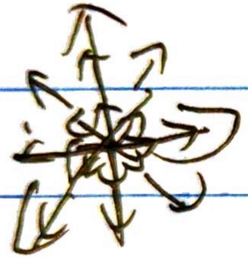
$= 2\pi h a\left(1-\frac{\sqrt{2}}{3}\right)$

PS11 2/4/2015

Lec 30

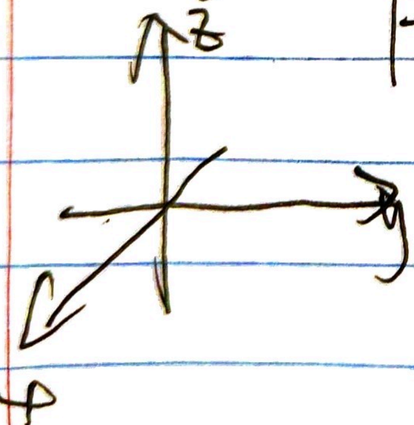
6A-1a)



b) point to the y axis, magnitude is the distance from the y axis.

6A-2 $\frac{-x\hat{i}-y\hat{j}-z\hat{k}}{2}$

6A-3



$\vec{F} = -z\hat{j} + y\hat{k}$

A-4 $\vec{F} = \langle 3, -4, 1 \rangle$

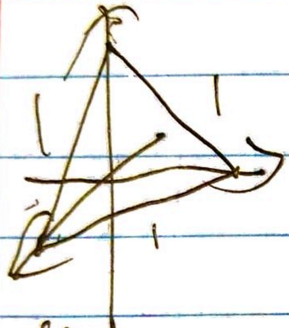
B-1 $\iint_S \vec{F}\cdot\hat{n}\, dS = \iint_S a \, a^2 \sin\phi \, d\phi \, d\theta$

$= \int_0^{2\pi} -a^3 \cos\phi \Big|_0^{\pi} d\theta$

$= 4\pi a^3$

B-2 0

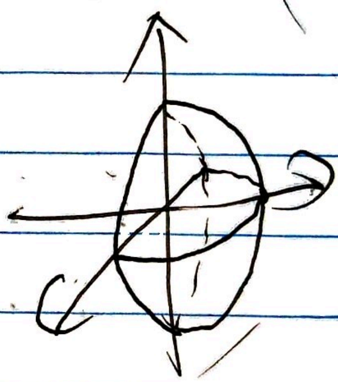
3



$\hat{i}\cdot\frac{\langle 1,1,1\rangle}{\sqrt{3}} = \frac{1}{\sqrt{3}}$

$\text{flux} = \frac{1}{2}\sin\frac{\pi}{3}\cdot\frac{1}{\sqrt{3}} = \frac{1}{4}$

4 $\iint_S \vec{F}\cdot\hat{n}\, dS = \iint_S y\hat{j}\cdot\frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}\, dS$



$= \iint_S \frac{y^2}{a}\, dS = \int_0^{\pi}\int_0^{\pi} \frac{y^2}{a} a^2 \sin\phi \, d\phi \, d\theta$

$= \int_0^{\pi}\int_0^{\pi} \frac{(a\sin\phi\sin\theta)^2}{a} a^2 \sin\phi \, d\phi \, d\theta$

$= \int_0^{\pi}\int_0^{\pi} a^3 \sin^3\phi \sin^2\theta \, d\phi \, d\theta$

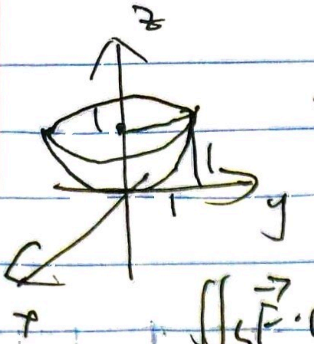
$\int_0^{\pi} 2a^3 \sin^2\theta \int_0^{\frac{\pi}{2}} \sin^3\phi \, d\phi \, d\theta$

