left[\frac{y}{2} + \frac{y^2}{2} + zy\right]_{-1}^1 dz$

$= \int_0^2 \left[(1+z) - \left(-\frac{1}{2} + \frac{1}{2} - z\right)\right] dz$

$= \int_0^2 (2z+1)\,dz = 4$

b) $\int_0^2 \int_0^{\sqrt{y}} \int_0^{xy} 2xy^2 z\,dz\,dx\,dy$

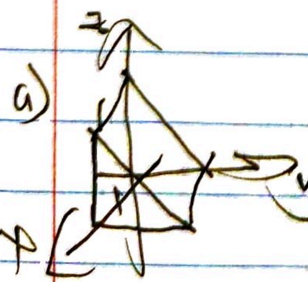
$= \int_0^2 \int_0^{\sqrt{y}} xy^2 z^2 \Big|_0^{xy} dx\,dy$

$= \int_0^2 \int_0^{\sqrt{y}} x^3 y^4\,dx\,dy = \int_0^2 \frac{y^4}{4} x^4 \Big|_0^{\sqrt{y}} dy$

$= \int_0^2 \frac{1}{4} y^6\,dy = \frac{1}{28} y^7 \Big|_0^2 = \frac{128}{28} = \frac{32}{7}$

$= \frac{32}{7}$

2 a)

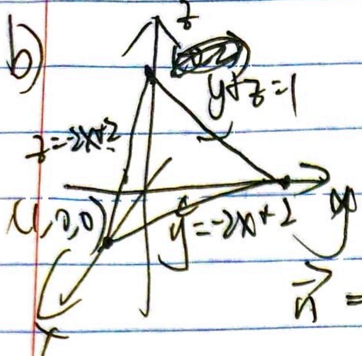


i) $\int_0^1 \int_0^1 \int_0^{1-y} dz\,dy\,dx$

ii) $\int_0^1 \int_0^{1-y} \int_0^1 dx\,dz\,dy$

iii) $\int_0^1 \int_0^1 \int_0^{1-z} dy\,dx\,dz$

b)



$$\vec{n} = \langle 0, 1, -1\rangle \times \langle 1, -2, 0\rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & -1 \\ 1 & -2 & 0 \end{vmatrix} = -2\hat{i} + \hat{j} - \hat{k}$$

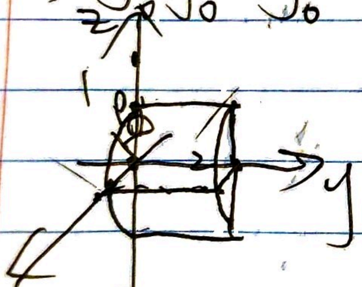
$\therefore$ plane : $-2(x-1)+y-z=0$

$$-2x+y-z=-2$$

$$z = 2-2x+y$$

$$\int_0^1 \int_0^{-2x+2} \int_0^{2-2x+y} dz\,dy\,dx$$

c)

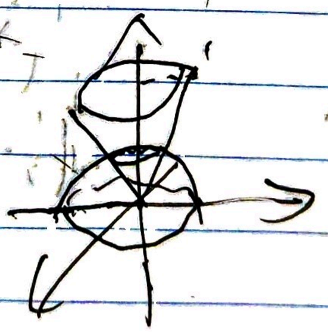


$$\int_0^{\frac{\pi}{2}} \int_0^1 \int_0^2 dy\,\rho\,d\rho\,d\phi$$

d)

$x^2+y^2+z^2=2$

$z^2 = 2-x^2-y^2 = x^2+y^2$

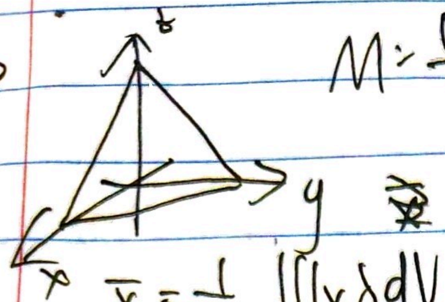


$r^2 = x^2+y^2 < 1$

$r^2 < z^2 < 2-r^2$

$$\iiint_R dV = \int_0^{2\pi} \int_0^1 \int_r^{\sqrt{2-r^2}} dz\,r\,dr\,d\theta$$

