PS4 24 Oct 2024
Lecture 14 24 Oct 2024

2G-1b

$f(x) = \ln x$

$f'(x) = \frac{\ln 2 - \ln 1}{1} = \ln 2$

$(\ln x)' = \frac{1}{x} = \ln 2$

$x = \frac{1}{\ln 2}$

2G-2b Let $f(x) = \sqrt{1+x}$. $f'(x) = \frac{1}{2\sqrt{1+x}}$

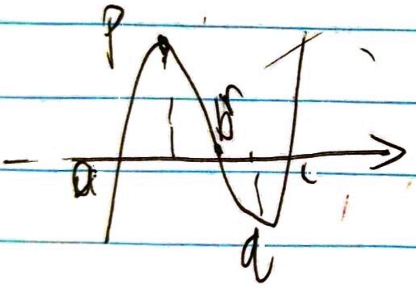
$f(0) = 1$ $f'(0) = \frac{1}{2}$

$\frac{f(x)}{} = f'(c) = \frac{f(x) - f(0)}{x}$, $c > 0$.

$f(x) = xf'(c) + f(0) = \frac{x}{2\sqrt{1+c}} + 1$

$\sqrt{1+x} = \frac{x}{2\sqrt{1+c}} + 1 > \frac{1}{2}x + 1$

5. a)



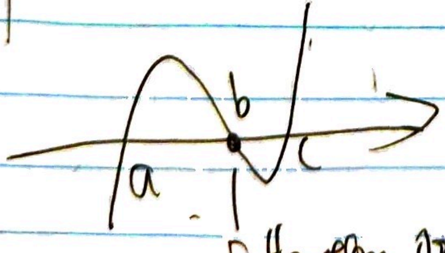
$\exists q,\ f'(q) = \frac{f(a) - f(b)}{a-b} = 0$

$\exists r,\ f'(r) = \frac{f(b) - f(c)}{b-c} = 0$.

$\exists p,\ f''(p) = \frac{f'(r) - f'(q)}{r-q} = 0$.

b) $f(x) = (x-a)(x-b)(x-c)$

$= x^3 - (a+b+c)x^2 + (ab+bc+ac)x - abc$



$f'(x) = 3x^2 - 2(a+b+c)x + (ab+bc+ac)$

$f''(x) = 6x - 2(a+b+c) = 0$

inflection point. $\therefore p = \frac{a+b+c}{3}$.

$f''(p) = 0$

6. form 2 of MVT: $f(b)=(b-a)f'(c)+f(a)$

a) $\because \forall x\in[a,b],\ f'(x)\geq 0$

$\therefore \forall x\in[a,b],\ \cancel{f(x)=(x-a)f'(c)+f(a)}$

$f(x)-f(a)=(x-a)f'(x)\geq 0.$

$\therefore f(x)\nearrow$ on $[a,b]$.

b) $\because \forall x\in[a,b],\ f'(x)=0.$

$\therefore \forall x\in[a,b],\ f(x)-f(a)=(x-a)f'(x)=0.$

$\Rightarrow f(x)=f(a).$

Lecture 15. 4 Nov.

A-1 d) $d(e^{3x}\sin x)=(3e^{3x}\sin x+e^{3x}\cos x)dx$

e) $\cancel{d(\sqrt{x}+\sqrt{y})=0}$

$\frac{d(\sqrt{x}+\sqrt{y})}{dx}=0$ $\quad \sqrt{y}=1-\sqrt{x}$

$y=x+1-2\sqrt{x}$

$\frac{1}{2\sqrt{x}}+\frac{y'}{2\sqrt{y}}=0$ $\quad \frac{dy}{2\sqrt{y}\,dx}=-\frac{1}{2\sqrt{x}}$

$dy=-\frac{\sqrt{y}}{\sqrt{x}}dx$

$=-\frac{\sqrt{x-2\sqrt{x}+1}}{\sqrt{x}}dx.$

2. acegik.

