PS6 25/3/2025

Lec 15

2J-1 a) $\left(\frac{\partial w}{\partial y}\right)_z = \left(\frac{\partial w}{\partial y}(z - y^2 + y^2 + z^2)\right) = 0$

b) $\left(\frac{\partial w}{\partial z}\right)_y = \left(\frac{\partial}{\partial z}\left(z - y^2 + y^2 + z^2\right)\right) = 1 + 2z$

2J-2 i) $w = x^2 + y^2 + z^2 \qquad z = x^2 + y^2$

$dw = 2x\,dx + 2y\,dy + 2z\,dz$

$\cancel{= 2x\,dx + 2y\,dy + 2z(2x\,dx + 2y\,dy)}$

$dz = 2x\,dx + 2y\,dy$

$dw = (2z+1)\,dz$

$\frac{dw}{dz} = 2z + 1$

$dw = dz - 2y\,dy + 2y\,dy + 2z\,dz, \quad dz = 0.$

$= 0$

$\left(\frac{\partial w}{\partial y}\right)_z = 0.$

$0 = \frac{\partial z}{\partial y} = 2x\left(\frac{\partial x}{\partial y}\right)_z + 2y$

ii) $\cancel{dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz}$

$\cancel{0 = \frac{dz}{dy} = 2x\,dx + 2y\,dy}$ $\qquad \left(\frac{\partial x}{\partial y}\right)_z = -\frac{y}{x}$

$\cancel{z = x + y \qquad dz = 0}$

$\cancel{0 = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy}$

$\left(\frac{\partial w}{\partial y}\right)_z = \frac{\partial w}{\partial x}\left(\frac{\partial x}{\partial y}\right)_z + \frac{\partial w}{\partial y}\left(\frac{\partial y}{\partial y}\right)_z + \frac{\partial w}{\partial z}\left(\frac{\partial z}{\partial y}\right)_z$

$= 2x \cdot \left(-\frac{y}{x}\right) + 2y = 0$ $\qquad \left(\frac{\partial x}{\partial z}\right)_y = \frac{1}{2x}$

$\cancel{\left(\frac{\partial w}{\partial y}\right)_y = \frac{\partial w}{\partial x}\left(\frac{\partial x}{\partial y}\right)_y}$ $\qquad 1 = 2x\left(\frac{\partial x}{\partial z}\right)_y + 2y\left(\frac{\partial y}{\partial z}\right)_y = 2x\left(\frac{\partial x}{\partial z}\right)_y$

$\left(\frac{\partial w}{\partial z}\right)_y = \frac{\partial w}{\partial x}\left(\frac{\partial x}{\partial z}\right)_y + \frac{\partial w}{\partial y}\left(\frac{\partial y}{\partial z}\right)_y + \frac{\partial w}{\partial z}\left(\frac{\partial z}{\partial z}\right)_y$

$= 1 + 2z$

2) $w = x^3y - z^2t$

$xy = zt$

$dw = 3yx^2dx + x^3dy - 2tz\,dz - z^2dt$

$ydx + xdy = zdt + tdz$

a) x, z constant

$dw = x^3dy - z^2dt$

$= x^2(zdt + tdz - ydx) - z^2dt$

$= (x^2z - z^2)dt$

$\left(\frac{\partial w}{\partial t}\right)_{x,z} = x^2z - z^2$

b) x, y constant

$dw = -2tz\,dz - z^2dt$

$= -2tz\,dz - z(-tdz)$

$= (-2zt + zt)dz$

$\left(\frac{\partial w}{\partial z}\right)_{x,y} = -zt$

4 Chain Rule:

a) $x\left(\frac{\partial y}{\partial t}\right)_{x,z} = z$

$\left(\frac{\partial y}{\partial t}\right)_{x,z} = \frac{z}{x}$

$\left(\frac{\partial w}{\partial t}\right)_{x,z} = x^3\left(\frac{\partial y}{\partial t}\right)_{x,z} - z^2$

$= x^2z - z^2$

b) ~~$\left(\frac{\partial w}{\partial z}\right)_{x,y} =$~~

$0 = z\left(\frac{\partial t}{\partial z}\right)_{x,y} + t$

$\left(\frac{\partial t}{\partial z}\right)_{x,y} = -\frac{t}{z}$

$\left(\frac{\partial w}{\partial z}\right)_{x,y} = -2tz - z^2\left(\frac{\partial t}{\partial z}\right)_{x,y}$

