PS8B Lec34 14/2/2025

6A-1 b) $\lim_{x\to 0} \frac{\cos(\frac{x}{2})-1}{x^2} = \lim_{x\to 0} \frac{-\sin\frac{x}{2}-1}{2x} = -\infty$

e) $\lim_{x\to 0} \frac{\tan^{-1}x}{5x} = \lim_{x\to 0} \frac{1}{(1+x^2)5} = \frac{1}{5}$

f) $\lim_{x\to 0} \frac{x-\sin x}{x^3} = \lim_{x\to 0} \frac{1-\cos x}{3x^2} = \lim_{x\to 0} \frac{1+\sin x}{6x} = +\infty$

g) $\lim_{x\to 1} \frac{x^a-1}{x^b-1} = \lim_{x\to 1} \frac{ax^{a-1}}{bx^{b-1}} = \frac{a}{b}$

j) $\lim_{x\to \pi} \frac{\ln\sin(\frac{x}{2})}{(x-\pi)^2} = \lim_{x\to \pi} \frac{\frac{\cos(\frac{x}{2})}{\sin(\frac{x}{2})}}{2(x-\pi)} = \lim_{x\to \pi} \frac{\cos(\frac{x}{2})}{2\sin(\frac{x}{2})(x-\pi)}$

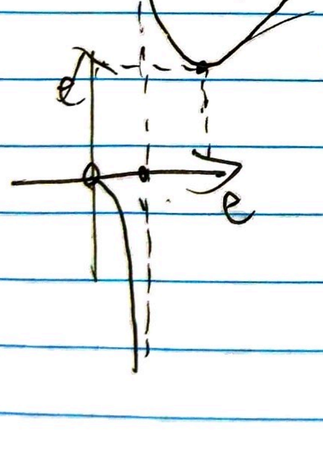
$= \lim_{x\to \pi} \frac{-\sin(\frac{x}{2})}{2\cos(\frac{x}{2})(x-\pi)+2\sin(\frac{x}{2})} = -\frac{1}{2}$

5 $\lim_{x\to 0} \frac{6x-4}{2-2x} \neq \lim_{x\to 0} \frac{6}{-2}$ since $6x-4 \neq 2-2x \to 0$ as $x \to 0$

$\lim_{x\to 0} \frac{3x^2-4x}{2x-x^2} = \lim_{x\to 0} \frac{6x-4}{-2x+2} = \lim_{x\to 0} \left(-3 + \frac{2}{-2x+2}\right)$

$= -3 + \lim_{x\to 0} \left(\frac{1}{1-x}\right) = -2$

6 c)




$\lim_{x\to 0} \frac{x}{\ln x} = 0$

$\lim_{x\to 1^-} \frac{x}{\ln x} = -\infty$ $\lim_{x\to 1^+} \frac{x}{\ln x} = +\infty$

$\lim_{x\to \infty} \frac{x}{\ln x} = \lim_{x\to\infty} \frac{1}{\frac{1}{x}} = \infty$

$y' = \frac{\ln x - 1}{(\ln x)^2}$

Lec35 17/2/2025

6B-1 $\int_1^\infty \frac{dx}{\sqrt{x^3+3}} < \int_1^\infty \frac{dx}{x^{\frac{3}{2}}} = -2x^{-\frac{1}{2}}\Big|_1^\infty = -\frac{1}{2\sqrt{\infty}} + 2$

as $N \to \infty$, $\int_1^N \frac{dx}{\sqrt{x^3+3}} < 2$

converges

6B-2 $\int_0^{\infty} \frac{x^2 dx}{x^3+2} = \int_0^{\infty} \frac{dx}{x+\frac{2}{x^2}} = \int_0^{\infty} \frac{dx}{x} = \ln x \Big|_0^{\infty}$

$\int_0^{\infty} = \frac{x^2 dx}{x^3+2} > \int_0^{\infty} \frac{dx}{x+x^2} = \int_0^{\infty} \frac{dx}{2x} = \lim_{N\to\infty} \ln x \Big|_0^{N}$

Diverges.

6B-3(a) $\int_0^{\infty} e^{-8x} dx = \lim_{N\to\infty} -\frac{1}{8} e^{-8x} \Big|_0^{N} = \lim_{N\to\infty} -\frac{1}{8} e^{-8N} + \frac{1}{8} = \frac{1}{8}$

Convergent

(f) $\int_e^{\infty} \frac{dx}{x(\ln x)^2} = \lim_{N\to\infty} -\frac{1}{\ln x} \Big|_e^{N} = \lim_{N\to\infty} -\frac{1}{\ln N} + 1 = 1$

Convergent.