a) $\int(2x^4+3x^2+x+8)dx=\frac{2}{5}x^5+x^3+\frac{1}{2}x^2+4x+C$

c) $\int\sqrt{8+9x}\,dx=\int u\,d\left(\frac{u^2-8}{9}\right)=\frac{1}{9}\int u\,d(u^2-8)=\frac{1}{9}\int u(2u)du=\frac{2\times u^3}{3\times 9}+C$

let $u=\sqrt{8+9x}$. $u^2=8+9x$ $\quad x=\frac{u^2-8}{9}$

$=\frac{2}{27}(8+9x)^{\frac{3}{2}}+C$

e) $\int\frac{x}{\sqrt{8-2x^2}}dx=\int\frac{1}{4\sqrt{2}\sqrt{4-x^2}}dx^2=\frac{1}{2\sqrt{2}}\int\frac{1}{u}d(-u^2-4)$

$=\frac{1}{2\sqrt{2}}\int\frac{1}{u}(-2u)du=-\frac{u}{\sqrt{2}}=-\sqrt{\frac{4-x^2}{2}}+C$

g) $\int 7x^4 e^{x^5}\,dx = \frac{7}{5}\int e^{x^5}\,dx^5 = \frac{7}{5}e^{x^5}+C$

i) $\int \frac{dx}{3x+2} = \int \frac{d(\frac{u-2}{3})}{u} = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u|+C = \frac{1}{3}\ln|3x+2|+C$

k) $\int \frac{x}{x+5}\,dx = \int (1-\frac{5}{x+5})\,dx = \int (1-\frac{5}{u})\,d(u-5) = u - 5\ln|u|$
$= x - 5\ln|x+5| + C$

3 a, c, e, g

a) $\int \sin(5x)\,dx = \cancel{\frac{1}{5}\int \sin} - \frac{1}{5}\cos 5x + C$

c) $\int \cos^3 x \sin x\,dx = \cancel{\int (1-\sin^2 x)\sin x\,dx = \int (\sin x - \sin^3 x)\,dx}$
$= \cancel{-\cos x + \frac{1}{4}\cos^4 x} = -\frac{1}{3}\cos^3 x$

e) $\int \sec^2(\frac{x}{5})\,dx = \cancel{\int \frac{1}{\cos^2(\frac{x}{5})}dx}\ 5\tan\frac{x}{5}$

g) $\int \sec^9 x \tan x\,dx = \int \frac{\sin x\,dx}{\cos^{10} x} = \int \frac{du}{10u\cdot\cos^9 x} = \frac{1}{10}\int \frac{du}{u^{\frac{19}{10}}}$

$u = \cos^{10} x,\ du = 10\cos^9 x \sin x\,dx$

$= \frac{1}{10}\cdot\frac{10}{9}u^{-\frac{9}{10}} + C$
$= \cancel{\frac{1}{9}\cos^9 x}$
$= \frac{1}{9\cos^9 x} + C$

Lecture 16. 4 Nov

3F-1 c, d.

c) $\frac{dy}{dx} = \frac{3}{\sqrt{y}}$
$\int \sqrt{y}\,dy = \int 3\,dx$.
$\frac{2}{3}y^{\frac{3}{2}} = 3x + C$
$x = \frac{2}{9}y^{\frac{3}{2}}$

d) $\frac{dy}{dx} = xy^2$.
$\int \frac{dy}{y^2} = \int x\,dx$
$-\frac{1}{y} = \frac{x^2}{2}$
$y = -\frac{2}{x^2}$

2 a) $\frac{dy}{dx} = 4xy$, $y(1)=3$. Find $y(3)$. (Assume $y>0$).
$\frac{dy}{y} = 4x\,dx$ $\ln y = 2x^2 + c$ $y = e^{2x^2+c}$ $e^{2+c} = 3$ $2+c = \ln 3$
$c = \ln 3 - 2$
$y = e^{2x^2+\ln 3-2}$
$y(3) = e^{18-2+\ln 3}$
$= e^{16+\ln 3}$

e) $\frac{dy}{dx} = e^y$ $y(3) = 0$ Find $y(0)$.