$= \frac{4}{3}a^3 \int_0^{\pi} \sin^2\theta \, d\theta$

$= \frac{8}{3}a^3 \int_0^{\frac{\pi}{2}} \sin^2\theta \, d\theta$

$= \frac{2\pi}{3}a^3$

6B-6



$\cancel{\hat{n} = \langle 0,0,1\rangle}$

$\cancel{\vec{F}\cdot\hat{n} =}$

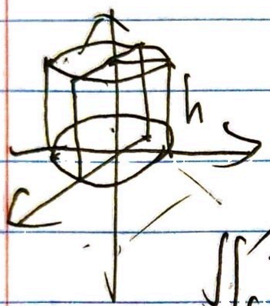
$d\vec{S} = (-2x\hat{i} - 2y\hat{j} + \hat{k})\,dx\,dy$

$\iint_S \vec{F}\cdot d\vec{S} = \iint_R (-2x^2 - 2y^2 + z)\,dx\,dy$

$= \int_0^{2\pi}\int_0^1 -r^2 r\,dr\,d\theta$

$= -\int_0^{2\pi} \frac{1}{4}\,d\theta = -\frac{\pi}{2}$

6B-8



$d\vec{S} = a\,dz\,d\theta$

$\hat{n} = \frac{x\hat{i} + y\hat{j}}{a}$

$\vec{F}\cdot\hat{n} = \frac{y^2}{a} = \frac{(a\cos\theta)^2}{a} = a\cos^2\theta$

$\iint_S \vec{F}\cdot d\vec{S} = \iint_R a^2\cos^2\theta\,dz\,d\theta$

$= \int_0^{2\pi}\int_0^h a^2\cos^2\theta\,dz\,d\theta = a^2h\int_0^{2\pi}\cos^2\theta\,d\theta$

$= 4a^2h\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta = \frac{h\pi}{2}a^2$

Lec 31 2/4/2025

6C-1(a) $\nabla\cdot\vec{F} = 2xy + x + x = 2xy + 2x$

2 $\nabla\cdot\vec{F} = \frac{\partial\rho^n}{\partial x}x + \rho^n + \frac{\partial\rho^n}{\partial y}y + \rho^n + \frac{\partial\rho^n}{\partial z}z + \rho^n$

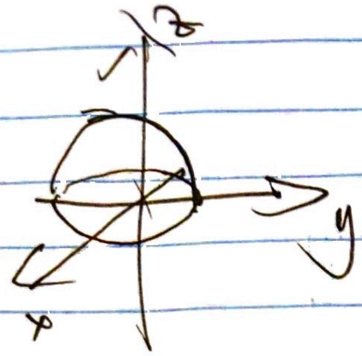
$= 3\rho^n + n\rho^{n-1}\frac{x^2}{\rho} + n\rho^{n-1}\frac{y^2}{\rho} + n\rho^{n-1}\frac{z^2}{\rho}$

$= 3\rho^n + n\rho^{n-1}\frac{(x^2+y^2+z^2)}{\rho}$ $\qquad n = -3\rho^2$

$\cancel{= \rho^{n-2}(3\rho^2 + n)(x+y+z)}$

$= (n+3)\rho^n = 0 \qquad n = -3.$

3



$\oiint_S \vec{F} \cdot d\vec{S}$

$= \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S}$

$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \langle x,y,z \rangle \cdot \frac{\langle x,y,z \rangle}{a} a^2 \sin\phi \, d\phi \, d\theta$

$+ \int_0^{2\pi} \int_0^{a} \langle x,y,z \rangle \cdot \langle 0,0,1 \rangle \, r \, dr \, d\theta$

$= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} a^3 \sin\phi \, d\phi \, d\theta$