(e) $\int_0^{\infty} e^{-2x} \cos x \, dx < \int_0^{\infty} e^{-2x} = \lim_{N\to\infty} -\frac{1}{2} e^{-2x} \Big|_0^{N} = -\frac{1}{2}$ $+4e^{-2x}\cos x$ $2e^{-2x}\sin x$

Convergent

$\int_0^{\infty} e^{-2x} \cos x \, dx =$ $(e^{-2x}\sin x)' = -2e^{-2x}\sin x + e^{-2x}\cos x$ $-2e^{-2x}\cos x$

$\int_0^{\infty} e^{-2x} \cos x \, dx = \frac{1}{5} e^{-2x} \sin x - \frac{2}{5} e^{-2x} \cos x \Big|_0^{\infty} = \frac{2}{5}$

(m) $\int_0^{\infty} \frac{dx}{(x+2)^3} = -\frac{1}{2}(x+2)^{-2} \Big|_0^{\infty} = 0 + \frac{1}{8} = \frac{1}{8}$ Convergent

6-c) $\lim_{x\to\infty} e^{x^2} \int_0^{x} e^{-t^2} dt$

$\int_0^{\infty} e^{-t^2} dt = \int_0^{1} e^{-t^2} dt + \int_1^{\infty} e^{-t^2} dt$

$\int_0^{1} e^{-t^2} dt + \int_1^{\infty} e^{-t^2} dt$

$\lim_{x\to\infty} e^{x^2}$

$\because \int_0^{1} e^{-t^2} dt$ converges $\therefore \int_0^{\infty} e^{-t^2} dt$ converges.

$\therefore \lim_{x\to\infty} e^{x^2} \int_0^{x} e^{-t^2} dt$ diverges.

Lec 36. 19/2/2025.

7A-1(a) $S = 1 + \frac{1}{4} + \cdots \frac{1}{4^n} + \cdots$

$\frac{1}{4}S = \frac{1}{4} + \frac{1}{4^2} + \cdots \frac{1}{4^n} + \cdots$

$\therefore S - \frac{1}{4}S = 1$

$S = \frac{4}{3}$

b) divergent

c) $\sum^{\infty} \frac{1}{n+1} \Leftrightarrow \int_0^{\infty} \frac{dx}{x+1} = \ln(n+1)$ diverges.

7B-1 a) $\sum_0^{\infty} \frac{n}{n^2+1} \Leftrightarrow \int_0^{\infty} \frac{dx}{x+\frac{1}{x}} \Leftrightarrow \int_0^{\infty} \frac{dx}{x} = \ln n \Big|_0^{\infty}$ diverges

b) $\sum_0^{\infty} \frac{1}{n^2+1} \Leftrightarrow \int_0^{\infty} \frac{dx}{x^2+1} = \arctan x \Big|_0^{\infty} = \frac{\pi}{2}$ convergent.

c) $\sum_0^{\infty} \frac{1}{\sqrt{n+1}} \Leftrightarrow \int_0^{\infty} \frac{dx}{\sqrt{x+1}} = 2\sqrt{x+1} \Big|_0^{\infty}$ diverges

f) $\sum_1^{\infty} \frac{1}{n^p}$ ~~$\Leftrightarrow \int_1^{\infty} \frac{1}{x^p}$~~

1° $p > 1$, $\sum_1^{\infty} \frac{1}{n^p} \Leftrightarrow \int \frac{1}{x^p} = \frac{x^{1-p}}{1-p} \Big|_1^{\infty} = \frac{1}{p-1}$

convergent

2° $p = 1$. $\sum_1^{\infty} \frac{1}{n} \Leftrightarrow \int_1^{\infty} \frac{dx}{x} = \ln x \Big|_1^{\infty} \to +\infty$

divergent.

3° $p < 1$ $\sum_1^{\infty} \frac{1}{n^p} \Leftrightarrow \frac{x^{1-p}}{1-p} \Big|_1^{\infty} = \infty$.

diverges.

$\therefore \sum^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \le 1$.

7B-2 a) $\sum_1^{\infty} \frac{1}{n^2+3n}$ ~~$\Leftrightarrow \int_1^{\infty} \frac{dx}{x^2+3x}$~~

$\frac{1}{n^2+3n} - \frac{1}{n^2} = \frac{3n}{(n^2+3n)n^2} \to 0$ as $n \to \infty$.

