PS2 27/2/2025

Lec 4 27/2/2025

1H-3 a) $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ -1 & c & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ has a non-trivial solution

$\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ -1 & c & 2 \end{vmatrix} = 0$

$\begin{vmatrix} 1 & 1 \\ c & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ -1 & c \end{vmatrix} = (2-c) + (4+1) + (2c+1)$

$= c + 8 = 0$

$c = -8.$

b) $\left[\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}\right]\begin{pmatrix} x \\ y \end{pmatrix} = 0$ non-trivial

$\begin{vmatrix} 2-c & 1 \\ 0 & -1-c \end{vmatrix} = 0$

$-2-2c+c+c^2 = 0$

$c = 2, c = -1$

c) $\langle 1, -1, 1 \rangle \times \langle 2, 1, 1 \rangle = \left\langle \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}, \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} \right\rangle$

$= \langle -2, 1, 3 \rangle$

$\langle -2, 1, 3 \rangle \cdot \langle -1, -8, 2 \rangle = 2 - 8 + 6 = 0$

7 c) $\begin{bmatrix} \cos x_1 & \sin x_1 \\ \cos x_2 & \sin x_2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ $\det\begin{bmatrix} \cos x_1 & \sin x_1 \\ \cos x_2 & \sin x_2 \end{bmatrix} = \cos x_1 \sin x_2 - \cos x_2 \sin x_1$

$= \sin(x_2 - x_1)$

$\therefore x_2 \neq x_1 + n\pi$

$\therefore \sin(x_2 - x_1) \neq 0$

$\therefore$ this is pr...

b) $y_2 = (-1)^n y_1$

15-1 a) $x-2+2(y-0)-2(z-(-1))=0$

$x-2+2y-2z-2=0$

$x+2y-2z=4$

b) $\vec{N} = \langle 1,1,0\rangle \times \langle 2,-1,3\rangle$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 2 & -1 & 3 \end{vmatrix} = 3\hat{i} - 3\hat{j} + (-1-2)\hat{k} = \langle 3,-3,-3\rangle$$

$\therefore$ plane $3x-3y-3z=0$

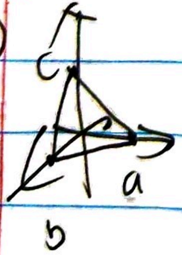
c) $\vec{N} = \langle 1,-1,1\rangle \times \langle -2,3,1\rangle$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ -2 & 3 & 1 \end{vmatrix} = (-1-3)\hat{i} - (1+2)\hat{j} + (3-2)\hat{k} = \langle -4,-3,1\rangle$$

$-4(x-1) - 3y + (z-1) = 0$

$-4x-3y+z = -3$

d)



$\vec{N} = \langle -a, b, 0\rangle \times \langle -a, 0, ac\rangle$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = \cancel{bc\hat{i} - (-ac)\hat{j} + (ab)\hat{k}} \quad bc\hat{i} + ac\hat{j} + ab\hat{k}$$

$= \cancel{\langle bc, ac, ab\rangle}$

$bc(x-a) + ac\,y + ab\,z = 0$

$bcx + \dots$

$x\ \ y + z = 1$

a) 

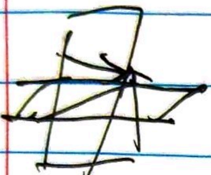
$\vec{N} = \langle -1,1,0\rangle \times \langle 1,-1,2\rangle$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 0 \\ 1 & -1 & 2 \end{vmatrix} = 2\hat{i} - (-2)\hat{j} + (1-1)\hat{k}$ 1

$= 2\hat{i} + 2\hat{j}$

$2(x-1)+2y=0$

$x+y=1$

2  

$\sin\theta = \frac{\langle 2,-1,1\rangle \cdot \langle 1,1,2\rangle}{\sqrt{4+1+1}\sqrt{1+1+4}}$

$= \frac{1}{6}(2-1+2) = \frac{1}{2}$

$\theta = \arcsin\frac{1}{2} = \frac{\pi}{6}$

b $ax+by+cz=d$ $\qquad \langle a,b,c\rangle \cdot \langle 0,-c,b\rangle = 0$

$-cy+bz=0$ $\qquad \langle a,b,c\rangle \cdot \langle -b,a,0\rangle = 0$

$-bx+ay=0$

$\cancel{ax+by+cz}=0$

$\cancel{-cy+bz}=0$

$\begin{bmatrix} a & b & c \\ 0 & -c & b \\ -b & a & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d \\ 0 \\ 0 \end{bmatrix}$