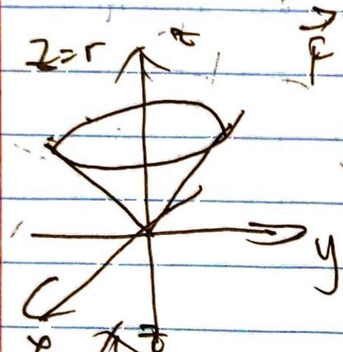
$\nabla \cdot \vec{F} = 3$

$= a^3 \int_0^{2\pi} -\cos\phi \Big|_0^{\frac{\pi}{2}} \, d\theta$

$= 2\pi a^3$

RHS $= \iiint_D \nabla \cdot \vec{F} \, dV = 3 \, Vol(D) = \frac{4\pi a^3}{2} = 2\pi a^3$

6



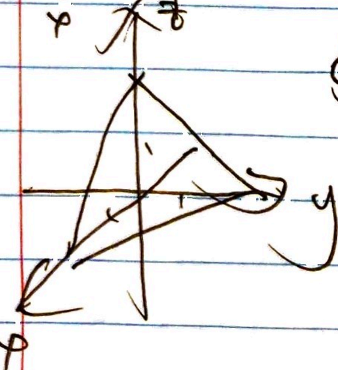
$\vec{F} = z\hat{k}$

$\oiint_S \vec{F} \cdot d\vec{S}$

$= \iiint_D \nabla \cdot \vec{F} \, dV = \iiint_D dV = Vol(D)$

$= \frac{1}{3}\pi$

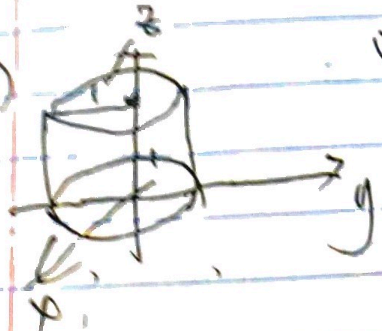
5



$\oiint_S \vec{F} \cdot d\vec{S}$

$= \iiint_D \nabla \cdot \vec{F} \, dV$

$= \iiint_D 1 \, dV = Vol(D) = \frac{1}{6}$

7 a) 

$\nabla \cdot \vec{F} = 2x + x = 3x.$

$LHS = \oint\!\!\oint_S \vec{F} \cdot d\vec{S}$

$= \oint\!\!\oint_S (x^2\hat{i} + xy\hat{j}) \cdot \langle x\hat{i} + y\hat{j} \rangle \; d\theta\, dz$

$= \int_0^{2\pi}\int_0^1 \cancel{x^3 + xy^2} (\cos^3\theta + \cos\theta \cdot \sin^2\theta)\, dz\, d\theta$

$= \int_0^{2\pi} \cos\theta\, d\theta \quad = 0$

$RHS = \iiint_D \nabla \cdot \vec{F}\, dV = \iiint_D 3x\, dV$

$= \int_0^{2\pi}\int_0^1\int_0^1 3r\cos\theta\, dz\, dr\, d\theta$

$= \int_0^{2\pi}\int_0^1 3r\cos\theta\, dr\, d\theta$

$= \int_0^{2\pi} \frac{3}{2} \cos\theta \quad d\theta = 0.$

8 a) $\iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S} = \oint\!\!\oint_S \vec{F} \cdot dS = \iiint_D \nabla \vec{F}\, dV = 0$

b) $\nabla \cdot \vec{F} = 0, \; \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 0$

Lec 33

6D-1 a) $\int_C \vec{F} \cdot d\vec{r} = \int_C y\,dx + z\,dy - x\,dz$

$= \int_0^1 t^2 dt + t^3 \cdot 2t\,dt - t \cdot 3t^2 dt = \int_0^1 (2t^4 - 3t^3 + t^2)\, dt$

$= \frac{2}{5}t^5 - \frac{3}{4}t^4 + \frac{1}{3}t^3 \Big|_0^1 = \frac{2}{5} - \frac{3}{4} + \frac{1}{3} = \frac{24 - 45 + 20}{60} = -\frac{1}{60}$

