$A = \iint_R dA = \iint_R \frac{1}{r^{29}} dA = \int_0 \int_0 \frac{1}{r^{29}} dr\, d\theta$

$$= \int_0^{2\pi} \frac{1}{29 \times 2} d\theta$$

$$= \frac{\pi}{29}$$

PS 8 28/3/2025

1-(b) ~~p~~ point to the origin, magnitude is ~~[illegible]~~ the distance to the origin.

d) Towards the origin, magnitude is 1

2a) $\nabla w = \langle a, b \rangle$

b) $\nabla w = \langle \frac{1}{x+y}, \frac{1}{x+y} \rangle$

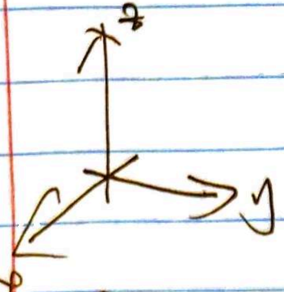
c) $\nabla w = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle = \langle \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x}, \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial y} \rangle = \langle f'(r), f'(r) \rangle$

3a) $\hat{i} + 2\hat{j}$

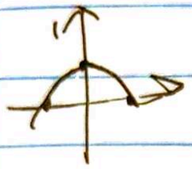
b) ~~[illegible]~~ $-rx\hat{i} - ry\hat{j}$

c) $\frac{-y\hat{i} + x\hat{j}}{r^3}$

d) $f(x,y)(\hat{i}+\hat{j})$

4 

$\frac{k}{r^2}(-y\hat{i}+x\hat{j})$

(4B-1a) 

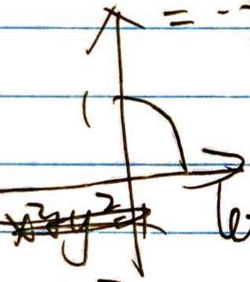
$\int_{C_1}\vec{F}\cdot d\vec{r}$

$= \int_{C_1}(x^2-y)\,dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3}$

$y=1-x^2 \quad dy=-2x\,dx$

$\int_{C_2}\vec{F}\cdot d\vec{r} = \int_{1}^{-1}(x^2-(1-x^2))\,dx - 4x^2\,dx = \int_{-1}^{1}(-2x^2-1)\,dx$

$= -\frac{2}{3}x^3 - x\Big|_{-1}^{1} = -\frac{10}{3}$

b) 

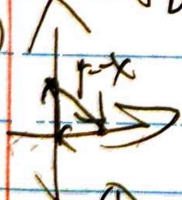
$\cancel{\int_C \vec{F}\cdot d\vec{r} = \int_{\frac{\pi}{2}}^{0}}$

$\cancel{x^2+y^2=1}$ Let $x=\cos t$, $y=\sin t$,

$dx=-\sin t\,dt$, $dy=\cos t\,dt$

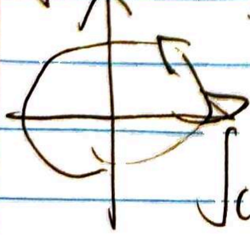
$\int_C \vec{F}\cdot d\vec{r} = \int_{\frac{\pi}{2}}^{0} xy\,dx - y^2\,dy = \int_{\frac{\pi}{2}}^{0} \cos t\sin t(-\sin t\,dt) - \cos^3 t\,dt$

$= \int_{\frac{\pi}{2}}^{0} \cos t\,dt = 1-0 = 1$

c) 

$\int_C \vec{F}\cdot d\vec{r} = \int_C y\,dx - x\,dy$

$= \int_0^1 0-0\,dy + \int_1^0 0-0 + \int_0^1 -x\,dx + x\,dx = 0$

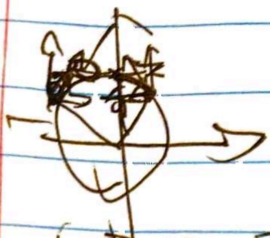
d) 

$x=2\cos t$, $y=\sin t$

$dx=-2\sin t\,dt$, $dy=\cos t\,dt$

$\int_C \vec{F}\cdot d\vec{r} = \int y\,dx = \int_0^{2\pi} \sin t\cdot(-2\sin t\,dt) = -2\int_0^{2\pi}\sin^2 t\,dt$

