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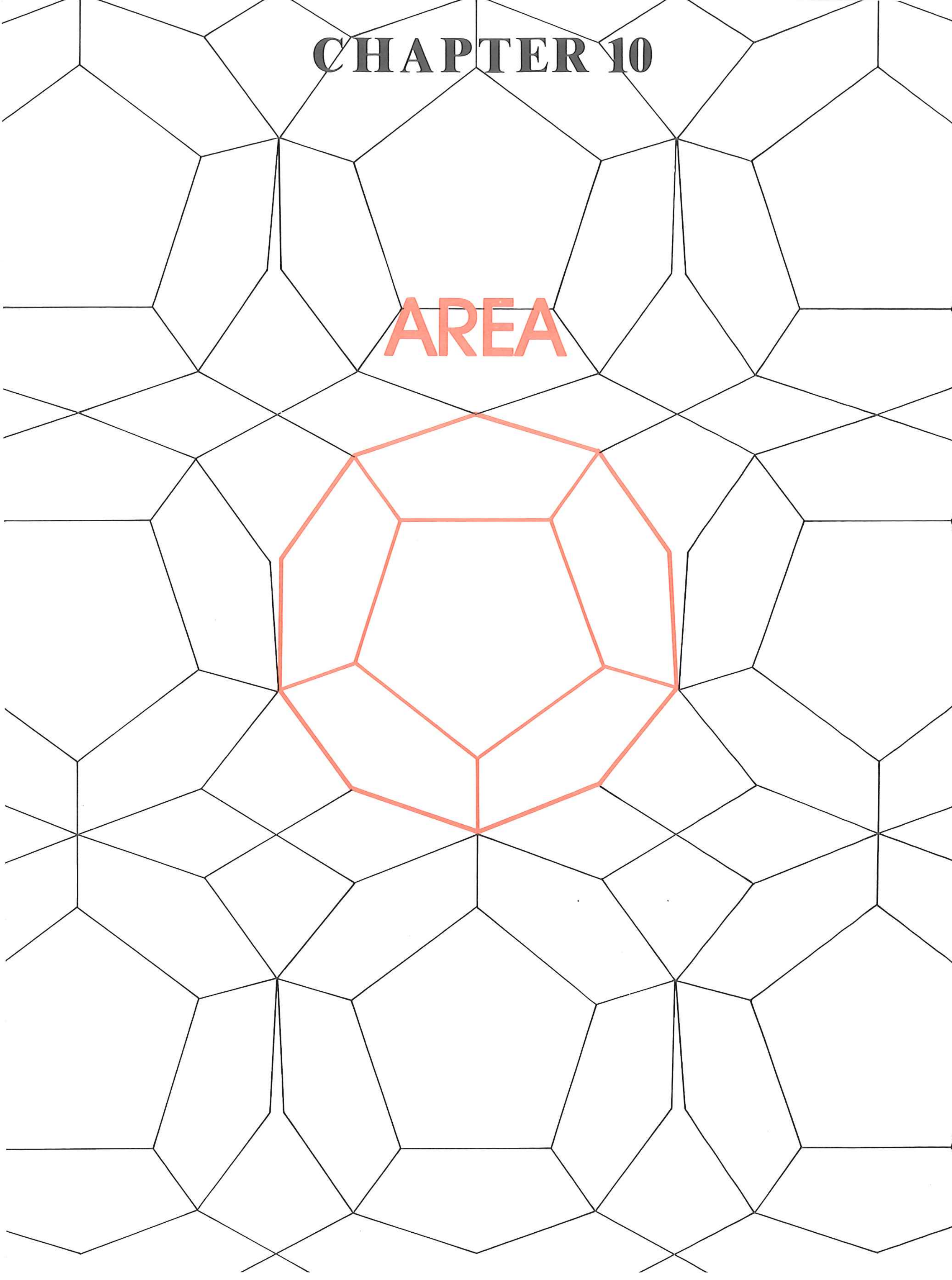
CHAPTER 10

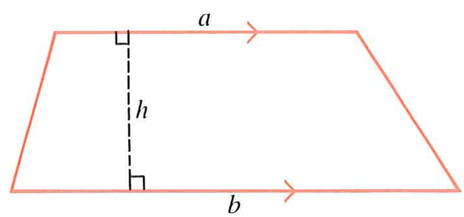
AREA



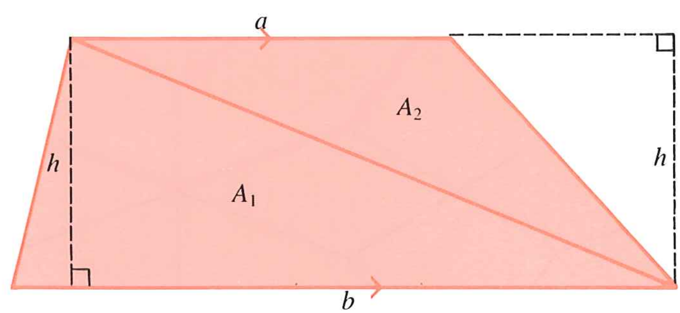
REVIEW AND PREVIEW TO CHAPTER 10

Area of a Trapezoid

A **trapezoid** is a quadrilateral with one pair of opposite sides parallel. The trapezoid in the diagram has parallel sides of lengths a and b. The perpendicular distance between these parallel sides is the height, h, of the trapezoid.



We calculate the area of the trapezoid by drawing a diagonal and summing the areas of the two triangles. Let the area of the trapezoid be A.

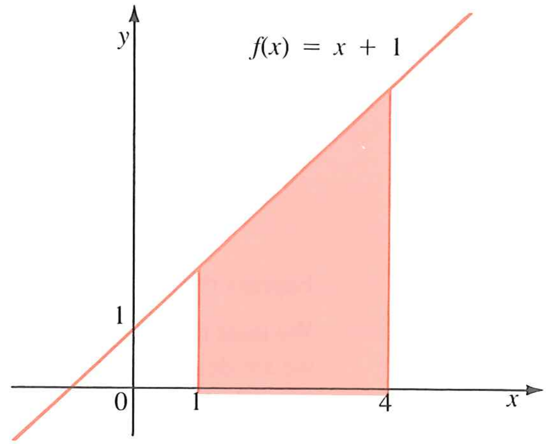


$$\begin{aligned} A &= A_1 + A_2 \\ &= \tfrac{1}{2}bh + \tfrac{1}{2}ah \\ &= \tfrac{1}{2}h(a + b) \end{aligned}$$

Area of a Trapezoid

$$A = \tfrac{1}{2}h(a + b)$$

Example 1 Find the area of the shaded region in the diagram.



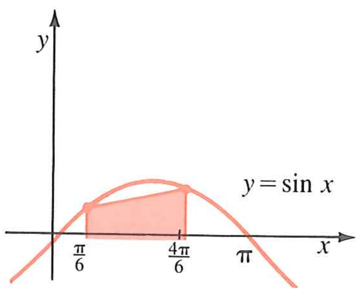
Solution The shaded region is a trapezoid with parallel sides of length $f(1) = 2$ and $f(4) = 5$. The distance between the parallel sides is $h = 4 - 1 = 3$. Therefore

$$
\begin{aligned}
A &= \tfrac{1}{2}(3)(2 + 5) \\
&= 10.5
\end{aligned}
$$

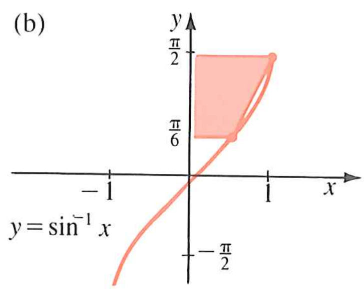
EXERCISE 1

1. Calculate the area of the shaded region.

(a)



(b)



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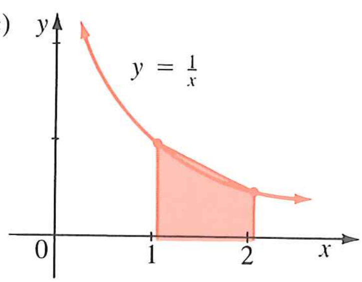


Sigma Notation

A series is the sum of a sequence. We can write a series using sigma notation.

$$\sum_{i=1}^{n} t_i = t_1 + t_2 + t_3 + \ldots + t_n$$

Example 1 Express the series $1 + 3 + 5 + 7 + 9$ in sigma notation.

Solution We need the general term to express the series in sigma notation. Since we are dealing with consecutive odd numbers, each term is of the form $2i - 1$, $i \in N$.

Therefore $\quad 1 + 3 + 5 + 7 + 9 = \sum_{i=1}^{5}(2i - 1)$

The properties of Sigma Notation that we use in this section are summarized in the chart.

Basic Properties of Sigma Notation

(1) $\sum_{i=1}^{n} c = c + c + c + \ldots + c = nc$, c is a constant.

(2) $\sum_{i=1}^{n} ct_i = c\sum_{i=1}^{n} t_i$, c is a constant.

(3) $\sum_{i=1}^{n} (t_i + s_i) = \sum_{i=1}^{n} t_i + \sum_{i=1}^{n} s_i$

Example 1 Use the basic properties of sigma notation to express $\sum_{i=1}^{n} (3i - 2)^2$ in terms of monomial summations.

Solution

$$\begin{aligned} \sum_{i=1}^{n} (3i - 2)^2 &= \sum_{i=1}^{n} (9i^2 - 12i + 4) \\ &= \sum_{i=1}^{n} 9i^2 + \sum_{i=1}^{n} (-12i) + \sum_{i=1}^{n} 4 \quad \text{(by Property 3)} \\ &= 9\sum_{i=1}^{n} i^2 - 12\sum_{i=1}^{n} i + 4n \quad \text{(by Properties 2 and 1)} \end{aligned}$$

EXERCISE 2

1. Write each series in expanded form.

(a) $\sum_{i=1}^{5}(i^2 + 1)$ (b) $\sum_{i=1}^{4}\frac{i}{4}f(i)$ (c) $\sum_{i=1}^{n}\frac{3}{n}f\left(1 + \frac{3}{4}i\right)$

2. Write each series in sigma notation.

(a) $1 + 4 + 7 + 10 + 13 + 16$

(b) $1 - 1 + 1 - 1 + 1 - 1 + 1$

(c) $x + x^2 + x^3 + \ldots + x^n$

(d) $\frac{1}{6}f\left(\frac{1}{6}\right) + \frac{2}{6}f\left(\frac{2}{6}\right) + \frac{3}{6}f\left(\frac{3}{6}\right) + \frac{4}{6}f\left(\frac{4}{6}\right) + \frac{5}{6}f\left(\frac{5}{6}\right) + f(1)$

(e) $\frac{1}{n}f\left(\frac{2(1) - 2}{n}\right) + \frac{2}{n}f\left(\frac{2(2) - 2}{n}\right) + \frac{3}{n}f\left(\frac{2(3) - 2}{n}\right) + \ldots$
$+ \frac{n}{n}f\left(\frac{2(n) - 2}{n}\right)$

3. Express in terms of monomial summations.

(a) $\sum_{i=1}^{n}(2 + i)^2$ (b) $\sum_{i=1}^{20}(3i^2 - 12i)$

(c) $\sum_{i=1}^{n}(2i^3 - 3i^2 + 5i - 12)$

Sum of a Series

In this chapter we need the sums of the special series that are listed below. Refer to Chapter 11 in Algebra and Geometry or Chapter 4 in Finite Mathematics for more detail.

1. Sum of an arithmetic series:

$$a + (a + d) + (a + 2d) + \ldots + [a + (n - 1)d]$$
$$= \frac{n}{2}(2a + (n - 1)d)$$

2. Sum of a geometric series:

$$a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$$

3. Sum of the natural numbers:

$$\sum_{i=1}^{n} i = \frac{n(n + 1)}{2}$$

4. Sum of the squares of the natural numbers:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

5. Sum of the cubes of the natural numbers:

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Example 1 Evaluate $\sum_{i=1}^{n} (3i^2 - 2i)$.

Solution

$$\begin{aligned}\sum_{i=1}^{n} (3i^2 - 2i) &= 3\sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i \\ &= 3\,\frac{n(n+1)(2n+1)}{6} - 2\,\frac{n(n+1)}{2} \\ &= \frac{3n(n+1)(2n+1) - 6n(n+1)}{6} \\ &= \frac{3n(n+1)[(2n+1) - 2]}{6} \\ &= \frac{n(n+1)(2n-1)}{2}\end{aligned}$$

EXERCISE 3

1. Evaluate.

(a) $3 + 7 + 11 + \ldots + (4n - 1)$

(b) $1 + 3 + 9 + 27 + \ldots + 3^{n-1}$

(c) $\sum_{i=1}^{n} (3i^2 - i)$

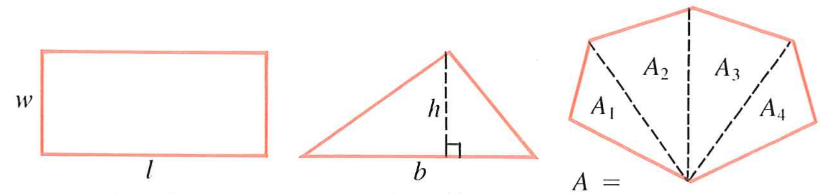
(d) $\sum_{i=1}^{n} (2i^3 + 3i - 2)$

(e) $\sum_{i=1}^{20} (i + 3)$

(f) $\sum_{i=41}^{100} (i^3 - 2i)$

INTRODUCTION

Finite regions with straight sides pose few problems when we have to calculate their areas.

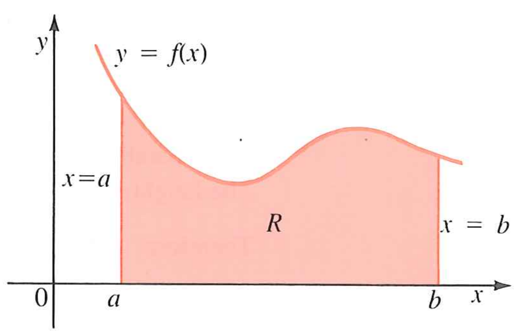


$A = lw$ $\qquad$ $A = \frac{1}{2}bh$ $\qquad$ $A = A_1 + A_2 + A_3 + A_4$

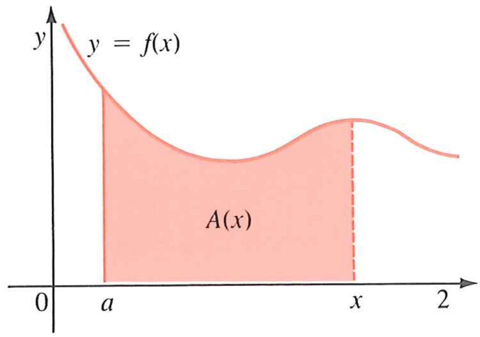
In this chapter we tackle the problem of calculating the area of a region with curved sides. It is interesting to note that, historically, the area problem preceded the tangent problem that we have already discussed in Section 1.1, and that there is a definite connection between the two concepts. Newton's teacher at Cambridge, Isaac Barrow (1630–1677), discovered that these two problems are closely related. Newton and Leibniz exploited the relationship and used it to develop calculus into a systematic mathematical method.

10.1 AREA UNDER A CURVE

If $y = f(x)$ is a positive function, the **area of the region under $y = f(x)$ from a to b** is the area of the region below $y = f(x)$ and above the x-axis ($y = 0$), to the right of the vertical line $x = a$ and left of $x = b$. Region R in the following diagram fulfills these requirements. The **area problem** involves finding the area of such a region.



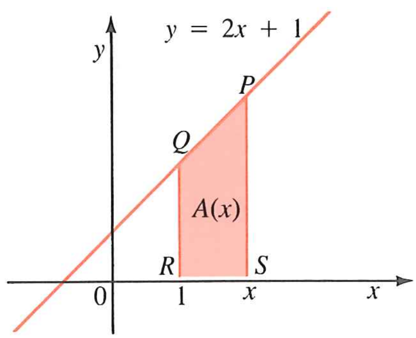
To find the area of such a region we need to create an area function. If a is a fixed value, then the distance x that we move to the right of a determines the area of the region. Thus the area is a function of x.