2 $\int_C \vec{F}\cdot d\vec{r}$ $\quad z = \rho\cos\phi = a\cos\phi$

$= \int_C x\,dx + y\,dy + \cancel{z\,dz}$ $\quad y = r\sin\theta$

$x = r\cos\theta$

$= \int_0^{2\pi} -r\cos\theta \cdot r\sin\theta\, d\theta + r\sin\theta \cdot r\cos\theta\, d\theta$

$= 0$

4 a) $\vec{F} = \nabla f = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

b) $x = \cos t,\ y = \sin t,\ z = t \quad t \in [0, 2n\pi]$. $\vec{F} = \cancel{2x\hat{i} + 2y\hat{j}} + x\hat{k}$

i) $\int_C \vec{F}\cdot d\vec{r} = \int_C 2x\,dx + 2y\,dy + x\,dz$

$= \int_0^{2n\pi} 2\cos t\cdot(-\sin t)\,dt + 2\sin t\cos t\,dt + 2t\,dt$

$= \int_0^{2n\pi} 2t\,dt = t^2\Big|_0^{2n\pi} = 4n^2\pi^2$

ii) $(x(0), y(0), z(0)) = (1, 0, 0)$

$(x(2n\pi), y(2n\pi), z(2n\pi)) = (1, 0, 2n\pi)$

$\int_C \vec{F}\cdot d\vec{r} = \int_{(1,0,0)}^{(1,0,2n\pi)} \vec{F}\cdot d\vec{r} = \int_C 2z\,dz = z^2 = 4n^2\pi^2$

iii) $\int_C \vec{F}\cdot d\vec{r} = f(1,0,2n\pi) - f(1,0,0)$

$= 4n^2\pi$

5 $\cancel{\vec{F} = \nabla f}$

$\vec{F} = \nabla f = yz\cos(xyz)\hat{i} + xz\cos(xyz)\hat{j} + xy\cos(xyz)\hat{k}$

$f_{max} = 1 \quad f_{min} = -1$

$\int_C \vec{F}\cdot d\vec{r} = f_{max} - f_{min} = 2$

6 B-1 a) $\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & y^2 & z^2 \end{vmatrix} = 0$ $\quad f(x,y,z) = \cancel{\int x^2dx + y^2dy + z^2dz}$

$\int (x^2)\,dx$

$= \frac{1}{3}x^3 + g(y,z)$.

$\therefore$ F is exact.

$df = x^2dx + y^2dy + z^2dz$

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = y^2$$

$$g = \int dg = \int y^2\,dy = \frac{1}{3}y^3 + h(z).$$

$$\frac{\partial g}{\partial z} = \frac{dh}{dz} = z^2$$

$$h = \int dh = \int z^2\,dz = \frac{1}{3}z^3 + C$$

$$\therefore f(x,y,z) = \frac{1}{3}x^3 + \frac{1}{3}y^3 + \frac{1}{3}z^3 + C$$

2 $$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & yz & xyz^2 \end{vmatrix} = + yz^2\,\hat{j} - x^2\,\hat{k} - (xz^2 - y)\,\hat{i}$$

3a) i) $$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = 0$$

ii) $$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy+z & x^2 & x \end{vmatrix} = 0\,\hat{i} + (1-1)\,\hat{j} + (2x-2x)\,\hat{k} = 0$$

b) i) Med 1

$$\int_{(0,0,0)}^{(x_1,y_1,z_1)} df = \int_0^{x_1} x\,dx + \int_0^{y_1} y\,dy + \int_0^{z_1} z\,dz$$

$$f\Big|_{(0,0,0)}^{(x_1,y_1,z_1)} = \frac{1}{2}x_1^2 + \frac{1}{2}y_1^2 + \frac{1}{2}z_1^2$$

$$\therefore f(x_1,y_1,z_1) = \frac{1}{2}x_1^2 + \frac{1}{2}y_1^2 + \frac{1}{2}z_1^2 + f(0,0,0).$$

$$f(x,y,z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2 + C$$

Med 2.