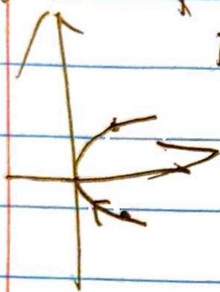
$= -2\int_0^{2\pi}\frac{1-\cos 2t}{2}\,dt = -2\left[t-\sin t\right]_0^{2\pi} = -4\pi$

2a)  $\int_C \vec{F}\cdot d\vec{r} = 0$

b) $\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi a} -a\,ds = -2a^2\pi$

Lec 22 30/3/2015

4(-1) a) $\vec{F} = \nabla f = 3x^2y\,\hat{i} + (x^3+3y^2)\,\hat{j}$

b) 

i) $dx = 2y\,dy$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{-1}^{1} 3y^5\,dx + (y^6+3y^2)\,dy$$

$$= \int_{-1}^{1} \cancel{6y^6\,dy} + \cancel{6y^6} + (7y^6+3y^2)\,dy$$

$$= y^7 + y^3 \Big|_{-1}^{1} = 2+2 = 4$$

ii) $\int_C \vec{F}\cdot d\vec{r} = \int_{(1,-1)}^{(1,1)} (x^3+3y^2)\,dy$

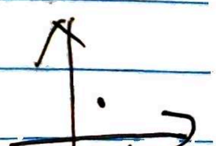
$$= \int_{-1}^{1} (1+3y^2)\,dy$$

$$= y + y^3 \Big|_{-1}^{1} = 4$$

iii) $\int_C \vec{F}\cdot d\vec{r} = f(1,1) - f(1,-1) = 2+2 = 4$

2a) $\vec{F} = \nabla f = (e^{xy} + xye^{xy})\,\hat{i} + x^2e^{xy}\,\hat{j}$

b) i) $\int_C \vec{F}\cdot d\vec{r} = \int_C e^{xy}(1+xy)\,dx + x^2e^{xy}\,dy$



$dy = -\frac{1}{x^2}\,dx$

$$= \int_1^0 2e\,dx + ex^2\cdot\left(-\frac{1}{x^2}\right)dx$$

$$= \int_1^0 e\,dx = ex\Big|_1^0 = -e$$

ii) $\int_C \vec{F}\cdot d\vec{r}$

$= f(0,\infty) - f(1,1) = 0 - e = -e$

3 a) $\vec{F} = \nabla f = \cos y \cos x\,\hat{i} - \sin x \sin y\,\hat{j}$

b) $\left(\int_C \vec{F}\cdot d\vec{r}\right)_{max} = f_{max}(P_1) - f_{min}(P_0)$

~~f_{max}~~ ~~$\frac{\partial f}{\partial x} = \cos y \cos x$~~, $\frac{\partial f}{\partial y} = -\sin x \sin y$

~~$\cos y \cos x = 0$~~

~~$-\sin x \sin y = 0$~~

$-1 \le f(x,y) \le 1$

$\left(\int_C \vec{F}\cdot d\vec{r}\right)_{max} = 1 - (-1) = 2$.

Lec 23 31/3/2025

4(-5a) $\frac{\partial F_x}{\partial y} = 2y$, $\frac{\partial F_y}{\partial x} = ay = 2y$, $a = 2$.

$f(x_1,y_1) = \int_0^{x_1} 2x\,dx + \int_0^{y_1} 2x_1 y\,dy + f(0,0)$

$= x_1^2 + x_1 y_1^2 + f(0,0)$

$f(x,y) = x^2 + xy^2 + c$

b) $\frac{\partial F_x}{\partial y} = (x+a)e^{x+y}$ $\quad \frac{\partial F_y}{\partial x} = e^{x+y} + xe^{x+y} = (x+a)e^{x+y}$

$a = 1$.

~~$f(x_1,y_1) = \int_0^{x_1} e^{x}(x+1)\,dx + \int_0^{y_1} x_1 e^{x+y}\,dy$~~

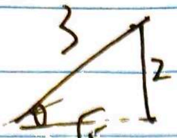
$\frac{\partial f}{\partial x} = e^{x+y}(x+1)$

$f = \int \frac{\partial f}{\partial x}\,dx = \int e^{x+y}(x+1)\,dx = xe^{x+y} + g(y)$

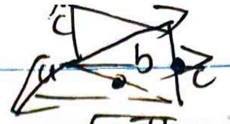
$\frac{\partial f}{\partial y} = xe^{x+y} + g'(y) = (x+1)e^{x+y}$