3



$$M = \frac{1\times1\times1}{2\times3} = \frac{1}{6}$$

$$\bar{x} = \frac{1}{M}\iiint x\,\delta\,dV$$

$$= 6\int_0^1\int_0^{1-z}\int_0^{1-y-z} x\,dx\,dy\,dz$$

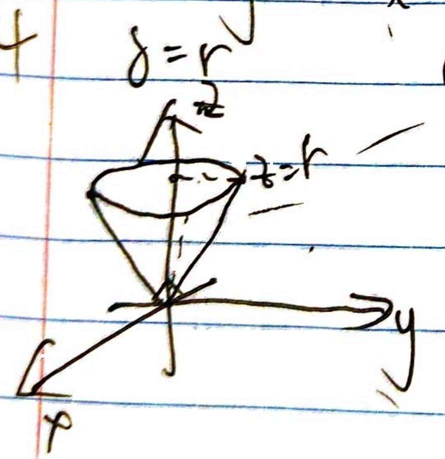
$$= 6\int_0^1\int_0^{1-z} \frac{(1-y-z)^2}{2}\,dy\,dz$$

$$= 6\int_0^1 \left.-\frac{(1-y-z)^3}{6}\right|_0^{1-z} dz$$

$$= 6\int_0^1 \frac{1}{6}(1-z)^3\,dz = -\frac{1}{4}(1-z)^4\Big|_0^1 = \frac{1}{4}$$

$$\bar{y} = \bar{z} = \bar{x} = \frac{1}{4}$$

4 $\delta = r$



a) ~~mass~~ mass $= \iiint_R \delta\,dV$

$$= \int_0^{2\pi}\int_0^h\int_r^h r^2\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^h (r^2 h - r^3)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{h}{3}r^3 - \frac{1}{4}r^4\right]_0^h d\theta$$

$$= \int_0^{2\pi} \frac{1}{12}h^4\,d\theta$$

$$= \frac{\pi}{6}h^4$$

b) ~~$\bar{x} = \bar{y} = 0.$~~

b) $\bar{x} = \bar{y} = 0$, $\bar{z} = \frac{1}{mass}\iiint_R z\delta\, dV$

$\iiint_R z\delta\, dV = \int_0^{2\pi}\int_0^h\int_r^h z r^2\, dz\, dr\, d\theta$

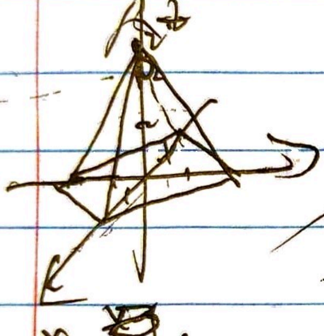
$= \int_0^{2\pi}\int_0^h \frac{1}{2}(r^2h^2 - r^4)\, dr\, d\theta$

$= \int_0^{2\pi} \frac{1}{2}\left(\frac{h^5}{3} - \frac{h^5}{5}\right) d\theta$

$= \int_0^{2\pi} \frac{1}{15}h^5\, d\theta$

$= \frac{2\pi}{15}h^5$ $\quad \bar{z} = \frac{1}{mass}\frac{2\pi}{15}h^5 = \frac{\cancel{6}^{2}}{\cancel{2\pi} h^4}\cdot\frac{\cancel{2\pi}}{\cancel{15}_{5}}h^5 = \frac{4}{5}h$

5



$I = \iiint_R (\sqrt{z^2+y^2})^2\, \delta\, dV$

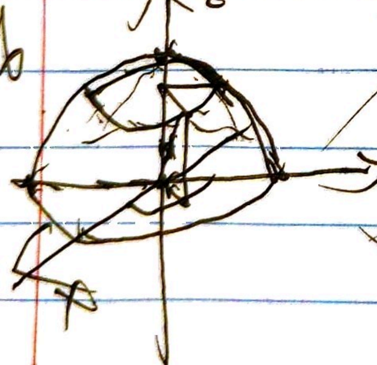
$= 4\int_0^1\int_0^{1-x}\int_0^{2-2x-2y} z^2 y^2\, dz\, dy\, dx$

