$\left(\frac{\partial w}{\partial z}\right)_y = -2$

PS7 26/3/2005

lec 18.

3A-1 a) $\int_0^2 \int_{-1}^1 (6x^2+2y)\,dy\,dx$

$= \int_0^2 \left[6x^2y + y^2\right]_{-1}^1 dx$

$= \int_0^2 (6x^2+1)-(-6x^2+1)\,dx$

$= \int_0^2 12x^2\,dx$

$= \left[4x^3\right]_0^2 = 32$

b) $\int_0^{\frac{\pi}{2}} \int_0^{\pi} (u\sin t + t\cos u)\,dt\,du$

$= \int_0^{\frac{\pi}{2}} \left[-u\cos t + (\tfrac{1}{2}\cos u)t^2\right]_0^{\pi} du$

$= \int_0^{\frac{\pi}{2}} \left[\left(u + \frac{\cos u}{2}\pi^2\right) - (-u)\right] du$

$= \int_0^{\frac{\pi}{2}} \left[2u + \frac{\pi^2}{2}\cos u\right] du$

$= \left[u^2 + \frac{\pi^2}{2}\sin u\right]_0^{\frac{\pi}{2}}$

$= \left(\frac{\pi^2}{4} + \frac{\pi^2}{2}\right) - 0 = \frac{3\pi^2}{4}$

c) $\int_0^1 \int_{\sqrt{x}}^{x^2} 2x^2 y\,dy\,dx$

$= \int_0^1 \left[x^2y^2\right]_{\sqrt{x}}^{x^2} dx$

$= \int_0^1 \left[x^6 - x^3\right] dx$

$= \left[\frac{1}{7}x^7 - \frac{1}{4}x^4\right]_0^1$

$= \frac{1}{7} - \frac{1}{4} = \frac{-3}{28}$

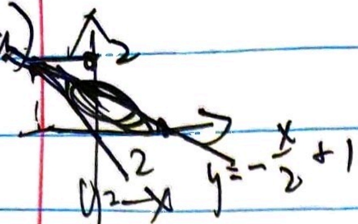
d) $\int_0^1 \int_0^u \sqrt{u^2+4}\,dv\,du$

$= \int_0^1 \left[\sqrt{u^2+4}\cdot v\right]_0^u du$

$= \int_0^1 u\sqrt{u^2+4}\,du$

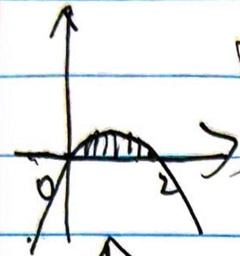
$= \int_0^1 \frac{\sqrt{u^2+4}}{2}\,du^2 = \left[\frac{2}{3}\cdot\frac{1}{2}(u^2+4)^{\frac{3}{2}}\right]_0^1 = \frac{1}{3}\times 5^{\frac{3}{2}} - \frac{1}{3}\times 8 = \frac{1}{3}(5^{\frac{3}{2}} - 8)$

2 a)



i) $\iint_R dydx = \int_{-2}^{0}\int_{-x}^{2} dydx$ ii) $\iint_R dxdy = \int_0^2\int_{-y}^{0} dxdy$

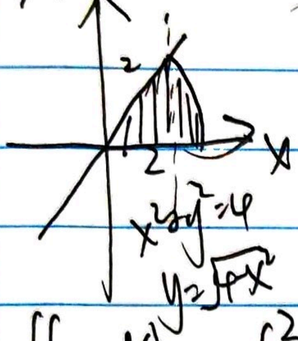
b)



i) $\iint_R dydx = \int_0^2\int_0^{2x-x^2} dydx$

ii) $\iint_R dxdy = \int_0^1\int_{1-\sqrt{1-y}}^{1+\sqrt{1-y}} dxdy$

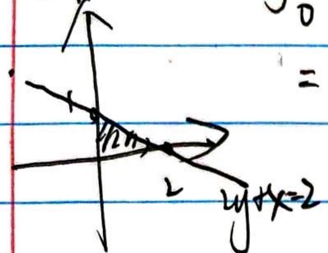
c)



i) $\iint_R dydx = \int_0^{\sqrt{2}}\int_0^{x} dydx + \int_{\sqrt{2}}^{2}\int_0^{\sqrt{4-x^2}} dydx$

ii) $\iint_R dxdy = \int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} dxdy$

3 a) $\iint_R x\,dA = \int_0^2\int_0^{1-\frac{x}{2}} x\,dydx$



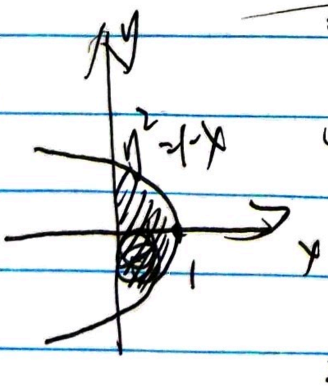
$= \int_0^2 [xy]_0^{1-\frac{x}{2}} dx$

$= \int_0^2 \left[x - \frac{x^2}{2}\right] dx$

$= \frac{x^2}{2} - \frac{x^3}{6}\Big|_0^2$

$= 2 - \frac{4}{3} = \frac{2}{3}$

b)