e) $\frac{dy}{dx} = e^{y}$ $y(3) = 0$ Find $y(4)$.

$$\int \frac{1}{e^{y}} dy = \int dx$$

$$-e^{-y} = x + C$$

$$\frac{1}{e^{y}} = -x + C$$

$$e^{y} = -\frac{1}{x} + C$$

$$y = \ln\left(-\frac{1}{x} + C\right)$$

$$\frac{1}{e^{y}} = \frac{4}{3}$$

$$e^{y} = \frac{3}{4}$$

$$y = \ln\left(\frac{3}{4}\right)$$

$$y(3) = \ln\left(-\frac{1}{3} + C\right) = 0$$

$$-\frac{1}{3} + C = 1$$

$$C = \frac{4}{3}$$

$$y = \ln\left(-\frac{1}{x} + \frac{4}{3}\right)$$

$\cancel{y(4) = \ln}$

4 b) $\cancel{\frac{dT}{dt} = k(T_e - T)}$

$\cancel{dT = k(T_e - T)\,dt}$

$$\frac{dT}{T_e - T} = k\,dt$$

$$kt = \int \frac{-du}{u}$$

$$kt = -\ln|T_e - T| + C$$

$$|T_e - T| = e^{-kt + C}$$

$$T_e - T = \pm e^{kt + C}$$

$$= \pm e^{C} e^{kt}$$

$$T = T_e \pm e^{C} e^{kt}$$

$$= T_e + (T_0 - T_e)e^{kt}$$

$$= T_e -$$

c) $\lim_{t\to\infty} T = T_e + (T_0 - T_e)\lim_{t\to\infty} e^{-kt} = T_e$

d) let $\Delta T = T - T_e$ ~~640°C~~ $\Delta T_0 = T_0 - T_e = 640°C$

$\Delta T = (\Delta T_0)e^{-kt} = 640\, e^{-kt}$ ~~160~~ $\Delta T(8) = 160$

$e^{-k\cdot 8} = \frac{1}{4}$

$-8k = \ln\frac{1}{4}$

$k = -\frac{\ln\frac{1}{4}}{8} = \frac{\ln 4}{8}$

$\Delta T = 10°$

$640e^{-\frac{\ln 4}{8}t} = 10$

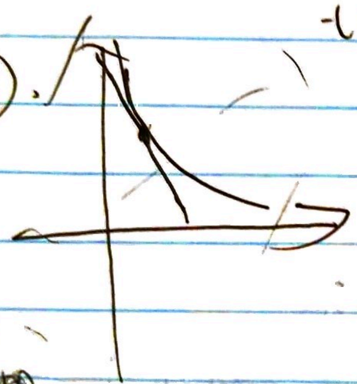
$e^{-\frac{\ln 4}{8}t} = \frac{1}{64}$

$-\frac{\ln 4 t}{8} = -\ln 64 = -6\ln 2$

$2\ln 2 t = 48\ln 2$

$\therefore t = 24$ hours.

8 b).



$\frac{d}{dx_0}f(x_0)\cdot(x - x_0) = y - f(x_0)$

let $y=0$.

$\frac{d}{dx_0}f(x_0)\cdot(x - x_0) = -f(x_0)$

$\because$ bisect, then $x = 2x_0$. so

$\frac{d}{dx_0}f(x_0)\cdot x_0 = -f(x_0)$.

put $x_0 = x$.

$\frac{df}{dx}x = -f$

$\int\frac{df}{f} = -\frac{dx}{x}$

$\ln|f| = -\ln|x|$

$\ln f = -\ln x + c$

$f = \frac{A}{x}$, $A>0$.

Lecture 18. 21 Nov

8B-2 a) $\sum_{i=1}^{3} 4i-1 - 4i-1$

b) $\sum_{i=1}^{\infty} \frac{1}{i^2}$

3b) ~~$\int_0^1 x^3 dx$~~ $\int_{-1}^{3} x^2 dx$

upper: $(-1)^2 + 1^2 + 2^2 + 3^2 = 1+4+9 = 15$

lower $= 0 + 0 + 1^2 + 2^2 = 5$

left $= (-1)^2 + 0 + 1^2 + 2^2 = 6$

right $= 0 + 1^2 + 2^2 + 3^2 = 14$.

