PS VIII 29/4/2025

5. i) $A\sin(\omega t+\delta) = -0.07$

$$\sin\delta = -\frac{7}{32}$$

$$\delta = \sin^{-1}\left(-\frac{7}{32}\right) = -0.22$$

$$\delta = -0.22 + \pi = 2.92$$

ii) $f = \frac{\omega}{2\pi}$

$$v = x' = A\omega\cos(\omega t+\delta)$$

$$A\omega\cos\delta = -2$$

$$\omega = \frac{-2}{0.32\cos\delta} \quad \frac{-2}{A\cos\delta}$$

$$f = \frac{\omega}{2\pi} = \frac{1}{\pi A\cos\delta} = 1.02\ \text{Hz}$$

iii) $E = \frac{1}{2}kx^2$

$k = \frac{2E}{x^2} = \frac{11.2}{(0.32)^2} = 109\ N/m$

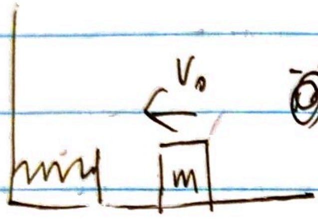
iv) $v_{max} = A\omega$

$\frac{1}{2}mv_{max}^2 = \frac{1}{2}kA^2$ $\quad mA^2\omega^2 = kA^2$

~~$m = \frac{kx^2}{v^2} = \frac{2E}{(A\omega)^2} = \frac{11.2}{(\ldots)^2} = \frac{5.6}{1.38}$~~

$m = \frac{k}{\omega^2} = \frac{2E A^2 1.38}{A^2 (1.2)^2} = \frac{1}{2}E\,1.38 = 27\ kg$

6.



$\frac{1}{2}kA^2 = \frac{1}{2}mv_0^2$

$A = \frac{v_0}{\omega} = v_0\sqrt{\frac{m}{k}}$

$t = \frac{T}{2} = \frac{\pi}{\omega} = \frac{\pi}{\sqrt{\frac{k}{m}}} = \pi\sqrt{\frac{m}{k}}$

7. $\omega_0 = \sqrt{\frac{K}{I_0}}$ $\quad f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{K}{I_0}}$

$I_0 = \frac{1}{6}ML^2$

$I = I_0 + 2\cdot m\cdot\frac{L^2}{2} = I_0 + mL^2$

$\omega = \sqrt{\frac{K}{I}} = \sqrt{\frac{K}{I_0+mL^2}}$ $\quad f = \frac{1}{2\pi}\sqrt{\frac{K}{I_0+mL^2}}$

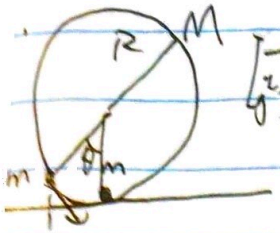
$f = 0.9f_0$

$0.81\frac{K}{I_0} = \frac{K}{I_0+mL^2}$

$0.19I_0 = 0.81mL^2$

$\frac{19}{6}ML^2 = 81mL^2$

$\frac{m}{M} = \frac{19}{6\times 81} = 0.039$



$\int_0^{2\pi} \frac{M}{2\pi R} R\,d\theta\, R^2$ $I_0 = \frac{1}{2}MR^2$

$= \frac{M}{2\pi}\int_0^{2\pi} R^2\,d\theta = MR^2$

$I = I_0 + MR^2$

$\tau = -mg\sin\theta \simeq -mg\theta$

$I = I_0 + MR^2 + m(R\theta)^2 \simeq \frac{3}{2}MR^2$

$K = -mg$

$\omega = \sqrt{\frac{K}{I}} = \sqrt{\frac{-mg}{\frac{3}{2}MR^2}}$

$T = 2\pi\cdot\frac{1}{\omega} = 2\pi\sqrt{\frac{3MR}{2mg}}$

1. $\frac{v}{2\pi} = f$

$v = 2\pi f = 5\,m/s$

0. $x = x_0\cos(\omega t - \varphi)$

$x_0 = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2 + b\omega^2}}$ $\omega_0^2 = \frac{k}{m}$ $\gamma = \frac{b}{m}$

$= \frac{F_0}{\sqrt{(k-m\omega^2)^2 + b^2\omega^2}}$

$\frac{d}{d\omega}\left((k-m\omega^2)^2 + b\omega^2\right) = 2(k-m\omega^2)(-2m\omega) + 2b^2\omega = 0$

$\rightarrow -2km\omega + 2m^2\omega^3 + b^2\omega = 0$

$-2\omega_0 + 2\omega^2 + \gamma^2 = 0$

$\omega = \sqrt{\omega_0^2 - b^2/2m^2}$ $\omega^2 = \frac{(2km - b^2)}{2m^2}$

$V = \dot{a} x_0 \omega \cos(\omega t + \varphi)$

$$x_0 = \frac{F_0 \cdot \omega / m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}}$$

$$= \frac{F_0}{\sqrt{\left(\frac{k}{\omega^2} - m\right)^2 + b^2}}$$

$$\frac{d}{d\omega}\left(\left(\frac{k}{\omega^2} - m\right)^2 + b^2\right) = 2\left(\frac{k}{\omega^2} - m\right)\left(-\frac{2k}{\omega^3}\right) = 0$$

$$(k - m\omega^2)k = 0$$

$$k^2 - mk\omega^2 = 0$$

$$\omega^2 = \frac{k}{m} \qquad \omega = \sqrt{\frac{k}{m}} = \omega_0$$

11. $x = Ce^{-\frac{\gamma}{2}t}\cos(\omega t + \varphi)$

$$Ce^{-\frac{\gamma}{2}T^*} = \frac{1}{2}C$$

$$-\frac{\gamma}{2}T^* = -\ln 2$$

$$T^* = 2\ln 2 \cdot \frac{1}{\gamma} = 2\ln 2 \frac{b}{m}$$

12. $m\ddot{x} = -kx - b\dot{x}$ let $\omega_0 = \sqrt{\frac{k}{m}}$, $\gamma = \frac{b}{m}$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = 0$$