$= -2tz + zt = -zt$

5 a) $pv = nRT$

$S = S(p, v, T)$

$d(pv) = d(nRT)$

$vdp + pdv = nRdT$

$v + \cancel{\frac{\partial v}{}} = nR\left(\frac{\partial T}{\partial p}\right)_v$

$\left(\frac{\partial T}{\partial p}\right)_v = \frac{v}{nR}$

$$\left(\frac{\partial S}{\partial p}\right)_v = \left(\frac{\partial S}{\partial p}\right)\left(\frac{\partial p}{\partial p}\right)_v + \frac{\partial S}{\partial v}\left(\frac{\partial v}{\partial p}\right)_v + \frac{\partial S}{\partial T}\left(\frac{\partial T}{\partial p}\right)_v$$

$$= \left(\frac{\partial S}{\partial p}\right) + \frac{\partial S}{\partial T}\cdot\frac{v}{nR}$$

$$= S_p + S_T \cdot \frac{v}{nR}$$

b) $w = u^3 - uv^2$ (1)

$u = xy$ (2)

$v = u + x$ (3)

i) take the derivative of (2) w.r.t. u

$1 = y\left(\frac{\partial x}{\partial u}\right)_x + x\left(\frac{\partial y}{\partial u}\right)_x \qquad \left(\frac{\partial y}{\partial u}\right)_x = \frac{1}{x}$

take the derivative of (3) w.r.t. u

$\left(\frac{\partial v}{\partial u}\right)_x = 1$

$$\left(\frac{\partial w}{\partial u}\right)_x = 3u^2 - v^2 - 2uv\left(\frac{\partial v}{\partial u}\right)_x$$

$$= 3u^2 - v^2 - 2uv$$

$\cancel{\left(\frac{\partial u}{\partial x}\right)_u}\ 0 = y + x\left(\frac{\partial y}{\partial x}\right)_u \qquad \left(\frac{\partial y}{\partial x}\right)_u = -\frac{y}{x}$

$\left(\frac{\partial v}{\partial x}\right)_u = 1$

$$\left(\frac{\partial w}{\partial x}\right)_u = -2uv\left(\frac{\partial v}{\partial x}\right)_u = -2uv$$

b) $dw = 3u^2 du - ((du)\cdot v^2 + u \cdot 2v dv)$

$\quad = \cancel{3u^2 du}\ (3u^2 - v^2)du + 2uv dv$

$du = y dx + x dy$

$dv = du + dx$

if x constant

$\cancel{dw =}\ dv = du$

$dw = (3u^2 - v^2)du + 2uv du$

$\left(\frac{\partial w}{\partial u}\right)_x = 3u^2 - v^2 + 2uv$

if u constant

$dv = dx$

$dw = 2uv dx$

$\left(\frac{\partial w}{\partial x}\right)_u = 2uv$

7 $\frac{\partial f}{\partial x} = 2, \frac{\partial f}{\partial y} = 1, \frac{\partial f}{\partial z} = -3$ $\quad df = 2dx + dy - 3dz$

$dy = \frac{\partial y}{\partial x} dx + \frac{\partial y}{\partial z} dz$

$0 = \frac{\partial y}{\partial x} + \frac{\partial y}{\partial z} \left(\frac{\partial z}{\partial x}\right)_y$

$dg = \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial z} dz$

$0 = \frac{\partial y}{\partial x} \left(\frac{\partial x}{\partial z}\right)_y + \frac{\partial y}{\partial z}$

$\nabla g = \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right\rangle$

$\left(\frac{\partial g}{\partial x}\right)_y = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial z}\left(\frac{\partial z}{\partial x}\right)_y$

$dz = 2x dx + dy$

$df = 2dx + dz - 2x dx - 3dz$

$\quad = (2-2x)dx - 2dz = g - 2dz = dg$

$\cancel{\nabla g = \langle 2-2x, -2 \rangle}$ $\quad \nabla g = \langle 0, -2 \rangle$

Lec 16 25/3/2025

1 $w(x,y) = \ln\sqrt{x^2+y^2} = \frac{1}{2}\ln(x^2+y^2)$

$\frac{\partial w}{\partial x} = \frac{2x}{2(x^2+y^2)} = \frac{x}{x^2+y^2}$ $\quad \frac{\partial w}{\partial y} = \frac{y}{x^2+y^2}$