4a) upper $=$

$= \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \cdot \left(\frac{1}{n}\right) = \frac{1}{n^3} \sum_{i=1}^{n} i^2$

lower $= \sum_{i=0}^{n} \left(\frac{i}{n}\right)^2 \cdot \frac{1}{n} = \frac{1}{n^3} \sum_{i=0}^{n-1} i^2$

$\sigma = \frac{1}{n^3} \sum_{i=1}^{n} i^2 - \frac{1}{n^3} \sum_{i=0}^{n} = \frac{1}{n^3}(n^2) = \frac{1}{n}$

as $n \to \infty$, $\frac{1}{n} \to 0$.

5. $f(b) = \sum_{i=1}^{n} \frac{1}{n} \sin \frac{ib}{n} = \frac{1}{n} \sum_{i=1}^{n} \sin \frac{ib}{n}$

$\therefore \lim_{n \to +\infty} f(b) = \int_0^1 \sin bx\, dx = -\frac{\cos bx}{b}\Big|_0^1 = \frac{\cos b + 1}{b} = \frac{1-\cos}{b}$

4J-1 $\pi \cdot \frac{1}{4} \cdot \Delta y$ $\Delta y = \frac{100}{n}$

$C = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{100}{n} \cdot \frac{i}{n} \cdot 100 \cdot \pi \frac{1}{4} \cdot k = \int_0^1 2500\pi k x\, dx$

$2500\pi k \frac{x^2}{2}\Big|_0^1 = 1250$

ent Objectives

PS4 PART II 6 Dec.

a) Assume $c \geq 0$

$f(c) - f(0) = f'(x) \cdot c \geq 0$

$0 = f(0) \leq f(c)$

put $b = c$ $f(x) \geq 0$.

b) let $f(x) = x - \ln(1+x)$

$f(0) = 0$

$f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} \geq 0$ as $x \geq 0$.

according to the (a)

$f(x) \geq 0$. $\therefore \ln(1+x) \leq x$ for $x \geq 0$.

c) let $f(x) = \ln(1+x) - x + \frac{x^2}{2} \geq 0$

$f(0) = 0 - 0 + 0 = 0$

$f'(x) = \frac{1}{1+x} - 1 + x$

$= \frac{x^2}{1+x} \geq 0$ for $x \geq 0$

>0 as

$\therefore f(x) \geq 0$.

let $f(x) = -\ln(1+x) + x - \frac{x^2}{2} + \frac{x^3}{3}$

$f'(x) = \frac{-1}{1+x} + 1 - x + x^2$

$= \frac{x^3}{1+x} \geq 0$ for $x \geq 0$.

$\therefore \ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3}$ for $x \geq 0$

4

1) $\sum_1^n (-1)^i \frac{x^i}{i} \le \ln(1+x) \le \sum^{n+1} (-1)^i \frac{x^i}{i}$

1) let $f(x) = \cancel{\ln(1+x) - x}$ $\quad f(x) = x - \ln(1+x)$

$f'(x) = -\frac{1}{1+x} + 1 = \frac{x}{1+x}$ $\quad \le 0.$

Assume $c \le x \le 0$

$f(0) = 0.$

$\cancel{f(0) - f(x) = f'(x) \cdot c}$

$f(0) - f(x) = f'(c) \cdot x$

$f(x) = f(0) - f'(x) \cdot x \ge 0.$

$\therefore x \ge \ln(1-x)$ for $-1 \le x \le 0.$

a) $\left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2}$

$\left(\frac{x}{1-x}\right)' = \frac{(1-x)+x}{(1-x)^2} = \frac{1}{(1-x)^2}$

$\frac{x}{1-x} = \frac{-1+x+1}{1-x} = -1 + \frac{1}{1-x}$

) $\left(\frac{\tan^2 x}{2}\right)' = \tan x \cdot \sec^2 x$

$\left(\frac{\sec^2 x}{2}\right)' = \left(\frac{\frac{1}{\cos^2 x}}{2}\right)' = \frac{-2\frac{1}{\cos^3 x}(-\sin x)}{2} = \frac{\sin x}{\cos^3 x} = \tan x \cdot \sec^2 x$

