18.02 Final Exam 5/4/2015 238/250 95% P:

P1 (a) ~~[illegible]~~

~~[illegible]~~

$\vec{V}_1 = \langle -1, 2, -3 \rangle$

$\vec{V}_2 = \langle -1, 1, 2 \rangle - \langle 1, 1, 2 \rangle = \langle -2, 0, 0 \rangle$

$\vec{V}_1 \times \vec{V}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & -3 \\ -2 & 0 & 0 \end{vmatrix} = 6\hat{j} + 4\hat{k}$

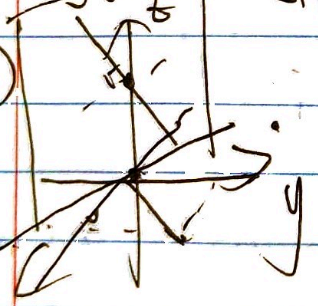
$\therefore 0(x+1) + 6(y-1) + 4(z-2) = 0$

$0x + 3y + 2z = 3+4$

$0x + 3y + 2z = 7$

b) ~~[illegible]~~

$\frac{\langle 2,1,1 \rangle \cdot \langle \ldots \rangle}{\sqrt{4+1+1} \cdot \sqrt{1+4+9}}$ [...] $\frac{-2+2-3}{\sqrt{84}} = \frac{-3}{\sqrt{84}}$

P2 a) 

$x(t) = t$

$y(t) = 2t$

$z = 1$

b) $x + 2y = 0$

c) i) $P^* = (t^*, 2t^*, 1)$, $t \neq 0$

ii) suppose $p = (t, 2t, 1)$

$|t^*| = 2t$

iii) $t^* = -t$

P3 a) $|A_2| = \begin{vmatrix} 1 & 0 & 3 \\ -2 & 1 & -1 \\ -1 & 1 & 2 \end{vmatrix} = 2+1+3(-2+1) = 2-2 = 0$

b) $A_{2,1} \times A_{2,2} = \langle 1,0,3 \rangle \times \langle -2,1,-1 \rangle$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 3 \\ -2 & 1 & -1 \end{vmatrix}$

$= -3\hat{i} - (-1+6)\hat{j} + \hat{k} = -3\hat{i} - 5\hat{j} + \hat{k}$

$\therefore$ line : $\begin{cases} x = -3t \\ y = 5t \\ z = t \end{cases}$

c) $A_1 = \begin{bmatrix} 1 & 0 & 3 \\ -2 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$ $|A_1| = 2+3(-2+1) = -1$

$\text{adj}(A) = |A_1| \cdot A_1^{-1} = \begin{bmatrix} * & * & * \\ 3 & -p & -5 \\ * & * & * \end{bmatrix}$

$\text{Cof}(A) = \text{adj}(A)^T = \begin{bmatrix} * & 3 & * \\ * & -p & * \\ * & -5 & * \end{bmatrix}$

$\therefore -p = \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 1+3 = 4$ $p = -4$

P4 a) $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\langle -\sin(e^t)\cdot e^t, \cos(e^t) e^t, e^t \rangle}{\sqrt{e^{2t}+e^{2t}}}$

$= \frac{\langle -\sin(e^t), \cos(e^t), 1 \rangle}{\sqrt{2}}$

b) $\vec{T}'(t) = \frac{1}{\sqrt{2}} \cdot \langle -\cos(e^t)\cdot e^t, -\sin(e^t)\cdot e^t, 0 \rangle$

P5 a) $\nabla F = \langle z \cdot \frac{1}{2} \cdot \frac{2x}{\sqrt{x^2+y}}, z \cdot \frac{1}{2\sqrt{x^2+y}} + \frac{3}{z}, \sqrt{x^2+y} - 2\frac{y}{z^2} \rangle$

at $P_0(1,3,2)$: $\nabla F = \langle \frac{2}{\sqrt{4}}, \frac{1}{\sqrt{4}} + \frac{3}{2}, \sqrt{4} - \frac{6}{4} \rangle$

$= \langle 1, \frac{3}{2}, \frac{1}{2} \rangle$

$T: x-1 + \frac{3}{2}(y-3) + \frac{1}{2}(z-2) = 0$

$2x + 3y + z = 2 + 9 + 2 = 13$

b) $\because F_y > F_x > F_z \therefore y$

-2

$\Delta F \approx F_x \Delta x + F_y \Delta y + F_z \Delta z$

$= 0.1 + 0.15 + 0.05 = 0.3$ Largest: 0.15

c) $\hat{u} = \frac{\langle -2,2,1 \rangle}{\sqrt{4+4+1}} = \frac{1}{3}\langle -2,2,1 \rangle$