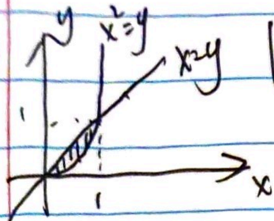


$\iint_R = \int_0^1\int_0^{1-y^2} (2x+y^2)\,dxdy$

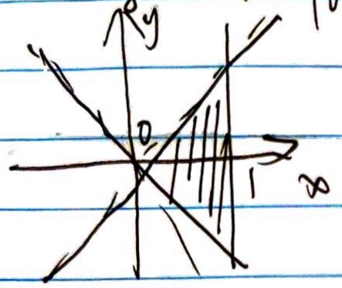
$= \int_0^1 [x^2+y^2x]_0^{1-y^2} dy$

$= \int_0^1 [(1-y^2)^2 + y^2 - y^4]\,dy$

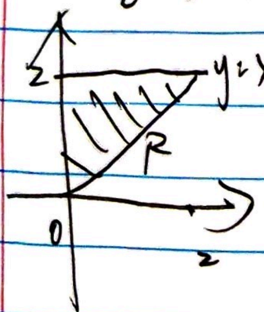
$= \int_0^1 y^4-2y^2+1+y^2-y^4\,dy = \int_0^1 -y^2+1\,dy = -\frac{1}{3}y^3+y\Big|_0^1 = \frac{2}{3}$

4b) 

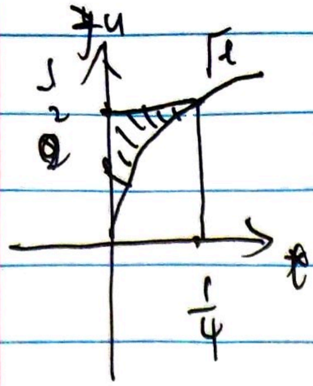
$$V=\iint_R z\,dA$$
$$=\int_0^1\int_{x^2}^{x} xy\,dy\,dx$$
$$=\int_0^1\left[\tfrac{1}{2}xy^2\right]_{x^2}^{x}dx$$
$$=\int_0^1\left[\tfrac{1}{2}x^3-\tfrac{1}{2}x^5\right]dx$$
$$=\tfrac{1}{8}x^4-\tfrac{1}{12}x^6\Big|_0^1=\tfrac{1}{8}-\tfrac{1}{12}=\tfrac{1}{24}$$

c) 

$(x+y)(x-y)$
$$V=\iint_R x^2-y^2\,dA$$
$$=\int_0^1\int_{-x}^{x} x^2-y^2\,dy\,dx$$
$$=\int_0^1\left[x^2y-\tfrac{1}{3}y^3\right]_{-x}^{x}dx \qquad \tfrac{4}{3}x^3$$
$$=\int_0^1\left[(x^3-\tfrac{1}{3}x^3)-(-x^3+\tfrac{1}{3}x^3)\right]dx=\int_0^1 \cancel{2x^3-\tfrac{2}{3}x^3}\,dx$$
$$=\tfrac{1}{3}x^4\Big|_0^1=\tfrac{1}{3}$$

5a) 

$$\int_0^2\int_x^2 e^{-y^2}dy\,dx$$
$$=\int_0^2\int_0^y e^{-y^2}dx\,dy=\int_0^2\left[e^{-y^2}x\right]_0^y dy$$
$$=\int_0^2 ye^{-y^2}dy=\int_0^2\frac{e^{-y^2}}{2}dy^2=-\tfrac{1}{2}e^{-y^2}\Big|_0^2$$
$$=-\tfrac{1}{2}(e^{-4}-1)=-\frac{e^{-4}-1}{2}$$

b) 

$\int_0^{\frac{1}{4}}\int_{\sqrt t}^{\frac{1}{2}} \frac{e^u}{u}\,du\,dt$

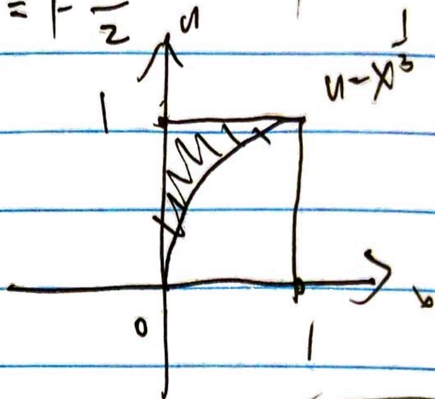
$= \int_0^{\frac{1}{2}}\int_0^{u^2} \frac{e^u}{u}\,dt\,du$

$= \int_0^{\frac{1}{2}} \left[\frac{t}{u} e^u\right]_0^{u^2} du$

$= \int_0^{\frac{1}{2}} u e^u\,du$

$= (u-1)e^u \Big|_0^{\frac{1}{2}} = 1 - \frac{\sqrt e}{2}$

c) $\int_0^1\int_{x^{1/3}}^1 \frac{1}{1+u^4}\,du\,dx$



$= \int_0^1\int_0^{u^3} \frac{1}{1+u^4}\,dx\,du$

$= \int_0^1 \left[\frac{x}{1+u^4}\right]_0^{u^3} du$

$= \int_0^1 \frac{u^3}{1+u^4}\,du = \frac{1}{4}\int_0^1 \frac{du^4}{1+u^4} = \frac{1}{4} \ln|1+u^4| \Big|_0^1 = \frac{1}{4}\ln 2$

6

$\iint_R x\,dA = 0$

$2\iint_S e^x\,dA$ S: Half of R.