let $x = Ae^{\alpha t}$

$$A(\alpha^2 + \alpha\gamma + \omega_0^2)e^{\alpha t} = 0$$

$$\alpha^2 + \alpha\gamma + \omega_0^2 = 0$$

$$\alpha = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2} \qquad \gamma^2 < 4\omega_0^2$$

$$\alpha_0 = \frac{-\gamma \pm \sqrt{4\omega_0^2 - \gamma^2}\,i}{2} = -\frac{\gamma}{2} \pm \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}\,i \quad \text{let } \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2} = \omega'$$

$x = Ae^{\alpha_+ t} + Be^{\alpha_- t}$

$= e^{-\frac{\gamma}{2}t}\left(e^{i\omega' t} + e^{-i\omega' t}\right)$

$= e^{-\frac{\gamma}{2}t} \cos(\omega' t + \varphi)$

$x(0) = e^{-\frac{\gamma}{2}t}\cos(\varphi) = 2$

$v = -e^{-\frac{\gamma}{2}t}\omega' \sin(\omega' t + \varphi)$

$v(0) = -e^{-\frac{\gamma}{2}t}\omega' \sin(\varphi) + \frac{\gamma}{2} e^{-\frac{\gamma}{2}t}\cos(\varphi) = 0$

$\cos(\varphi)\left(\omega' \sin\varphi + \frac{\gamma}{2}\cos\varphi\right) = 0$

$\omega_0 \left(\sin\varphi + \frac{\gamma}{2\omega_0}\cos\varphi\right) = 0$

$\varphi = -\sin^{-1}\left(\frac{\gamma}{8}\right) = -0.125$

$\gamma = \frac{b}{m} = 2 \quad \omega' = \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}$

$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{b}{m} - 1} = \sqrt{3}$

$x(0) = \frac{3\sqrt{7}}{8} \cdot 2$

$C = \frac{16}{3\sqrt{7}} = 2.02$

$x = 2.02 e^{-t}\cos(7.94t - 0.125)$

$x_d = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}} \cos(\omega t + \varphi)$

$\ddot{x}_d + \gamma \dot{x}_d + \omega_0^2 x_d = \frac{F_0}{m}\cos\omega t$

$-x_0\omega^2\cos(\omega t + \varphi) - x_0\gamma\omega\sin(\omega t + \varphi) + \omega_0^2 x_0 \cos(\omega t + \varphi) = \frac{F_0}{m}\cos\omega t$

$(\omega_0^2 - \omega^2)\cos(\omega t + \varphi) - \gamma\omega\sin(\omega t + \varphi) = \frac{F_0}{m x_0}\cos\omega t$

at $t = 0$: $(\omega_0^2 - \omega^2)\cos\varphi - \gamma\omega\sin\varphi = \sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$

$$\cos\varphi = \frac{\omega_0^2-\omega^2}{\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}} \qquad \sin\varphi = \frac{\gamma\omega}{\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}$$

$$= \frac{-192}{\sqrt{192^2+1024}} = -0.99 \qquad = \frac{32}{\sqrt{\quad}} = 0.16$$

$$\varphi = 2.98.$$

$$x = x_d + x_L = x_0\cos(\omega t-\varphi) + Ae^{-\frac{\gamma}{2}t}\cos(\omega' t-\varphi_0)$$

$$x(0) = 0.1\cos(-0.99) + A\cos(-\varphi_0) \Rightarrow x_0 = \frac{F_0/m}{\sqrt{192^2+1024}} = 0.10$$

$$v(0) = x_0\omega\sin\varphi + A\omega'\sin\varphi_0 + A\frac{\gamma}{2}\cos\varphi_0$$

$$= 1.6\times\sin(-0.99) - A\left(\omega'\sin\varphi_0 + \frac{\gamma}{2}\cos\varphi_0\right) = 0 \qquad \omega' = 7.94$$

$\omega_0 = 8$, $\frac{\gamma}{2}$, $\Delta\varphi$, ω'

$$A\cos(\varphi_0-\varphi) = \frac{1.6\times(-0.99)}{\omega}$$

$$A\cos(\varphi_0) = 2 - 0.1(-0.99)$$

$$\omega'\tan\varphi_0 = \frac{1.6\times(-0.99)}{2-0.1(-0.99)} - 1$$

$$\tan\varphi_0 = -0.22$$

$$\varphi_0 = 0.22$$

$$A\cos\varphi_0 = 2 - x_0\cos\varphi$$

$$A\left(\omega\sin\varphi_0 + \frac{\gamma}{2}\cos\varphi_0\right) = \omega x_0\sin\varphi$$

$$\omega'\tan\varphi_0 = \frac{\omega x_0\sin\varphi}{2 - x_0\cos\varphi} - \frac{\gamma}{2}$$

$$\tan\varphi_0 = \frac{1.6\times 0.16}{2+0.1\times 0.99} - 1 \Big/ 7.94 = 0.11$$

$$\varphi_0 = 0.11$$

$$A = \frac{v - x_0 \omega \sin\phi}{\omega \sin\phi_0} = \frac{2 - 0.1(-0.94)}{\omega \sin(0.11)} = 2.1$$

$$x(t) = 0.10 \cos(16t - 3.0) + 2.1 e^{-t} \cos(7.94t - 0.11)$$