$\left.\frac{df}{ds}\right|_{\hat{u}} = \nabla f \cdot \hat{u} = -\frac{2}{3} + 1 + \frac{1}{6} = \frac{1}{6}$

$0.1 = \Delta F \simeq \left.\frac{df}{ds}\right|_{\hat{u}} \cdot \Delta s$

$\Delta s \simeq 0.6$

P6 a) $f_x = 1 + \frac{2}{y} \cdot \left(-\frac{1}{x^2}\right) = 1 - \frac{2}{x^2 y}$

$f_y = 4 - \frac{2}{y^2 x}$

$\begin{cases} x - \frac{2}{xy} = 0 \\ 4y - \frac{2}{yx} = 0 \end{cases} \Rightarrow \begin{cases} 4y - x = 0 \\ 4y - \frac{1}{2y^2} = 0 \end{cases} \Rightarrow \begin{cases} x = 2 \\ y = \frac{1}{2} \end{cases}$

b) $A = f_{xx}(2, \frac{1}{2}) = -\frac{2}{y}\left(\frac{-2}{x^3}\right)\Big|_{(2,\frac{1}{2})} = -4\left(-\frac{1}{4}\right) = 1$

$B = f_{xy}(2, \frac{1}{2}) = -\frac{2}{x^2}\left(\frac{-1}{y^2}\right)\Big|_{(2,\frac{1}{2})} = -\frac{1}{2}(-4) = 2$

$C = f_{yy}(2, \frac{1}{2}) = -\frac{2}{x}\left(\frac{-2}{y^3}\right)\Big|_{(2,\frac{1}{2})} = -1\cdot(-16) = 16$

$\because AC - B^2 = 16 - 4 = 12 > 0, \quad A > 0$

$\therefore (2, \frac{1}{2})$ is a local min

7) $f(x,y,z) = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2$

$g(x,y,z) = Ax + By + Cz = D$

$\nabla f = \lambda \nabla g$

$$\begin{cases} 2(x-x_0) = \lambda A \\ 2(y-y_0) = \lambda B \\ 2(z-z_0) = \lambda C \end{cases}$$

8 a) $x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

when $(\rho, \phi, \theta) = (2, \frac{\pi}{4}, -\frac{\pi}{4})$,

$(x, y, z) = (1, -1, \sqrt{2}) \quad \therefore F_x = 1, F_y = 2, F_z = -2$

$$\frac{\partial F}{\partial \phi} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \phi} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \phi} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial \phi}$$

$$= \frac{\partial x}{\partial \phi} + 2\frac{\partial y}{\partial \phi} - 2\frac{\partial z}{\partial \phi}$$

$$= \rho\cos\theta\cos\phi + 2\rho\sin\theta\cos\phi + 2\rho\sin\phi = 1 - 4\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} + 2\sqrt{2}$$

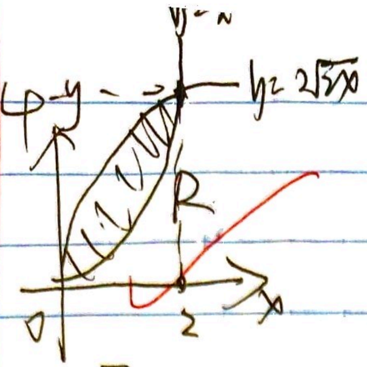
$$= 2\sqrt{2} - 1$$

b) $\frac{\partial y}{\partial x} = 0 \; 1, \; \frac{\partial x}{\partial y} = 1$

yes. No

-5

P9 (a)



b) $\int_0^4\int_{\frac{y^2}{8}}^{\sqrt{y}} f(x,y)\,dx\,dy$

P10 let $u=x^2y$, $v=\frac{y}{x}$

$y=\sqrt[3]{uv^2}$ $\quad x=\sqrt[3]{\frac{u}{v}}$

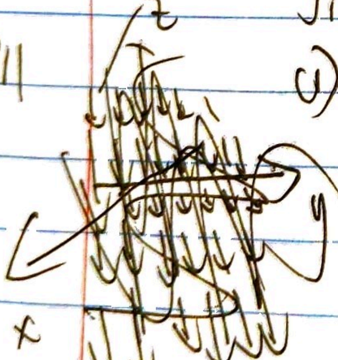
$\cancel{dxdy} = dudv = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dxdy = \begin{vmatrix} 2xy & x^2 \\ -\frac{y}{x^2} & \frac{1}{x}\end{vmatrix} dxdy = (2y+y)\,dxdy = 3y\,dxdy$