$A(x)$ is the area of the region under $y = f(x)$ from a to x. It is important to note that $A(a) = 0$.

Example 1 Sketch the region under $y = 2x + 1$ from 1 to x and
(a) find the area function,
(b) compare the derivative of the area function with the equation of the straight line.

Solution



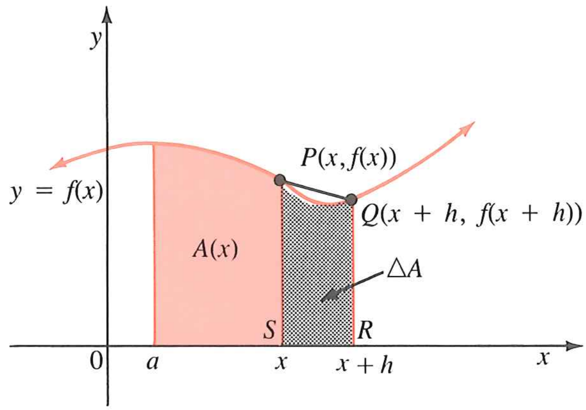
(a) The required area $A(x)$ is the area of trapezoid $PQRS$. The lengths of the parallel sides are $RQ = f(1) = 3$ and $SP = f(x) = 2x + 1$. The height is $RS = x - 1$.

Therefore
$$\begin{aligned} A(x) &= \tfrac{1}{2}(x - 1)(3 + 2x + 1) \\ &= \tfrac{1}{2}(x - 1)(2x + 4) \\ &= \tfrac{1}{2}(2x^2 + 2x + 4) \\ &= x^2 + x + 2 \end{aligned}$$

(b)
$$\begin{aligned} A'(x) &= 2x + 1 \\ &= f(x) \end{aligned}$$

In Example 1 the derivative of the area function turned out to be the function that defined the straight line. The area problem can be solved if we can establish that the relationship $A'(x) = f(x)$ is generally true.

Let $A(x)$ be the area under the function $y = f(x)$ from a to x, where $y = f(x)$ is continuous and positive.



The area is uniquely determined by the position of point $P(x, f(x))$ on the curve. Select a point $Q(x+h,\ f(x+h))$ very close to P. The area of the curved region $PQRS$ is the change in area (ΔA) as x changes to $x + h$.

$$\begin{aligned} A'(x) &= \lim_{h\to 0} \frac{\Delta A}{h} \\ &= \lim_{h\to 0} \frac{A(x + h) - A(x)}{h} \end{aligned}$$

Now $A(x + h) - A(x)$, the area of the curved region $PQRS$, can be approximated by the area of trapezoid $PQRS$. The smaller h gets, the better the approximation becomes, with equality in the limit.

Thus $\quad A(x + h) - A(x) \doteq \frac{1}{2}h[f(x) + f(x + h)]$

and

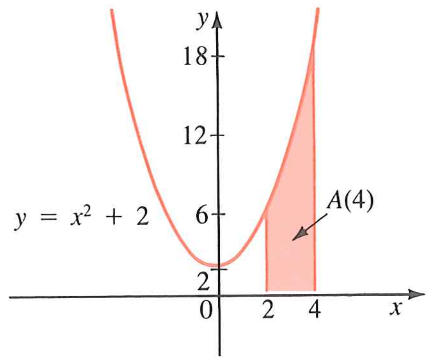
$$\begin{aligned} A'(x) &= \lim_{h\to 0} \frac{\frac{1}{2}h[f(x) + f(x + h)]}{h} \\ &= \lim_{h\to 0} \frac{f(x) + f(x + h)}{2} \\ &= \frac{f(x) + f(x)}{2} \end{aligned}$$

$$\boxed{A'(x) = f(x)}$$

In the next example we find the area of a region by solving the differential equation $A'(x) = f(x)$ subject to a particular initial condition.

Example 2 Find the area under $y = x^2 + 2$ from $x = 2$ to $x = 4$.

Solution The required area $A(4)$ is shown in the diagram.



Since $\quad A'(x) = x^2 + 2$

we have $\quad A(x) = \frac{1}{3}x^3 + 2x + C$

The initial condition is $\quad A(2) = 0$

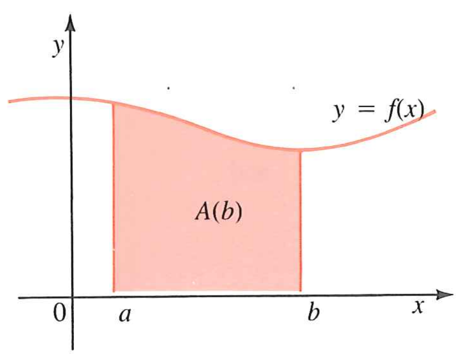
Therefore $\quad 0 = \frac{8}{3} + 4 + C$

$$C = -\frac{20}{3}$$

and $\quad A(x) = \frac{1}{3}x^3 + 2x - \frac{20}{3}$

The required area $A(4) = \frac{64}{3} + 8 - \frac{20}{3} = \frac{68}{3}$.

Now we are ready to tackle the area problem. We want to find the area under $y = f(x)$ from a to b.



The required area $A(b)$ is shown in the above diagram.

Now $\quad A'(x) = f(x)$

Therefore $\qquad A(x) = F(x) + C$, where $F(x)$ is any antiderivative of $f(x)$

The initial condition

$$A(a) = 0$$

gives us $\qquad 0 = F(a) + C$

and $\qquad C = -F(a)$

Therefore $\qquad A(x) = F(x) - F(a)$

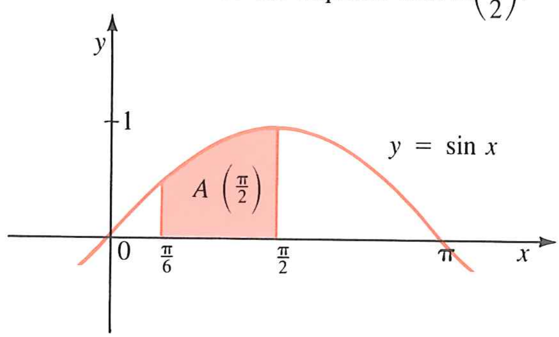
and $\qquad A(b) = F(b) - F(a)$

> If F is any antiderivative of the positive function f, the area under $y = f(x)$ from a to b is $A(b) = F(b) - F(a)$.

In practice, we choose the antiderivative with constant 0. If $f(x) = e^{2x}$ we would select $F(x) = \frac{1}{2}e^{2x}$ not $F(x) = \frac{1}{2}e^{2x} + C$.

Example 3 Find the area between $y = \sin x$ and the x-axis from $x = \dfrac{\pi}{6}$ to $x = \dfrac{\pi}{2}$.

Solution We sketch the curve and locate the required area $A\left(\dfrac{\pi}{2}\right)$.



We choose the antiderivative

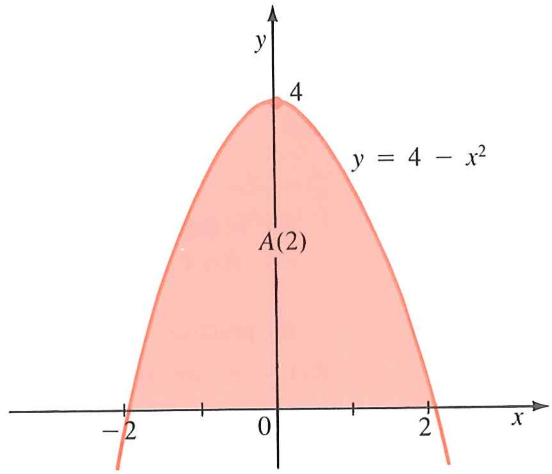
$$F(x) = -\cos x$$

The required area is

$$\begin{aligned} A\left(\frac{\pi}{2}\right) &= F\left(\frac{\pi}{2}\right) - F\left(\frac{\pi}{6}\right) \\ &= -\cos\frac{\pi}{2} - \left(-\cos\frac{\pi}{6}\right) \\ &= 0 + \frac{\sqrt{3}}{2} \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

Example 4 Find the area of the region below $y = 4 - x^2$ and above the x-axis.

Solution First we set $4 - x^2 = 0$ and solve for the x-intercepts ± 2. This helps us sketch the curve and locate the required area $A(2)$.



We choose the antiderivative

$$F(x) = 4x - \frac{x^3}{3}$$

The required area is

$$\begin{aligned} A(2) &= F(2) - F(-2) \\ &= 8 - \tfrac{8}{3} - \left(-8 + \tfrac{8}{3}\right) \\ &= 8 - \tfrac{8}{3} + 8 - \tfrac{8}{3} \\ &= \tfrac{32}{3} \end{aligned}$$

EXERCISE 10.1

B **1.** Find the area under the given curve from a to b.

(a) $y = x^2 + 1$ from 0 to 4

(b) $y = -x^2 + 1$ from $-\frac{1}{2}$ to $\frac{1}{4}$

(c) $y = x^2 - 1$ from -4 to -2

(d) $y = \dfrac{1}{x}$ from e to e^2

(e) $y = 2 \cos x$ from $-\dfrac{\pi}{2}$ to 0

(f) $y = \sqrt{x}$ from 0 to 4

(g) $y = -\sin x$ from $-\pi$ to 0

(h) $y = \sec^2 x$ from $-\dfrac{\pi}{4}$ to $\dfrac{\pi}{3}$

(i) $y = e^{-x}$ from -2 to 4

(j) $y = x^3$ from 1 to 3

(k) $y = x^2 - x + 2$ from -2 to 1

(l) $y = 2e^{-2x}$ from 0 to 1

(m) $y = \sin\left(\dfrac{x}{2}\right)$ from 0 to $\dfrac{3\pi}{4}$

(n) $y = 3\cos(2x)$ from $-\dfrac{\pi}{4}$ to $\dfrac{\pi}{8}$

(o) $y = \dfrac{1}{(x+1)^2}$ from 0 to 10

(p) $y = x(x^2 + 1)^4$ from 1 to 2

2. Find the area below the given curve and above the x-axis.

(a) $y = 4x - x^2$
(b) $y = 9 - x^2$
(c) $y = x^2 - x^3$ from -2 to 1
(d) $y = x^2 - x^4$
(e) $y = -\cos x$ from $-\pi$ to π
(f) $y = 10 - 11x - 6x^2$
(g) $y = x^3 - 3x^2 - 9x + 27$
(h) $y = 4 + 3x - x^2$

3. Calculate the area between $y = \dfrac{1}{x}$ and the x-axis from $x = 1$ to the given line.

(a) $x = 2$ (b) $x = 3$ (c) $x = e$ (d) $x = n$

4. Calculate the area between $y = \dfrac{1}{x}$ and the x-axis from the given line to $x = 1$.

(a) $x = \frac{1}{2}$ (b) $x = \frac{1}{3}$ (c) $x = \frac{1}{e}$ (d) $x = \frac{1}{10}$

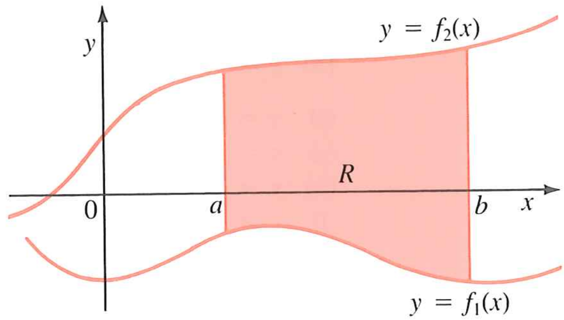
C **5.** Find the area between $y = x^3 - 1$ and the x-axis from $x = -1$ to $x = 1$.

6. Find the area between $y = x^2 - 4$ and the x-axis from -1 to 3.

7. Find the area between $y = \sin x \cos x$ and the x-axis from $-\dfrac{\pi}{2}$ to $\dfrac{3\pi}{4}$.

10.2 AREA BETWEEN CURVES

In this section we examine a more general area problem, that of finding the area between two curves in a particular interval. Region R in the diagram is between $y = f_1(x)$ and $y = f_2(x)$ from a to x. If a is fixed, the area of region R is a function of x and is denoted by $A(x)$.



In the interval $[a, x]$, $f_2(x) > f_1(x)$ and an argument similar to that in Section 10.1 establishes the result

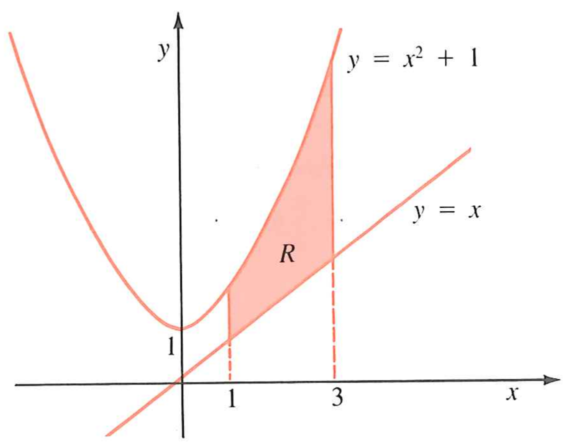
$$A'(x) = f_2(x) - f_1(x), \quad f_2(x) - f_1(x) \geqslant 0$$

or $$A'(x) = f(x), \text{ where } f(x) = f_2(x) - f_1(x)$$

Thus, the problem of finding the area between two curves has been reduced to our original problem of finding the area under a curve. Generally, if f and g are continuous functions in $[a, b]$ and $f \geqslant g$ in $[a, b]$, the area between f and g from a to b is the area under $f - g$ from a to b.

Example 1 Find the area between $y = x^2 + 1$ and $y = x$ from $x = 1$ to $x = 3$.

Solution 1 We sketch both curves, identify the required region R, and note that $y = x^2 + 1$ is always above $y = x$.



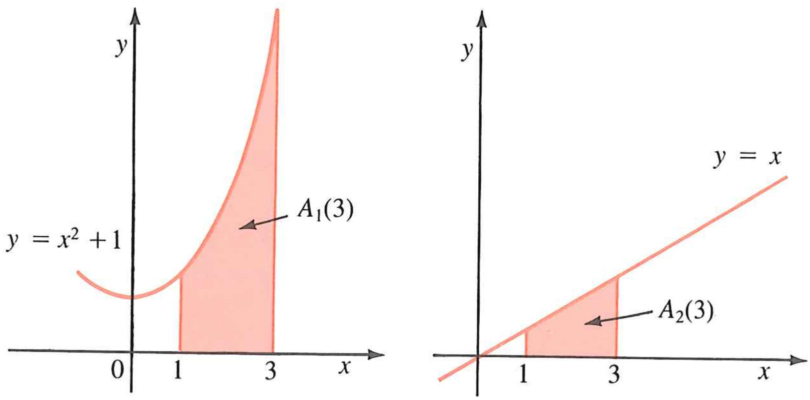
Since $$A'(x) = x^2 + 1 - x$$

we choose the antiderivative $$F(x) = \frac{x^3}{3} + x - \frac{x^2}{2}$$

The area of R is

$$\begin{aligned} A(3) &= F(3) - F(1) \\ &= 9 + 3 - \tfrac{9}{2} - \left(\tfrac{1}{3} + 1 - \tfrac{1}{2}\right) \\ &= \tfrac{20}{3} \end{aligned}$$

Solution 2 $A(3)$ can be calculated using the areas $A_1(3)$ and $A_2(3)$ in the diagrams.