$abx+b^2y+cbz=db$

$\cancel{-c^2y}+bcz=0$

$abx+(b^2+c^2)y=db$

$-abx+a^2y=0$

$(a^2+b^2+c^2)y=db$

$x=\frac{ad}{(a^2+b^2+c^2)}$

$y=\frac{bd}{a^2+b^2+c^2}$

$z=\frac{cd}{a^2+b^2+c^2}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (0-ab) & -(b^2) & -bc \\ +ac & +bc & -(a^2+b^2) \\ b^2+c^2 & -ab & ac \end{bmatrix} \cdot \frac{1}{(-ab)-b^2-bc} \cdot \begin{bmatrix} d \\ 0 \\ 0 \end{bmatrix}$

$D=\sqrt{x^2+y^2+z^2} = \sqrt{\frac{a^2d^2+b^2d^2+c^2d^2}{(a^2+b^2+c^2)^2}}$

$= \frac{|d|}{\sqrt{a^2+b^2+c^2}}$

[redacted]

Lec5 4/3/2025

16/3a) $(\cancel{2x+1, -y})$

$(2t+1, -t, 3t-1)$

b) $(t+2, -t-1, 2t-1)$

c) $\langle a, b, c\rangle \cdot \langle 1, 2, -1\rangle$

$= a + 2b - c = 0$

$(at+1, bt+1, ct+1)$

14 $x = 2t$, $y = -t+1$, $z = t+2$

$2t + 4(t+1) + t + 2 = 4$

$7t + 6 = 4$

$t = -\frac{2}{7}$

$x = -\frac{4}{7}$, $y = \frac{9}{7}$, $z = \frac{12}{7}$

$\therefore$ point $\left(-\frac{4}{7}, \frac{9}{7}, \frac{12}{7}\right)$

5 $x = t+1$, $y = 2t+1$, $z = -t-1$

$2(t+1) - (2t+1) - t - 1 = 1$

$2t + 2 - 2t - 1 - t - 1 = 1$

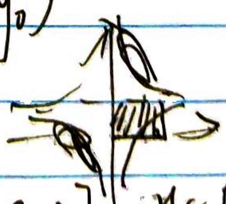
$-t = 1$

$t = -1$

$(0, -1, 0)$

17-1 $\vec{r}(t) = \left(\frac{av}{\sqrt{a^2+b^2}}t + x_0, \frac{bv}{\sqrt{a^2+b^2}}t + y_0\right)$

3a) $x = 2\cos^2 t$ $y = \sin^2 t$



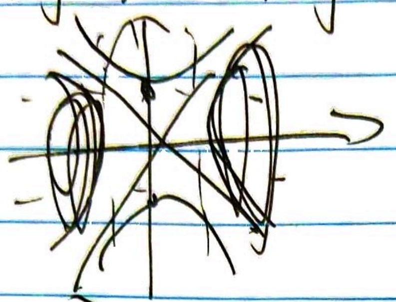
$\frac{x}{2} + y = 1$ $x \in [0, 2]$, $y \in [0, 1]$

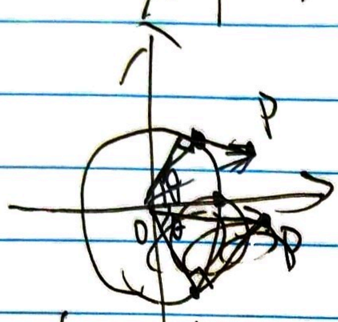
b) $x = \cos 2t = 2\cos^2 t - 1$ $y = \cos t$

$x = 2y^2 - 1$ $x \in [-1, 1]$, $y \in [-1, 1]$

d) $x = \tan t \quad y = \sec t$

$1 + x^2 = y^2$





$\vec{OP} = \cancel{A}\langle \cos\theta, \sin\theta\rangle + \langle \sin\theta, -\cos\theta\rangle \cdot \cancel{A}$

$= \langle \cancel{A}(\theta \sin\theta + \cos\theta), -\cancel{A}(\theta\cos\theta + \sin\theta)\rangle$

Lec 6 4/3/2025

1)-1 a) $\vec{v} = \frac{d\vec{r}}{dt} = e^t \hat{i} - e^{-t}\hat{j}$

$|v| = \sqrt{e^{2t} + e^{-2t}}$

$\hat{v} = \frac{\vec{v}}{|v|} = \frac{e^t\hat{i} - e^{-t}\hat{j}}{\sqrt{e^{2t} + e^{-2t}}}$

