PS 9 31/3/2025

Lec 24

4D-1a) $\oint_C \vec{F}\cdot d\vec{r} = \oint_C 2y\,dx + x\,dy$

$= \oint_C -2\sin^2\theta\, d\theta + \cos^2\theta\, d\theta$

let $x = \cos\theta$

$y = \sin\theta$

$dx = -\sin\theta\, d\theta$

$dy = \cos\theta\, d\theta$

$= \int_0^{2\pi} (\sin^2\theta + \cos 2\theta)\, d\theta$

$= \int_0^{2\pi} \left(\frac{1-\cos 2\theta}{2} + \cos 2\theta\right) d\theta$

$\nabla\times\vec{F} = \frac{\partial}{\partial x}x - \frac{\partial}{\partial y}2y$

$= -1$

$= \frac{1}{2}\int_0^{2\pi} (1 + \cos 2\theta)\, d\theta$

$= \frac{1}{2}\left(2\pi + \frac{1}{2}\sin 4\pi - \frac{1}{2}\sin 0\right) = -\pi$

$\oint_{C_0} \vec{F}\cdot d\vec{r} = \frac{1}{2}\int_{2\pi}^{0} (1+\cos 2\theta)\, d\theta = \pi$

$\oint_C \vec{F}\cdot d\vec{r} = \iint_R \nabla\times\vec{F}\, dA = \int_0^{2\pi}\int_0^1 -r\, dr\, d\theta$

$= -\pi$

2 $\vec{F} = 4x^3y + x^4$ $\quad \nabla\times\vec{F} = 4x^3 - 4x^3 = 0$

~~$\nabla\vec{F} = \nabla f$~~

$\oint_C 4x^3y\,dx + x^4\,dy = \iint_R \nabla\times\vec{F}\, dA = 0$

3 $\iint_R dA = \oint_C y\,dx$

~~$\nabla\vec{F} = 1$~~

~~Assume $\vec{F} = y\vec{i}$~~

$dx = 3\cos^2\theta\sin\theta\, d\theta$

$= \int_0^{2\pi} \sin^3\theta\, 3\cos^2\theta\sin\theta\, d\theta$

$= \int_0^{2\pi} 3\sin^4\theta(1 - \sin^2\theta)\, d\theta$

$= \int_0^{2\pi} (3\sin^4\theta - 3\sin^6\theta)\, d\theta =$

$A = \iint_R dA = \frac{1}{2}\oint_C \cancel{y\,dx - x\,dy}\ \cancel{xd}\ -y\,dx + x\,dy$

$= \frac{3}{2}\int_0^{2\pi} \sin^4\theta\cos^2\theta\,d\theta + 3\sin^2\theta\cos^4\theta\,d\theta$

$= \frac{3}{2}\int_0^{2\pi} \sin^2\theta\cos^2\theta\,d\theta$

$= \frac{3}{8}\int_0^{2\pi} \sin^2 2\theta\,d\theta = \frac{3}{16}\int_0^{2\pi} 1-\cos 4\theta\,d\theta$

$= \frac{3}{16}(2\pi - 1 + 1 -) = \frac{3\pi}{8}$

4 $\oint_C -y^3\,dx + x^3\,dy = \iint_R (3x^2+3y^2)\,dA > 0$

5 $\oint_C xy^2\,dx + (x^2y+2x)\,dy = \iint_R (2yx + 2 - 2xy)\,dA = \iint_R 2\,dA$

$= 2\iint_R dA = 2\,\text{Area}.$

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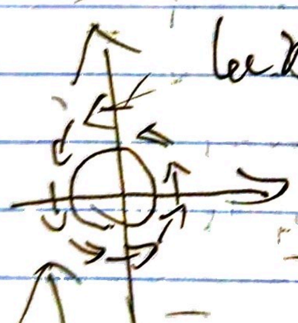
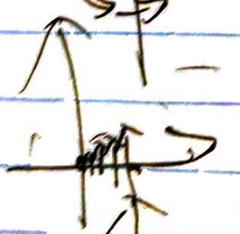
4E-1a)