$\frac{\partial^2 w}{\partial x^2} = \frac{x^2+y^2-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$ $\quad \frac{\partial^2 w}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$

$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2} = 0$ as $(x,y)=(0,0)$, $\frac{\partial^2 w}{\partial x^2}$ and $\frac{\partial^2 w}{\partial y^2}$ DNE

2 $w_x = n(x^2+y^2+z^2)^{n-1} 2x$

$w_y = n(x^2+y^2+z^2)^{n-1} 2y$

$w_z = n(x^2+y^2+z^2)^{n-1} 2z$

$w_{xx} = n(n-1)(x^2+y^2+z^2)^{n-2} 4x^2 + n(x^2+y^2+z^2)^{n-1} \cdot 2$

$w_{yy} = n(n-1)(x^2+y^2+z^2)^{n-2} 4y^2 + n(x^2+y^2+z^2)^{n-1} 2$

$w_{zz} = n(n-1)(x^2+y^2+z^2)^{n-2} 4z^2 + n(x^2+y^2+z^2)^{n-1} 2$

$w_{xx}+w_{yy}+w_{zz}$

$= 6n(x^2+y^2+z^2)^{n-1} + 4n(n-1)(x^2+y^2+z^2)^{n-2}(x^2+y^2+z^2)$

$= 2n(3+2(n-1))(x^2+y^2+z^2)^{n-1}$ $\quad 2n+1$

$n = -\frac{1}{2}, 0$

3a) $w_x = 2ax+b$ $\quad w_{xx} = 2a$

$w_y = bx+2cy$ $\quad w_{yy} = 2c$

$w_{xx}+w_{yy} = 2a+2c = 0$

$a = -c$

$w = ax^2+bxy-ay^2$ $\quad f_1(x,y) = (x^2-y^2)$

$= a(x+y)(x-y)+bxy$ $\quad f_2(x,y) = xy$

2K-4 $w(x,t) = f(x+ct) + g\cdots$

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} = f''(x+ct) + g''(x-ct)$$

$y(1) = 0$

$$f_{xx} = \frac{1}{c^2} f_{tt}$$

5 $w(0,t) = 0 \quad w(1,t) = \sin k\, e^{rt} = 0 \Rightarrow k = n\pi$

$w_{xx} = -e^{rt} k^2 \sin kx$

$w_t = r e^{rt} \sin kx = -\frac{r}{k^2} w_{xx}$

PART B 26/3/2015

P1 $0 = \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy \qquad dy = -\frac{g_x}{g_y}$

$$dw = w_x dx + w_y dy + w_z dz$$
$$= f_x dx - \frac{g_x}{g_y} dx + f_z dz$$
$$= \left(f_x - \frac{g_x}{g_y}\right) dx + f_z dz$$

$$\left(\frac{\partial w}{\partial x}\right)_z = f_x - \frac{g_x}{g_y}$$

P2 (i) $dt = \cos(x+y)dx + \cos(x+y)dy \qquad dy = \frac{dt - \cos(x+y)}{\cos(x+y)}$

$$dw = 3ytx^2 dx + x^3 t\, dy + x^3 y\, dt$$
$$= 3ytx^2 dx + \frac{x^3 t\, dt}{\cos(x+y)} - x^3 t\, dx + x^3 y\, dt$$

$$\left(\frac{\partial w}{\partial t}\right)_x = \frac{x^3 t}{\cos(x+y)} + x^3 y$$

b) $(x, y, z) = (1, 1, 2)$.

$x^5 + yz = 3$

$xy^2 + yz^2 + zx^2 = 7$

$5x^4 dx + z dy + y dz = 0 \qquad dz = -\frac{5x^4}{y} dx - \frac{z}{y} dy$

$(y^2 + 2zx) dx + (2xy + z^2) dy + (2yz + x^2) dz = 0$

$(y^2 + 2zx) dx + (2xy + z^2) dy + (2yz + x^2) \left(-\frac{5x^4}{y} dx - \frac{z}{y} dy \right) = 0$

$(x, y, z) = (1, 1, 2)$.

$5dx + 6dy + 5(-5dx - 2dy) = 0$

$-20dx - 4dy = 0$

$\frac{dx}{dy} = -\frac{4}{20} = -\frac{1}{5}$

Exam 2 2/12/2015 100

P1