$\vec{a} = \frac{d\vec{v}}{dt} = e^t\hat{i} + e^{-t}\hat{j}$

b) $\vec{v} = \frac{d\vec{r}}{dt} = 2t\hat{i} + 3t^2\hat{j}$

$|v| = \sqrt{4t^2 + 9t^4}$

$\hat{v} = \frac{2t\hat{i} + 3t^2\hat{j}}{\sqrt{4t^2 + 9t^4}}$

$\vec{a} = \frac{d\vec{v}}{dt} = 2\hat{i} + 6t\hat{j}$

c) $\vec{v} = \frac{d\vec{r}}{dt} = -4t\hat{i} + 2t\hat{j} + 4t\hat{k}$

$|v| = \sqrt{16t^2 + 4t^2 + 16t^2} = \sqrt{36t^2} = 6t$

$\hat{v} = \frac{-4t\hat{i} + 2t\hat{j} + 4t\hat{k}}{6t} = \frac{4}{6}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}$

$\vec{a} = \frac{d\vec{v}}{dt} = -4\hat{i} + 2\hat{j} + 4\hat{k}$

2a) $\vec{v} = \frac{d\vec{r}}{dt} = (-1)(1+t^2)^{-2}\cdot(2t)\hat{i} + \frac{1+t^2-t(2t)}{(1+t^2)^2}\hat{j}$

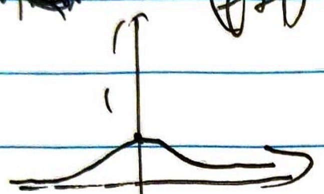
$= \frac{-2t}{(1+t^2)^2}\hat{i} + \frac{1-t^2}{(1+t^2)^2}\hat{j}$

$|v| = \frac{\sqrt{4t^2+t^4-2t^2+1}}{(1+t^2)^2} = \frac{t^2+1}{(1+t^2)^2} = \frac{1}{t^2+1}$

$\vec{T} = \frac{\vec{v}}{|v|} = (t^2+1)\left(\frac{-2t\hat{i}+(1-t^2)\hat{j}}{(1+t^2)^2}\right) = \frac{-2t\hat{i}+(1-t^2)\hat{j}}{1+t^2}$

b) $|v|' = \frac{-(2t)}{(t^2+1)^2} = 0$

$t = 0$

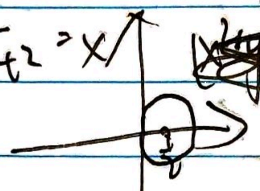


P(1, 0) |v| greatest.

|v| → smallest as $t \to \pm\infty$

c) $x = \frac{1}{1+t^2}$ $y = \frac{t}{1+t^2}$

$x^2+y^2 = \frac{1}{1+t^2} = x$



$x^2+y^2 = x$

$x^2 - x + \frac{1}{4} + y^2 = \frac{1}{4}$

3 $\vec{r}\cdot\vec{s} = x_1x_2 + y_1y_2$

$\frac{d}{dt}\vec{r}\cdot\vec{s} = x_1'x_2 + x_1x_2' + y_1'y_2 + y_1y_2'$

$\frac{d\vec{r}}{dt}\cdot\vec{s} = (x_1'\hat{i} + y_1'\hat{j})\cdot(x_2\hat{i} + y_2\hat{j}) = x_1'x_2 + y_1'y_2$

$\vec{r}\cdot\frac{d\vec{s}}{dt} = (x_1\hat{i} + y_1\hat{j})\cdot(x_2'\hat{i} + y_2'\hat{j}) = x_1x_2' + y_1y_2'$

∴ LHS = RHS ∎

4a) ~~$\vec{r}\ \frac{d\vec{r}}{dt}$~~

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ $\sqrt{x^2+y^2+z^2} = |r| =$ constant

$\vec{v} = \frac{d\vec{r}}{dt} = x'\hat{i} + y'\hat{j} + z'\hat{k}$ $x^2+y^2+z^2 =$ constant

$\vec{r}\cdot\vec{v} = xx' + yy' + zz'$ $2xx' + 2yy' + 2zz' = 0$

$= 0$ ∴ $\vec{r} \perp \vec{v}$

b) $\frac{d}{dt}\vec{r}\cdot\vec{v} =$

$\vec{v}\cdot\vec{v} = \frac{d\vec{r}}{t}\cdot\vec{r} = \frac{d}{dt}\cdot r^2 - \vec{r}\cdot\frac{d\vec{r}}{dt}$

$= -\vec{r}\cdot\vec{v}$

$\vec{v}\cdot\vec{r} + \vec{r}\cdot\vec{v} = 0$

$\vec{v}\cdot\vec{r} = 0$

c) $\vec{r}\cdot\frac{d\vec{r}}{dt} = 0 = \frac{d}{dt}\vec{r}\cdot\vec{r} - \frac{d\vec{r}}{dt}\cdot\vec{r}$

$= \frac{d}{dt} r^2$

$\therefore \frac{d}{dt}|r| = 0$, P moves on a sphere.