Now $\quad A_1{}'(x) = x^2 + 1 \qquad\qquad A_2{}'(x) = x$

We choose the antiderivatives

$$F_1(x) = \tfrac{1}{3}x^3 + x \qquad\qquad F_2(x) = \tfrac{1}{2}x^2$$

and

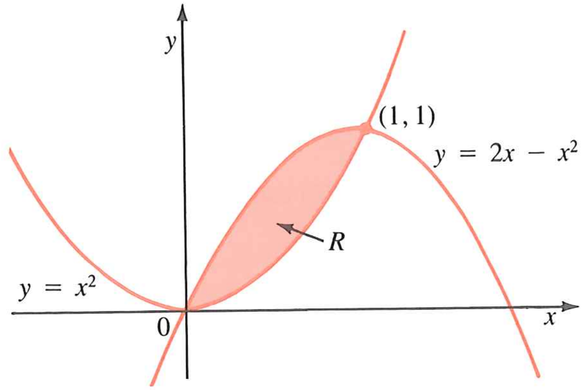
$$\begin{aligned} A_1(3) &= F_1(3) - F_1(1) \\ &= 9 + 3 - \tfrac{1}{3} - 1 \\ &= \tfrac{32}{3} \end{aligned} \qquad\qquad \begin{aligned} A_2(3) &= F_2(3) - F_2(1) \\ &= \tfrac{9}{2} - \tfrac{1}{2} \\ &= 4 \end{aligned}$$

Therefore,

$$\begin{aligned} A(3) &= A_1(3) - A_2(3) \\ &= \tfrac{32}{3} - 4 \\ &= \tfrac{20}{3} \end{aligned}$$

Example 2 Find the area of the region bounded by the parabolas $y = x^2$ and $y = 2x - x^2$.

Solution First we find the points of intersection of the two curves to determine the required interval. They also help us to sketch the curves and identify the region R whose area we wish to calculate.



To find the points of intersection set

$$x^2 = 2x - x^2$$

Therefore, $$2x^2 - 2x = 0$$

$$2x(x - 1) = 0$$

and $$x = 0 \quad \text{or} \quad x = 1$$

The points of intersection are $(0, 0)$ and $(1, 1)$.

Since $y = 2x - x^2$ is above $y = x^2$, for $0 < x < 1$,

$$\begin{aligned} A'(x) &= 2x - x^2 - x^2 \\ &= 2x - 2x^2 \end{aligned}$$

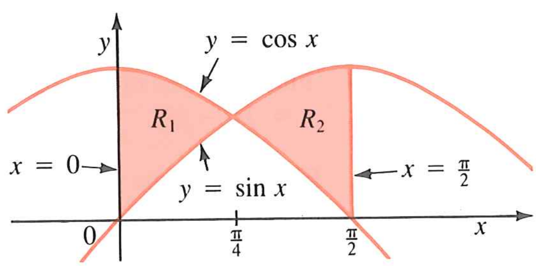
We choose the antiderivative

$$F(x) = x^2 - \tfrac{2}{3}x^3$$

The area of R is $$\begin{aligned} A(1) &= F(1) - F(0) \\ &= 1 - \tfrac{2}{3} - 0 \\ &= \tfrac{1}{3} \end{aligned}$$

Example 3 Find the area of the region between the curves $y = \sin x$ and $y = \cos x$ from 0 to $\dfrac{\pi}{2}$.

Solution Two separate regions, R_1 and R_2, fall within the required boundaries. Their areas must be evaluated separately and the results added.



To find the points of intersection set

$$\sin x = \cos x$$

Therefore $\quad \tan x = 1,\ 0 \leqslant x \leqslant \dfrac{\pi}{2}$

and $\quad x = \dfrac{\pi}{4}$

The point of intersection is $\left(\dfrac{\pi}{4}, \dfrac{1}{\sqrt{2}}\right)$.

Since region R_1 has $y = \cos x$ above $y = \sin x$,

$$A_1'(x) = \cos x - \sin x$$

We choose the antiderivative

$$F_1(x) = \sin x + \cos x$$

Therefore the area of region R_1 is

$$\begin{aligned}
A_1\left(\frac{\pi}{4}\right) &= F_1\left(\frac{\pi}{4}\right) - F_1(0) \\
&= \sin\frac{\pi}{4} + \cos\frac{\pi}{4} - \sin 0 - \cos 0 \\
&= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1 \\
&= \frac{2}{\sqrt{2}} - 1 \\
&= \sqrt{2} - 1
\end{aligned}$$

Since region R_2 has $y = \sin x$ above $y = \cos x$,

$$A_2'(x) = \sin x - \cos x$$

We choose the antiderivative

$$F_2(x) = -\cos x - \sin x$$

Therefore, the area of region R_2 is

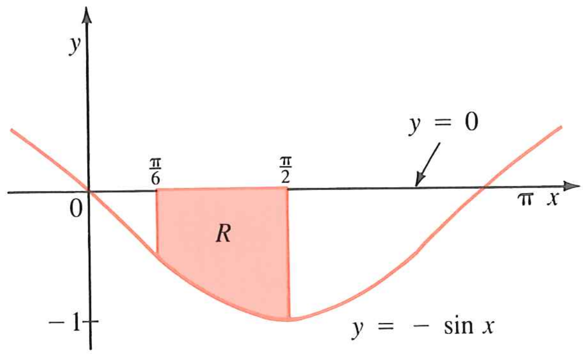
$$\begin{aligned}
A_2\left(\frac{\pi}{2}\right) &= F_2\left(\frac{\pi}{2}\right) - F_2\left(\frac{\pi}{4}\right) \\
&= -\cos\frac{\pi}{2} - \sin\frac{\pi}{2} + \cos\frac{\pi}{4} + \sin\frac{\pi}{4} \\
&= 0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \\
&= \frac{2}{\sqrt{2}} - 1 \\
&= \sqrt{2} - 1
\end{aligned}$$

The required area is $\sqrt{2} - 1 + \sqrt{2} - 1 = 2\sqrt{2} - 2$

If at the outset we had noticed that R_2 is the reflection of R_1 in the line $x = \frac{\pi}{4}$, then the required area is $2A_1(\frac{\pi}{4})$ and our calculations would have been considerably reduced.

Example 4 Find the area between $y = -\sin x$ and the x-axis from $\dfrac{\pi}{6}$ to $\dfrac{\pi}{2}$.

Solution First we sketch the curve and identify the required region R. We treat this as the area between two curves with one of the curves being the x-axis ($y = 0$).



Since region R has $y = 0$ above $y = -\sin x$,

$$A'(x) = 0 - (-\sin x) = \sin x$$

We choose the antiderivative

$$F(x) = -\cos x$$

The area of R is

$$\begin{aligned} A\left(\frac{\pi}{2}\right) &= F\left(\frac{\pi}{2}\right) - F\left(\frac{\pi}{6}\right) \\ &= -\cos\frac{\pi}{2} - \left(-\cos\frac{\pi}{6}\right) \\ &= 0 + \frac{\sqrt{3}}{2} \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

An alternative solution would be to reflect the region in the x-axis. This would not change its area. The image of $y = -\sin x$ is $y = \sin x$ and we would find the area under $y = \sin x$ from $\frac{\pi}{6}$ to $\frac{\pi}{2}$. See Example 3 in Section 10.1.

EXERCISE 10.2

B **1.** Find the area of the region between the given curves. Include a sketch of the region.

(a) $y = x^2 + 3$ and $y = x + 1$ from 2 to 4
(b) $y = 2 - x^2$ and $y = -2x + 3$ from -1 to 1
(c) $y = x^2$ and $y = 2x$
(d) $y = 4 - x^2$ and $2x - y + 1 = 0$
(e) $y = 4 - x^2$ and $y = 2x^2 - 8$
(f) $y = x^2$ and $y = 8\sqrt{x}$
(g) $y = 2x - x^2$ and $y = -x$
(h) $y = x^2$ and $y = x^3$
(i) $y = x^3 + 8$ and $y = 4x + 8$
(j) $y = \frac{4}{x^2}$ and $y = 5 - x^2$
(k) $y^2 = 4x$ and $x^2 = 4y$
(l) $y = x^3 - x$ and $y = 0$
(m) $y = x$ and $y = -\frac{x}{2}$ and $y = 5x - 44$
(n) $x + y = 1$ and $x + y = 5$ and $y = 2x + 1$ and $y = 2x + 6$
(o) $y = \sin x$ and $y = \cos x$ from $-\pi$ to π
(p) $y = \sec^2 x$ and $y = x - 1$ from 0 to $\frac{\pi}{4}$
(q) $y = e^{-x}$ and $y = -x$ from $-\ln 3$ to $\ln 3$
(r) $y = \frac{1}{x}$ and $x + y = 2$ from $\frac{1}{2}$ to 3
(s) $y = \sin x$ and $y = \cos 2x$ from $\frac{\pi}{6}$ to $\frac{5\pi}{6}$
(t) $y = 3 \sin x$ and $y = \sin 3x$ from 0 to π

C **2.** Sketch the area bounded by the given curves and find the area of the region.

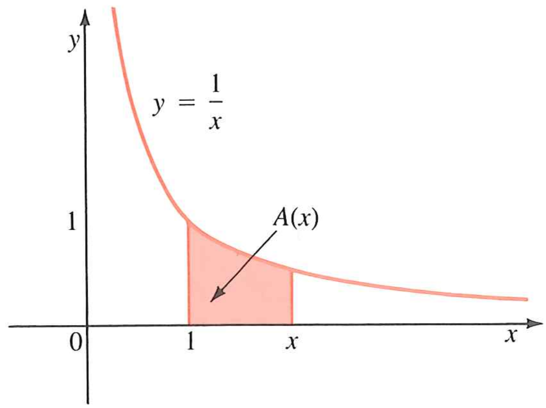
(a) $y = |x - 1| + |x + 1|$ and $y = 3 - x^2$
(b) $y = |x| - x$ and $y = -x^2 - x + 2$
(c) $xy - 3y + 1 = 0$ and $x^2y + y = 1$

3. Sketch the region bounded by the given curves and find the area of the region.

(a) $y = \frac{1}{x}$, $x = 0$, $y = 1$, and $y = 2$
(b) $x = y^2$ and $x = -2y^2 + 1$
(c) $y = x - 1$ and $y^2 = 2x + 6$

10.3 THE NATURAL LOGARITHM AS AN AREA

In Chapter 8 we defined $y = \ln x$ as the inverse of the function $y = e^x$ and found its derivative, namely $\frac{d}{dx} \ln x = \frac{1}{x}$. The function $y = \frac{1}{x}$ is continuous and positive in the interval $(0, \infty)$. We find the area under $y = \frac{1}{x}$ from 1 to x.



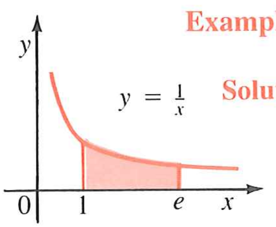
The required area $A(x)$ is shown in the diagram above.

Since
$$A'(x) = \frac{1}{x}$$

we choose the antiderivative
$$F(x) = \ln x$$

Therefore
$$\begin{aligned} A(x) &= F(x) - F(1) \\ &= \ln x - \ln 1 \\ &= \ln x \end{aligned}$$

> If $x > 1$, the natural logarithm $\ln x$ is the area under the curve $y = \dfrac{1}{x}$ from 1 to x.

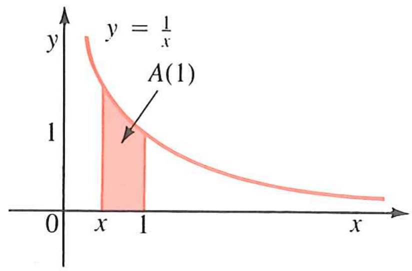


Example 1 Find the area under $y = \dfrac{1}{x}$ from $x = 1$ to $x = e$.

Solution The required area is $\ln e = 1$.

Example 2 Find the area under $y = \dfrac{1}{x}$ from x to 1, $0 < x < 1$.

Solution The required area $A(1)$ is shown in the following diagram.



Since $\qquad A'(x) = \dfrac{1}{x}$

we choose the antiderivative

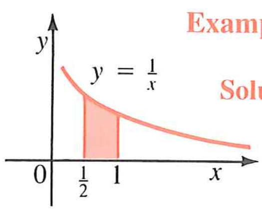
$$F(x) = \ln x$$

Therefore

$$\begin{aligned} A(1) &= F(1) - F(x) \\ &= \ln 1 - \ln x \\ &= -\ln x \end{aligned}$$

Since $\ln x = -A(1)$ in Example 2, we draw the following conclusion:

> The natural logarithm $\ln x$ is the negative of the area under the curve $y = \dfrac{1}{x}$ from x to 1 for $0 < x < 1$.



Example 3 Find the area under $y = \dfrac{1}{x}$ from $x = 0.5$ to $x = 1$.

Solution Since $0 < 0.5 < 1$, the required area is $-\ln 0.5 \doteq 0.693\ 147$.

Example 4 (a) Prove that the area of region $R = \{(x, y) \mid 4 \leqslant x \leqslant 9,\ 0 \leqslant xy \leqslant 1\}$ is $2 \ln 1.5$.

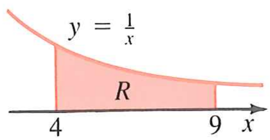
(b) Use an area to prove that $\ln 1.5 < \frac{65}{144}$.

Solution (a) R is the region under $y = \dfrac{1}{x}$ from 4 to 9.

The area under $y = \dfrac{1}{x}$ from 1 to 9 is ln 9.

The area under $y = \dfrac{1}{x}$ from 1 to 4 is ln 4.

Therefore the area of R is



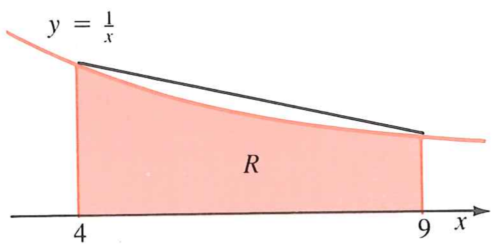
$$\begin{aligned}\ln 9 - \ln 4 &= \ln\left(\tfrac{9}{4}\right)\\ &= \ln 2.25\\ &= \ln(1.5)^2\\ &= 2\ln 1.5\end{aligned}$$

(b) The area of R is less than the area of the trapezoid. The area A of the trapezoid is

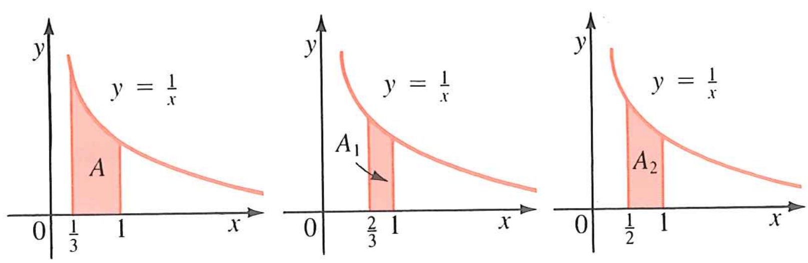
$$\begin{aligned}A &= \tfrac{1}{2}(9-4)\left(\tfrac{1}{4}+\tfrac{1}{9}\right)\\ &= \tfrac{1}{2}(5)\left(\tfrac{13}{36}\right)\\ &= \tfrac{65}{72}\end{aligned}$$

Therefore $\quad 2\ln 1.5 < \frac{65}{72}$

and $\quad \ln 1.5 < \frac{65}{144}$



Example 5 Prove that $A = A_1 + A_2$ if A, A_1, and A_2 are the areas in the diagrams.