~~x+y+z=1~~

$2x+2y+z=2$

$z = 2-2x-2y$

6



$I = \iiint_R r^2\, dV$

$= \int_0^{2\pi}\int_0^a\int_0^{\sqrt{a^2-r^2}} r^2\, dz\, r\, dr\, d\theta$

$= \int_0^{2\pi}\int_0^a r^3\sqrt{a^2-r^2}\, dr\, d\theta$

$= 2\pi\int_0^a r^3\sqrt{a^2-r^2}\, dr$

$\int_0^a r^3\sqrt{a^2-r^2} = \int_0^a r^2 r\sqrt{a^2-r^2} = \int_0^a \cancel{2r^2\sqrt{a^2-r^2}}$

$= r^2\int r\sqrt{a^2-r^2}\, dr\Big|_0^a - \int_0^a \cancel{2r^3\sqrt{a^2-r^2}\, dr}$

$u = r^2 \quad v' = r\sqrt{a^2-r^2}$

$u' = 2r \quad v = \int r\sqrt{a^2-r^2}\,dr = -\frac{1}{2}\int \sqrt{a^2-r^2}\,dr^2 = -\frac{1}{2}\cdot\frac{2}{3}\cdot(a^2-r^2)^{\frac{3}{2}}$

$= -\frac{1}{3}(a^2-r^2)^{\frac{3}{2}}$

$\int_0^a r^2 r\sqrt{a^2-r^2}$

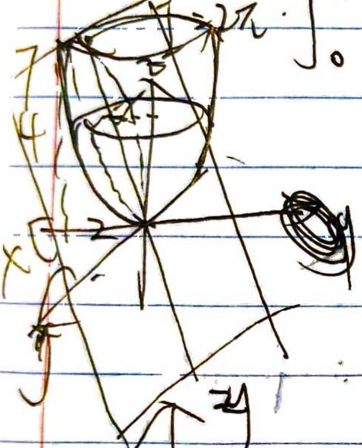
$= r^2\int r\sqrt{a^2-r^2}\,dr\Big|_0^a + \int_0^a \frac{2}{3} r(a^2-r^2)^{\frac{3}{2}}\,dr$

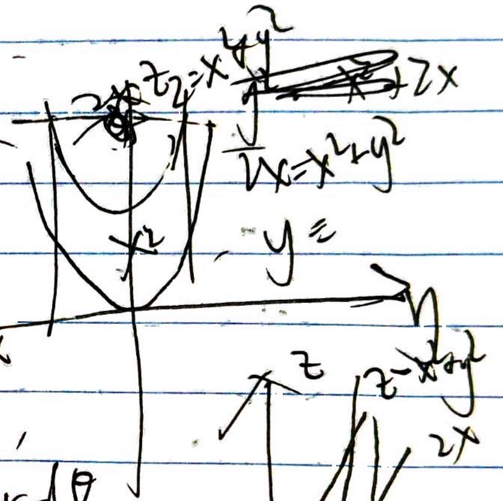
$= -\frac{r^2}{3}(a^2-r^2)^{\frac{3}{2}}\Big|_0^a + \frac{1}{3}\int_0^a (a^2-r^2)^{\frac{3}{2}}\,dr^2$

$= -\frac{a^2}{3}(a^2-a^2)^{\frac{3}{2}} - \frac{1}{3}\cdot\frac{2}{5}(a^2-r^2)^{\frac{5}{2}}\Big|_0^a$

$= \frac{2}{15}a^5$

$2\pi\cdot\int_0^a r^3\sqrt{a^2-r^2} = \frac{4\pi}{15}a^5$





$I = \iiint r^2\,dV$

$= \int_0^2 \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} \int_{r^2}^{2x} r^2\,dz\,dy\,dx$

$= \int_0^{\pi} \int_0^{2\cos\theta} \int_{r^2}^{2r\cos\theta} r^3\,dz\,dr\,d\theta$

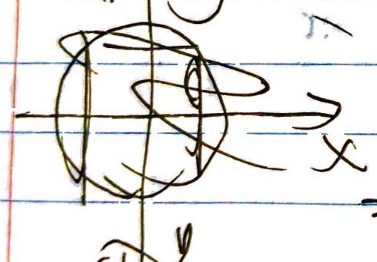
$= \int_0^{\pi}\int_0^{2\cos\theta} r^3(2r\cos\theta - r^2)\,dr\,d\theta$