$\frac{\tan^2 x}{2} = \frac{\sin^2 x}{2\cos^2 x} = \frac{\sin^2 x}{2\cos^2 x} + \frac{1}{2} - \frac{1}{2} = \frac{1}{2\cos^2 x} - \frac{1}{2} = \frac{\sec^2 x}{2} - \frac{1}{2}$

3.

a) $y = e^{\sqrt{x}} + \sqrt{e^x}$ $\quad \frac{dy}{dx} = e^{\sqrt{x}} \cdot \frac{1}{2}\frac{1}{\sqrt{x}} + \frac{1}{2}\frac{1}{\sqrt{e^x}} \cdot e^x$

$= \frac{1}{2}\left(\frac{e^{\sqrt{x}}}{\sqrt{x}} + \sqrt{e^x}\right)$

b) $y = \sqrt{e^{2x} + 2x}$ $\quad \frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{\sqrt{e^{2x}+2x}} \cdot (2e^{2x} + 2)$ $\quad$ 6

$= \frac{e^{2x} + 1}{\sqrt{e^{2x}+2x}}$

No mea... ...ed to determine the...

4. a) $\frac{dp}{dh} = -0.13p$

$\int \frac{dp}{p} = \int -0.13\, dh$

$\ln p = -0.13h + c$

$\ln 1 = c$

$c = 0$

$p = e^{-0.13h} = e^{-1.3}$

b) $\Delta p = 1 - e^{-0.13 \times 0.1} = 1 - e^{-0.013} \approx 0.0129$ (3 s.f.)

$e^x \approx e^0 + e^0 \cdot x = x + 1$

$\Delta p = 1 - e^{-0.013} \approx 1 - (1 - 0.013) = 0.013\, kg/cm^2$

c) $\Delta p = \frac{d}{dh} p(0) \cdot \Delta h = -0.13 \Delta h = -0.013\, kg/cm^2$

$dh = 10\, km$, $\Delta p = \frac{d}{dh} p(0) \Delta h = -1.3\, kg/cm^2$

5. $\int_0^1 e^x dx = \lim_{n \to \infty} \frac{1}{n} \sum_0^{n-1} e^{\frac{i}{n}}$

$= \lim_{n \to \infty} \frac{1}{n} \cdot \frac{1 - (e^{\frac{1}{n}})^n}{1 - e^{\frac{1}{n}}}$

$= \lim_{n \to \infty} \frac{1 - e}{n(1 - e^{\frac{1}{n}})}$

$\lim_{n \to \infty} n(1 - e^{\frac{1}{n}}) = n(1 - (1 + \frac{1}{n})) = -1$

$\int_0^1 e^x dx = \frac{1-e}{-1} = e - 1$

6. a) $m_0(x - x_0) = y - y_0$

Let $x = 0$, $y = y_0 - m_0 x_0$

Let $y = 0$, $m_0 x - m_0 x_0 = -y_0$, $x = x_0 - \frac{y_0}{m_0}$

$L^2 = (y_0 - m_0 x_0)^2 + (x_0 - \frac{y_0}{m_0})^2 = y_0^2 - 2m_0 x_0 y_0 + m_0^2 x_0^2 + x_0^2 + \frac{y_0^2}{m_0^2} - 2\frac{x_0 y_0}{m_0}$