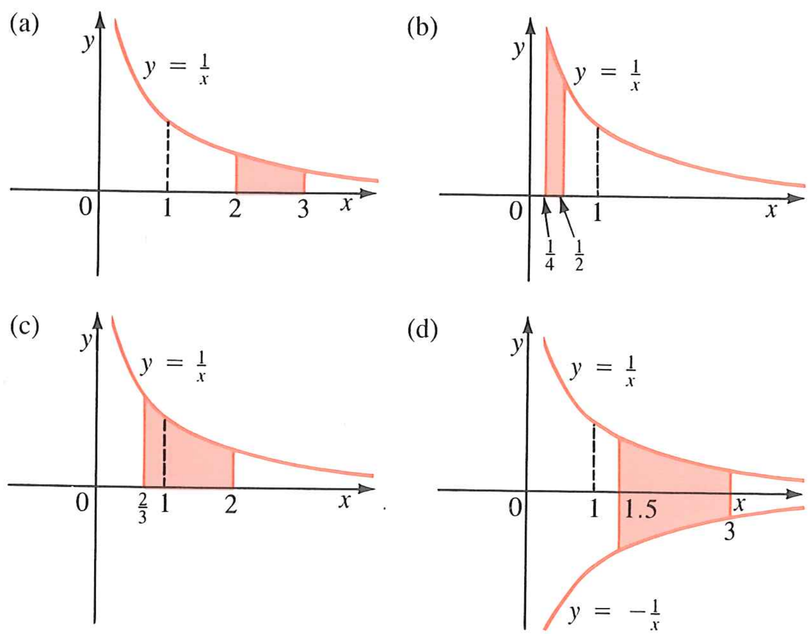


Solution Since the three areas are under the curve $y = \dfrac{1}{x}$ and have $x = 1$ as the right-hand boundary, each can be expressed as a natural logarithm.

$$\begin{aligned}\text{Now}\quad A &= -\ln \tfrac{1}{3}\\ &= -\ln\left(\tfrac{2}{3} \times \tfrac{1}{2}\right)\\ &= -\left(\ln \tfrac{2}{3} + \ln \tfrac{1}{2}\right)\\ &= -\ln \tfrac{2}{3} - \ln \tfrac{1}{2}\\ &= A_1 + A_2\end{aligned}$$

EXERCISE 10.3

B **1.** State the area of the shaded region.



2. Sketch an area represented by each of the following:

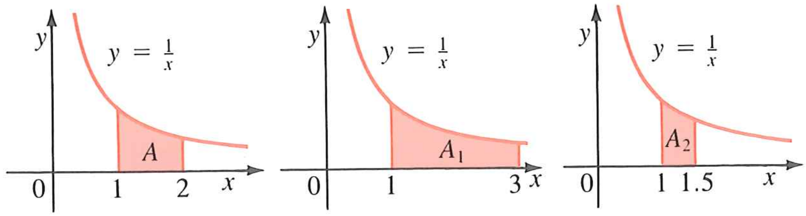
(a) $\ln 4$ (b) $-\ln \frac{1}{4}$ (c) $\ln 2 + \ln 4$

(d) $\ln 6 - \ln 3$ (e) $2 \ln 2$ (f) $-\ln 0.75$

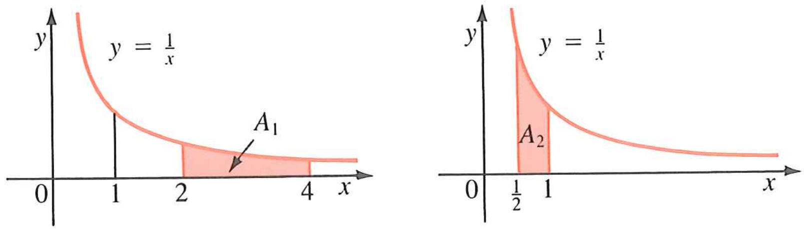
(g) $-\ln 0.5 - \ln 0.25$ (h) $-\frac{1}{2} \ln \frac{1}{9}$

(i) $-\ln \frac{1}{3} + \ln 3$

3. Prove that $A = A_1 - A_2$ if A, A_1, and A_2 are the areas in the diagrams.



4. Prove that $A_1 = A_2$ if A_1 and A_2 are the areas in the diagrams.



5. Refer to Example 4 of this section and

(a) find the equation of the tangent to $y = \dfrac{1}{x}$ that is parallel to the slanted side of the trapezoid;

(b) use an area to prove that $\ln 1.5 > \frac{55}{144}$.

6. (a) Prove that the area of the region $R = \{(x, y) \mid 3 \leqslant x \leqslant 6,\ 0 \leqslant xy \leqslant 1\}$ is $\ln 2$.

(b) Use an area to establish an upper bound for the value of $\ln 2$.

7. Repeat Problem 6 for the region $R = \{(x, y) \mid 18 \leqslant x \leqslant 36,\ 0 \leqslant xy \leqslant 1\}$.

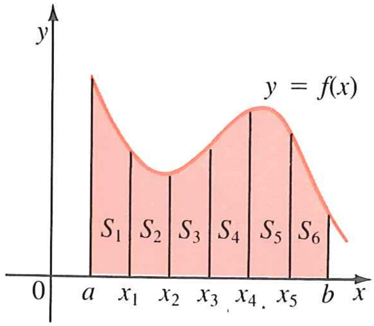
8. By comparing areas show that

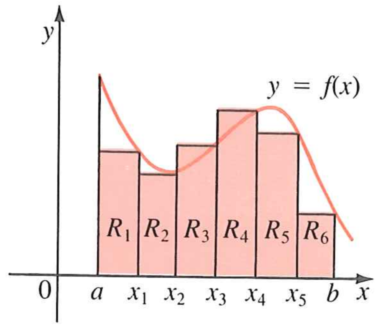
(a) $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \dfrac{1}{n} < \ln n < 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \dfrac{1}{n - 1}$

(b) $\ln 2 < 1 < \ln 3$

10.4 AREAS AS LIMITS

In this section we examine a different method of calculating area. The region whose area we wish to calculate is divided into narrow strips and a rectangle is used to approximate the area of each strip. The sum of the areas of these rectangles approximates the area of the region.





The region under $y = f(x)$ from a to b has been divided into six strips, $S_1, S_2, \ldots, S_6$. Rectangles, $R_1, R_2, \ldots, R_6$, have been constructed to approximate the area of each strip. In this case, the base of the rectangle is the width of the strip and the height of the rectangle is the value of the function at the right-hand endpoint of each strip. Rectangle R_3 has width $x_3 - x_2$ and height $f(x_3)$.

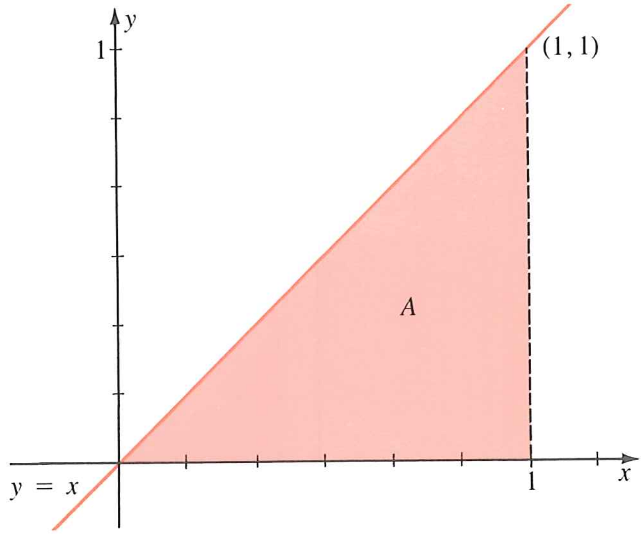
The left-hand endpoint or any point in between, such as the midpoint of the base, could have been chosen to determine the height of a rectangle but we will be using the right-hand endpoint exclusively.

Example 1 (a) Calculate the area under $y = x$, from 0 to 1.

(b) Approximate the same area by subdividing the region into six strips of equal width and finding the sum of the areas of the rectangles determined by the right-hand endpoint of each interval.

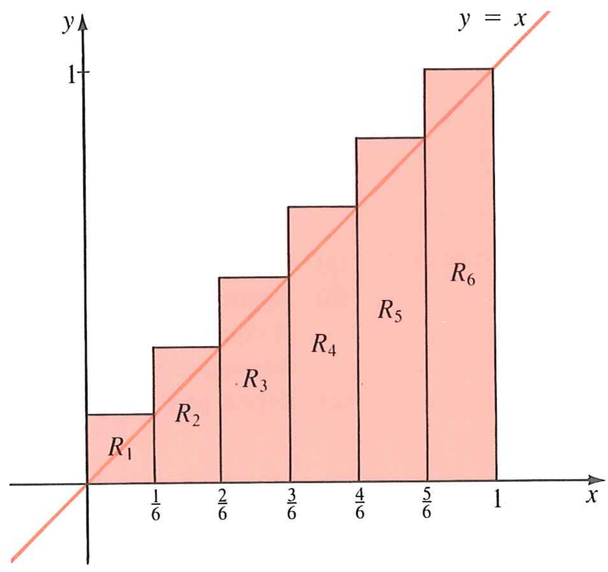
(c) Repeat part (b) using twelve strips of equal width.

Solution (a)



The required area is a right triangle with base 1 and height 1. Since the formula for the area of a triangle is $\frac{1}{2}bh$, the required area is $\frac{1}{2}(1)(1) = \frac{1}{2} = 0.5$.

(b)



The required area is approximately the sum of the areas of the rectangles R_1, R_2, R_3, R_4, R_5, and R_6.

Now the width of each rectangle is $\frac{1}{6}$.

So, $\quad \text{Area} \doteq \frac{1}{6}f\left(\frac{1}{6}\right) + \frac{1}{6}f\left(\frac{2}{6}\right) + \frac{1}{6}f\left(\frac{3}{6}\right) + \frac{1}{6}f\left(\frac{4}{6}\right) + \frac{1}{6}f\left(\frac{5}{6}\right) + \frac{1}{6}f\left(\frac{6}{6}\right)$

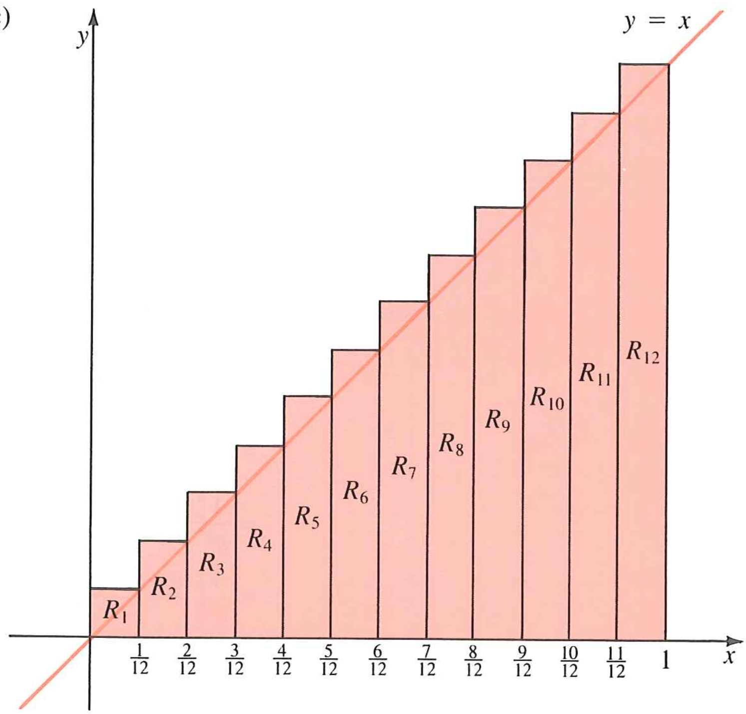
$$= \frac{1}{6}\left(\frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6}\right)$$

$$= \frac{1}{36}(1 + 2 + 3 + 4 + 5 + 6)$$

$$= \frac{21}{36}$$

$$\doteq 0.583$$

(c)



The required area is approximately the sum of the areas of the rectangles $R_1, R_2, R_3, \ldots, R_{12}$.

Now the width of each rectangle is $\frac{1}{12}$.

So, $\quad \text{Area} \doteq \frac{1}{12}\left[f\left(\frac{1}{12}\right) + f\left(\frac{2}{12}\right) + f\left(\frac{3}{12}\right) + \ldots + f\left(\frac{12}{12}\right)\right]$

$$= \frac{1}{12}\left(\frac{1}{12} + \frac{2}{12} + \frac{3}{12} + \ldots + \frac{12}{12}\right)$$

$$= \frac{1}{144}(1 + 2 + 3 + \ldots + 12)$$

$$= \frac{1}{144}\left(\frac{12 \times 13}{2}\right)$$

$$= \frac{78}{144}$$

$$\doteq 0.542$$

Intuitively, the more rectangles that are constructed the better the approximation becomes. Suppose we constructed 100 rectangles of equal width $\frac{1}{100}$.

$$\begin{aligned}
\text{Area} &\doteq \frac{1}{100}\left[f\left(\tfrac{1}{100}\right) + f\left(\tfrac{2}{100}\right) + f\left(\tfrac{3}{100}\right) + \ldots + f\left(\tfrac{100}{100}\right)\right] \\
&\doteq \tfrac{1}{100}\left(\tfrac{1}{100} + \tfrac{2}{100} + \tfrac{3}{100} + \ldots + \tfrac{100}{100}\right) \\
&= \tfrac{1}{100^2}(1 + 2 + 3 + \ldots + 100) \\
&= \tfrac{1}{100^2}\left(\tfrac{100 \times 101}{2}\right) \\
&= \tfrac{101}{200} \\
&= 0.505
\end{aligned}$$

This is much closer to the actual area of 0.5 calculated in Example 1(a).

If we subdivide the region into n strips of equal width $\frac{1}{n}$ we could construct n rectangles and

$$\begin{aligned}
\text{Area} &\doteq \frac{1}{n}\left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + f\left(\frac{3}{n}\right) + \ldots + f\left(\frac{n}{n}\right)\right] \\
&= \frac{1}{n}\left(\frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \ldots + \frac{n}{n}\right) \\
&= \frac{1}{n^2}(1 + 2 + 3 + \ldots + n) \\
&= \frac{1}{n^2}\left[\frac{n(n + 1)}{2}\right] \\
&= \frac{n^2 + n}{2n^2}
\end{aligned}$$

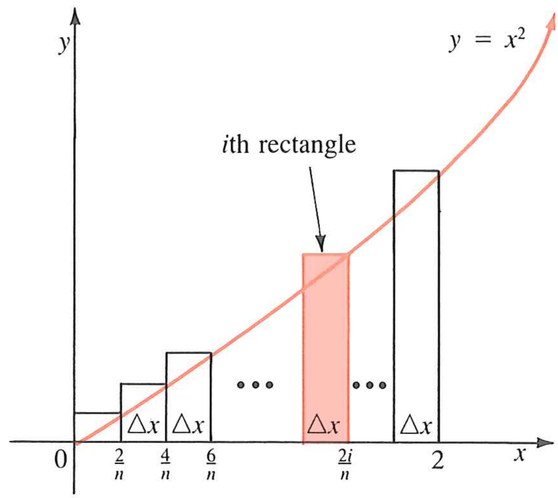
Now as $n \to \infty$ the number of rectangles increases and their width $\frac{1}{n}$ approaches 0 and the limit of the sum of the rectangles as $n \to \infty$ produces the actual area.

$$\text{Area} = \lim_{n\to\infty} \frac{n^2 + n}{2n^2} = \lim_{n\to\infty} \frac{1 + \frac{1}{n}}{2} = \tfrac{1}{2} = 0.5$$

We use sigma notation to write the series in the rest of the examples.

Example 2 Find the area under $y = x^2$ from $x = 0$ to $x = 2$.

Solution Subdivide the region into n strips of equal width, $\frac{2}{n}$. Consider a general rectangle, which we call the ith rectangle, having width $\Delta x = \frac{2}{n}$ and height $f\left(\frac{2i}{n}\right)$ determined by the right-hand endpoint.



Using sigma notation, the sum of the areas of the n rectangles is

$$\sum_{i=1}^{n} \frac{2}{n} f\left(\frac{2i}{n}\right) = \sum_{i=1}^{n} \left(\frac{2}{n}\right)\left(\frac{4i^2}{n^2}\right)$$

$$= \frac{8}{n^3} \sum_{i=1}^{n} i^2$$

$$= \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{4(2n^2 + 3n + 1)}{3n^2}$$

Therefore,

$$\text{Area} = \frac{4}{3} \lim_{n \to \infty} \frac{2n^2 + 3n + 1}{n^2}$$

$$= \frac{4}{3} \lim_{n \to \infty} \left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

$$= \tfrac{4}{3}(2 + 0 + 0)$$

$$= \frac{8}{3}$$

Example 3 Find the area under $y = x^3 + x$ from $x = 1$ to $x = 4$.