$\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy$

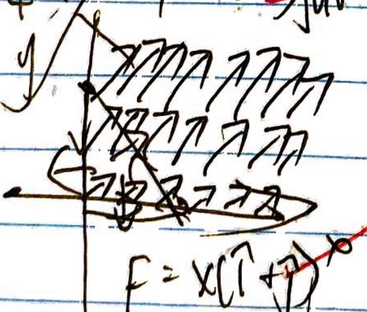
$= \iint_{R'} f\left(\sqrt[3]{\frac{u}{v}}, \sqrt[3]{uv^2}\right)\frac{1}{3\sqrt[3]{uv^2}}\,du\,dv$

$= \int_1^2\int_4^9 f\left(\sqrt[3]{\frac{u}{v}}, \sqrt[3]{uv^2}\right)\frac{1}{3\sqrt[3]{uv^2}}\,du\,dv$

P11



a)



positive

$F = x(\vec{i}+\vec{j})$

$y=1-x$

b) $\oint_C \vec{F}\cdot\hat{n}\,ds = \oint_C$ ~~x dx + y dy~~

$= \int_C x\,dy - x\,dy = \int_0^1 x\,dx + \int_1^0 -x\,dx = 0$

C

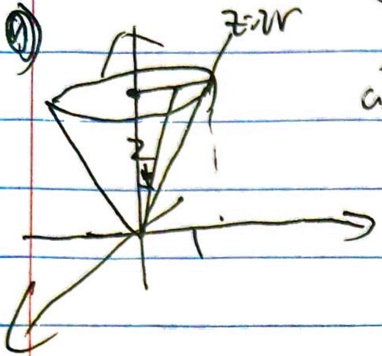
P12

P

c) $\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_R \nabla\cdot\vec{F}\,dA$

$= \int_0^1\int_0^{1-x} dy\,dx = \int_0^1 (1-x)\,dx = 1-\frac{1}{2} = \frac{1}{2}$

P12 (a)



a) $M = \iiint_R z\,dV$

$= \int_0^{2\pi}\int_0^1\int_{2r}^2 z\,dz\,r\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^1 \frac{1}{2}(4-4r^2)\,r\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^1 2r-2r^3\,dr\,d\theta = \int_0^{2\pi}\left(r^2-\frac{1}{2}r^4\Big|_0^1\right)d\theta$

$= \pi$

b) $\bar{z} = \iiint_R z^2\,dV$

$= \int_0^{2\pi}\int_0^1\int_{2r}^2 z^2 r\,dz\,dr\,d\theta$

-5

c) $\bar{z} = \iiint_R \frac{1}{z^2}\,dV = \int_0^{2\pi}\int_0^{\tan^{-1}\frac{1}{2}}\int_0^{2\sec\phi} \frac{1}{z^2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$ $\quad (\rho\cos\phi)^2$

P13 a) $\begin{cases} F_{x,y} = F_{y,x} \\ F_{y,z} = F_{z,y} \\ F_{x,z} = F_{z,x} \end{cases} \Leftarrow \begin{cases} 1+2yz = 1+2yz \\ -1+2xy = -1+2xy \\ y^2 = y^2 \end{cases}$

b) $f(x,y,z) = \int \frac{\partial f}{\partial x}\,dx = \int y+y^2z\,dx = yx+xy^2z+g(y,z)$

$\frac{\partial f}{\partial y} = x+2xyz+\frac{\partial g}{\partial y} = x-z+2xyz \Rightarrow \frac{\partial g}{\partial y} = -z$

$g(y,z) = \int \frac{\partial g}{\partial y}\,dy = \int -z\,dy = -yz+h(z)$.