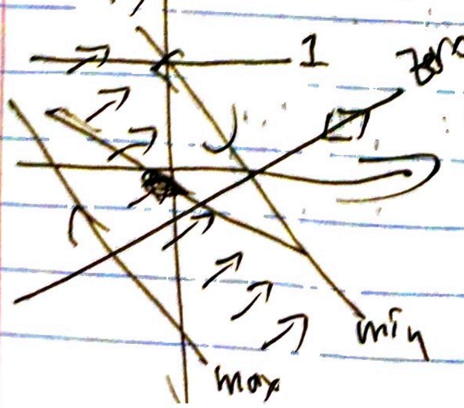
$\int_C \vec{F}\cdot\hat{n}\,ds = 0$

b) ①



$\int_C \vec{F}\cdot\hat{n}\,ds = \int_0^1 -x\,dx = -\frac{x^2}{2}\Big|_0^1 = -\frac{1}{2}$

2



e) max $\int_C \vec{F}\cdot\hat{n}\,ds$

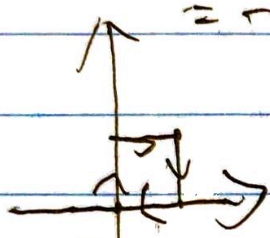
$= \int_0^1 \sqrt{2}\,ds = \sqrt{2}.$

min $= -\sqrt{2}$.

3 $y=(x-1)^2$

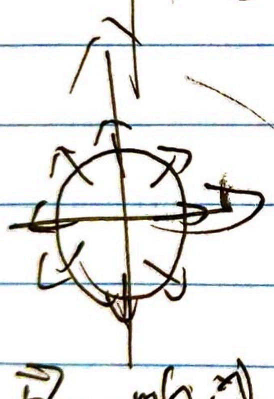
$$\int_C \vec{F}\cdot\hat{n}\,ds = \int_C xy\,dx - x^2\,dy$$
$$= \int_0^1 (t+1)t^2\,dt - (t+1)^2\cdot 2t\,dt$$
$$= \int_0^1 (t^3+t^2)\,dt - (2t^3+4t^2+2t)\,dt$$
$$= \int_0^1 (-t^3-3t^2-2t)\,dt$$
$$= -\frac{1}{4} - 1 - 1 = -\frac{9}{4}$$

4



$$\int_C \vec{F}\cdot\hat{n}\,ds = -\iint_R 2\,dA = -2$$

5 a)



$$\int_C \vec{F}\cdot\hat{n}\,ds \quad \cancel{dx = -\sin\theta\,d\theta \quad dy = \cos\theta\,d\theta}$$
$$\cancel{\int_C \cos\theta\,dx\,a^2\sin\theta\,dx - a^2\cos\theta}\,dy$$
$$\cancel{= \int_C \int_0^{2\pi} a^2\sin^2\theta\,d\theta - a^2\cos^2\theta\,d\theta}$$
$$\cancel{= 2\pi - 2a^2\pi}$$

$\vec{F} = r^m(\hat{i}+\hat{j})$

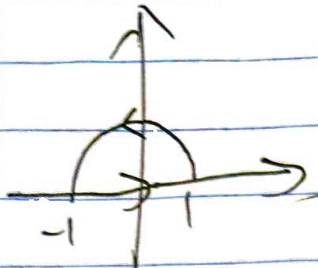
$dx = -r\sin\theta\,d\theta \quad dy = r\cos\theta\,d\theta.$

$$\int_C \vec{F}\cdot\hat{n}\,ds = \int_0^{2\pi} -a^{m+1}\,d\theta = -2a^{m+1}\pi$$

b) $\dfrac{\partial F_x}{\partial y} = m r^{m-1}\dfrac{2yx}{\sqrt{x^2+y^2}} \qquad \dfrac{\partial F_y}{\partial x} = m r^{m-1}\dfrac{2xy}{\sqrt{x^2+y^2}}$

$\therefore m \neq -1$

-3



$\vec{F} = x\hat{i} + y\hat{j}$

$\nabla \cdot \vec{F} = 2$

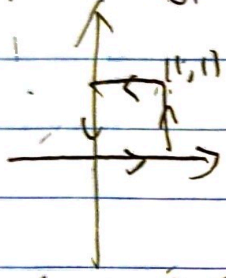
$LHS = \oint_C \cancel{y\,dx - x\,dy}\ x\,dy - y\,dx$

$= \int_0^{2\pi} \sin\theta \cdot (-\sin\theta)\,d\theta - \cos\theta\cos\theta\,d\theta + \int_{-1}^{1} 0 \cdot dx$