Solution Subdivide the region into n strips of equal width

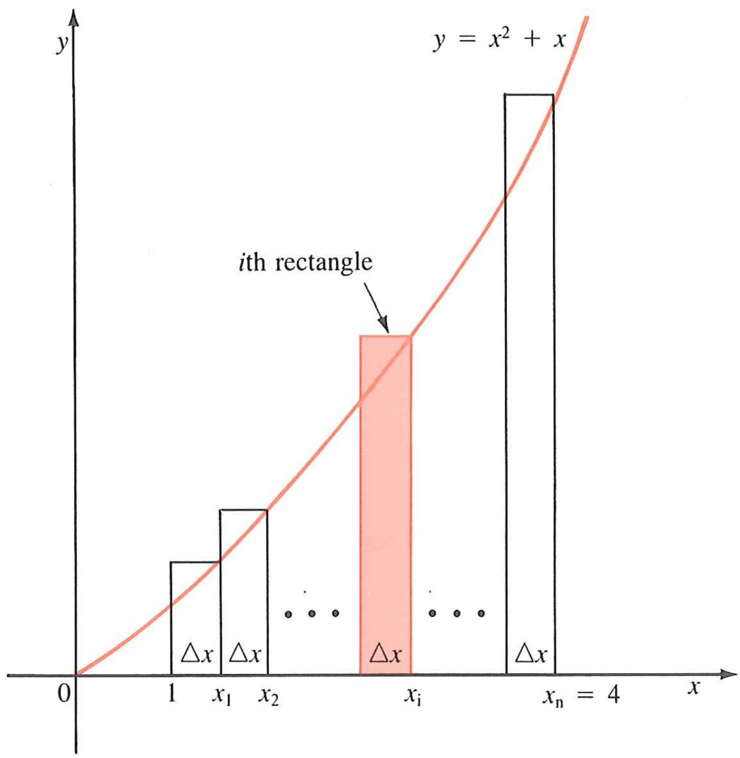
$$\Delta x = \frac{4-1}{n} = \frac{3}{n}$$

$$x_1 = 1 + \frac{3}{n}$$
$$x_2 = 1 + \frac{6}{n}$$
$$x_3 = 1 + \frac{9}{n}$$
$$\vdots$$

The ith rectangle has right-hand endpoint

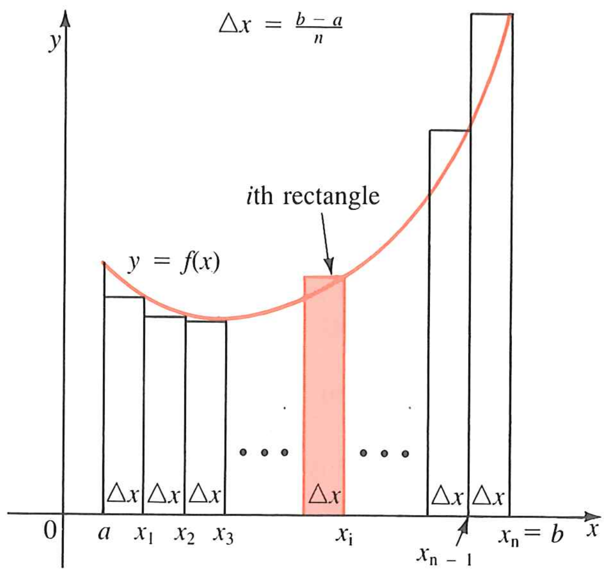
$$x_i = 1 + \frac{3i}{n}$$

and height $f\left(1 + \frac{3i}{n}\right)$



$$\begin{aligned}
\text{Area} &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n} f\left(1 + \frac{3i}{n}\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left[\left(1 + \frac{3i}{n}\right)^3 + \left(1 + \frac{3i}{n}\right)\right] \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{3}{n} + \frac{27i}{n^2} + \frac{81i^2}{n^3} + \frac{81i^3}{n^4} + \frac{3}{n} + \frac{9i}{n^2}\right) \\
&= \lim_{n\to\infty} \left(\frac{81}{n^4}\sum_{i=1}^{n} i^3 + \frac{81}{n^3}\sum_{i=1}^{n} i^2 + \frac{36}{n^2}\sum_{i=1}^{n} i + \frac{6}{n}\sum_{i=1}^{n} 1\right) \\
&= \lim_{n\to\infty} \left[\frac{81n^2(n+1)^2}{4n^4} + \frac{81n(n+1)(2n+1)}{6n^3} + \frac{36n(n+1)}{2n^2} + \frac{6n}{n}\right] \\
&= \lim_{n\to\infty} \left[\frac{81}{4}\left(1 + \frac{1}{n}\right)^2 + \frac{81}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) + 18\left(1 + \frac{1}{n}\right) + 6\right] \\
&= \frac{81}{4} + \frac{81}{3} + 18 + 6 \\
&= \frac{285}{4}
\end{aligned}$$

We develop a formula to find the area under $y = f(x)$ from a to b for f continuous and positive. We subdivide the region into n strips of equal width



From the diagram we see that the right-hand endpoints of the intervals are

$$\begin{aligned} x_1 &= a + \Delta x \\ x_2 &= a + 2\Delta x \\ x_3 &= a + 3\Delta x \\ &\vdots \end{aligned}$$

The right-hand endpoint of the ith interval is

$$x_i = a + i\Delta x$$

The height of the ith rectangle is $f(x_i)$, so its area is

$$\text{height} \times \text{width} = f(x_i)\Delta x$$

To find the required area, we take the limit of the sums of the areas of the rectangles.

$$\boxed{\begin{gathered} \text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\ \text{where } \Delta x = \frac{b-a}{n} \text{ and } x_i = a + i\Delta x \end{gathered}}$$

EXERCISE 10.4

B **1.** R is the region under $y = x + 1$ from 0 to 6.
  (a) Calculate the area of R using the formula for the area of a trapezoid.
  (b) Approximate the area of R by dividing it into six subintervals of equal width and summing the areas of rectangles.

2. R is the region under $y = x^2 + 1$ from 1 to 3.
  (a) Calculate the area of R using the differential equation $A'(x) = f(x)$.
  (b) Approximate the area of R by dividing it into ten subintervals of equal width and summing the areas of rectangles.

3. Use methods of this section to calculate the area of the given region.
  (a) under $y = x^3$ from 0 to 4
  (b) under $y = 2 + x^2$ from 0 to 3
  (c) under $y = x + 2x^3$ from 0 to 2
  (d) under $y = 3x^3 + 2x^2 + x$ from 0 to 1

4. Use methods of this section to calculate the area of the given region.
(a) $y = -x^2 + 16$ from 1 to 3
(b) $y = x^2 + 3x - 2$ from 1 to 4
(c) $y = \frac{1}{2}x^3$ from 2 to 4
(d) $y = x^2 + x + 1$ from -1 to 3

5. Approximate the area under $y = \sin x$ from 0 to π by summing the areas of six rectangles of equal width.

C 6. Find the area between the given curves by summing the areas of n rectangles of equal width.
(a) $y = x^2 + 4$ and $y = x + 2$ from $x = 0$ to $x = 2$
(b) $y = x^3 - 4x$ and $y = 5x$

PROBLEMS PLUS

(a) Show that

$$2 \sin \tfrac{1}{2}x \cos ix = \sin\left(i + \tfrac{1}{2}\right)x - \sin\left(i - \tfrac{1}{2}\right)x$$

(b) Use the identity in part (a) to show that

$$\sum_{i=1}^{n} \cos ix = \frac{\sin\left(n + \frac{1}{2}\right)x - \sin \frac{1}{2}x}{2 \sin \frac{1}{2}x}$$

(c) Deduce from part (b) that

$$\sum_{i=1}^{n} \cos ix = \frac{\sin \frac{1}{2}nx \cos \frac{1}{2}(n + 1)x}{\sin \frac{1}{2}x}$$

(d) Use part (c) to find the area under the curve $y = \cos x$ from 0 to b, $0 \leqslant b \leqslant \dfrac{\pi}{2}$, as a limit of sums.

10.5 NUMERICAL METHODS

We can accurately evaluate an area under $y = f(x)$ if we can find an antiderivative of f. Sometimes it is difficult or even impossible to find such an antiderivative. In such cases we can only approximate the area under the curve. In Section 10.4, we approximated the area of a region by subdividing it into narrow strips and approximating the area of each strip with a rectangle. The sum of the areas of the rectangles approximated the area of the region. As the widths of the strips became narrower, the number of rectangles increased and the approximation became better.

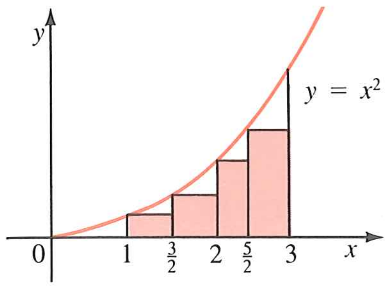
In this section we examine other methods of approximating the area of a curved region. In particular, we are looking for better approximations using fewer subintervals.

Example 1 Find the area under $f(x) = x^2$ from $x = 1$ to $x = 3$ by dividing the interval into four subintervals of equal width and approximating the area using

(a) rectangles with height determined by the left-hand endpoint;
(b) rectangles with height determined by the right-hand endpoint;
(c) trapezoids;
(d) rectangles with height determined by the midpoint.

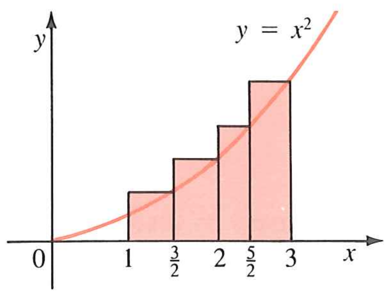
Solution (a)

$$\begin{aligned}\text{Area} &\doteq \tfrac{1}{2}f(1) + \tfrac{1}{2}f\left(\tfrac{3}{2}\right) + \tfrac{1}{2}f(2) + \tfrac{1}{2}f\left(\tfrac{5}{2}\right)\\ &= \tfrac{1}{2}\left(1 + \tfrac{9}{4} + 4 + \tfrac{25}{4}\right)\\ &= \tfrac{1}{2}\left(\tfrac{54}{4}\right)\\ &= 6.75\end{aligned}$$



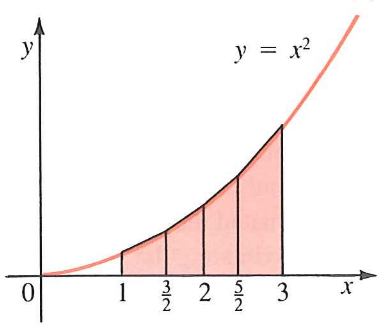
(b)

$$\begin{aligned}\text{Area} &\doteq \tfrac{1}{2}f\left(\tfrac{3}{2}\right) + \tfrac{1}{2}f(2) + \tfrac{1}{2}f\left(\tfrac{5}{2}\right) + \tfrac{1}{2}f(3)\\ &= \tfrac{1}{2}\left(\tfrac{9}{4} + 4 + \tfrac{25}{4} + 9\right)\\ &= \tfrac{1}{2}\left(\tfrac{86}{4}\right)\\ &= 10.75\end{aligned}$$

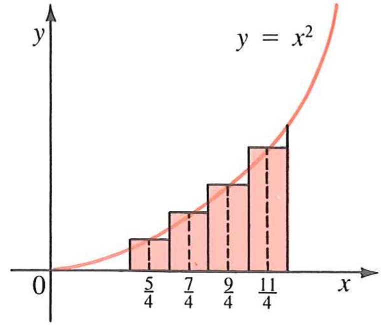


(c)

$$\begin{aligned}A &\doteq \tfrac{1}{2}\left(\tfrac{1}{2}\right)\left[f(1) + f\left(\tfrac{3}{2}\right)\right] + \tfrac{1}{2}\left(\tfrac{1}{2}\right)\left[f\left(\tfrac{3}{2}\right) + f(2)\right] + \tfrac{1}{2}\left(\tfrac{1}{2}\right)\left[f(2) + f\left(\tfrac{5}{2}\right)\right]\\ &\quad + \tfrac{1}{2}\left(\tfrac{1}{2}\right)\left[f\left(\tfrac{5}{2}\right) + f(3)\right]\\ &= \tfrac{1}{4}\left[f(1) + 2f\left(\tfrac{3}{2}\right) + 2f(2) + 2f\left(\tfrac{5}{2}\right) + f(3)\right]\\ &= \tfrac{1}{4}\left(1 + \tfrac{9}{2} + 8 + \tfrac{25}{2} + 9\right)\\ &= \tfrac{1}{4}\left(\tfrac{70}{2}\right)\\ &= 8.75\end{aligned}$$



(d)



$$
\begin{aligned}
A &\doteq \frac{1}{2}f\left(\frac{1+\frac{3}{2}}{2}\right) + \frac{1}{2}f\left(\frac{\frac{3}{2}+2}{2}\right) + \frac{1}{2}f\left(\frac{2+\frac{5}{2}}{2}\right) + \frac{1}{2}f\left(\frac{\frac{5}{2}+3}{2}\right) \\
&= \tfrac{1}{2}\left(\left(\tfrac{5}{4}\right)^2 + \left(\tfrac{7}{4}\right)^2 + \left(\tfrac{9}{4}\right)^2 + \left(\tfrac{11}{4}\right)^2\right) \\
&= \tfrac{1}{32}(25 + 49 + 81 + 121) \\
&= \tfrac{276}{32} \\
&= 8.625
\end{aligned}
$$

We can find the area in Example 1 by solving the differential equation

$$A'(x) = x^2$$

We choose the antiderivative

$$F(x) = \frac{x^3}{3}$$

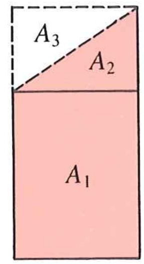
The required area is

$$
\begin{aligned}
A(3) &= F(3) - F(1) \\
&= 9 - \tfrac{1}{3} \\
&= 8\tfrac{2}{3} \\
&\doteq 8.667
\end{aligned}
$$

Neither of the first two sets of rectangles produces a good approximation, but the average of the two results, $\frac{10.75 + 6.75}{2} = 8.75$, is a good approximation. In fact, we get the approximation determined by the trapezoids. This is not surprising. The diagram illustrates that the area of the trapezoid is the average of the areas of the upper and lower rectangles.