$$\int (x^3)^{\frac{1}{3}} = \int (y^3)^{\frac{1}{3}}$$

$$\frac{3}{2}(x^3)^{\frac{2}{3}}$$

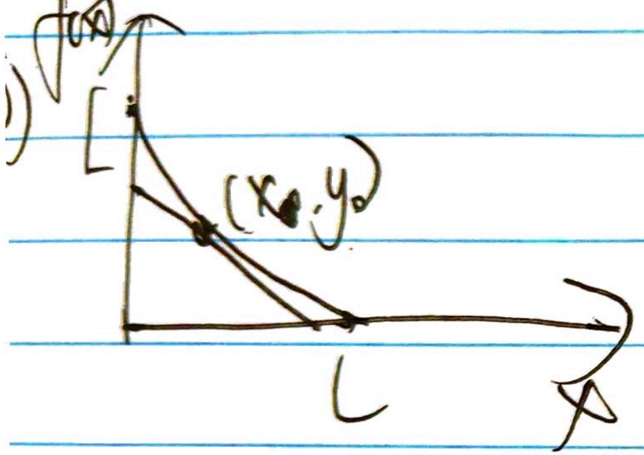
$$\int 3x\,dx = \int 3y\,dy$$

$$\frac{3}{2}x^2 = \frac{3}{2}y^2 + C$$

$$\frac{3}{2}y^2 + C$$

$$\frac{3}{2}L^2 + C = 0$$

$$C = -\frac{3}{2}L^2$$



$$f'(x_0)(x - x_0) = y - y_0$$

$$-x_0 f'(x_0) = y - y_0$$

$$y = y_0 - x_0 f'(x_0)$$

$$-y_0 = (x - x_0) f'(x_0)$$

$$x = \frac{-y_0 + f'(x_0)x_0}{f'(x_0)}$$

$$y + xy'^3$$

$$y y' + x y'^4$$

$$x y'^3 + 2 y y'^4$$

$$(y - y'x)^2 + \left(x - \frac{y}{y'}\right)^2 = L^2$$

$$(y - y'x)^2\left(1 + \frac{1}{y'}\right)^2 = L^2$$

$$y^2 - 2yy'x + (y')^2x^2 + x^2 - \frac{2xy}{y'} + \frac{y^2}{(y')^2} = L^2$$

$$2yy' - 2(y''x + y') + 2y'y''x^2 + 2(y')^2x + 2x -$$

b) $f'(x_0)(x-x_0)=y-y_0$

$$L^2 = (y-y'x)^2+(x-\frac{y}{y'})^2$$

c) $2(y-y'x)(y'-y''x-y')+2(x-\frac{y}{y'})(1-\frac{y'^2-yy''}{y'^2})=L^2$

$$y''(xy'-y)(2x+\frac{2y}{y'^3})=0$$

d) 1° $x+\frac{y}{(y')^3}=0$

$$(y')^3=\frac{y}{-x}$$

$$\frac{y'}{y^{\frac{1}{3}}}=\frac{1}{-(x)^{\frac{1}{3}}}$$

$$\int\frac{dy}{y^{\frac{1}{3}}}=\int\frac{dx}{x^{\frac{1}{3}}}$$

$$\frac{3}{2}y^{\frac{2}{3}}=-\frac{3}{2}x^{\frac{2}{3}}+C$$

$$\frac{3}{2}L^{\frac{2}{3}}=C$$

$$\frac{3}{2}x^{\frac{2}{3}}+\frac{3}{2}y^{\frac{2}{3}}=\frac{3}{2}L^{\frac{2}{3}}$$

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=L^{\frac{2}{3}}$$

2° $y''=0$ $\quad y'=c \quad y=cx+d$

$C<0$. $\quad L^2=d^2+(\frac{d}{c})^2 \quad C^2=\frac{d^2}{L^2-d^2} \quad C=\frac{-d}{\sqrt{L^2-d^2}}$

3° $xy'-y=0$

$$x\frac{dy}{dx}-y=0$$

$$\frac{x}{dx}=\frac{y}{dy} \quad \text{let } x\neq 0, y\neq 0$$

$$\int\frac{dy}{y}=\int\frac{dx}{x}$$

$$\ln y=\ln x+C$$

$y=Ax$ Tangent does not exist, so ~~the~~ no meaning.