$= \cancel{2\pi}\ 2\pi$

$RHS = \iint_R \nabla \cdot \vec{F}\,dA = 2A = 2\pi$

4



$LHS = \oint_C x^2\,dy - xy\,dx$

$= \int_0^1 0\,dx + \int_0^1 y\,dy + \int_1^0 x\,dx + \int_1^0 0\,dy$

$= \frac{3}{2}$

$RHS = \iint_R \nabla \cdot \vec{F}\,dA = \iint_R (2x + x)\,dA = \int_0^1 \int_0^1 3x\,dx\,dy$

$= \int_0^1 \frac{3}{2}\,dy = \frac{3}{2}$

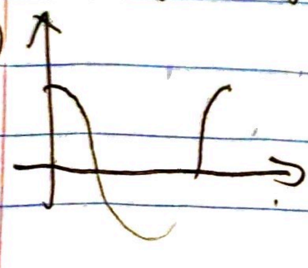
PART B 1/4/2015

a) ~~Area(R) =~~ $\iint$

$\oint_C x\,dy = \iint_R \nabla \times \langle 0, x \rangle\,dA = \iint_R dA = \text{Area}(R)$

$\oint_C -y\,dx = \iint_R \nabla \times \langle -y, 0 \rangle\,dA = \iint_R dA = \text{Area}(R)$

b)



$y = a(1 - \cos t) \geq 0$

$a > 0 \quad |\cos t| \leq 1$

$t \in [0, 2\pi]$

$dx = a(1 - \cos t)\,dt$

$A = \iint_R dA$

$= \oint -y\,dx = \int_0^{2\pi} a^2(1 - \cos t)^2\,dt$

$= a^2 \int_0^{2\pi} (-2\cos t + 1)\,dt + \frac{a^2}{2} \int_0^{2\pi} (\cos 2t + 1)\,dt$

$A = a^2\int_0^{2\pi}(-2\cos t+1)\,dt + \frac{a^2}{2}\int_0^{2\pi}(\cos 2t+1)\,dt$

$= a^2\left[-2\sin t + t\right]_0^{2\pi} + \frac{a^2}{2}\left[\frac{1}{2}\sin 2t + t\right]_0^{2\pi}$

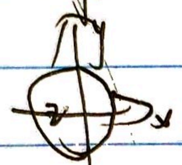
$= a^2[2\pi] + \frac{a^2}{2}[2\pi] = 3\pi a^2$

P2 a) $\oint_C (x^2y + y^3 - y)\,dx + (3x + 2y^2x + e^y)\,dy$

$= \iint_R \nabla\times\vec{F}\,dA = \iint_R\left[(\cancel{3x}\,3+2y^2) - (x^2+3y^2-1)\right]dA$

$= \iint_R (4-x^2-y^2)\,dA$ $\qquad 4-x^2-y^2=0$

$\qquad x^2+y^2=4$



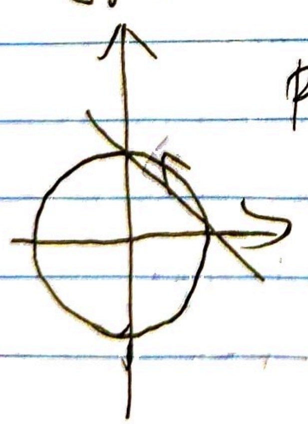
b) $\iint_R (4-x^2-y^2)\,dA$

$= \iint_R (4-r^2)\,dA = \int_0^{2\pi}\int_0^2 (4-r^2)\,r\,dr\,d\theta$

$= \int_0^{2\pi}\left[2r^2-\frac{1}{4}r^4\right]_0^2 d\theta$

$= \int_0^{2\pi}[8-4]\,d\theta = \int_0^{2\pi} 4\,d\theta = 8\pi$

P3 a)



Flux $= \oint -y^2\,dx + xy\,dy$

$-y^2+xy>0$

$\cancel{-y(y}\; y(x-y) < 0$

$y>0,\ x>y$

or $y<0,\ x<y$

← the portion in the first quadrant contributes negatively.

b) $\oint_C -y^2 dx + xy\,dy$ let $x = \cos t$, $y = \sin t$

$dx = -\sin t\,dt$ $dy = \cos t\,dt$

$= \int_0^{2\pi} -\sin^2 t(-\sin t\,dt) + \cos t \sin t \cdot \cos t\,dt$

$= \int_0^{2\pi} \sin t(1)\,dt = -\cos t \Big|_0^{2\pi} = 0$

c) $\oint_C -y^2 dx + xy\,dy = \iint_R y + 2y\,dA = 0$