A_1 = area of *lower rectangle*

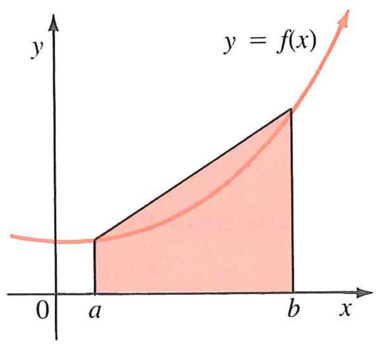
$A_1 + A_2 + A_3$ = area of the *upper rectangle*



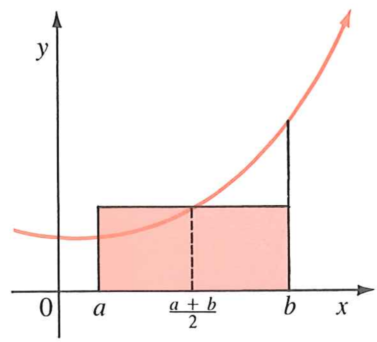
$$\mathrm{A}_2 = \mathrm{A}_3$$

$$\frac{\mathrm{A}_1 + (\mathrm{A}_1 + \mathrm{A}_2 + \mathrm{A}_3)}{2} = \frac{2\mathrm{A}_1 + \mathrm{A}_2 + \mathrm{A}_2}{2} = \mathrm{A}_1 + \mathrm{A}_2 = \text{area of } \textit{trapezoid}$$

The trapezoids and the rectangles with height determined by the midpoint of the interval produced very good approximations with a small number of subintervals. We shall concentrate on these two methods in this section. Note the similarity in the following approximations.



$$\text{Area} \doteq (b - a)\frac{(f(a) + f(b))}{2}$$



$$\text{Area} \doteq (b - a)\, f\!\left(\frac{a + b}{2}\right)$$

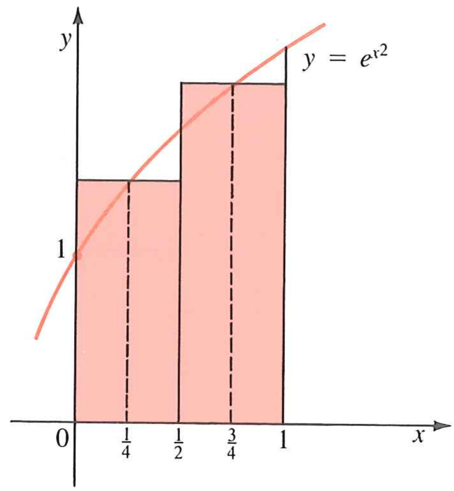
Example 2 (a) Approximate the area of the region under $y = e^{x^2}$ from 0 to 1 using 2 rectangles of equal width with height determined by the midpoint of the base.

(b) Repeat part (a) using 4 rectangles.

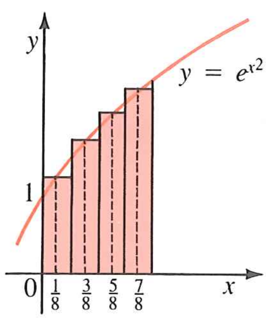
Solution (a) The width of each rectangle is $\dfrac{1 - 0}{2} = \dfrac{1}{2}$.

The midpoint of the first interval is $\dfrac{0 + \frac{1}{2}}{2} = \dfrac{1}{4}$.

The midpoint of the second interval is $\dfrac{\frac{1}{2} + 1}{2} = \dfrac{3}{4}$.



$$\begin{aligned}\text{Area} &\doteq \frac{1}{2}f\left(\frac{1}{4}\right) + \frac{1}{2}f\left(\frac{3}{4}\right)\\ &= \frac{1}{2}\left(e^{\frac{1}{16}} + e^{\frac{9}{16}}\right)\\ &\doteq 1.409\,775\end{aligned}$$



(b) The width of each interval is $\dfrac{1 - 0}{4} = \dfrac{1}{4}$.

The successive midpoints are $\frac{1}{8}, \frac{3}{8}, \frac{5}{8}$, and $\frac{7}{8}$.

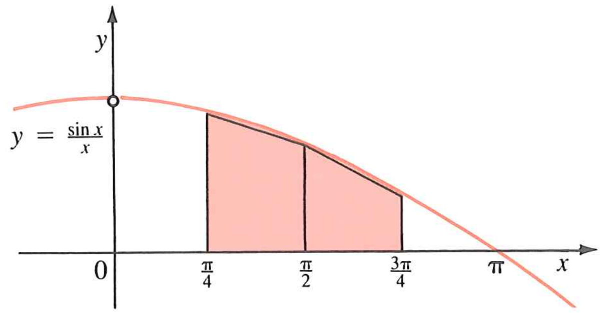
$$\begin{aligned}\text{Area} &\doteq \frac{1}{4}\left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right]\\ &= 0.25\left(e^{\frac{1}{64}} + e^{\frac{9}{64}} + e^{\frac{25}{64}} + e^{\frac{49}{64}}\right)\\ &\doteq 1.448\,745\end{aligned}$$

Example 3 (a) Approximate the area of the region under $y = \dfrac{\sin x}{x}$ from $\dfrac{\pi}{4}$ to $\dfrac{3\pi}{4}$ using two trapezoids of equal height.

(b) Repeat part (a) using 4 trapezoids.

Solution (a) The width of each interval, hence the height of each trapezoid, is

$$\frac{\frac{3\pi}{4} - \frac{\pi}{4}}{2} = \frac{\pi}{4}$$



The first trapezoid has parallel sides with lengths $f\left(\frac{\pi}{4}\right)$ and $f\left(\frac{\pi}{2}\right)$.

The second trapezoid has parallel sides of length $f\left(\frac{\pi}{2}\right)$ and $f\left(\frac{3\pi}{4}\right)$.

$$\begin{aligned}
\text{Area} &\doteq \frac{1}{2}\left(\frac{\pi}{4}\right)\left[f\left(\frac{\pi}{4}\right) + f\left(\frac{\pi}{2}\right)\right] + \frac{1}{2}\left(\frac{\pi}{4}\right)\left[f\left(\frac{\pi}{2}\right) + f\left(\frac{3\pi}{4}\right)\right] \\
&= \frac{\pi}{8}\left[f\left(\frac{\pi}{4}\right) + 2f\left(\frac{\pi}{2}\right) + f\left(\frac{3\pi}{4}\right)\right] \\
&= \frac{\pi}{8}\left(\frac{\sin\frac{\pi}{4}}{\frac{\pi}{4}} + 2\,\frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} + \frac{\sin\frac{3\pi}{4}}{\frac{3\pi}{4}}\right) \\
&\doteq 0.971\,405
\end{aligned}$$

(b) The width of each interval, hence the height of each trapezoid, is

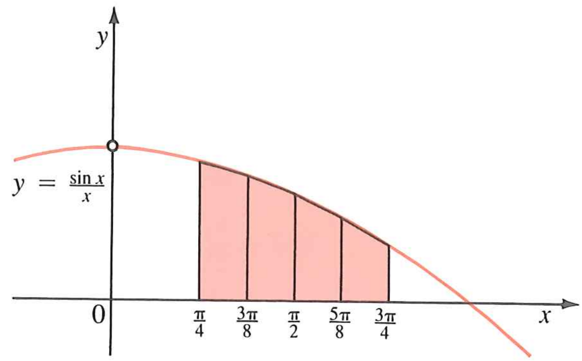
$$\frac{\frac{3\pi}{4} - \frac{\pi}{4}}{4} = \frac{\pi}{8}$$

The lengths of the parallel sides of the first trapezoid are $f\left(\frac{\pi}{4}\right)$ and $f\left(\frac{3\pi}{8}\right)$.

The lengths of the parallel sides of the second trapezoid are $f\left(\frac{3\pi}{8}\right)$ and $f\left(\frac{\pi}{2}\right)$.

The lengths of the parallel sides of the third trapezoid are $f\left(\frac{\pi}{2}\right)$ and $f\left(\frac{5\pi}{8}\right)$.

The lengths of the parallel sides of the fourth trapezoid are $f\left(\frac{5\pi}{8}\right)$ and $f\left(\frac{3\pi}{4}\right)$.



$$\begin{aligned}\text{Area} &\doteq \frac{1}{2}\left(\frac{\pi}{8}\right)\left[f\left(\frac{\pi}{4}\right) + 2f\left(\frac{3\pi}{8}\right) + 2f\left(\frac{\pi}{2}\right) + 2f\left(\frac{5\pi}{8}\right) + f\left(\frac{3\pi}{4}\right)\right] \\ &= \frac{\pi}{16}\left(\frac{\sin\frac{\pi}{4}}{\frac{\pi}{4}} + 2\,\frac{\sin\frac{3\pi}{8}}{\frac{3\pi}{8}} + 2\,\frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} + 2\,\frac{\sin\frac{5\pi}{8}}{\frac{5\pi}{8}} + \frac{\sin\frac{3\pi}{4}}{\frac{3\pi}{4}}\right) \\ &\doteq 0.978\,438\end{aligned}$$

In general, increasing the number of intervals, and hence the number of trapezoids or rectangles, increases the accuracy of the approximation. This is true up to a point. If many arithmetic operations are required, the accumulated round-off error may start to reduce the accuracy of the approximation.

If we divide the region under $y = f(x)$ from a to b into n subintervals of equal width, $\frac{b-a}{n}$, and approximate the area of each subinterval by the area of a trapezoid, then we can approximate the area of the region using the formula below.

Trapezoidal Rule

$$A \doteq \tfrac{1}{2}\left(\frac{b-a}{n}\right)[f(a) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(b)]$$

where $\quad x_i = a + \dfrac{b-a}{n}i$

Example 2 Use the Trapezoidal Rule with ten subintervals to approximate the value of ln 2.

Solution Recall that ln 2 is the area under $f(x) = \frac{1}{x}$ from $x = 1$ to $x = 2$. We set $n = 10$, $a = 1$, and $b = 2$ in the Trapezoidal Rule.

$$\begin{aligned}\ln 2 &\doteq \frac{1}{2}\left(\frac{2-1}{10}\right)[f(1) + 2f(1.1) + 2f(1.2) + \dots + 2f(1.9) + f(2)] \\ &= \frac{1}{20}\left(1 + \frac{2}{1.1} + \frac{2}{1.2} + \dots + \frac{2}{1.9} + \frac{1}{2}\right) \\ &\doteq 0.693\ 771\end{aligned}$$

EXERCISE 10.5

B

Round off all approximations to six decimal places.

1. (a) Estimate the area of the region under $y = e^x$ from 0 to 2, (i) using 4 trapezoids, and (ii) using 4 rectangles with height determined by the midpoint of the base.
   (b) Compare the approximations in part (a) with the actual area.

2. Repeat Question 1 for the region $y = \sin x$ from $\frac{\pi}{4}$ to π.

3. Use six trapezoids to estimate the area of the region under $y = xe^x$ from 1 to 3.

4. Use three rectangles with height determined by the midpoint of the base to approximate the area of the region under $y = \tan x$ from 0 to $\frac{\pi}{4}$.

5. Sum the areas of four trapezoids to approximate the value of ln 3.

6. Use the trapezoidal rule with $n = 12$ to approximate the area under $y = \frac{x}{e^x}$ from 1 to 2.

COMPUTER APPLICATION

The following pseudocode and BASIC program illustrate the use of the Trapezoidal Rule in finding the area under $y = e^{x^2}$ from 0 to 3

Pseudocode

```
input the number of trapezoids, N
calculate the width of each trapezoid
set the x-coordinates to the corners of
  the base of the first trapezoid

loop N times to calculate the areas of the N
trapezoids.
  calculate the area of the trapezoid:
    (the lengths of the parallel sides are the y-
    coordinates of the points on the function whose
    x-coordinates are the corners of the base of the
    trapezoid)
  sum the areas
  reset the variables for next trapezoid.
endloop

output the SUM.
```

A BASIC Version

```
DEF function(x)=exp(x^2)
LET a=0
LET b=3

PRINT ``Enter the number of parts in which to divide
the region'';
INPUT n
LET width=(b-a)/n
LET xleft=a
LET xright=a+width

FOR x=1 to n
      LET area=1/2*width*(function(xleft)
             +function(xright))
      LET sum=sum+area
      LET xleft=xright
      LET xright=xright+width
NEXT x

PRINT USING ``the area is ############ '':N sum
END
```

Output

Number of Parts N	Area
3	4 109.358 398 437 500
30	1 484.775 634 765 625
300	1 444.932 128 906 250
3000	1 444.368 652 343 750

10.6 REVIEW EXERCISE

B *Round all approximations to six decimal places.*

1. Find the area of the region under:

(a) $y = \frac{1}{x^3}$ from 1 to 3 (b) $y = 1 - 4x^2$ from $-\frac{1}{2}$ to $\frac{1}{4}$

(c) $y = 1 + \sin x$ from $\frac{\pi}{6}$ to $\frac{5\pi}{6}$

(d) $y = x^2 + \cos\left(\frac{1}{2}x\right)$ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

(e) $y = 3e^x - x$ from 0 to 2 (f) $y = \frac{1}{x} - 1$ from $\frac{1}{2}$ to 1

2. Find the area bounded by the given curve and the x-axis.
(a) $y = 4 - x^2$ (b) $y = x^3 - x^4$
(c) $y = x^2 - x - 12$

3. Find the area between the curve and the x-axis in the given interval.

(a) $y = x^2 - 4$ from -1 to 3 (b) $y = 2\sin x$ from $-\frac{\pi}{2}$ to π

(c) $y = (x - 1)^3$ from -1 to 2 (d) $y = -\cos 2x$ from $-\pi$ to $\frac{\pi}{2}$

4. Find the area between the given curves.
(a) $y = x^2 - 6x$, $y = 12x - 2x^2$

(b) $y = \frac{4}{x^2}$, $y = 5 - x^2$

(c) $y = \sin x$, $y = \cos 2x$, $-\frac{\pi}{2} \leqslant x \leqslant \frac{5\pi}{6}$

(d) $y = \frac{1}{x} - 1$, $y = 1 - \frac{1}{x}$, $x = 4$

5. Sketch the area determined by each of the following.

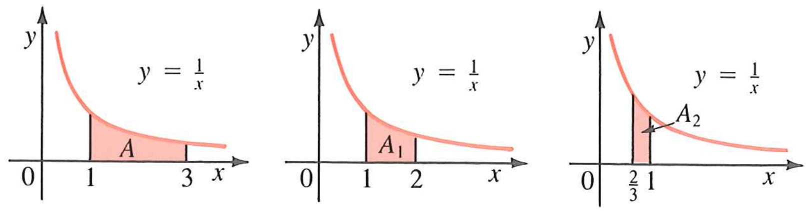
(a) $-\ln \frac{3}{4}$ (b) $\ln 12 - \ln 4$

6. (a) Calculate the area of the region

$$R = \{(x, y) \mid 4 \leqslant x \leqslant 12,\ 0 \leqslant xy \leqslant 1\}.$$

(b) Use an area to establish an upper bound for the value of ln 3.

7. Prove that $A = A_1 + A_2$ if A, A_1, and A_2 are the areas in the diagrams.



8. Calculate the required area by finding the limit of the sum of an infinite number of rectangles.
   (a) The area under $y = x^2 - 2x$ from 0 to 2.
   (b) The area under $y = x^3 - 1$ from 1 to 3.
   (c) The area between $y = x^3 + x^2 + 1$ from 2 to 6.

9. (a) Use six trapezoids to estimate the area under $y = \dfrac{1}{x^2}$ from 1 to 4.

   (b) Use three trapezoids to estimate the area under $y = \dfrac{x}{\sin x}$ from $\dfrac{\pi}{4}$ to $\dfrac{3\pi}{4}$.

10. Use four rectangles with heights determined by the midpoint of the base to approximate the area under $y = \dfrac{e^x}{x}$ from 1 to 5.

11. Approximate the value of ln 10 by summing the areas of four rectangles.

12. (a) Find the value of $\lim\limits_{n \to \infty} \dfrac{1^5 + 2^5 + 3^5 + \dots + n^5}{n^6}$ by showing that the limit is the area under $y = x^5$ from $x = 0$ to $x = 1$ and calculating the area.

    (b) Find the value of $\lim\limits_{n \to \infty} \dfrac{1}{n^4}(1^3 + 2^3 + 3^3 + \dots + n^3)$.

13. (a) Find the area bounded by $y = x^2$ and $y = 4$.
    (b) The line $y = c$ cuts the region in part (a) into two equal parts. Find the value of c.

10.7 CHAPTER 10 TEST

1. Find the area under $y = (x - 1)^3 + 1$ from 1 to 3.
2. Find the area between $y = x^2 - 6x$ and $y = 2x - x^2$.
3. Find the area enclosed by $y = x^3 - 4x$ and the x-axis.
4. Find the area between $y = x^4 - 2x^2$ and $y = 2x^2$.
5. Find the area between $y = 2 \sin x$ and $y = 1$ from $x = -\dfrac{\pi}{4}$ to $x = \pi$.
6. (a) Find the area of the region
$$R = \{(x, y) \mid 2 \leqslant x \leqslant 5,\ 0 \leqslant xy \leqslant 1\}$$
(b) Use an area to prove that $\ln 2.5 < 1.05$.
7. Find the exact area under $y = x^2 + 2x$ from 0 to 3 by taking the limit of the sum of areas of rectangles.
8. Use four trapezoids to approximate the area under $y = x \cos x$ from 0 to $\dfrac{\pi}{2}$. Give your answer to six decimal places.
9. Approximate the area under $y = \dfrac{e^x}{x}$ from 2 to 6 using three rectangles with height determined by the midpoint of the base. Give your answer to six decimal places.
10. Find the area between $y = |x^2 - 1|$ and $y = -|x|$ from $x = -2$ to $x = 3$.