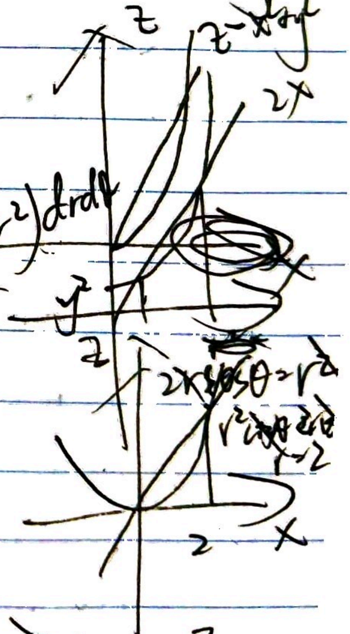
$= \int_0^{\pi}\left[\frac{2}{5}r^5\cos\theta - \frac{1}{6}r^6\right]_0^{2\cos\theta} d\theta$

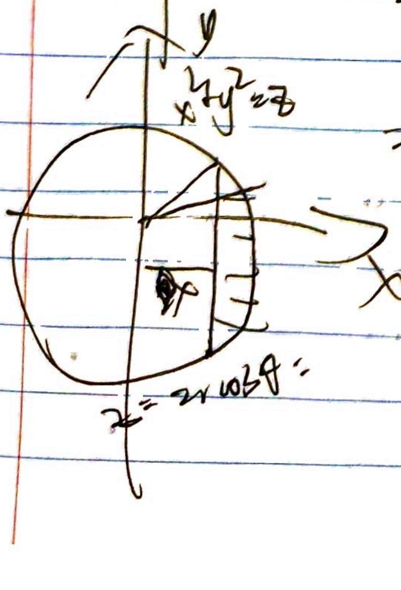
$= \frac{32}{15}\int_0^{\pi}\cos^6\theta\,d\theta$

$= \frac{32}{15}\cdot 2\int_0^{\frac{\pi}{2}}\cos^6\theta\,d\theta$

$= \frac{2\pi}{3}$

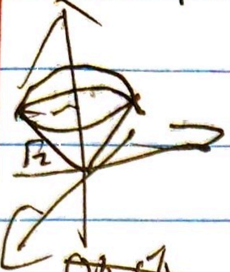






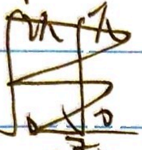
Lec 29 1/4/2025

513-1 a)



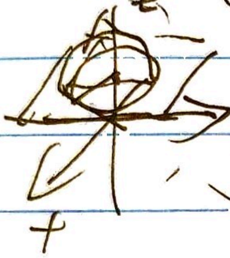
$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{2}} \rho^2 \sin\phi \, d\phi \, d\theta$$

b)



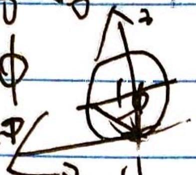
$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\infty} d\rho \, d\phi \, d\theta$$

c)



$x=\rho\sin\phi$

$z=r\cos\phi$



$x^2+(z-1)^2=1$

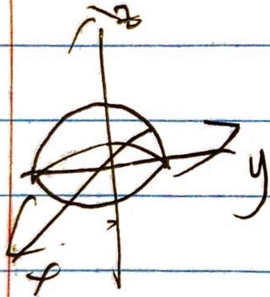
$\rho^2-2\rho\cos\phi+1=1$

$\rho=2\cos\phi$

$2\cos\phi=\rho$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_{\sqrt{2}\sec\phi}^{2\cos\phi} d\rho \, d\phi \, d\theta$$

2



$\bar{x}=\bar{y}=0$

$$\iiint_R z \, dV$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{a} \rho\cos\phi \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

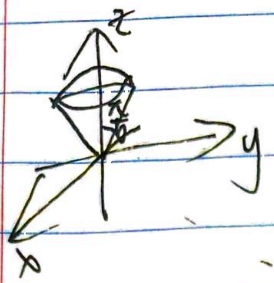
$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \frac{1}{4}\rho^4 \cdot \cos\phi\sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\left[\frac{1}{4}a^4 \cdot \frac{1}{2}\sin^2\phi\right]_0^{\frac{\pi}{2}} d\theta$$

$$= \frac{1}{8}a^4 \cdot 2\pi = \frac{\pi}{4}a^4$$

$\text{Mass} = \delta V = \frac{2}{3}\pi a^3$

$$\bar{z} = \frac{1}{\text{Mass}}\iiint_R z \, dV = \frac{3}{2\pi a^3}\cdot\frac{\pi}{4}a^4 = \frac{3}{8}a$$