$4\iint_Q x^2\,dA$ Q: first quadrant

$\iint_R x^2 y\,dA = 0$.

$4\iint_Q x^2\,dA$

$\iint_R xy\,dA = 1$.

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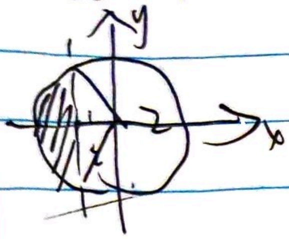
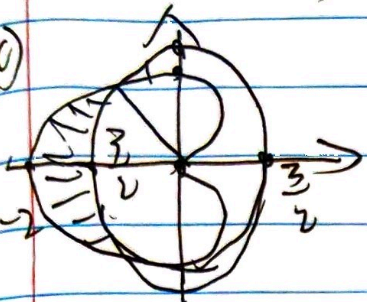
4B-1a)



$\iint_R dA = \int \int_0 dA$

$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \int_{-\sec\theta}^{2} r\,dr\,d\theta$

c)



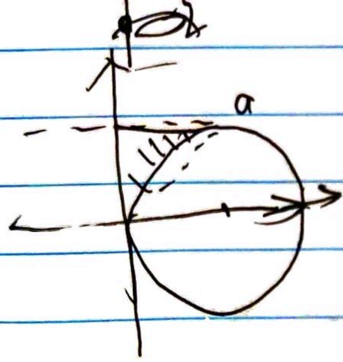
$r = 1 - \cos\theta = \frac{3}{2}$

$\cos\theta = -\frac{1}{2}$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

$\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \int_{\frac{3}{2}}^{1-\cos\theta} r\,dr\,d\theta$

d)



~~$r = 2a\cos\theta$~~

$(x-a)^2 + y^2 = a^2$

$r^2\cos^2\theta + a^2 - 2ar\cos\theta + r^2\sin^2\theta = a^2$

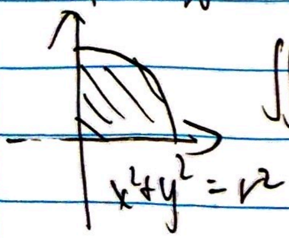
$r^2 = 2ar\cos\theta$

$r = 2a\cos\theta.$

$y = a = r\sin\theta \qquad r = \csc\theta \cdot a$

$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{2a\cos\theta}^{a\csc\theta} r\,dr\,d\theta$

2b)



$\iint_R \frac{dx\,dy}{1+x^2+y^2} = \int_0^{\frac{\pi}{2}} \int_0^a \frac{r\,dr\,d\theta}{1+r^2}$

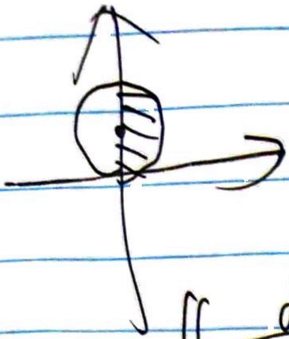
$= \int_0^{\frac{\pi}{2}} \int_0^a \frac{1}{\cancel{1+\ldots}}$

~~$= \arctan a = \frac{\pi}{2}$~~

$\int_0^{\frac{\pi}{2}} \left[\ldots\right]_0^a d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{\ln(1+a^2)}{2}\,d\theta$

$= \frac{\pi}{4}\ln(1+a^2)$

d) 

$x^2+(y-\frac{1}{2})^2=\frac{1}{4}$

$r^2-r\sin\theta=0$

$r=\sin\theta$

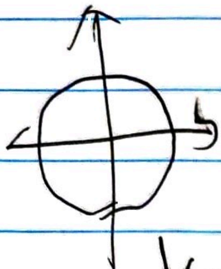
$\iint_R \frac{dx\,dy}{\sqrt{1-x^2-y^2}} = \int_0^{\frac{\pi}{2}}\int_0^{\sin\theta} \frac{r}{\sqrt{1-r^2}}\,dr\,d\theta$

$=\int_0^{\frac{\pi}{2}} \left[-\sqrt{1-r^2}\right]_0^{\sin\theta} d\theta$

$=\int_0^{\frac{\pi}{2}} \cancel{\cos\theta-1}\; 1-\cos\theta \; d\theta$

$= \cancel{\sin\theta-\theta}\; \theta-\sin\theta \Big|_0^{\frac{\pi}{2}}$

$= \cancel{1-\frac{\pi}{2}}\; \frac{\pi}{2}-1$

3 a) 

$z^2+x^2+y^2=a^2$

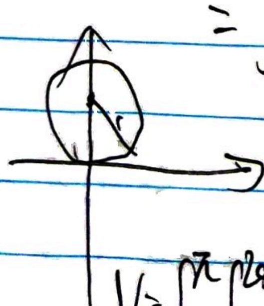
$z=\sqrt{a^2-x^2-y^2}$

$=\sqrt{a^2-r^2}$

$V=\iint_r z\,dA = \int_0^{2\pi}\int_0^1 \sqrt{a^2-r^2}\,r\,dr\,d\theta$

$=\int_0^{2\pi}\int_0^a \frac{\sqrt{a^2-r^2}\,dr^2}{2}\,d\theta = \int_0^{2\pi}\left[-\frac{2}{3}\frac{1}{2}(a^2-r^2)^{\frac{3}{2}}\right]_0^a d\theta$