CUMULATIVE REVIEW FOR CHAPTERS 8 TO 10

1. (a) Graph $y = 1 - e^x$ starting from the graph of $y = e^x$.
(b) State the domain, range, and asymptote of the function in part (a).

2. (a) Graph $y = \ln(x + 6)$ starting from the graph of $y = \ln x$.
(b) State the domain, range, and asymptote of the function in part (a).

3. Evaluate.
(a) $\lim_{x\to\infty} \left(1 + e^{-x^2}\right)$ (b) $\lim_{x\to 0^+} \ln(\sin x)$

4. Evaluate.
(a) $\ln(e^2)$ (b) $e^{2 \ln 3}$

5. Solve each equation. State your answer exactly and also correct to six decimal places.
(a) $e^{2x+1} = 20$ (b) $\ln(1 - x) = -2$

6. Differentiate.
(a) $y = (x + 1)e^{3-4x}$ (b) $y = \dfrac{\ln(x^2 + 1)}{x}$
(c) $y = \ln(1 + e^{x^2})$ (d) $y = 10^{-\sqrt{x}}$
(e) $y = \ln\sqrt{\dfrac{x}{1 - x^3}}$ (f) $y = x^{\tan x}$

7. If f is a differentiable function, find the derivatives of the following functions.
(a) $g(x) = e^{f(x)}$ (b) $h(x) = f(e^x)$

8. Find the equation of the tangent line to the curve $y = e^x$ that is parallel to the line $3x - y = 6$.

9. For the function $f(x) = xe^x$, determine
(a) the intervals of increase and decrease,
(b) the maximum and minimum values,
(c) the intervals of concavity and inflection points.

10. Find the general antiderivative of f on $(-\infty, \infty)$.
(a) $f(x) = 12x^3 - 9x^2 + 8x + 31$
(b) $f(x) = 4 \sin 2x + 5 \cos(3x + 1)$
(c) $f(x) = -2e^{3x} + \frac{1}{3}e^{-4x}$

11. Find the general antiderivative of f on $(0, \infty)$.
(a) $f(x) = \dfrac{\sqrt{2}}{x + 1} - \dfrac{\sqrt{3}}{x}$ (b) $f(x) = \sqrt{2x} + \sqrt{5x} + \sqrt{8x}$

12. Find the function F given that the point $(2, 3)$ is on the graph $y = F(x)$, where
(a) $F'(x) = 3x^2 + 2x$ (b) $F'(x) = \dfrac{1}{\sqrt{x}} - x$
(c) $F'(x) = 3e^{4x}$

13. A stone is hurled straight down at 20 m/s from the edge of a bridge 155 m above the bay below. How many seconds later does the splash occur?

14. A bacteria culture starts with 1200 bacteria. After an hour the estimated count is 4000.
(a) Find the number of bacteria after t hours.
(b) Find the number of bacteria after 3 h.
(c) Find the rate of growth after 3 h.
(d) When will the bacteria population reach 10 000?

15. An isotope of bismuth, ^{214}Bi, has a half-life of 19.7 min. A sample of ^{214}Bi has a mass of 50 g.
(a) Find the mass that remains after t minutes.
(b) Find the mass that remains after 2 h.
(c) Find the rate of decay after 2 h.
(d) How long does it take the sample to decay until its mass is 1 g?

16. Bread is removed from a 150°C oven and placed in a cooling rack maintained at 30°C. The bread cools 50° in the first 3 min. How many more minutes will it take to reach a temperature of 40°C?

17. Three kilograms of salt are dissolved in 450 L of water. A brine solution having a salt concentration of 17 g/L is pumped into the tank at a rate of 6 L/min. The water is well stirred and the tank has an overflow mechanism so that there is always 450 L of salt water in the tank. Find the amount of salt in the tank after half an hour.

18. Two hundred fish are put into a lake whose carrying capacity is 6000. The number of fish triples in the first year. When will there be 3000 fish in the lake?

19. Solve the differential equation

$$\frac{d^2s}{dt^2} + 1.44s = 0$$

with $s = 1.7$ and $\dfrac{ds}{dt} = 1.8$ when $t = 0$.

20. A spring with mass 1.2 kg has natural length 0.50 m. A force of 9 N is required to stretch it to a length of 0.80 m. The spring is compressed to a length of 0.35m and then released from rest.
(a) How long does it take to return to that position?
(b) Find its speed when its displacement is zero.

21. Find the area of the given region.
(a) Under $y = -x^2 + 16$ from -1 to 3
(b) Under $xy = 1$ from 3 to 5
(c) Under $y = e^{-x}$ from 0 to 7

22. Find the area of the given region.
(a) Between $y = x^4$ and $y = 2x - x^2$
(b) Between $y = x^2 - 4x + 3$ and $x - y - 1 = 0$
(c) Between $y = \sin x$ and $y = -\cos x$ from $-\pi$ to π
(d) Between $y = -\cos x$ and the x-axis from 0 to π
(e) Between $y = x^3 - 2x^2 - 5x + 6$ and the x-axis from -1 to 2

23. Find the area of the given region by taking the limit of the sum of areas of rectangles.
(a) Under $y = mx$ from 0 to a
(b) Under $y = -x^2 + 2x$ from 0 to 2

24. Approximate the area under $y = xe^x$ from 2 to 4 using four trapezoids. Give your answer to 6 decimal places.

25. Estimate the area under $y = \dfrac{\sin x}{x}$ from 0 to π using four rectangles with height determined by the mid-point of the base. Give your answer to six decimal places.

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ANSWERS

CHAPTER 10 AREA

REVIEW AND PREVIEW TO CHAPTER 10

EXERCISE 1

1. (a) $\frac{\pi}{8}(1 + \sqrt{3})$ (b) $\frac{\pi}{4}$ (c) $\frac{3}{4}$

EXERCISE 2

1. (a) $(1^2 + 1) + (2^2 + 1) + (3^2 + 1) + (4^2 + 1) + (5^2 + 1) = 60$

(b) $\frac{1}{4}f(1) + \frac{1}{2}f(2) + \frac{3}{4}f(3) + f(4)$

(c) $\frac{3}{n}f\left(1 + \frac{3}{4}\right) + \frac{3}{n}f\left(1 + \frac{6}{4}\right) + \frac{3}{n}f\left(1 + \frac{9}{4}\right) + \dots + \frac{3}{n}f\left(1 + \frac{3n}{4}\right)$

2. (a) $\sum_{i=1}^{6}(3i - 2)$ (b) $\sum_{i=1}^{7}(-1)^{i-1}$ (c) $\sum_{i=1}^{n}x^i$

(d) $\sum_{i=1}^{6}\frac{i}{6}f\left(\frac{i}{6}\right)$ (e) $\sum_{i=1}^{n}\frac{i}{n}f\left(\frac{2i-2}{n}\right)$

3. (a) $\sum_{i=1}^{n}i^2 + 4\sum_{i=1}^{n}i + 4n$ (b) $3\sum_{i=1}^{20}i^2 - 12\sum_{i=1}^{20}i$

(c) $2\sum_{i=1}^{n}i^3 - 3\sum_{i=1}^{n}i^2 + 5\sum_{i=1}^{n}i - 12n$

EXERCISE 3

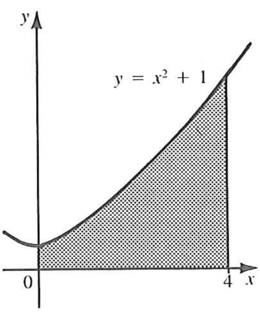
1. (a) $n(2n + 1)$ (b) $\frac{1}{2}(3^n - 1)$ (c) $n^2(n + 1)$

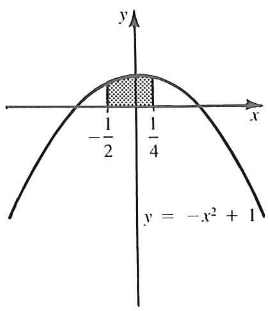
(d) $\frac{n}{2}(n^3 + 2n^2 + 4n - 1)$ (e) 270

(f) 24 821 640

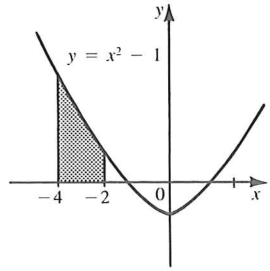
EXERCISE 10.1

1. (a) $\frac{76}{3}$ (b) $\frac{45}{64}$

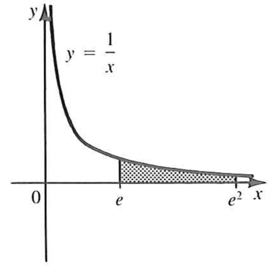




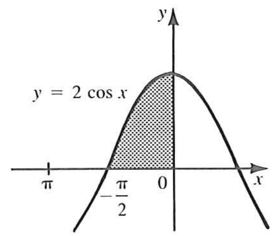
(c) $\frac{50}{3}$



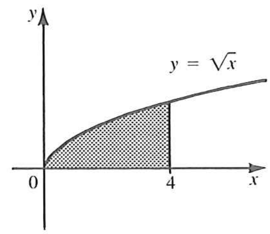
(d) 1



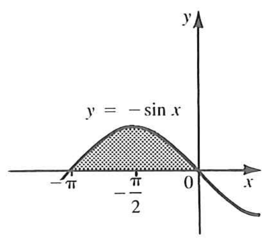
(e) 2



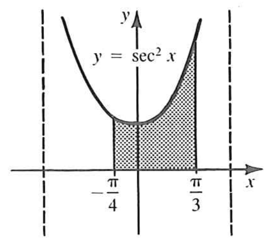
(f) $\frac{16}{3}$



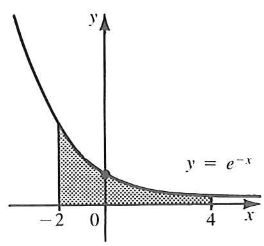
(g) 2



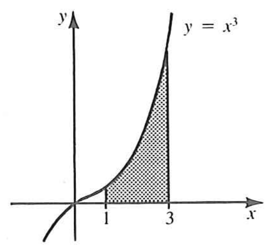
(h) $1 + \sqrt{3}$



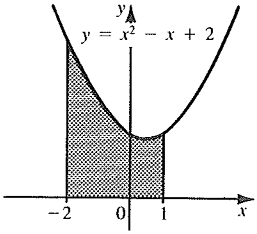
(i) $\frac{e^6 - 1}{e^4}$



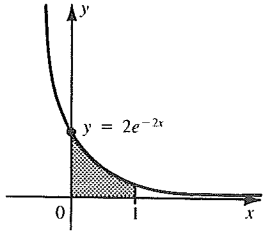
(j) 20



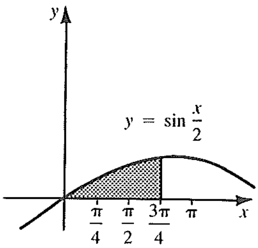
(k) 10.5



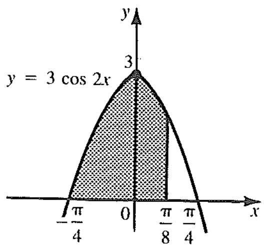
(l) $\dfrac{e^2 - 1}{e^2}$



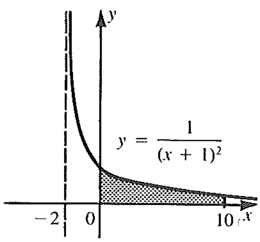
(m) $-2\sqrt{\dfrac{\sqrt{2} - 1}{2\sqrt{2}}} + 2$



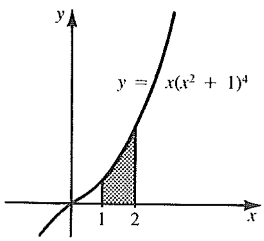
(n) $\dfrac{3(1 + \sqrt{2})}{2\sqrt{2}}$



