$$= \begin{bmatrix} \sin\theta & \frac{\cos\theta}{r} \end{bmatrix} \begin{bmatrix} \cdots \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\frac{1}{5} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 10 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

PS5 25/3) ~~203~~ 2015

Lec 13

2D-1 a) $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (3x^2, 6y^2)$ $\quad \nabla f\big|_{(1,1)} = \langle 3, 6\rangle$

$\hat{u} = \frac{\vec{A}}{|A|} = \frac{\hat{i}-\hat{j}}{\sqrt{2}}$ $\quad \cancel{\frac{df}{ds}\Big|_{\hat{u}} = \frac{df}{ds}\Big|_{(1,1)}\Big|_{\hat{u}}}$

$\frac{df}{ds}\Big|_{\hat{u}} = \nabla f \cdot \frac{d\vec{r}}{ds} = \nabla f \cdot \hat{u} = \frac{3}{\sqrt{2}} - \frac{6}{\sqrt{2}} = -\frac{3}{\sqrt{2}} = -\frac{3\sqrt{2}}{2}$

b) $\nabla w = \left\langle \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}\right\rangle = \left\langle \frac{y}{z}, \frac{x}{z}, -\frac{xy}{z^2}\right\rangle$

$(\nabla w)_p = -\hat{i} + 2\hat{j} + 2\hat{k}$ $\quad \hat{u} = \frac{\hat{i}+2\hat{j}-2\hat{k}}{\sqrt{1+4+4}} = \frac{\hat{i}+2\hat{j}-2\hat{k}}{3}$

$\frac{df}{ds}\Big|_{\hat{u}} = (\nabla w)_p \cdot \hat{u} = -\frac{1}{3} + \frac{4}{3} - \frac{4}{3} = -\frac{1}{3}$

e) $\frac{\partial f}{\partial u} = 2(u+2v+3w)$ $\quad \frac{\partial f}{\partial v} = 2f \cdot 2$ $\quad \frac{\partial f}{\partial w} = 2f \cdot 3$ $\quad f(1,-1,1) = 2$

$(\nabla f)_p = \langle 4, 8, 12\rangle$ $\quad \hat{u} = \frac{-2\hat{i}+2\hat{j}-\hat{k}}{3}$

$\frac{df}{ds}\Big|_{\hat{u}} = (\nabla f)_p \cdot \hat{u} = -\frac{8}{3} + \frac{16}{3} - \frac{12}{3} = -\frac{4}{3}$

2b) $\nabla w = \left\langle \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}\right\rangle = \langle y, z, x\rangle = \langle -1, 2, 1\rangle$

$\hat{u} = \frac{\nabla w}{|\nabla w|} = \frac{y\hat{i}+z\hat{j}+x\hat{k}}{\sqrt{x^2+y^2+z^2}}$ ~~maximum~~ $= \frac{-\hat{i}+2\hat{j}+\hat{k}}{2}$ max.

$\left(\frac{dw}{ds}\Big|_{\hat{u}}\right)_{max} = \nabla w \cdot \hat{u} = \frac{x^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}} = \sqrt{x^2+y^2+z^2} = 2$

$$\hat{u} = -\frac{\nabla w}{|\nabla w|} = -\frac{y\hat{i}+z\hat{j}+x\hat{k}}{\sqrt{x^2+y^2+z^2}} = -\frac{-\hat{i}+2\hat{j}+\hat{k}}{2}$$

$$\left(\frac{dw}{ds}\Big|_{\hat{u}}\right)_{min} = \nabla w \cdot \hat{u} = -\frac{x^2+y^2+z^2}{\sqrt{x^2+y^2+z^2}} = -\sqrt{x^2+y^2+z^2} = -2$$

$\hat{u} \cdot \nabla w = 0$. let $\hat{u} = \langle a, b, c \rangle$, $a^2+b^2+c^2=1$

$ay + bz + cx = 0$

$-a + 2b + c = 0$ $\qquad$ $\cancel{4b^2+c^2+4bc+b^2+c^2=1}$

$a = 2b + c$

$$\therefore \hat{u} = \frac{(2b+c)\hat{i} + b\hat{j} + c\hat{k}}{\sqrt{a^2+b^2+c^2}}, \quad a, b, c \in \mathbb{R}$$