$=\int_0^{2\pi}\frac{2}{3}\frac{1}{2}\cdot a^3\,d\theta = \frac{1}{3}a^3\theta\Big|_0^{2\pi} = \frac{2\pi}{3}a^3$

c) 

$x^2+(y-1)^2=1$

$r^2-2r\sin\theta=0$

$r=2\sin\theta$

$V=\int_0^{\pi}\int_0^{2\sin\theta}\sqrt{r^2}\;r\,dr\,d\theta = \int_0^{\pi}\left[\frac{1}{3}r^3\right]_0^{2\sin\theta}d\theta$

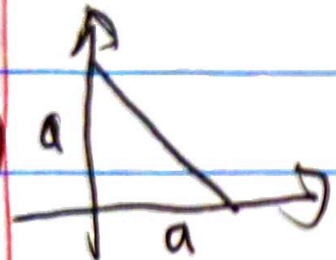
$=\int_0^{\pi}\frac{8}{3}\sin^3\theta\,d\theta = \frac{8}{3}\int_0^{\pi}(1-\cos^2\theta)\,d(-\cos\theta)$

$$= \frac{8}{3}\int_0^{\pi} \cos^2\theta - 1 \, d\cos\theta$$

$$= \frac{8}{3}\left[\frac{\cos^3\theta}{3} - \cos\theta\right]_0^{\pi} = \frac{8}{3}\left[0 - \left(\frac{1}{3} - 1\right)\right] = \frac{16}{9}$$

5.1-1

b)



$$I = \iint_R x^2 \, \delta \, dA$$

$$= \int_0^a \int_0^{a-x} x^2 \, dy \, dx$$

$$= \int_0^a x^2 y \Big|_0^{a-x} dx$$

$$= \int_0^a \cancel{\frac{x^3}{3}} \; ax^2 - x^3 \, dx$$

$$= \cancel{\frac{x^4}{12}\Big|_0^a} \; \frac{ax^3}{3} - \frac{x^4}{4}\Big|_0^a$$

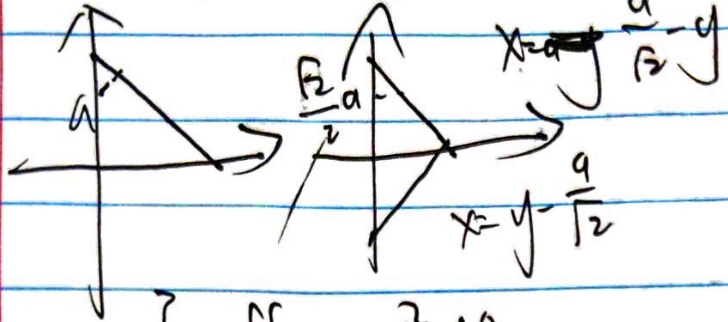
$$= a^4$$

b) $I = \iint_R x^2 + y^2 \, dA$

$= \iint_R x^2 \, dA + \iint_R y^2 \, dA$

$= 2 \times \frac{1}{12} a^4 = \frac{1}{6} a^4$

c)



$I = \iint_R x^2 \, dA$

$= 2\int_0^{\frac{a}{\sqrt{2}}}\int_0^{\frac{a}{\sqrt{2}} - x} x^2 \, dy \, dx$

$= 2\int_0^{\frac{a}{\sqrt{2}}} x^2 y \Big|_0^{\frac{a}{\sqrt{2}} - x} dx$

$= 2\int_0^{\frac{a}{\sqrt{2}}} \frac{a x^2}{\sqrt{2}} - x^3 \, dx$

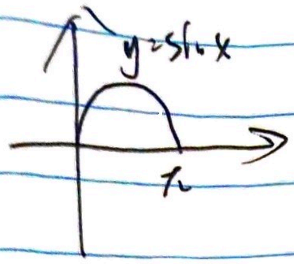
$= 2\left[\frac{a x^3}{3\sqrt{2}} - \frac{1}{4} x^4\right]_0^{\frac{a}{\sqrt{2}}}$

$= 2\left(\frac{a}{3\sqrt{2}} - \frac{1}{4}x\right) x^3 \Big|_0^{\frac{a}{\sqrt{2}}}$

$= 2\left[\frac{4a}{12\sqrt{2}} - \frac{3a}{12\sqrt{2}}\right]\frac{a^3}{2\sqrt{2}}$

$= 2 \, \frac{a^4}{48} = \frac{a^4}{24}$

2



$$\text{mass} = \iint_R dA$$
$$= \int_0^{\pi}\int_0^{\sin x} dy\,dx$$
$$= \int_0^{\pi} \sin x\,dx$$
$$= -\cos x\Big|_0^{\pi} = 2$$

$$\bar{x} = \frac{1}{\text{mass}}\iint_R x\delta\,dA \qquad \bar{y} = \frac{1}{\text{mass}}\iint_R x\delta\,dA$$

a) $\delta = 1$

$$\bar{x} = \frac{1}{2}\iint_R x\,dA = \frac{\pi}{2}$$
$$\cancel{= \frac{1}{2}\int_0^{\pi}\int_0^{\sin x} x\,dy\,dx} \qquad \cancel{= \frac{1}{2}\int_0 \int}$$
$$\cancel{= \frac{1}{2}\int_0^{\pi} xy\Big|_0^{\sin x}}\,dx$$
$$\cancel{= \frac{1}{2}\int_0^{\pi} x\sin x}\,dx$$