5 a)

$\frac{d}{dt} v^2 = 0$

$\frac{d}{dt}\vec{v}\cdot\vec{v} = 0 = \frac{d\vec{v}}{dt}\cdot\vec{v} + \vec{v}\cdot\frac{d\vec{v}}{dt} = 2\vec{v}\cdot\vec{a}$

$\therefore \vec{v}\cdot\vec{a} = 0$, $\vec{v}\perp\vec{a}$

b) $\vec{v}\cdot\vec{a} = 0 = (\vec{v}\cdot\vec{v})' - \vec{a}\cdot\vec{v}$

$= (\vec{v}\cdot\vec{v})' = (v^2)'$

$\therefore |v|$ constant.

6 a) $\vec{v} = \frac{d\vec{r}}{dt} = -a\sin t\,\hat{i} + a\cos t\,\hat{j} + b\hat{k}$

$\vec{a} = \frac{d\vec{v}}{dt} = -a\cos t\,\hat{i} - a\sin t\,\hat{j}$

$\vec{T} = \frac{\vec{v}}{|v|} = \frac{\vec{v}}{\sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2}} = \frac{-a\sin t\,\hat{i} + a\cos t\,\hat{j} + b\hat{k}}{\sqrt{a^2+b^2}}$

$\left|\frac{ds}{dt}\right| = \left|\frac{d\vec{r}}{dt}\right|$ $\quad |v| = \sqrt{a^2+b^2}$

9 a) $|r| = \sqrt{9\cos^2 t + 25\sin^2 t + 16\cos^2 t} = \sqrt{25} = 5$

b) $|\vec{v}| = \cancel{\left|\frac{d\vec{r}}{dt}\right|} =$

$\vec{v} = \frac{d\vec{r}}{dt} = -3\sin t\,\hat{i} + 5\cos t\,\hat{j} - 4\sin t\,\hat{k}$

$|v| = \sqrt{9\sin^2 t + 25\cos^2 t + 16\sin^2 t}$

$= \sqrt{25} = 5$

c) $\vec{a} = \frac{d\vec{v}}{dt} = -3\cos t\,\hat{i} - 5\cos t\,\hat{j} - 4\cos t\,\hat{k}$

$= -\vec{r}$

d) $t=0$ $\vec{r}_1 = \langle 3, 0, 4 \rangle$

$t = \frac{\pi}{2}$ $\vec{r}_2 = \langle 0, 5, 0 \rangle$

$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & 4 \\ 0 & 5 & 0 \end{vmatrix} = -20\hat{i} + 15\hat{k}$

Let $\vec{N} = -4\hat{i} + 3\hat{k}$

$\vec{N} \cdot \vec{r} = -12\cos t + 12\cos t = 0$

$\therefore \vec{r} \perp \vec{N}$.

$\therefore P$ in plain $-4x + 3z = 0$

e) A circle centered on origin in plain $-4x + 3z = 0$. radius $= 5$.

10 a) $\kappa = \left|\frac{d\vec{T}}{ds}\right| = \frac{d\vec{T}}{dt} \cdot \frac{dt}{ds}$

$\kappa \cdot \frac{ds}{dt} = \frac{d\vec{T}}{dt} = \left| \frac{-a\cos t\,\hat{i} - a\sin t\,\hat{j}}{\sqrt{a^2+b^2}} \right|$

$\kappa \cdot \sqrt{a^2+b^2} = \frac{-a\cos t\,\hat{i} - a\sin t\,\hat{j}}{\sqrt{a^2+b^2}}$

$\kappa = \frac{|-a\cos t\,\hat{i} - a\sin t\,\hat{j}|}{a^2+b^2} = \frac{|a|}{a^2+b^2}$

b) $b=0$, $\kappa = \frac{1}{|a|}$

1K-3 Given $\vec{r} \times \vec{a} = 0$

$\vec{r} \times \vec{a} = \frac{d}{dt}(\vec{r} \times \vec{v})$, $\vec{v} \times \vec{v} = 0$

$\frac{d}{dt}(\vec{r} \times \vec{v}) = 0$

$\therefore \vec{r} \times \vec{v} =$ constant vector

$\therefore |\vec{r} \times \vec{v}| =$ constant.