$\frac{\partial f}{\partial x} = x.$

$f(x,y,z) = \int \frac{\partial f}{\partial x} dx = \frac{1}{2}x^2 + g(y,z).$

$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} = y$

$g(y,z) = \int \frac{\partial g}{\partial y} dy = \int y\, dy = \frac{1}{2}y^2 + h(z)$

$\frac{\partial g}{\partial z} = \frac{\partial h}{\partial z} = z$

$h(z) = \int \frac{\partial g}{\partial z} dz = \int z\, dz = \frac{1}{2}z^2 + C.$

$\therefore f(x,y,z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2 + C$

ii) Med 1.

$f(x,y,z) - f(0,0,0) = \int_0^{x_1} 2xy + z\, dx + \int_0^{y_1} x^2\, dy + \int_0^{z_1} x\, dz$

$= 0 + x_1^2 y_1 + x_1 z$

$f(x,y,z) = x^2 y + xz + C$

Med 2

$\frac{\partial f}{\partial x} = 2xy + z$

$f(x,y,z) = \int \frac{\partial f}{\partial x} dx = x^2 y + zx + g(y,z).$

$\frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y} = x^2 \qquad \frac{\partial g}{\partial y} = 0.$

$g(y,z) = \int \frac{\partial g}{\partial y} dy = 0 + h(z).$

$\frac{\partial f}{\partial z} = x + \frac{\partial g}{\partial z} = x + \frac{\partial h}{\partial z} = x \qquad \frac{\partial h}{\partial z} = 0$

$h(z) = \int \frac{\partial h}{\partial z} dz = C$

$f(x,y,z) = x^2 y + zx + C$

5

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz^2 & xz^2+ayz & bxyz+y^2 \end{vmatrix}$$

$$= (bxz+2y-2xz-ay)\hat{i} + (byz-2yz)\hat{j} + (z^2-z^2)\hat{k} = 0$$

$$\begin{cases} (b-2)xz+(2-a)y=0 \\ (b-2)yz=0 \end{cases} \Rightarrow \begin{cases} a=2 \\ b=2 \end{cases}$$

PART B 4/4/2025

P1.667 $x^2+y^2=1$

$\rho = \sqrt{x^2+y^2+z^2}$

$$d\rho = \frac{1}{2}\frac{2xdx+2ydy+2zdz}{\sqrt{x^2+y^2+z^2}} = \frac{xdx+ydy+zdz}{\rho}$$

$|\vec{F}| = \frac{1}{\rho}$

$$\iint_S \vec{F}\cdot d\vec{S}$$

$d\rho =$

$\vec{F}\cdot\vec{n} = \sin\phi$

$$= \iint_S \frac{\sin\phi}{\rho} dS$$

$\rho\sin\phi = 1$

$\rho = \csc\phi$

$$= \iint_S \frac{\sin\phi}{\rho} d\theta\, dz$$

$z = \rho\cos\phi = \cot\phi$

$dz = d\cot\phi$

$dz = -\csc^2\phi\, d\phi$

$$= \iint_S \sin^2\phi = 2\pi\int_{-\infty}^{\infty} \frac{dz}{1+z^2}$$

$$= \int_0^{2\pi}\int_{-\infty}^{\infty} = \int_0^{2\pi}\int_0 \sin^2\phi(-\csc^2\phi)\,d\phi\, d\theta$$

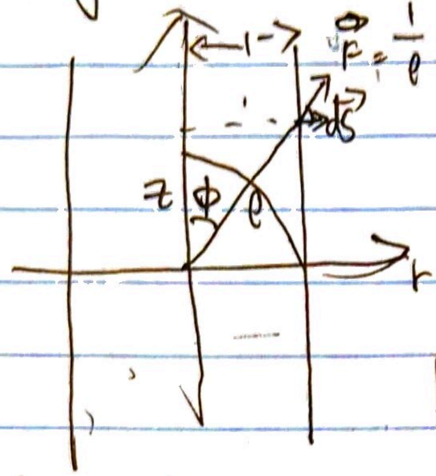
$$= 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\tan\theta}{\sqrt{1+\tan^2\theta}} = 2\pi\left[\ln(\tan\theta+\sec\theta)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sec^3\theta\, d\theta$$

PART B 4/4/2015.