$1 - 2\sin^2 x = \cos 2x$

$\sin^2 x = \frac{1-\cos 2x}{2}$

$$\bar{y} = \frac{1}{2}\iint_R y\,dA = \frac{1}{2}\int_0^{\pi}\int_0^{\sin x} y\,dy\,dx$$
$$= \frac{1}{2}\int_0^{\pi} \frac{1}{2}y^2\Big|_0^{\sin x} dx$$
$$= \frac{1}{2}\int_0^{\pi} \frac{1}{2}\sin^2 x\,dx$$
$$= \frac{1}{4}\int_0^{\pi} \sin^2 x\,dx$$
$$= \frac{1}{8}\int_0^{\pi} 1-\cos 2x\,dx$$
$$= \frac{1}{8}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi}$$
$$= \frac{1}{8}(\pi - 0) = \frac{\pi}{8}$$

mass $= \iint_R y^2 \, dA$

b) $\bar{x} = \frac{\pi}{2}$ $\quad \bar{y} = \frac{4}{\pi}\iint_R y^2 \, dA$

$= \frac{4}{\pi}\int_0^{\pi}\int_0^{\sin x} y^2 \, dy \, dx$

$= \frac{4}{\pi}\int_0^{\pi} \frac{1}{3}y^3 \Big|_0^{\sin x} dx$

$= \frac{4}{3\pi}\int_0^{\pi} \sin^3 x \, dx$

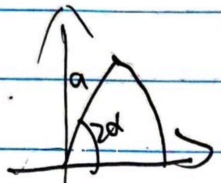
$= \frac{4}{3\pi}\int_0^{\pi} -(1-\cos^2 x) \, d\cos x$

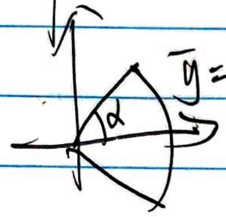
$= \frac{4}{3\pi}\int_0^{\pi} \cos^2 x - 1 \, d\cos x$

$= \frac{4}{3\pi}\left(\frac{1}{3}\cos^3 x - \cos x \Big|_0^{\pi}\right)$

$= \frac{4}{3\pi}\left(\frac{2}{3}+\frac{2}{3}\right) = \frac{16}{9\pi}$

4





$\bar{y} = 0$

mass $= \iint_R dA$

$= \int_0^{2\alpha}\int_0^{a} r \, dr \, d\theta$

$= \int_0^{2\alpha} \frac{a^2}{2} \, d\theta$

$= \frac{a^2\theta}{2}\Big|_0^{2\alpha} = \frac{2a^2\alpha}{2} = a^2\alpha$

$\cancel{\bar{x} = \iint_R r \, dA}$

$\cancel{= \int\int_0^{a} r^2 \, dr \, d\theta = \int_{-\alpha}^{\alpha} \frac{a^3}{3} \, d\theta = \frac{a^4}{12}\Big|_{-\alpha}^{\alpha}}$

$\bar{x} = \frac{1}{m}\iint_R r\cos\theta \, dA = \frac{1}{m}\int_{-\alpha}^{\alpha}\int_0^{a} r^2 \cos\theta \, dr \, d\theta$

$= \frac{1}{m}\int_{-\alpha}^{\alpha} \frac{\cos\theta}{3} \cdot a^3 \, d\theta = \frac{1}{m}\left(r^3 \frac{\sin\theta}{3}\Big|_{-\alpha}^{\alpha}\right) = \frac{2}{3}a\sin\alpha$

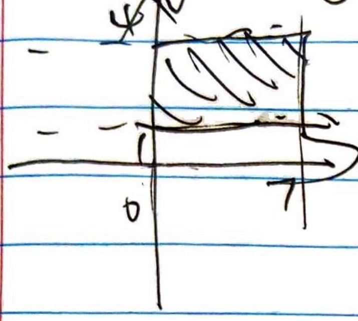
$= \frac{2}{3\alpha}\sin\alpha$

Lec20 27/3/2025

3D-1

$J = \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}$ $|J| = 1+6 = 7$

$du\,dv = 7\,dx\,dy$



$v=1 \quad v=4$

$$\iint_R \frac{u}{v}\,dA$$

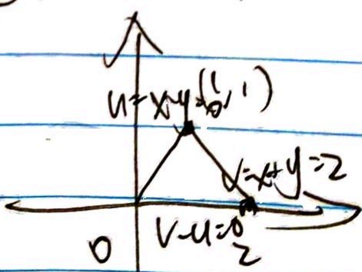
$$= \int_1^4 \int_0^7 \frac{u}{v}\,du\,dv$$

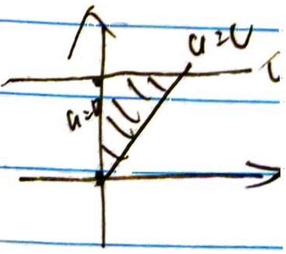
$$= \int_1^4 \left[\frac{u^2}{2v}\right]_0^7 dv = \int_1^4 \frac{49}{2v}\,dv$$

$$= \frac{49}{2}\left[\ln v\right]_1^4 = 49\ln 2$$

$$\iint_R \frac{x-3y}{2x+y}\,dx\,dy = \frac{1}{7}\iint_R \frac{u}{v}\,du\,dv = 7\ln 2$$

3D-2





let $u = x-y$, $v = x+y$.