3 a) $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle y^2z^3, 2xz^3y, 3xy^2z^2 \rangle$

as $(x, y, z) = (3, 2, 1)$

$\nabla f = \langle 4, 12, 36 \rangle$

Tangent plane: $4(x-3) + 12(y-2) + 36(z-1) = 12$

$x - 3 + 3y - 6 + 9z - 9 = 3$

$x + 3y + 9z = 21$

c) $\nabla f = \langle 2x, 2y, -2z \rangle$

$\nabla f(x_0, y_0, z_0) = \langle 2x_0, 2y_0, -2z_0 \rangle$

Tangent plane: $2x_0(x-x_0) + 2y_0(y-y_0) - 2z_0(z-z_0) = 0$

$x_0x - x_0^2 + y_0y - y_0^2 - z_0z + z_0^2 = 0$

$x_0x + y_0y - z_0z = x_0^2 + y_0^2 - z_0^2$.

8 $P(x,y,z) = 30 + (x+1)(y+2)e^z$

~~$g(x,y,z) = x^2+y^2+z^2$~~

$g(x,y,z) = x^2+y^2+z^2$

$\nabla g = \langle 2x, 2y, 2z \rangle$

$(\nabla P)_{P_{min}} = \lambda(\nabla g)$

$\frac{\partial P}{\partial x} = (y+2)e^z \quad \frac{\partial P}{\partial y} = (x+1)e^z \quad \frac{\partial P}{\partial z} = (x+1)(y+2)e^z$

$\nabla P = \langle (y+2)e^z, (x+1)e^z, (x+1)(y+2)e^z \rangle$

$P(0,0,0) = 32$

$P = 31.1, \quad (x+1)(y+2)e^z = 1.1, \quad P(0,0,0) = 32$

$\Delta P = -0.9 \approx \nabla P_0 \cdot \Delta \vec{r} \qquad \nabla P_0 = \langle 2, 1, 2 \rangle$

$= \nabla P_0 \cdot \langle x, y, z \rangle \qquad \langle x, y, z \rangle = \lambda \nabla P_0$

$= 2x + y + 2z$

$= 4\lambda + \lambda + 4\lambda$

$\lambda = -0.1$

$\therefore (x, y, z) \approx (-0.2, -0.1, -0.2)$

2(f)7 $\cancel{\nabla f(x(u,v), y(u,v))}$

$$\cancel{= \left[\frac{\partial f}{\partial x(u,v)} \quad \frac{\partial f}{\partial y(u,v)}\right]}$$

$$RHS = \nabla f(x,y) \cdot J$$

$$= \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right] \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

$$df = \frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} \cdot dy$$

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \qquad dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv$$

$$df = \frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv\right) + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv\right)$$

$$= \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial f} \cdot \frac{\partial y}{\partial u}\right) du + \left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v}\right) dv$$

$$LHS = \nabla f(x(u,v), y(u,v))$$

$$= \left[\frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u} \quad \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v}\right]$$

$$= \left[\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y}\right] \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = RHS$$

$$\therefore \nabla f(x(u,v), y(u,v)) = \nabla f(x,y) \cdot J$$

Lec 14 25/3/2025

4-1 a) $f(x,y,z) = x + 2y + 3z = 18$.

Maximise: $g(x,y,z) = xyz$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \langle 1, 2, 3 \rangle$$

$$\nabla g = \langle yz, xz, xy \rangle$$

$$\nabla f = \lambda \nabla g$$

P1

$\begin{cases} \lambda yz = 1 \\ \lambda xz = 2 \\ \lambda xy = 3 \end{cases} \Rightarrow \begin{cases} \lambda xyz = x \\ \lambda xyz = 2y \\ \lambda xyz = 3z \end{cases}$ $x = 2y = 3z = t$

$3t = 18$ $t = 6$.