P1. $\vec{F} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{x^2+y^2+z^2}$ $|F| = \frac{1}{\rho}$

$x^2+y^2=1$



$\vec{F} d\vec{S} = |\vec{F}| \cdot |d\vec{S}| \cdot \sin\phi$

$\cancel{d\vec{S} = \langle -f_x, -f_y, 1 \rangle \cdot d\vec{r}}$

$\cancel{\langle -2x, -2y, 1 \rangle \cdot d\vec{r}}$

$\cancel{|d\vec{S}| = \sqrt{4x^2+4y^2+1}}$

$= |\vec{F}| \cdot \sin\phi \, |\vec{n}| \cdot dS$

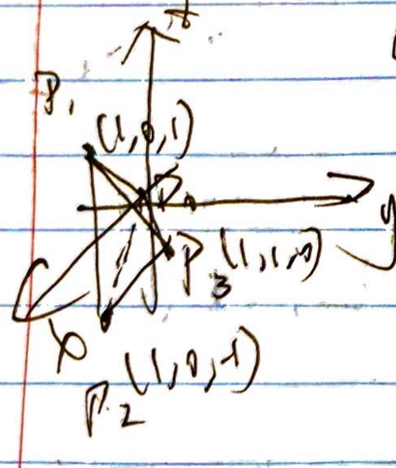
$dS = d\theta dz$ $z = \frac{1}{\tan\phi}$ $dz = -\frac{1}{\tan^2\phi} \cancel{\sec^2\phi} \, d\phi = -\csc^2\phi \, d\phi$

$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \frac{\sin\phi}{\rho} d\theta dz = \iint_S \frac{1}{z^2+1} dz \, d\theta$

$\cancel{= \iint_S \sin^2\phi} \, (-\csc^2\phi) \, d\phi \, d\theta = 2\pi \int_{-\infty}^{\infty} \frac{1}{z^2+1} dz = 2\pi^2$

$= 2\pi \int_{0}^{\pi} \sin^2\phi \, |d\cot\phi| = 2\pi^2$

P2



a). $P_1P_0P_3$ & $P_2P_0P_3$

b) $\cancel{\vec{P_0P_2} \times \vec{P_0P_3} = \langle 0, 1, 0 \rangle}$

$\cancel{\vec{P_3P_1} \times \vec{P_3P_2} =}$

$\vec{n_1} = \frac{\vec{P_0P_2} \times \vec{P_0P_1}}{|P_0P_2||P_0P_1|} = (0,1,0)$

$\vec{n_2} = \frac{\vec{P_3P_1} \times \vec{P_3P_2}}{|P_3P_1||P_3P_2|} = \langle 1, 0, 0 \rangle$

$\vec{n_3} = \vec{P_0P_1} \times \vec{P_0P_3} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -\hat{i} + \hat{j} + \hat{k}$ $\vec{n_4} = -\hat{i} - \hat{j} - \hat{k}$

c) $\Phi_1 = \cancel{\iint_{S_1} \vec{F} \cdot d\vec{S}}\ 0$

$\Phi_2 = \iint_{S_2} \vec{F} \cdot d\vec{S} = \cancel{\iint} y\hat{i} \cdot (1,0,0)\, dydz$

$= \int_0^1 \int_{y-1}^{1-y} y\, dz dy = \int_0^1 2y(1-y)\, dy = y^2 - \frac{2}{3}y^3 \Big|_0^1 = \frac{1}{3}$