$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ $|J| = 1+1 = 2$.

$du\,dv = 2\,dx\,dy$

$$\iint_R \frac{x-y}{x+y}\,dA = \frac{1}{2}\iint_R \frac{u}{v}\cdot 2\,dA = \frac{1}{2}\int_0^2\int_0^v \frac{u}{v}\,du\,dv$$

$$= \frac{1}{2}\int_0^2 \left[\frac{u^2}{2v}\right]_0^v dv = \frac{1}{2}\int_0^2 \left(\frac{v}{2}\right) dv = \frac{1}{2}\left[\frac{v^2}{4}\right]_0^2 = \frac{1}{2}$$

$$\iint_R \cos\left(\frac{x-y}{x+y}\right) dA$$

$$= \frac{1}{2}\iint_R \cos\frac{u}{v}\, 2\,dA$$

$$= \frac{1}{2}\int_0^2\int_0^v \cos\frac{u}{v}\, du\, dv \quad \cancel{= \frac{1}{2}\int_0^2 \frac{u^2}{2v}\Big|_0^v dv = \frac{1}{2}\int_0^2 \frac{v}{2}\, dv}$$

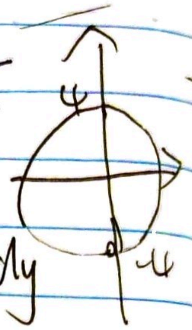
$$\cancel{\frac{1}{2}\left[\frac{v^2}{4}\right]_0^2 =} \quad = \frac{1}{2}\int_0^2 v \frac{\sin u}{v}\Big|_0^v dv$$

$$= \frac{1}{2}\int_0^2 v\sin 1\, dv \quad = \frac{1}{2}\left[\frac{\sin 1}{2} v^2\right]_0^2 = \sin 1$$

3)-3 $16 - x^2 - 4y^2 \geq 0$ 3D-

$R: x^2 + 4y^2 \leq 16$, $-\sqrt{16-4y^2} \leq x \leq \sqrt{16-4y^2}$, $\sqrt{16-v^2}$

let $u = x,\ v = 2y$ $\quad -4 \leq v \leq 4$



$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad |J| = 2 \quad du\,dv = 2\,dx\,dy$$

$$V = \iint_R 16 - x^2 - 4y^2\, dA = \frac{1}{2}\iint_R 16 - u^2 - v^2\ 2\,dA$$

$$\cancel{= \frac{1}{2}\int_{-4}^{4}\int_{-\sqrt{16-v^2}}^{\sqrt{16-v^2}} 16 - u^2 - v^2} = \cancel{4\iint}\ 2\int_0^4\int_0^{\sqrt{16-v^2}} 16 - u^2 - v^2\, du\, dv$$

$$\cancel{= 2\int_0^4 \left[16u - \frac{u^3}{3} - v^2 u\right]_0^{\sqrt{16-v^2}} dv}$$

$$= 2\int_0^4 \left[\cancel{(16-v^2)\sqrt{16-v^2}} - \frac{(16-v^2)\sqrt{16-v^2}}{3}\right] dv$$

$$= \frac{4}{3}\int_0^4 (16-v^2)^{\frac{3}{2}} dv \quad \text{let } \cancel{v = 4\sin t} \quad \cancel{dv = 4\cos t\, dt}$$

$$= \frac{4}{3}\int_0^{\frac{\pi}{2}} (16\cos^2 t)^{\frac{3}{2}}\ \cancel{4\cos t\, dt = \frac{4}{3}\int_0^4 [illegible]\, dt}$$

let ~~r=4~~ $u = r\cos\theta$, $v = r\sin\theta$.

$$V = 2\int_0^{\frac{\pi}{2}}\int_0^4 (16-r^2)\,r\,dr\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{1}{2}\int_0^4 (16-r^2)\,dr^2\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[16r^2 - \frac{r^4}{2}\right]_0^4 d\theta = \int_0^{\frac{\pi}{2}} \left[16^2 - \frac{16^2}{2}\right] d\theta$$