$\therefore (x, y, z) = (6, 3, 2)$

b) $f(x,y,z) = x^2 + 2y^2 + 4z^2 = 12$

$g(x,y,z) = xyz$

$\nabla f = \langle 2x, 4y, 8z \rangle$

$\nabla g = \langle yz, xz, xy \rangle$

$\nabla f = \lambda \nabla g$

$\begin{cases} \lambda xyz = 2x^2 \\ \lambda xyz = 4y^2 \\ \lambda xyz = 8z^2 \end{cases}$ let $2x^2 = 4y^2 = 8z^2 = 2t$

$\sqrt{2}x = 2y = 2\sqrt{2}z = \sqrt{2t}$

$x = \sqrt{2}y = 2z = \sqrt{t}$

$3t = 12$ $t = 4$.

$(x, y, z) = (2, \sqrt{2}, 1)$

2I-3 $A(x,y,z) = 3xy + 4xz + 2yz$

$V(x,y,z) = xyz = 1$

$\nabla A = \langle 3y + 4z, 3x + 2z, 4x + 2y \rangle$

$\nabla V = \langle yz, xz, xy \rangle$

$\nabla A = \lambda \nabla V$

$\begin{cases} \cancel{3xy + 4xz} = \lambda xyz \\ 3xy + \cancel{2yz} = \lambda xyz \\ 4xz + \cancel{2yz} = \lambda xyz \end{cases}$ $\begin{cases} \frac{3}{z} + \frac{4}{y} = \lambda \\ \frac{3}{z} + \frac{2}{x} = \lambda \\ \frac{4}{y} + \frac{2}{x} = \lambda \end{cases}$

let $\frac{1}{x} = a$, $\frac{1}{y} = b$, $\frac{1}{z} = c$.

$3c + 4b = 3c + 2a = 4b + 2a$

$a = 2b$, $2a = 3c$ let $2a = 3c = 4b = t$ $\therefore t = \sqrt[3]{?}$

$abc = 1$, $abc = \ldots$ $= 2\ldots$

$x = 3^{-\frac{1}{3}}$ $\quad z = 2 \times 3^{\frac{2}{3}}$ $\quad y = 2 \times 3^{-\frac{1}{3}}$

PART B 25/3/2015

P1 a) $\frac{\partial f}{\partial x} = 3x^2 - y^2 - 8x + 3 + 2yx$

$\frac{\partial f}{\partial y} = -2xy + x^2$

$\nabla f(\frac{1}{2}, 1) = \langle \frac{3}{4} - 1 - 4 + 3 + 1, -1 + \frac{1}{4} \rangle$

$= \langle -\frac{1}{4}, -\frac{3}{4} \rangle$

$\left.\frac{df}{ds}\right|_{\hat{u}} = \nabla f \cdot \hat{u}$

$\left(\left.\frac{df}{ds}\right|_{\hat{u}}\right)_{max} = |\nabla f| = \frac{1}{16} + \frac{9}{16} = \frac{10}{16} = \frac{5}{8}$

$\left(\left.\frac{df}{ds}\right|_{\hat{u}}\right)_{min} = -|\nabla f| = -\frac{5}{8}$

b) max: $\hat{u} = \frac{\nabla f}{|\nabla f|}$

min: $\hat{u} = -\frac{\nabla f}{|\nabla f|}$

P2 a) $\nabla g = \langle 2x, z, y \rangle$

$\nabla g(2, -1, 1) = \langle 4, 1, -1 \rangle$ $\quad |\nabla g| = 3\sqrt{2}$

$\hat{u} = \frac{\nabla g}{|\nabla g|} = \frac{4\hat{i} + \hat{j} - \hat{k}}{\sqrt{16+1+1}} = \frac{4\hat{i} + \hat{j} - \hat{k}}{3\sqrt{2}}$

$-\hat{u} = -\frac{4\hat{i} + \hat{j} - \hat{k}}{3\sqrt{2}}$

b) $g(2, -1, 1) = 4 - 1 = 3$ $\quad \Delta g = -1$

$\Delta g \approx |\nabla g| \cdot \Delta s = -1$

$\Delta s \approx \frac{-1}{|\nabla g|} = -\frac{1}{3\sqrt{2}}$

$\Delta \vec{x} =$

$\Delta \vec{r} = 0.5 \cdot \hat{u}$

$= \frac{t}{3\sqrt{2}} \cdot \frac{4\hat{i}+\hat{j}-\hat{k}}{3\sqrt{2}} = \frac{-4\hat{i}-\hat{j}+\hat{k}}{18}$