3 

$$I = \iiint_D r^2 z \, dV$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^a \rho\cos\phi \, \rho^2\sin^2\phi \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{6}} \frac{1}{6}a^6 \cdot \cos\phi\sin^3\phi \, d\phi \, d\theta$$

$$= \cancel{\int_0^{2\pi} \frac{1}{6}a^6} \quad 2\pi \cdot \frac{1}{6}a^6 \frac{1}{4}\sin^4\phi \Big|_0^{\frac{\pi}{6}}$$

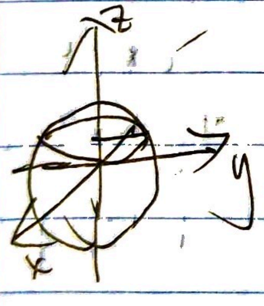
$$= 2\pi \cdot \frac{1}{6}a^6 \cdot \frac{1}{\cancel{4}_2} \cdot \frac{1}{16} = \frac{\pi}{194}a^6$$

4 a) $$\bar{d} = \frac{1}{V}\iiint_D \rho \, dV$$

$$= \frac{3}{4\pi a^3}\int_0^{2\pi}\int_0^{\pi}\int_0^a \rho \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{3}{4\pi a^3} 2\pi \frac{1}{4}a^4 (-\cos\phi)\Big|_0^{\pi}$$

$$= \frac{3}{4\pi a^3} \cdot \frac{\pi}{2}a^4 \cdot 2 = \frac{3}{4}a$$

b) 

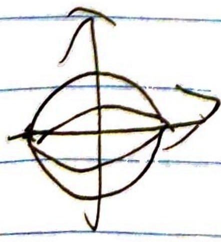
$$\bar{d} = \frac{1}{V}\iiint_D r \, dV$$

$$= \frac{3}{4\pi a^3}\int_0^{2\pi}\int_0^{\pi}\int_0^a \rho\sin\phi \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{3}{4\pi a^3}\int_0^{2\pi}\int_0^{\pi} \frac{1}{4}a^4 \sin^2\phi \, d\phi \, d\theta$$

$$= \frac{3}{4\pi a^3} 2\pi \, 2\int_0^{\frac{\pi}{2}} \frac{1}{4}a^4 \sin^2\phi \, d\phi \, d\theta$$

$$= \frac{3}{4\pi a^3} \pi a^4 \cdot \frac{\pi}{4} = \frac{3a\pi}{16}$$

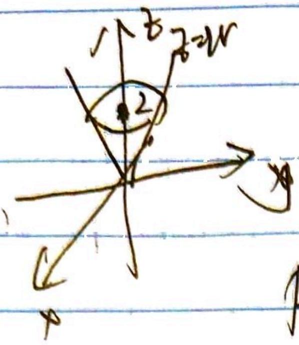
c) 

$$\bar{d} = \frac{1}{V} \cdot 2\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^a \rho\cos\phi \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{3}{4\pi a^3} \cdot 2\pi \cdot \frac{1}{4}a^4 \cdot \cancel{\frac{1}{2}\sin^2\phi \Big|_0^{\frac{\pi}{2}}} \ \frac{1}{2}$$

$$= \frac{3}{8}a$$

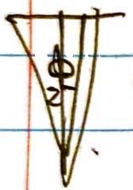
5C-2



$$\vec{F} = \iiint_R \frac{Gm\langle x,y,z\rangle}{\rho^3}\,\delta\, dV$$

$F_x = 0$, $F_y = 0$

$$F_z = \iiint_R \frac{Gmz}{\rho^3}\, dV$$



$$= \int_0^{2\pi}\int_0^{\tan^{-1}\frac{1}{2}}\int_0^{2\sec\phi} \frac{Gm\rho\cos\phi}{\rho^3}\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