$$= 8\times 16\theta \Big|_0^{\frac{\pi}{2}} = 64\pi$$

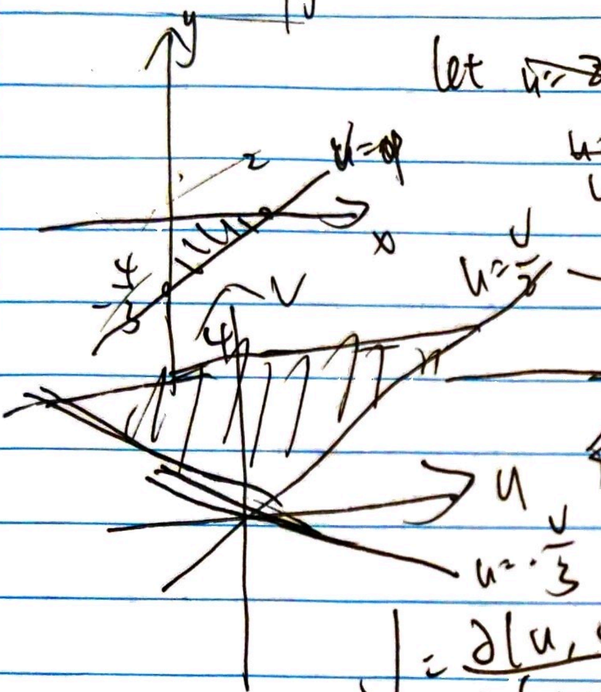
2.4

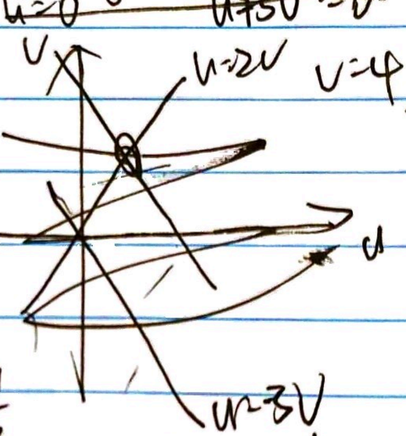
let ~~u = 2x-3y, v = x+y~~ ~~u=2x-3y, v=x+y~~ $u = x+y$, $v = 2x-3y$

$u=0$ ~~$u+5v=0$~~ ~~$2v-u=0$~~

$v=4$, $v+3u=0$

$2u-v=0$





$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -3-2 = -5$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{5} \quad V = \iint_R v^2u^2 \frac{du\,dv}{5}$$

$$= \frac{1}{5}\int_0^4\int_{-\frac{v}{3}}^{\frac{v}{2}} v^2u^2\,du\,dv = \frac{1}{5}\int_0^4 \frac{v^2}{3}u^3\Big|_{-\frac{v}{3}}^{\frac{v}{2}} dv$$

$$= \frac{1}{5}\int_0^4 \frac{v^5}{24} + \frac{v^5}{81}\,dv = \frac{1}{5}\cdot\frac{1}{6}\cdot\left(\frac{1}{24}+\frac{1}{81}\right)v^6\Big|_0^4$$

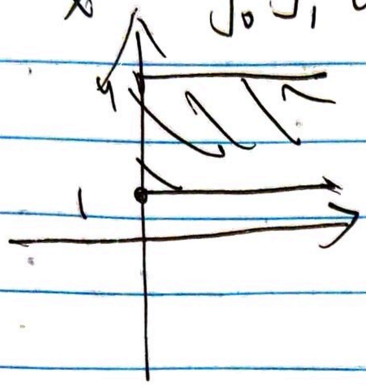
$$= \frac{1}{30}\left(\frac{1}{24}+\frac{1}{81}\right)4^6$$

PART B 28/3/2025

P1 a) $\int_1^a e^{-xy}\,dy = -\frac{1}{x}e^{-xy}\Big|_1^a = -\frac{1}{x}e^{-ax} + \frac{1}{x}e^{-x}$

$= \cancel{\frac{1}{x}\left(1 - \frac{e^{a}}{e^{x}}\right)}$

b) $\int_0^\infty \frac{e^{-x}-e^{-ax}}{x} = \int_0^\infty \int_1^a e^{-xy}\,dy\,dx$



$= \int_1^a \int_0^\infty e^{-xy}\,dx\,dy$

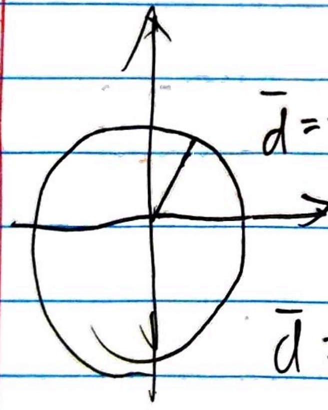
$= \int_1^a -\frac{1}{y}e^{-xy}\Big|_0^\infty dy$

$= \int_1^a \left(\lim_{x\to\infty} -\frac{1}{y}e^{-xy} + \frac{1}{y}\right) dy$

$= \int_1^a \frac{1}{y}\,dy$

$= \ln y\Big|_1^a = \ln a$

P2



$\bar{d} = \frac{1}{A}\iint_R r\,dA$

$A = \pi a^2$

$\bar{d} = \frac{1}{A}\iint_R r\,dA$

$= \frac{1}{A}\int_0^{2\pi}\int_0^a r \cdot r\,dr\,d\theta$

$= \frac{1}{A}\int_0^{2\pi} \frac{r^3}{3}\Big|_0^a d\theta = \frac{1}{A}\cdot\frac{a^3}{3}\cdot 2\pi$

$= \frac{1}{\pi a^2}\cdot\frac{a^3}{3}\cdot 2\pi = \frac{2a}{3}$

P3 $\bar{x} = \frac{1}{M}\iint_R x\delta\, dA$

$\bar{I} = \iint_R x^2 \delta\, dA$

$\bar{I} = \iint_R (x-\bar{x})^2 \delta\, dA$

$$\bar{I} + M\bar{x}^2 = \iint_R (x-\bar{x})^2 \delta\, dA + \cancel{\frac{1}{M}\iint_R x\delta\, dA}\ \overset{M\bar{x}^2}{\cancel{\bar{x}^2\iint_R \delta\, dA}}$$

$$= \cancel{\iint_R (x-\bar{x})^2 \delta\, dA + \frac{\bar{x}}{M}\iint_R x\delta\, dA}$$

