



The nice thing about  $z$  is that we can guess a solution

$$z(t) = z_0 \exp^{i\omega t} \quad (8)$$

where  $z_0$  is a constant (in time). Feeding in this guess (and remembering that every differentiation just brings down an  $i\omega$ ), we find

$$(j\omega^2 + i\omega\gamma + \omega_0^2)z_0 \exp^{i\omega t} = \frac{F_0}{m} \exp^{i\omega t} \quad (9)$$

Cancelling the  $e^{i\omega t}$  (which is never zero) we find that  $z_0$  is fully determined by the above to be

$$z_0 = \frac{F_0/m}{(j\omega^2 + i\omega\gamma + \omega_0^2)} = \frac{F_0/m}{I(\omega)} \quad (10)$$

where the impedance  $I(\omega)$  is

$$I(\omega) = (j\omega^2 + i\omega\gamma + \omega_0^2): \quad (11)$$

Let us write  $I$  in polar form

$$I(\omega) = jIj e^{iA} \quad (12)$$

where

$$jIj = \frac{1}{(j\omega^2 + \omega_0^2)^2 + (\omega\gamma)^2} \quad \tan A = \frac{\omega\gamma}{(j\omega^2 + \omega_0^2)} \quad (13)$$

This means

$$z_0 = \frac{F_0}{mjIj e^{iA}} \quad (14)$$

and that

$$z(t) = \frac{F_0}{mjIj} e^{i(\omega t - A)} \quad (15)$$

and finally

$$x(t) = \frac{F_0}{mjIj} \cos(\omega t - A) = x_0 \cos(\omega t - A) \quad (16)$$

(There is one more point about this answer I will return to shortly.)

Thus the driven oscillator vibrates at the frequency of the driving force, lags in phase by  $A$  and has an amplitude  $\frac{F_0}{mjIj}$ . Both the amplitude and phase are frequency dependent.

The amplitude  $x_0$  is largest where  $jIj$  is smallest:

$$x_0 = \frac{F_0/m}{(j\omega^2 + \omega_0^2)^2 + (\omega\gamma)^2} \quad (17)$$

If  $\gamma = 0$ , this clearly occurs at  $\omega = \omega_0$ . At this point  $x_0$ , the amplitude of vibrations diverges. This is however an un-physical case since there is always some friction or  $\gamma$ . In the presence of nonzero  $\gamma$ , the maximum in  $x_0$  occurs near  $\omega = \omega_0$ . This is called

resonance and is more pronounced, the smaller the value of  $\gamma$ . Note that at  $\omega = 0$ ,  $x_0 = (F_0/m\omega_0^2) = F_0/k$  which makes sense. The function then rises, peaks near  $\omega_0$  and vanishes as  $\omega \rightarrow \infty$ . See the book for some graphs, for different values of friction.

Once we have  $x(t)$  we can take derivatives and get the answer of the velocity. The amplitude of velocity oscillations will not peak where  $x_0$  does, though the two points will be close if  $\gamma$  is small.

Now for what is lacking in Eq. 16. Note it has no free parameters: both  $x_0$  (the amplitude) and  $A$  (the phase) are determined by  $m, \gamma, \omega_0$  and  $\omega$ . How then do we arrange to have  $x(0)$  and  $v(0)$  equal to some arbitrary initial conditions? The answer is that to the  $x(t)$  in Eqn.(16), which we will henceforth refer to as the *particular solution*  $x_p(t)$  we can always add the complimentary function  $x_c(t)$  from Eqn. (1) to get the answer

$$x(t) = x_p(t) + x_c(t) \quad (18)$$

$$= \frac{F_0}{m\sqrt{\omega_0^2 - \gamma^2}} \cos(\omega t + A) + x_0 \cos(\omega t + A) + Ce^{i\frac{\gamma}{2}t} \cos(\omega_0 t + A_0) \quad (19)$$

$$\omega_0 = \sqrt{\omega_0^2 - \frac{\gamma^2}{2}} \quad (20)$$

Adding  $x_c$  will not affect the fact that Eqn (4) is satisfied since

$$\frac{d^2 x_c}{dt^2} + \gamma \frac{dx_c}{dt} + \omega_0^2 x_c = 0 \quad (21)$$

Thus  $x = x_p + x_c$  obeys the requisite equation and has the two free parameters that allow us to choose our initial position and velocity at will.

Note however that due to the exponentially falling factor  $e^{i\frac{\gamma}{2}t}$  in  $x_c$ , it will die down after some time. Thus  $x_c$  is called the transient solution and  $x_p$ , which goes on and on the steady-state solution. We will focus on the steady-state part from now on.

Let us admire some fine points. By using complex numbers we have managed to convert a differential equation Eq. (4) into an algebraic equation, Eq. (9). Next, note that the response  $x_p(t)$  is obtained from the cause  $F_0 \cos \omega t$  by (i) dividing by  $\sqrt{\omega_0^2 - \gamma^2}$  and (ii) changing the phase by  $A$ . This cannot be readily done in the world of real variables. However once problem is cast in terms of a complex force  $F_0 e^{i\omega t}$  and its complex response,  $z = z_0 e^{i\omega t}$ , the two are related by

$$z_0 = \frac{F_0}{m\sqrt{\omega_0^2 - \gamma^2}} \quad (22)$$

and division by a single complex number  $I = \sqrt{\omega_0^2 - \gamma^2} e^{iA}$  re-scales and shift the (amplitude of the) applied force to give the (amplitude of the) response.

**For those of you who want the bottom line here it is. The driven oscillator has a complete solution given by**

$$x(t) = x_p(t) + x_c(t) = \frac{F_0}{m\sqrt{\omega_0^2 - \gamma^2}} \cos(\omega t + A) + x_0 \cos(\omega t + A) + Ce^{i\frac{\gamma}{2}t} \cos(\omega_0 t + A_0)$$

$$\omega_0 = \sqrt{\omega_0^2 - \frac{\gamma^2}{2}}$$

$$j/Ij = \frac{q}{(j^2 + I_0^2)^2 + (I^2 - I_0^2)}$$

$$\tan A = \frac{I^0}{I_0^2 j^2}$$

$C$  and  $A_0$  are free parameters chosen to fit initial conditions