$\Phi_3 = \Phi_4 = \iint_{S_3} \vec{F} \cdot d\vec{S} = \iint_S y\hat{i} \cdot \frac{-\hat{i} - \hat{j} + \hat{k}}{1}\, dxdy = \iint_S -y\, dxdy$

$= \int_0^1 \int_0^x -y\, dy dx = \int_0^1 -\frac{x^2}{2} dx = -\frac{1}{6}$

~~$z = -x-y$~~ ~~$z = x+y$~~

d) $\nabla \cdot \vec{F} = 0$

$RHS = \iiint_D \nabla \cdot \vec{F}\, dV = 0.$

$LHS = \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 = \frac{1}{3} - \frac{1}{6} - \frac{1}{6} = 0 = RHS$

a) ~~$\vec{F} = \nabla f$~~ =

$d\rho = -\frac{2x\,dx + 2y\,dy + 2z\,dz}{\sqrt{x^2+y^2+z^2}} = -\frac{2(x\,dx + y\,dy + z\,dz)}{\rho}$

$\vec{F} = \nabla f = \langle \frac{\partial}{\partial x}\frac{1}{\rho}, \frac{\partial}{\partial y}\frac{1}{\rho}, \frac{\partial}{\partial z}\frac{1}{\rho} \rangle$

$= \cancel{\frac{1}{\rho^2} \cdot \frac{\partial \rho}{\partial x}} \quad -\frac{1}{\rho^2} \cdot \langle \frac{\partial \rho}{\partial x}, \frac{\partial \rho}{\partial y}, \frac{\partial \rho}{\partial z} \rangle$

$= -\frac{1}{\rho^2} \langle -\frac{x}{\rho}, -\frac{y}{\rho}, -\frac{z}{\rho} \rangle$

$= \frac{1}{\rho^3} \langle x, y, z \rangle$ Inverse square to distance to origin

b) $\Phi = \iint_S \vec{F} \cdot \vec{n}\, dS = \iint_S \frac{\vec{r}}{\rho} \cdot \frac{1}{a^2} \cdot a^2 \sin\phi\, d\phi\, d\theta$

$= \int_0^{2\pi} \int_0^{\pi} \sin\phi\, d\phi\, d\theta = 4\pi$

c) $\nabla \cdot \vec{F} = \frac{\partial}{\partial x}\frac{x}{\rho^3} + \frac{\partial}{\partial y}\frac{y}{\rho^3} + \frac{\partial}{\partial z}\frac{z}{\rho^3} = \frac{\rho^3 + x \cdot 3\rho^2 \cdot (\frac{x}{\rho})}{\rho^6} + \ldots$

$= \frac{\rho^2 - 3x^2}{\rho^5} + \frac{\rho^2 - 3y^2}{\rho^5} + \frac{\rho^2 - 3z^2}{\rho^5} = \frac{3\rho^2 - 3(x^2+y^2+z^2)}{\rho^5} = 0$

Gauss's thm cannot be apply since $\vec{F}$ is NDF at origin

P4 a) $\oiint_S \vec{F}\, d\vec{S} = \iiint_D \nabla\vec{F}\, dV$

let $\vec{F} = \nabla f$, $\nabla\cdot(\nabla\cdot f) = \nabla\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f$

$\oiint_S \nabla f\, d\vec{S} = \iiint_D \nabla^2 f\, dV$

b) $\nabla(f\nabla f) = \nabla(f f_x + f f_y + f f_z) = (f_x f_x + f f_{xx} + f_y f_y + f f_{yy} + f_z f_z + f f_{zz})$

$= \cancel{(\nabla f)^2}\ |\nabla f|^2 + f\nabla^2 f$

let $\vec{F} = \nabla f$

$\oiint_S f\nabla f\, d\vec{S} = \iiint_D (f\nabla^2 f + |\nabla f|^2)\, dV$

c) $\cancel{\forall (x,y,z) \in D}$, $\nabla^2 f = 0$

$\because \forall (x,y,z) \in D$, $\nabla^2 f = 0$, $f = 0$

$\therefore \cancel{\oiint_S f\nabla f\, d\vec{S}} = \iiint_D |\nabla f|^2\, dV = \oiint f\nabla f \cdot d\vec{S} = 0$