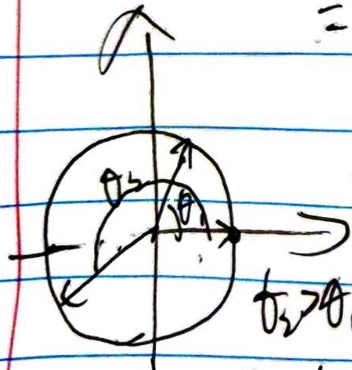
$$= \cancel{\iint_R (x^2 - 2x\bar{x})\delta\, dA}$$

$$= \iint_R (x^2 - 2x\bar{x} + \bar{x}^2)\delta\, dA + \cancel{\iint \bar{x}^2\delta\, dA}\ M\bar{x}^2$$

$$= \iint_R x^2 \delta\, dA - 2\bar{x}\iint_R x\delta\, dA \overset{+\iint \bar{x}^2 \delta\, dA}{+\cancel{\iint \bar{x}^2 \delta\, dA}} + M$$

$$= I - 2\bar{x}\cdot M\bar{x} + 2M\bar{x}^2$$

$$= I = LHS \qquad \square$$



$$\int_0^{2\pi}\int_0^{2\pi} d\theta_1\, d\theta_2 = \int_0^{2\pi} 2\pi\, d\theta_2 = 4\pi^2$$

$\theta_2 > \theta_1$ $\iint_R \frac{1}{2}\sin\theta_1 + \frac{1}{2}\sin(\theta_2-\theta_1) - \frac{1}{2}\sin\theta_2\ d\theta_1 d\theta_2$

$$= \frac{1}{2}\int_0^{2\pi}\int_{\theta_1}^{2\pi} \sin\theta_1 - \sin\theta_2 + \sin(\theta_2-\theta_1)\, d\theta_2\, d\theta_1$$

$$= \frac{1}{2}\int_0^{2\pi}\left[\cos\theta_2 - \cos(\theta_2-\theta_1)\right]_0^{2\pi} d\theta_1$$

$= \frac{1}{2}\int_0^{2\pi} \left[\cos\theta_2 - \cos(\theta_2-\theta_1)\right]_{\theta_1}^{2\pi} d\theta_1$

$= \frac{1}{2}\int_0^{2\pi} \left[(1-\cos(\theta_1)) - (\cos\theta_1 - 1)\right] d\theta_1$

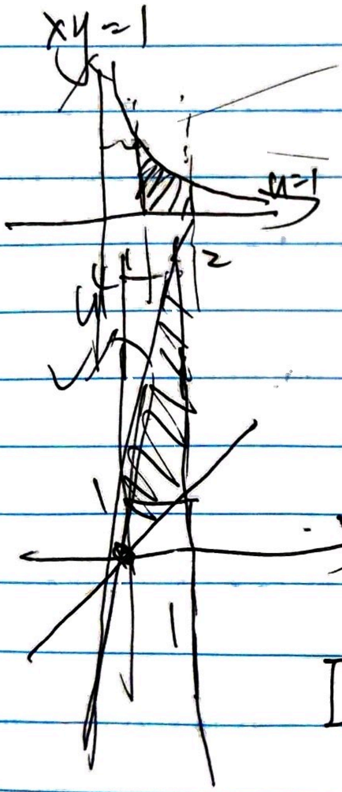
$= \frac{1}{2}\int_0^{2\pi} 2 - 2\cos\theta_1 \, d\theta_1$

$= \theta_1 - \sin\theta_1 \Big|_0^{2\pi} = 2\pi$

$\bar{A} = \frac{2\pi}{4\pi^2} = \frac{1}{2\pi}$

P531D-7

$xy = 1$



let $u = xy$

$v = \frac{y}{x}$

$\sqrt{\frac{y}{x}} = 1 \quad \sqrt{\frac{y}{x}} = 2$

$u = v \quad u = 4v$

$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{y}{x} + \frac{y}{x} = \frac{2y}{x}$

$= 2v$

$du\,dv = 2v\,dx\,dy$

$I = \iint_R x^2 + y^2 \, dA$

$= \iint_D \left(\frac{u}{v} + uv\right) \cdot |J| \, dA$

$= \int_0^1 \int_v^{4v} \frac{u}{2v^2} + \frac{u}{2} \, du\,dv$

$$= \int_0^{4} \int_{\frac{u}{4}}^{u} \frac{u}{2v} + \frac{u}{2} \, dv\, du + \int_1^{4} \int_{\frac{u}{4}}^{1} \frac{u}{2v} + \frac{u}{2} \, dv\,du$$

P6 ellipse $(2x+5y-3)^2 + (3x-7y+8)^2 < 1$,

let $u = 2x+5y-3$, $v = 3x-7y+8$.

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2 & 5 \\ 3 & -7 \end{vmatrix} = -14-15 = -29$$

$$du\,dv = |J|\,dx\,dy = 29\,dx\,dy$$

$u^2+v^2<1$ $\qquad r\,dr\,d\theta = du\,dv = 29\,dx\,dy$

$$A = \iint_R dA = \iint_{R'} \frac{1}{29}\,du\,dv = \int_0^{2\pi}\int_0^1 \frac{r}{29}\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{29 \times 2}\,d\theta$$

$$= \frac{\pi}{\ }$$