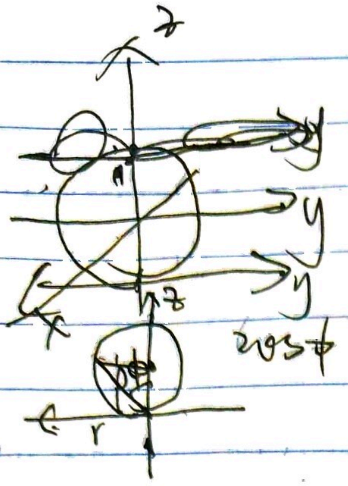
$$= G\int_0^{2\pi}\int_0^{\tan^{-1}\frac{1}{2}} \rho\cos\phi\sin\phi \Big|_0^{2\sec\phi} d\phi\, d\theta$$

$$= G\cdot 2\pi \int_0^{\tan^{-1}\frac{1}{2}} 2\sin\phi\, d\phi$$

$$= 2\pi G\left[-2\cos\phi\right]_0^{\tan^{-1}\frac{1}{2}}$$

$$= 2\pi G\left(2 - \frac{4}{\sqrt{5}}\right) = 4\pi G\left(1 - \frac{2}{\sqrt{5}}\right)$$

3.



$F_x = 0,\ F_y = 0$

$F_z = \iiint_R \frac{Gmz}{\rho^3} \cdot \rho^{-\frac{1}{2}}\, dV$

$= G\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2\cos\phi} \frac{\rho\cos\phi}{\rho^3}\cdot\rho^{-\frac{1}{2}}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

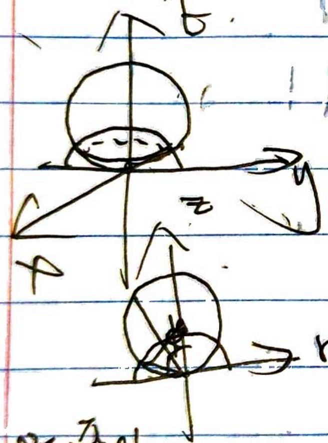
$= G\int_0^{2\pi}\int_0^{\frac{\pi}{2}} 2\rho^{\frac{1}{2}}\cos\phi\sin\phi \Big|_0^{2\cos\phi} d\phi\, d\theta$

$= 2\pi G\int_{\frac{\pi}{2}}^{0} -2\sqrt{2}\cos^{\frac{3}{2}}\phi\, d\cos\phi$

$= 2\pi G\left(-2\sqrt{2}\cdot\frac{2}{5}\cos^{\frac{5}{2}}\phi\right)_0^{\frac{\pi}{2}}$

$= 2\pi G\left(\frac{4\sqrt{2}}{5} - 0\right) = \frac{8\sqrt{2}\pi}{5}G$

4.



$F_x = 0,\ F_y = 0.$

$F_z = \iiint_R \frac{Gm\rho\cos\phi}{\rho^3}\, dV$

$\cancel{= \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^{2\cos\phi} \frac{\rho\cos\phi}{\rho^3}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta}$

$\cancel{= \int_0^{2\pi}\int_0^{\frac{\pi}{3}} \cos\phi\sin\phi\cdot 2\cos\phi\, d\phi\, d\theta}$

$= \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^1 \cos\phi\sin\phi\, d\rho\, d\phi\, d\theta\ \cancel{= 2\pi\int_0^{\frac{\pi}{3}} -2\cos^2\phi\, d\cos\phi} + \int_0^{2\pi}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\int_0^{2\cos\phi} \cos\phi\sin\phi\, d\rho$

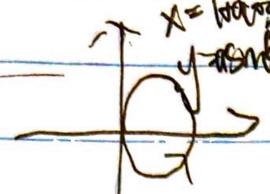
$= 2\pi G\cdot\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^2 + 2\pi G\cdot\frac{2}{3}\left[\frac{1}{2}\right]^3\ \cancel{2\pi\left[-\frac{2}{3}\cos^3\phi\right]_0^{\frac{\pi}{3}}}$

$= \frac{3\pi G}{4} + \frac{1}{6}\pi G\ \cancel{= 2\pi\left[\frac{2}{3}\cdot\frac{4}{4} - \frac{2}{3}\cdot\frac{1}{8}\right] = \frac{7\pi}{6}}$