$\therefore \iiint_D \cancel{|\nabla f|^2\, dV = 0}$ $|\nabla f|^2 \geq 0$ $\because f = 0$ on S

$\therefore \cancel{|\nabla f|}$ $\nabla f = 0$ $\therefore f$ is constant in D. $\therefore f = 0$ in D.

d) $\because \cancel{f = g \text{ on } S}$ $\forall (x,y,z) \in D$, $\nabla^2 f = \nabla^2 g = 0$, $\forall (x,y,z) \in S$, $f = g$.

$\cancel{\therefore \oiint_S f\nabla f\, d\vec{S}}$

$\oiint_S (f-g)\nabla(f-g)\, d\vec{S} = \iiint_D ((f-g)\nabla^2(f-g) + |\nabla(f-g)|^2)\, dV$

$0 = \iiint_D |\nabla(f-g)|^2\, dV$

$\therefore \cancel{\nabla|f-g|}$ $\nabla(f-g) = 0$ $\forall (x,y,z) \in D$

$\therefore f - g = 0$ in D.

P5 a) $\int_C (a\sin z + bxy^2)\,dx + 2x^2y\,dy + (x\cos z - z^2)\,dz$

$= \int_C (a\sin t + b\cos t\sin^2 t)(-\sin t)\,dt + 2\cos^2 t\sin t\cdot\cos t\,dt + (\cos^2 t - t^2)\,dt$

$= \int_0^{2\pi} (2\cos^3 t\sin t - b\cos t\sin^3 t - a\sin^2 t + \cos^2 t - t^2)\,dt$

$= \left[-\frac{1}{2}\cos^4 t - \frac{b}{4}\sin^4 t + t - \frac{1}{3}t^3\right]_0^{2\pi} - (a+1)\cdot 4\cancel{\int_0^{\frac{\pi}{2}}\sin^2 t}\,\frac{\pi}{4}$

$= 2\pi - \frac{1}{3}\cdot 8\pi^3 - (a+1)\pi$

b) $\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ a\sin z + bxy^2 & 2x^2y & x\cos z - z^2 \end{vmatrix} = (\cos z + a\cos z)\,\hat{j} + (4xy - 2bxy)\,\hat{k} = 0$

$\begin{cases} \cos z = a\cos z \\ 4xy = 2bxy \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = 2 \end{cases}$

c) $\vec{F} = (\sin z + 2xy^2)\,\hat{i} + 2x^2y\,\hat{j} + (x\cos z - z^2)\,\hat{k}$

$f(x,y,z) = \int \frac{\partial f}{\partial x}\,dx = \int (\sin z + 2xy^2)\,dx = x\sin z + x^2y^2 + g(y,z)$

$\frac{\partial f}{\partial y} = 2x^2y + \frac{\partial g}{\partial y} = 2x^2y \Rightarrow \frac{\partial g}{\partial y} = 0$

$\cancel{\frac{\partial f}{\partial z} = x\cos z + \frac{\partial g}{\partial z}}$

$g(y,z) = \int \frac{\partial g}{\partial y}\,dy = 0 + h(z)$

$\frac{\partial f}{\partial z} = x\cos z + \frac{\partial h}{\partial z} = x\cos z - z^2 \Rightarrow \frac{\partial h}{\partial z} = -z^2$

$h(z) = \int \frac{\partial h}{\partial z}\,dz = \int -z^2\,dz = -\frac{1}{3}z^3$

$\therefore f(x,y,z) = x\sin z + x^2y^2 - \frac{1}{3}z^3$

$\int_C \vec{F}\cdot d\vec{r} = f(1,0,2\pi) - f(1,0,0)$

$= -\frac{8}{3}\pi^3$