$\therefore \sum_1^{\infty} \frac{1}{n^2+3n} \sim \sum_1^{\infty} \frac{1}{n^2} \sim \int_1^{\infty} \frac{1}{n^2} = -\frac{1}{n} \Big|_1^{\infty} = 1$.

$\therefore$ converges.

b) $\sum_1^\infty \frac{1}{n+\sqrt{n}}$

$\frac{n}{n+\sqrt{n}} = \frac{1}{1+\frac{1}{\sqrt{n}}}$ as $n \to \infty$ $\frac{1}{1+\frac{1}{\sqrt{n}}} \to 1$

$\therefore \sum_1^\infty \frac{1}{n+\sqrt{n}} \sim \sum_1^\infty \frac{1}{n}$ divergent.

c) $\sum_1^\infty \frac{n}{\sqrt{n^2+n}} = \frac{1}{\sqrt{1+\frac{1}{n}}}$ as $n \to \infty$ $\frac{1}{\sqrt{1+\frac{1}{n}}} \to 1.$

$\therefore \sum_1^\infty \frac{n}{\sqrt{n^2+n}} \sim \sum_1^\infty \frac{1}{n}$ Divergent

d) $\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n^2}\right)}{\frac{1}{n^2}} = \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = 1$

$\sum_1^\infty \sin\left(\frac{1}{n^2}\right) \sim \sum_1^\infty \frac{1}{n^2}$, Convergent.

e) $\sum \frac{\cancel{\sqrt{n}}}{\cancel{n^2+1}}$ $\frac{n^{\frac{3}{2}}\sqrt{n}}{n^2+1} = \frac{1}{1+\frac{1}{n^2}}$ as $n\to\infty$ $\to 1$.

$\therefore \sum_1^\infty \frac{\sqrt{n}}{n^2+1} \sim \sum_1^\infty \frac{1}{n^{\frac{3}{2}}} \sim \int_1^\infty \frac{1}{n^{\frac{3}{2}}} = -2n^{-\frac{1}{2}}\Big|_1^\infty = 2.$

$\therefore$ Convergent.

Lec 37. 20/2/2015.

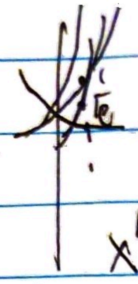
PART II 20/2/2015.

1 a) $\lim_{x\to\infty} x^m e^{-x}$ $\quad m<0$ $\lim_{x\to\infty} \frac{x^m}{e^x} = 0.$

$= \lim_{x\to\infty} \frac{x^m}{e^x}$ $m>0$

$= \lim_{x\to\infty} \left(\frac{x}{e^{\frac{x}{m}}}\right)^m = \left(\lim_{x\to\infty} \frac{1}{\frac{1}{m}e^{\frac{x}{m}}}\right)^m = 0.$

b) $\int_0^\infty x^n e^{-x} dx$



$x^n \le e^{\frac{x}{2}}$ for $x \ge a$.

$\int_0^\infty x^n e^{-x} dx = \int_0^a x^n e^{-x} dx \quad \int_a^\infty e^{-\frac{x}{2}} dx$

$\because \int_a^\infty e^{-\frac{x}{2}} dx$ converges

$\therefore \int_0^\infty x^n e^{-x} dx$ converges.

c) $A(n) = \int_0^\infty x^n e^{-x} dx$ $\quad u = x^{n+1} \quad u' = (n+1)x^n$

$\quad v' = e^{-x} \quad v = -e^{-x}$

$$\begin{aligned} A(n+1) &= \int_0^\infty x^{n+1} e^{-x} dx \\ &= -e^{-x}x^{n+1}\Big|_0^\infty - \int_0^\infty (n+1)x^n(-e^{-x})\,dx \\ &= \left(\lim_{x\to\infty} -e^{-x}x^{n+1}\right) + (n+1)\int_0^\infty x^n e^{-x} dx \\ &= (n+1)\,A(n). \end{aligned}$$

d) $A(0) = \int_0^\infty e^{-x} dx = -e^{-x}\Big|_0^\infty = 1$

$\because A(n+1) = (n+1)A(n)$

$\therefore A(1) = 1A(0) = 1$

$A(2) = 2A(1) = 2\times 1$

$A(3) = 3A(2) = 3\times 2\times 1$

$\dots$

$A(n) = nA(n-1) = n!$

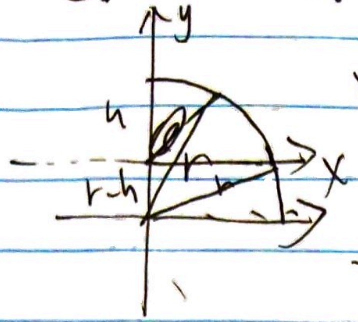
e) $A(-\frac{1}{2}) = \int_0^\infty x^{-\frac{1}{2}} e^{-x} dx = \int_0^\infty \frac{dx}{\sqrt{x}e^x} = \int_0^\infty \frac{2u\,du}{u e^{u^2}}$