$g'(y) = e^{x+y}$ $\quad \therefore f(x,y) = xe^{x+y} + e^{x+y} + c$

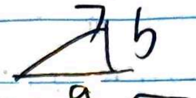
$g(y) = \int g'(y)\,dy = e^{x+y} + c$

6 $\csc\theta = \frac{3}{2}$ 

$\sin\theta = \frac{2}{3}$ $\quad \cos\theta = \frac{\sqrt{5}}{2}$



$\sqrt{a^2+b^2}$ $\quad a^2+b^2+c^2$



(a, b). $\sqrt{a^2+b^2}$

7 $(a+bi)^4 + 4(a+bi)^3 + 8(a+bi)^2 + 80(a+bi) + 400$

$= a^4 + 4a^3bi - 6a^2b^2 - 4ab^3i + b^4 + 4(a^3 + 3a^2bi - 3ab^2 - b^3i)$

$+ 8(a^2 + 2abi - b^2) + 80a + 80bi + 400$

$= a^4 + 4(1+bi)a^3 + (-6b^2 + 3bi + 8)a^2 - 4b^3i$

8 $\lim_{x\to 0}\left(\frac{\arctan 2x}{\tan 3x}\right) = \lim_{x\to 0}\left(\frac{\frac{2}{1+4x^2}}{3\sec^2 3x}\right) = \lim_{x\to 0}\left(\frac{2\cos^2 3x}{3+12x^2}\right) = \frac{2}{3}$

9(a) 60. $\quad 4\cdot 1 \quad 5$ $\quad 3\cdot1\cdot1 + 3\cdot 2 = 420.$

~~$1\times4\times5\times2$~~ $(\dots)\times5 + 1\times4\times5) =$ ~~$\dots = 70 \quad 3\cdot1\cdot1 + 3\cdot 2$~~

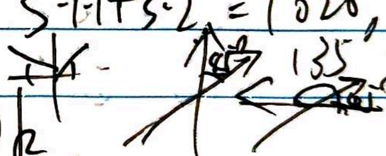
$6\times5\times4\times3 \dots + \dots \binom{6}{2} =$ $\quad A \in = 6\times5\times5\times3\times2$

$\binom{6}{2}$ $\quad = 6\times6\times5\times5$

$3\cdot1\cdot1 = 6\times5\times4 = 120$

$\binom{5}{2}\times$ ~~$\dots$~~ $+ \binom{4}{2}\times5\times4\times5$ $\quad 3\cdot 2 + 3\cdot1\cdot1 = 900$

$3\cdot1\cdot1 + 3\cdot 2 = 1020,$

11 $(2, 1, -1)k \quad k\in R$.  $135°$

$\vec{r} = (-1, 0, 3) + (2, 1, -1)k$.

$\begin{pmatrix} a \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 2a + 1 + 1 = 2a + 2 = \sqrt{2^2+1^2+1}\cdot\sqrt{a^2+1+1}\cdot \cos 45°$

$= \pm 2\cdot\sqrt{a^2+2}\cdot\frac{\sqrt{2}}{2}$

12 a) $f'(x) = \frac{1}{2}\frac{1}{\sqrt{1+x}}$ $\quad f''(x) = \frac{1}{2}(1+x)^{-\frac{3}{2}}\cdot -\frac{1}{2} = -\frac{1}{4}(1+x)^{-\frac{3}{2}}$.

b) $k \in [2, n)$, if $f^{(k)}(x) = \left(-\frac{1}{4}\right)^{k-1}\frac{(2k-3)!}{(k-2)!}\left(\frac{1}{\sqrt[4]{(1+x)^3}}\right)(1+x)^{\frac{1}{2}-k}$.

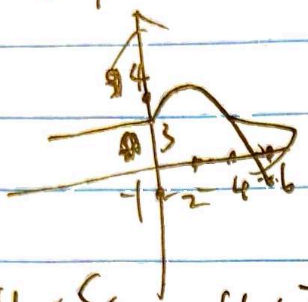
$f^{(k+1)}(x) = \left(-\frac{1}{4}\right)^{k-1}\frac{(2k-3)!}{(k-2)!}\,(1+x)^{\frac{1}{2}-k-1}\left(\frac{1}{2}-k\right)$ $\quad \frac{1}{2}-k$

$= \left(-\frac{1}{4}\right)^{k}\frac{(2k-3)!}{(k-2)!}\cdot\frac{(2k-1)(2k-2)}{(k-1)}(1+x)^{-k-\frac{1}{2}} = \left(-\frac{1}{4}\right)^{k}\frac{(2k-1)!}{(k-2)!}(1+\dots$

1 a) i) $f(2)=6$

ii) $f(f(2)) = f(6) = -2$

b)



$\frac{5}{6}$

2 $u_8 = S_8$ $u_1+7d = \cancel{\frac{u_1+u_8}{2}}$ $8u_1 + \frac{8\times 7}{2}d = 8u_1+28d = 8$

$u_1 = -3d$ $4d=8$

$d=2$. $u_1=-6$.