$= \frac{11}{12}\pi G$

PART B 2/4/2015

P1



$x^2+y^2 = a^2+2a\cos\theta+1 = 2a^2$

$r^2 = 2a\cos\theta$

$$\oint_C \vec{F}\cdot\hat{n}\,ds = \oint_C \frac{x}{r^2}dy - \frac{y}{r^2}dx$$

$dx = a\sin\theta\, d\theta$

$dy = -a\cos\theta\, d\theta$

$$= \oint_C \frac{1}{r^2}\left[(1+a\cos\theta)(\cancel{a\cos\theta\,d\theta}) - a\sin\theta\sin\theta\,d\theta\right]$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\left(-(1+a\cos\theta)d\theta - a\sin\theta\,d\theta\right)$$

$$= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(-a(\cos\theta+\sin\theta) - 1\right) d\theta$$

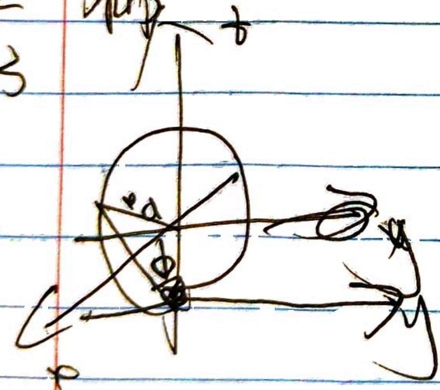
$$= \frac{1}{2}\left[-a(\sin\theta-\cos\theta) - \theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[-a(1) - \frac{\pi}{2} + a(-1) - \frac{\pi}{2}\right]$$

$$= \frac{1}{2}(-2a - \pi) = -a - \frac{\pi}{2}$$

P2 strip

P3



$$V(D) = \frac{4}{3}\pi a^3$$

$$\iiint_D \sqrt{x^2+y^2+z^2}\, dV$$

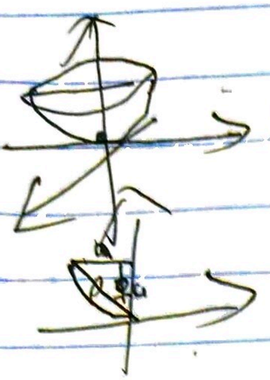
$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2a\cos\phi} \rho\cdot\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= 2\pi\int_0^{\frac{\pi}{2}} -a^4\cos^4\phi\, d\cos\phi$$

$$= -2\pi a^4\cdot\frac{1}{5}\cos^5\phi\Big|_0^{\frac{\pi}{2}}$$

$$= -2\pi a^4\left(-\frac{1}{5}\right) = \frac{2\pi}{5}a^4$$

P4



$F_x = 0$, $F_y = 0$,

$F_z = \iiint_D G \frac{z}{\rho^3}\, dV$

$= G\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\frac{a}{\cos\phi}} \frac{\rho\cos\phi}{\rho^3}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

$+ G\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2a\cos\phi} \frac{\rho\cos\phi}{\rho^3}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

$= 2\pi G\left(\int_0^{\frac{\pi}{4}} a\sin\phi\, d\phi + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} -2a\cos^2\phi\, d\cos\phi\right)$

$= 2\pi G\left([-a\cos\phi]_0^{\frac{\pi}{4}} + [-\frac{2}{3}a\cos^3\phi]_{\frac{\pi}{4}}^{\frac{\pi}{2}}\right)$

$= 2\pi G\left(a - \frac{\sqrt{2}}{2}a + \left(\frac{1}{2\sqrt{2}}\cdot\frac{2}{3}a\right)\right)$

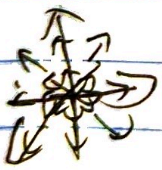
$= 2\pi G\left(a - \frac{\sqrt{2}}{3}a\right)$

$= 2\pi G a\left(1 - \frac{\sqrt{2}}{3}\right)$

PS11 2/4/2015

Lec 30

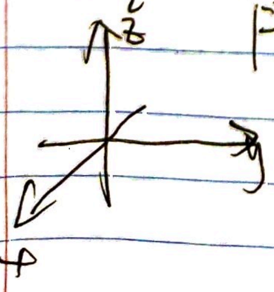
1A-1a)



b) point to the y axis, magnitude is the distance from the y axis.

1A-2 $-x\hat{i} - y\hat{j} - z\hat{k}$

1A-3



$\vec{F} = -z\hat{j} + y\hat{k}$