$\le \int_0^a x^{-\frac{1}{2}} e^{-x} + \int_a^\infty e^{-\frac{x}{2}}$ $\therefore$ converges.

$$\cancel{\int_0^\infty \frac{1}{\sqrt{x}\,e^x}dx}$$

$$\int_0^\infty \frac{dx}{\sqrt{x}\,e^x} = 2\int_0^\infty \frac{d\sqrt{x}}{e^x}$$

2. (a)




$$x = \sqrt{r^2-y^2}$$

$$= \int_{r-h}^{r} \pi(r^2-y^2)\,dy = \pi\int_{r-h}^{r} r^2-y^2\,dy$$

$$= \pi r^2 y\Big|_{r-h}^{r} - \pi\frac{1}{3}y^3\Big|_{r-h}^{r}$$

$$= \pi r^2(h) - \frac{\pi}{3}\left(3r^2h - 3rh^2 + h^3\right)$$

$$= \pi r^2 h - \pi r^2 h + \pi r h^2 - \frac{\pi}{3}h^3$$

$$= \pi\left(rh^2 - \frac{1}{3}h^3\right).$$

1° $h=0$, $V=0$

2° $h=r$, $V = \frac{2\pi r^3}{3}$

3° $h=2r$, $V = \pi\left(4r^3 - \frac{8r^3}{3}\right) = \frac{4\pi r^3}{3}$

The formula has no meaning if $h>2r$ or $h<0$.

b)

$$ds = \cancel{\sqrt{1+\left(\frac{dy}{dx}\right)^2}(dx)}$$

$$= \sqrt{\left(\frac{1}{2}\cdot\frac{1}{\sqrt{r^2-y^2}}\cdot -2y\right)^2 + (dy)^2}$$

$$= \sqrt{\frac{y^2 dy^2}{r^2-y^2} + (dy)^2}$$

$$= \sqrt{\frac{r^2}{r^2-y^2}}\,dy$$

1° $h=0$, $A=0$

2° $h=r$, $A = 2\pi r^2$

3° $h=2r$, $A = 4\pi r^2$

$$A = \int_{r-h}^{r} 2\pi x \cdot ds = \int_{r-h}^{r} 2\pi\sqrt{r^2-y^2}\sqrt{\frac{r^2}{r^2-y^2}}\,dy$$

$$= \int_{r-h}^{r} 2\pi r\,dy = 2\pi r y\Big|_{r-h}^{r} = 2\pi r^2 - 2\pi r^2 + 2\pi r h = 2\pi r h$$

i) $\pi\left(rh^2 - \frac{h^3}{3}\right) = V$

$$rh^2 - \frac{h^3}{3} = \frac{V}{\pi}$$

$$\cancel{h^2\left(r - \frac{h}{3}\right) = \frac{V}{\pi}}$$

$$r = \frac{\frac{V}{\pi} + \frac{h^3}{3}}{h^2}$$

$$= \frac{h}{3} + \frac{V}{\pi h^2}$$

$$A = 2\pi r h$$

$$= 2\pi\left(\frac{h}{3} + \frac{V}{\pi h^2}\right)h$$

$$= 2\pi\left(\frac{h^2}{3} + \frac{V}{\pi h}\right)$$

$$A' = 2\pi\left(\frac{2}{3}h + \frac{V}{\pi}\cdot\left(-\frac{1}{h^2}\right)\right) = 0$$

$$\frac{2}{3}h = \frac{V}{\pi h^2}$$

$$h^3 = \frac{3V}{2\pi}$$

$$h = \sqrt[3]{\frac{3V}{2\pi}}$$

$$\cancel{r = \frac{\sqrt[3]{\frac{3V}{2\pi}}}{3} + \frac{V}{\pi\left(\frac{3V}{2\pi}\right)^{\frac{2}{3}}}}$$

$$V = \frac{2\pi h^3}{3}$$

$$\pi r h^2 - \frac{\pi h^3}{3} = \frac{2\pi h^3}{3}$$

$$3\pi r h^2 - \pi h^3 = 2\pi h^3$$

$$3\pi r - \pi h = 2\pi h$$

$$3\pi r = 3\pi h$$