$S = 1+x+x^2+x^3+x^4+x^5+\dots$

3 a) 40 $xS = x+x^2+\dots$ $S(1-x)=1$

$S-xS=1$ $S=\frac{1}{1-x}$

b) $40-20=20$.

4 a) $f(x) = -(x^2-2hx+h^2)+2k = -x^2+2hx-h^2+2k$.

$f'(x) = -2x+2h$.

b) $f(3) = -(3-h)^2+2k$

$g(3) = e+k = -(3-h)^2+2k$

$g'(x) = e^{x-2}$ $g'(3) = e = f'(3) = -6+2h$

$h = \frac{e+6}{2} = \frac{e}{2}+3$

c) $f(3) = -\frac{e^2}{4}+2k = e+k$

$k = e+\frac{e^2}{4}$

5 a) $\sin 2x + \cos 2x - 1$

$= 2\sin x\cos x + 2\sin^2 x$

$= 2\sin x(\cos x + \sin x)$.

b) $\sin 2x + \cos 2x - 1 + \cos x - \sin x$

$= (\cos x - \sin x)(2\sin x + 1) = 0$

$\tan x = 1$ or $\sin x = \frac{1}{2}$. $x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{\pi}{6}, \frac{11\pi}{6}$

6 a) $\frac{\partial y}{\partial y} = 1 \quad \frac{\partial (-x)}{\partial x} = -1 \neq \frac{\partial y}{\partial y}$ ∴ not exact.

b) $\frac{\partial y(2x+y)}{\partial y} = (2x+y) + y = 2x+2y$

$\frac{\partial}{\partial x} x(2y+x) = (2y+x) + x = 2x+2y = \frac{\partial}{\partial y} y(2x+y)$ ∴ exact.

$\frac{\partial f}{\partial x} = y(2x+y)$; $\frac{\partial f}{\partial y} = x(2y+x)$

Med. 1
$f(x_1, y_1) = \int_C \nabla f \, d\vec{r} + f(0,0) = \int_0^{x_1} 0 \, dx + \int_0^{y_1} x_1(2y+x_1) \, dy + f(0,0)$

$= x_1 y^2 + x_1^2 y \Big|_0^{y_1} + f(0,0) = x_1 y_1^2 + x_1^2 y_1 + f(0,0)$

$f(x,y) = xy^2 + x^2 y + C$

Med. 2 $f = \int \frac{\partial f}{\partial x} dx = \int (2yx + y^2) dx = yx^2 + y^2 x + g(y)$

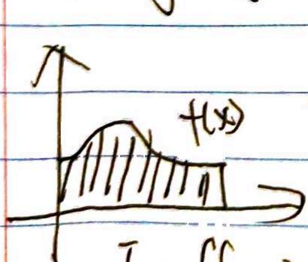
$\frac{\partial f}{\partial y} = 2xy + x^2 + g'(y) = 2xy + x^2$

$g'(y) = 0$

$\int g'(y) \, dy = 0 + C = C$

$f(x,y) = yx^2 + y^2 x + C$

P1



$\int_C \vec{F} \cdot d\vec{r} = \int_C x^2 y + \frac{1}{3} y^3 \, dx$

$= \int_{x_1}^{x_2} x^2 f(x) + \frac{1}{3} f^3(x) \, dx$

$I = \iint_R x^2 + y^2 \, dA = \int_{x_1}^{x_2} \int_0^{f(x)} x^2 + y^2 \, dy \, dx$

$= \int_{x_1}^{x_2} \left[x^2 y + \frac{1}{3} y^3 \right]_0^{f(x)} dx = \int_{x_1}^{x_2} x^2 f(x) + \frac{1}{3} f^3(x) \, dx = \int_C \vec{F} \cdot d\vec{r}$

P2 a) $\nabla\theta(x,y) = \frac{\partial}{\partial x}\tan^{-1}\left(\frac{y}{x}\right)\hat{i} + \frac{\partial}{\partial y}\tan^{-1}\left(\frac{y}{x}\right)\hat{j}$