$\therefore (x,y,z) \approx \left(2-\frac{2}{9}, -1-\frac{1}{18}, 1+\frac{1}{18}\right)$

$g(x,y,z) \approx 2.164$

P3 a) $g(x,y,z) = x^2+yz = 2$

$f(x,y,z) = (x-2)^2+(y+1)^2+(z-1)^2$

$\nabla g = \langle 2x, z, y \rangle$ $\quad \cancel{\nabla g(2,-1,1) = \langle 4, 1, 1\rangle}$

$\nabla f = \langle 2x-4, 2y+2, 2z-2 \rangle$

$\nabla f = \lambda \nabla g$

$$\begin{cases} 2x-4 = 2\lambda x \\ 2y+2 = \lambda z \\ 2z-2 = \lambda y \end{cases}$$

$\frac{2}{1-\lambda}$

$$= \frac{4}{(1-\lambda)^2} + \cancel{\frac{(2\lambda+1)(\lambda+2)}{4(\lambda^2-1)^2}} \to \cancel{2\lambda^2+5\lambda+2}$$

b) $2x - 4 = 2\lambda x$

$x = \frac{2}{1-\lambda}$

$$\begin{cases} 2y + 2 = \lambda z \\ 2z - 2 = \lambda y \end{cases}$$

$2y + 2z = \lambda y + \lambda z$

$(2-\lambda) y = (\lambda - 2) z$

$\therefore y = -z$ or $\lambda = 2$

1) $y = -z$ $\quad 2y + 2 = \lambda(-y)$

$(2+\lambda) y = -2$

$y = -\frac{2}{2+\lambda}$

$z = \frac{2}{2+\lambda}$

$$x^2 + yz = \left(\frac{2}{1-\lambda}\right)^2 + \frac{-4}{(2+\lambda)^2}$$

$$= \frac{4}{(1-\lambda)^2} + \frac{-4}{(2+\lambda)^2}$$

$$= \frac{4(2+\lambda)^2 - 4(1-\lambda)^2}{(1-\lambda)^2(2+\lambda)^2}$$

$$= \frac{4(\lambda^2+4\lambda+4) - 4(1-2\lambda+\lambda^2)}{(\lambda^2+4\lambda+4)(1-2\lambda+\lambda^2)}$$

$$\frac{-20\lambda + 12}{\lambda^4 + 2\lambda^3 - 3\lambda^2 - 4\lambda + 4} = 2$$

$2\lambda^4 + 4\lambda^3 - 6\lambda^2 - 32\lambda - 4 = 0$

$\lambda^4 + 2\lambda^3 - 3\lambda^2 - 16\lambda - 2 = 0$

$\lambda_1 = -0.148$ $\quad \lambda_2 = 2.25$

2) ~~If $\lambda = 2$,~~

~~$x^2 + y^2 + z^2 = \left(\frac{2}{1-2}\right)^2 + \frac{4}{(2+2)^2}$~~

~~$= 4 + \frac{1}{4} + \frac{?}{4}$ far away from~~

i) $\lambda \approx 0.148$

$x = \frac{2}{1-\lambda} \approx 1.742$

$y = \frac{-2}{2+\lambda} \approx -1.080$

$z = -y \approx 1.080$

$(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = 0.079$

ii) $\lambda \approx 2.256$

$x = \frac{2}{1-\lambda} \approx -1.592$

$y = \frac{-2}{2+\lambda} \approx -0.470$

$z = -y = 0.470$

$(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = 13.464$ ✗

2) $\lambda = 2$:

$x = -2$

$y = -\frac{1}{2}$

$z = \frac{1}{2}$ $\quad (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = 16.5$ ✗

(3 d.p.)

$\therefore (x, y, z) \approx$ ~~$(2.347, -0.951, 0.951)$~~ $(1.742, -1.080, 1.080)$

result from P2: $(x, y, z) = \left(2 - \frac{2}{9}, -1 - \frac{1}{16}, 1 + \frac{1}{16}\right)$

$\frac{\Delta x}{x} = \frac{0.03578}{1.742} \approx 0.0205 \approx 2\%$

$\frac{\Delta y}{y} = \frac{0.02444}{1.080} \approx 0.0226 \approx 2\%$