$\frac{\partial f}{\partial z} = xy^2 - y + \frac{dh}{dz} = -y+xy^2 \Rightarrow \frac{dh}{dz} = 0 \quad h=C.$

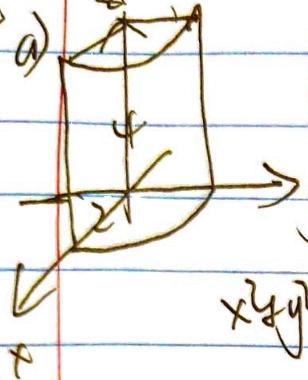
$\therefore f(x,y,z) = xy^2z+xy-yz+C$

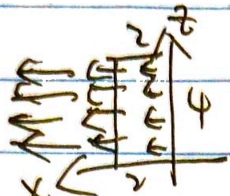
$f(x,y,z) = xy$

c) $\int_C \vec{F}\cdot d\vec{r} = f(1,1,2) - f(2,2,1)$

$= 2 - 1 + 2 - (8 + 4 - 2) = 3 - 10 = -7$ 6

P14 a)





$x^2 + y^2 = 4$

b) $\hat{n} = \frac{x\hat{i} + y\hat{j}}{r} = \frac{x\hat{i} + y\hat{j}}{2}$

$\iint_S \vec{F}\cdot\hat{n}\, dS = \iint_S \frac{x^2}{2}\cdot 2\, dz\, d\theta = \int_0^{2\pi}\int_0^4 4\cos^2\theta\, dz\, d\theta$

$= \int_0^{2\pi} 16\cos^2\theta\, d\theta = 4\pi$

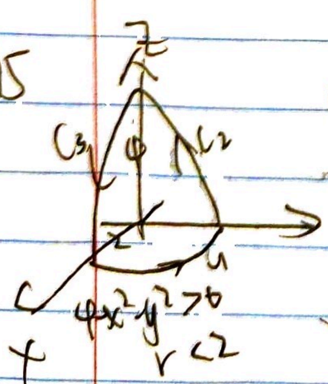
c) $\oiint_S \vec{F}\cdot\vec{n}\, dS = \iiint_G \nabla\cdot\vec{F}\, dV = \iiint_G dV$

$= \mathrm{Vol}(G) = \frac{1}{4}\cdot\pi(2)^2\cdot 4 = 4\pi$

d) $\iint_{S'-S} \vec{F}\cdot\hat{n}\, dS = \oiint_S \vec{F}\cdot\hat{n}\, dS - \iint_S \vec{F}\cdot\hat{n}\, dS$

$= 4\pi - 4\pi = 0$

P15



$4 - x^2 - y^2 > 0$

$r < 2$

a) $\oint_C \vec{F}\cdot d\vec{r} = \iint_S \nabla\times\vec{F}\cdot d\vec{S}$

$= \iint_S \langle x, y, -2z\rangle \cdot \langle -\frac{\partial}{\partial x}(4-x^2-y^2), -\frac{\partial}{\partial y}(4-x^2-y^2), 1\rangle\, dx\, dy$

$= \iint_S \langle x, y, -2z\rangle\cdot\langle 2x, 2y, 1\rangle\, dx\, dy$

$= \iint_S 2x^2 + 2y^2 - 2(4 - x^2 - y^2)\, dx\, dy$

$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ yz & -xz & 1 \end{vmatrix}$

$= x\hat{i} + y\hat{j} + (-z - z)\hat{k}$

$= x\hat{i} + y\hat{j} - 2z\hat{k}$

$= \iint_S (4x^2 + 4y^2 - 8)\, dy\, dx = \cancel{\int_0^{\frac{\pi}{2}}\int_0^4 (4z - 8)\, 2\, dz\, d\theta} = \cancel{\int_0^{\frac{\pi}{2}}\left[\ldots\right]_0^4 d\theta}$

$= \int_0^{\frac{\pi}{2}}\int_0^2 (4r^2 - 8)\, r\, dr\, d\theta$

$= 4\int_0^{\frac{\pi}{2}} 0\, d\theta = 0$

$= \cancel{\ldots 64\pi}$

b) $\oint_C \vec{F} \cdot d\vec{r}$

$= \oint_C yz\,dx - xz\,dy + dz$

$= \int_{C_1} 0 + \int_{C_2} dz + \int_{C_3} dz$

$= \int_0^4 dz + \int_4^0 dz = 0$

$C_1 : \begin{cases} x = 2\cos\theta \\ y = 2\sin\theta \end{cases}$

$C_2 : \ z = 4 - y^2$

$C_3 : \ z = 4 - x^2$