$= \frac{1}{1+\frac{y^2}{x^2}}\cdot\left(\frac{y}{x^2}\right)\hat{i} + \frac{1}{1+\frac{y^2}{x^2}}\cdot\left(\frac{1}{x}\right)\hat{j}$

$= \frac{-y\hat{i} + x\hat{j}}{x^2+y^2} = \vec{F}(x,y)$

b) $\cancel{\vec{F}(x,y) = \frac{-r\sin\theta\hat{i} + r\cos\theta\hat{j}}{r^2}} = \cancel{\frac{-\sin\theta\hat{i} + \cos\theta\hat{j}}{r}}$

$\int_C \vec{F}\cdot d\vec{r} = f(x_2,y_2) - f(x_1,y_1)$

$= \tan^{-1}\left(\frac{y_2}{x_2}\right) - \tan^{-1}\left(\frac{y_1}{x_1}\right)$

$= \tan^{-1}(\tan\theta_2) - \tan^{-1}(\tan\theta_1)$

$= \theta_2 - \theta_1$

c) $\int_{C_1}\vec{F}\cdot d\vec{r} = \int_{C_2}\vec{F}\cdot d\vec{r} = \theta_2 - \theta_1 = \pi$

d) $\nabla\times\vec{F} = \frac{\partial}{\partial x}\frac{x}{x^2+y^2} - \frac{\partial}{\partial y}\frac{-y}{x^2+y^2}$

$= \frac{x^2+y^2 - x\cdot 2x}{(x^2+y^2)^2} + \frac{x^2+y^2 - 2y^2}{x^2+y^2}$

$= 0$

e) Yes

P3 a) $\nabla\times\vec{F} = \frac{\partial}{\partial x}r^n y - \frac{\partial}{\partial y}r^n x$

$= ynr^{n-1}\frac{\partial r}{\partial x} - xnr^{n-1}\frac{\partial r}{\partial y}$

$\cancel{= 2xynr^{n-1} - 2xynr^{n-1} = 0}$

$= \frac{1}{2\sqrt{x^2+y^2}}\cdot 2xynr^{n-1} - \frac{1}{2\sqrt{x^2+y^2}}2xynr^{n-1} = 0.$

c) i) $n = 0$

$\vec{F} = x\hat{i} + y\hat{j}$

$\frac{\partial g}{\partial x} = x \qquad g = \int \frac{\partial g}{\partial x}\,dx = \frac{1}{2}x^2 + \cancel{g(y)}\; c(y)$

$\frac{\partial g}{\partial y} = c'(y) = y \qquad c(y) = \int c'(y)\,dy = \frac{1}{2}y^2 + C$

$\therefore g(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + C$

$\int uv' = uv - \int u'v$

ii) $n \in \mathbb{R},\ n \neq -2$

$\frac{\partial g}{\partial x} = r^n x \qquad g(x,y) = \int \frac{\partial g}{\partial x}\,dx = \int r^n x\,dx \cancel{= r^n x - \int} \int \sqrt{x^2+y^2}^{\,n}\,dx$

$r = \sqrt{x^2+y^2}$

$dr = \frac{1}{2}\frac{1}{\sqrt{x^2+y^2}}\,2x\,dx \qquad \cancel{= r^n x - \frac{2}{n}(x^2+y^2)\int r^{n+1}} = \int r^{n+1}\,dr$

$= \frac{1}{n+2} r^{n+2} + c(y)$

$= \frac{x}{r}\,dx \qquad \cancel{\frac{1}{n+2} r^{n+2} + c(y) = r^n}$

$r\,dr = x\,dx \qquad \frac{\partial g}{\partial y} = r^{n+1}\cdot\frac{y}{r} + c'(y) = r^n y$

$c'(y) = 0$

$c(y) = \int c'(y)\,dy = C$

$g(x,y) = \frac{1}{n+2} r^{n+2} + C$

iii) $n = -2$

$\frac{\partial g}{\partial x} = \frac{1}{r^2}x \qquad g(x,y) = \int \frac{\partial g}{\partial x}\,dx = \int \frac{1}{r^2}x\,dx = \int \frac{1}{r}\,dr = \ln r + c(y)$

$\frac{\partial g}{\partial y} = \frac{1}{r}\cdot\frac{y}{r} + c'(y) = r^{-2}y$

$c'(y) = 0$

$c(y) = \int c'(y)\,dy = C$

$g(x,y) = \ln r + C$