PART B 8/3/2023

P1 a) $\vec{N} = \langle a, b, c \rangle$

$\begin{cases} a - 3b + c = 2 \\ -a + 4b - 2c = 0 \end{cases}$ $b - c = 0$, $b = c$. $a - 3b + c = 2$ $\therefore \langle 2, 1, 1 \rangle = \vec{N}$

$\vec{AC} = \langle -1, -3, 3 \rangle$ $\vec{B} = \langle -3, 4, 0 \rangle$

$d = \left| \vec{AC} \cdot \frac{\vec{n}}{|n|} \right| = \frac{-2-3+3}{\sqrt{6}} = \frac{2}{\sqrt{6}}$

b) $\vec{L_1} = \langle -1, -4, -2 \rangle$

$P_1(2+t_1, 3t_1, t_1)$, $P_2(1-2t_2, 1-4t_2, 3-2t_2)$

$\vec{P_1P_2} = \langle 1-2t_2-2-t_1, 1-4t_2+3t_1, 3-2t_2-t_1 \rangle$

$= \langle -t_1-2t_2-1, +3t_1-4t_2-3, -t_1-2t_2+3 \rangle$

$\begin{cases} -t_1-2t_2-1 - 9t_1+12t_2+9 - t_1-2t_2+3 = 0 \\ t_1+2t_2+1-12t_1+16t_2+12+2t_1+4t_2-6 = 0 \end{cases}$

$\begin{cases} -11t_1+8t_2+11 = 0 \\ -9t_1+22t_2+7 = 0 \end{cases}$

$t_1 = -11 - \frac{88}{85}$

$\begin{cases} -99t_1+72t_2+99 = 0 \\ -99t_1+242t_2+77 = 0 \end{cases}$

$170t_2 = 22$ $t_2 = \frac{11}{85}$

$$|P_1P_2| = \left|\left\langle \frac{-88+72}{85} - 1,\ \frac{\sqrt{188}-4\times 36}{85} - 2,\ \frac{-88-72}{85}+3\right\rangle\right|$$

$$= \left|\left\langle \frac{-85}{85},\ \frac{184-144-120}{85},\ \frac{255-160}{85}\right\rangle\right|$$

$$= \left|\left\langle \frac{15}{17},\ \frac{10}{17},\ \frac{19}{17}\right\rangle\right|$$

$$= \sqrt{\frac{15^2+10^2+19^2}{(17)^2}} = \sqrt{\frac{325+361}{(17)^2}} = \frac{\sqrt{687}}{17}$$

$19 \times 19 = 171,\ 361$

$$\vec{P_1P_2} = \left\langle \frac{93-22}{85} - 1,\ \frac{\sqrt{79}-44}{85} - 3,\ \frac{-115}{85}+3\right\rangle$$

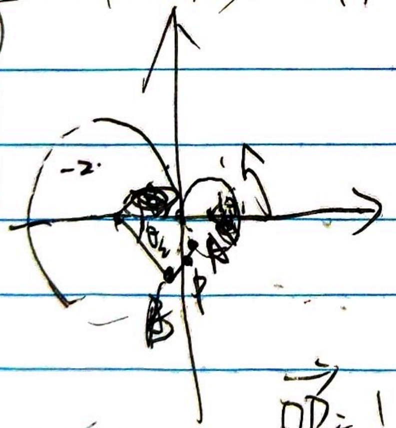
-100

$$= \left\langle \frac{-100}{85\ 17},\ \frac{235-255}{85},\ \frac{140}{85}\right\rangle$$

$$= \left\langle -\frac{100}{17},\ \frac{-4}{85\ 17},\ \frac{28}{17}\right\rangle$$

$$|P_1P_2| = \frac{\sqrt{10793}}{17}$$

P2 a)



$2t = 2\theta_2 = \theta_1$

$A = (1-\cos\theta_1,\ -\sin\theta_1)$

$B = (2\cos\theta_2 - 2,\ -2\sin\theta_2)$

$$\vec{OP} = \frac{1}{2}\vec{AB} + \vec{OA} = \left\langle \frac{1-\cos 2t + 2\cos t - 2}{2},\ \frac{-\sin 2t - 2\sin t}{2}\right\rangle$$

b) $$\frac{d\vec{OP}}{dt} = \langle \sin 2t - \sin t,\ -2\cos 2t - 2\cos t\rangle$$

$$\left.\frac{d\vec{OP}}{dt}\right|_0 = \langle 0, -4\rangle$$