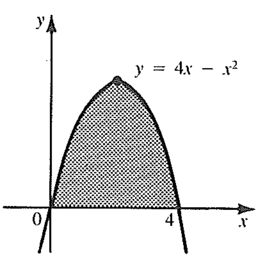
(o) $\frac{10}{11}$



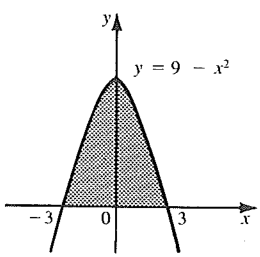
(p) 309.3



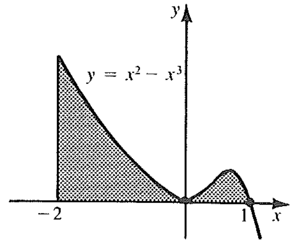
2. (a) $10\frac{2}{3}$



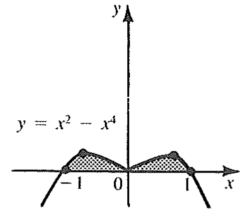
(b) 36



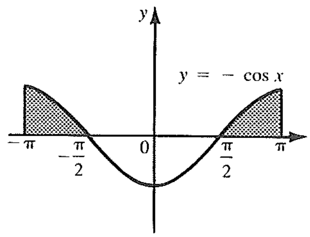
(c) 6.75

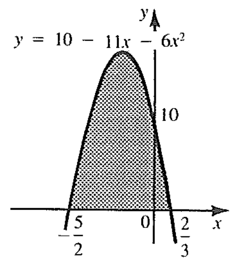


(d) $\frac{4}{15}$

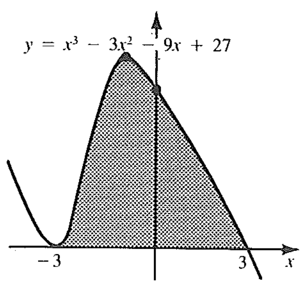


(e) 2

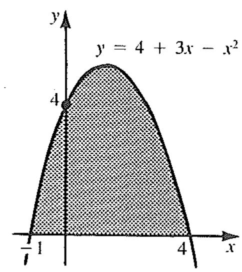




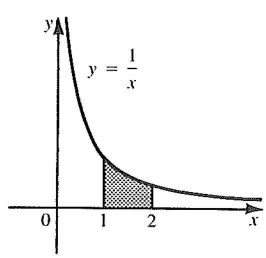
(g) 108



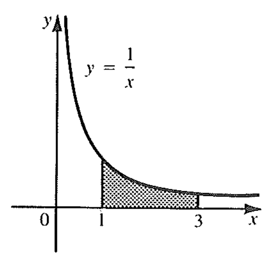
(h) $\frac{125}{6}$



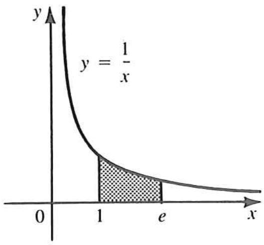
3. (a) ln 2

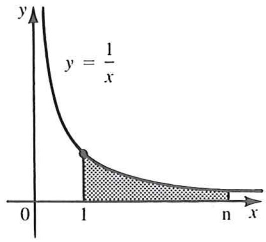


(b) ln 3

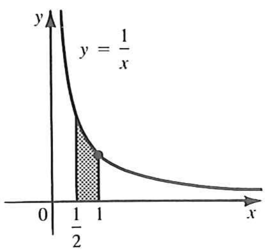


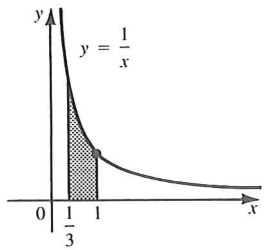
(c) $\ln e = 1$ (d) $\ln n$



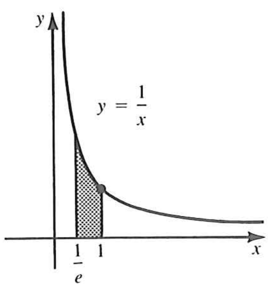


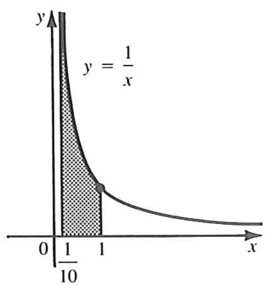
4. (a) $\ln 2$ (b) $\ln 3$



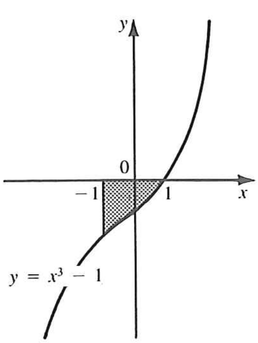


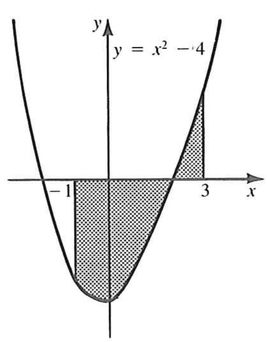
(c) $\ln e = 1$ (d) $\ln 10$



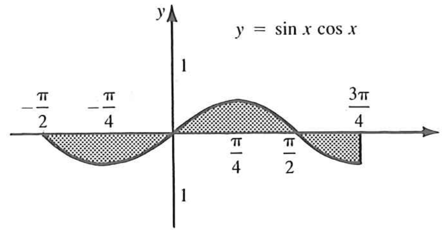


5. 2 **6.** $11\frac{1}{3}$



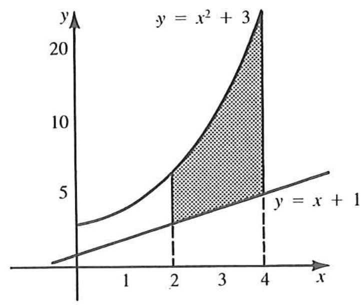


7. 1.25

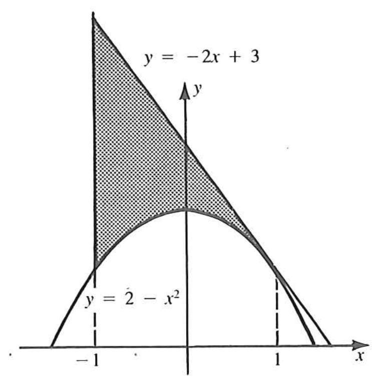


EXERCISE 10.2

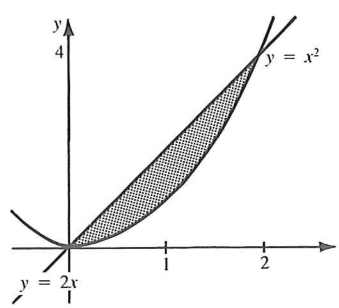
1. (a) $\frac{50}{3}$



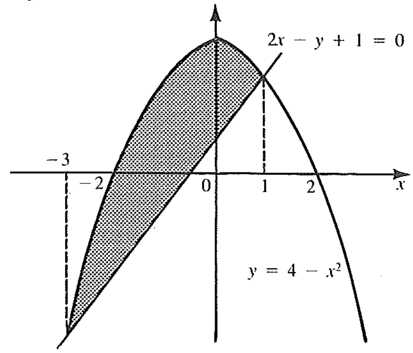
(b) $\frac{8}{3}$



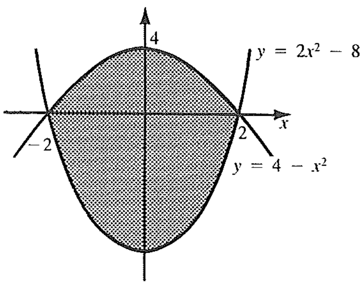
(c) $\frac{4}{3}$



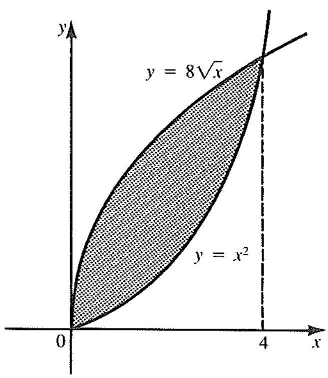
(d) $\frac{32}{3}$



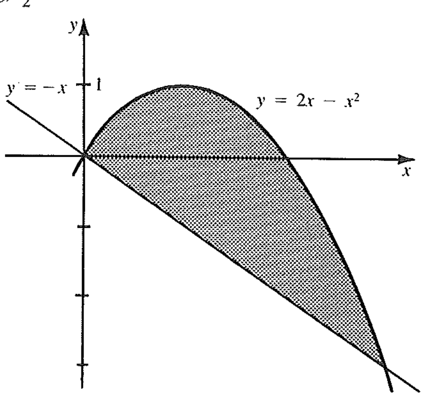
(e) 32



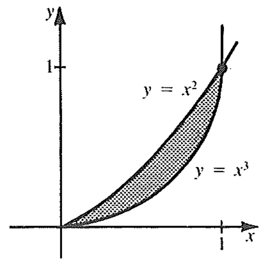
(f) $\frac{64}{3}$



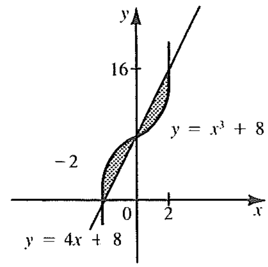
(g) $\frac{9}{2}$



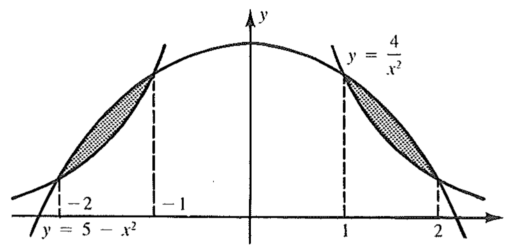
(h) $\frac{1}{12}$



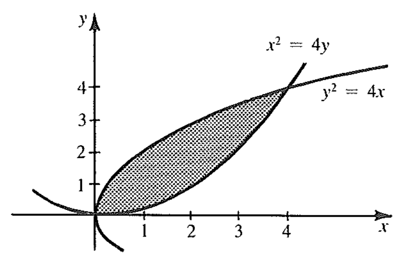
(i) 8



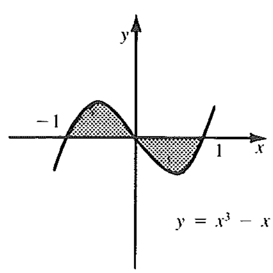
(j) $\frac{4}{3}$



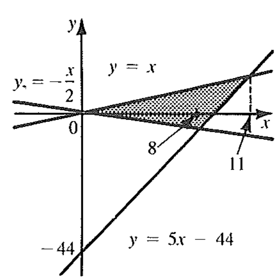
(k) $\frac{16}{3}$



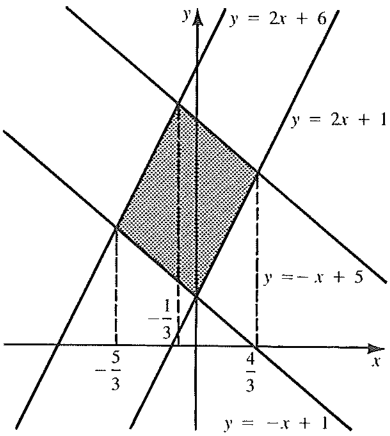
(l) $\frac{1}{2}$



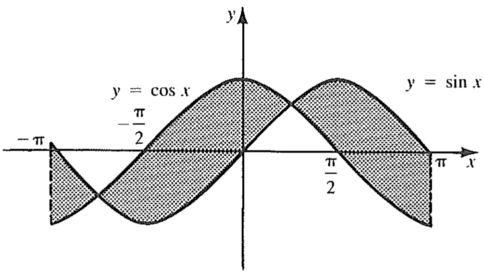
(m) 66



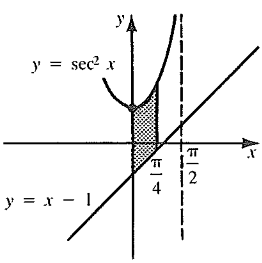
(n) $\frac{20}{3}$



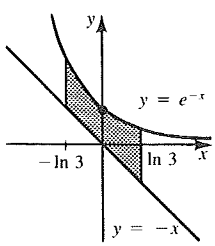
(o) $4\sqrt{2}$



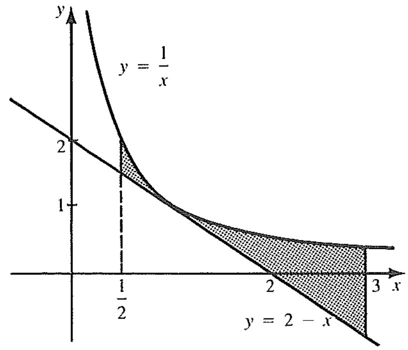
(p) $\dfrac{32 - \pi^2 + 8\pi}{32}$



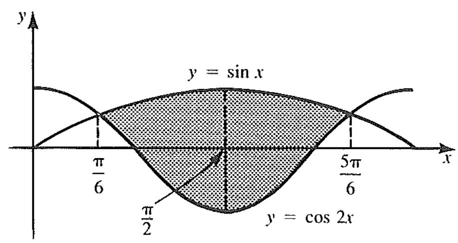
(q) $\frac{8}{3}$



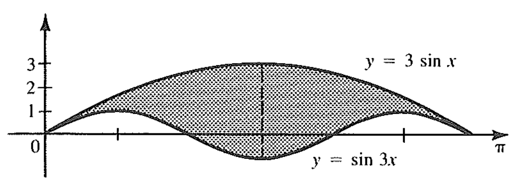
(r) $\ln 3 + \ln 2 - \frac{5}{8}$



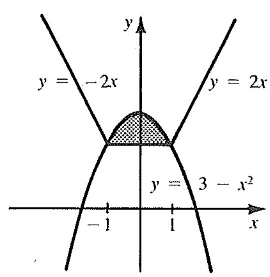
(s) $\dfrac{3\sqrt{3}}{2}$

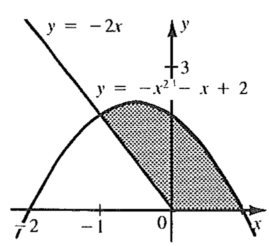


(t) $\frac{16}{3}$

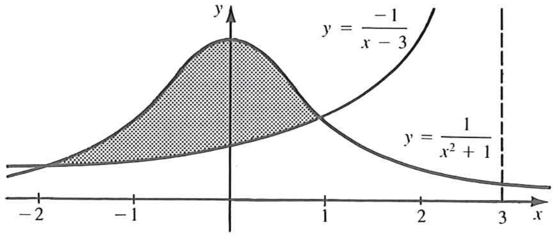


2. (a) $\frac{4}{3}$ (b) $\frac{7}{3}$

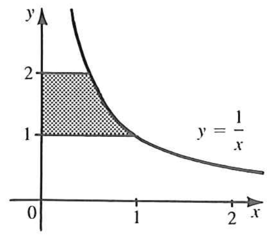




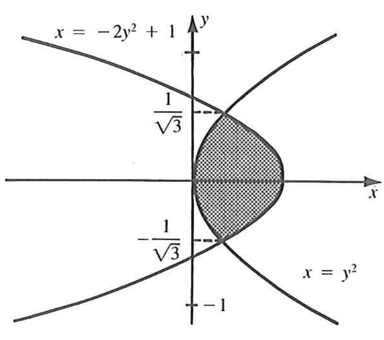
(c) $\frac{\pi}{4} + \tan^{-1} 2 + \ln \frac{2}{5}$



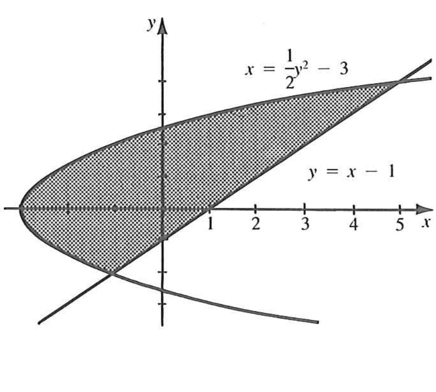
3. (a) ln 2



(b) $\frac{4\sqrt{3}}{9}$



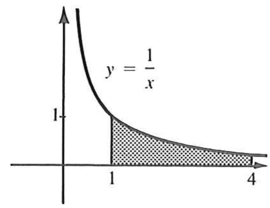
(c) 18



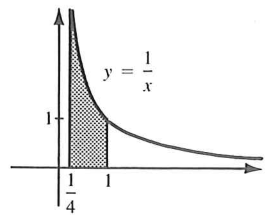
EXERCISE 10.3

1. (a) ln 1.5 (b) ln 2 (c) ln 3 (d) ln 4

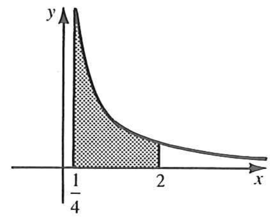
2. (a)



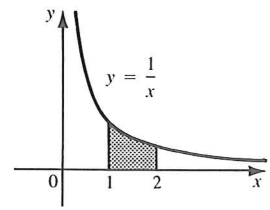
(b)



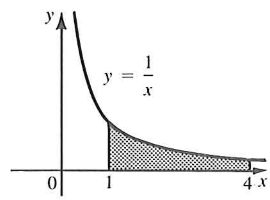
(c)



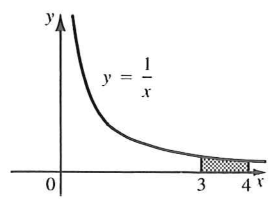
(d)



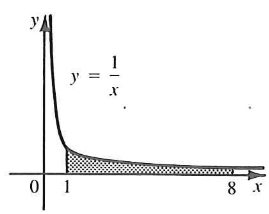
(e)



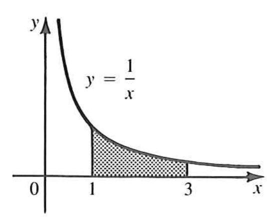
(f)



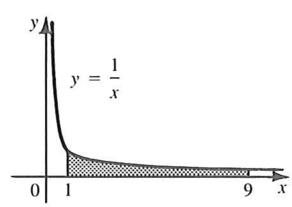
(g)



(h)



(i)



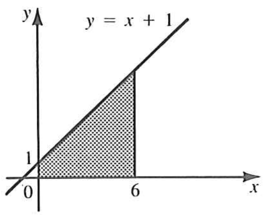
5. (a) $x + 36y - 12 = 0$ (b) Area of the trapezoid determined by the tangent in (a).

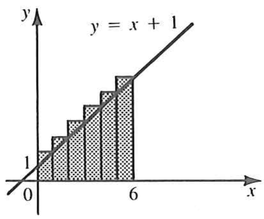
6. (b) Area of the trapezoid with vertices $(3, 0)$, $(6, 0)$, $\left(6, \frac{1}{6}\right)$, $\left(3, \frac{1}{3}\right)$

7. (b) Area of the trapezoid with vertices $(18, 0)$, $(36, 0)$, $\left(36, \frac{1}{36}\right)$, $\left(18, \frac{1}{18}\right)$

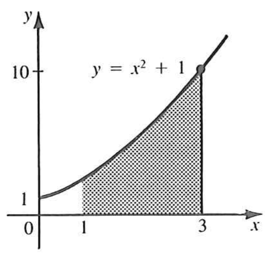
EXERCISE 10.4

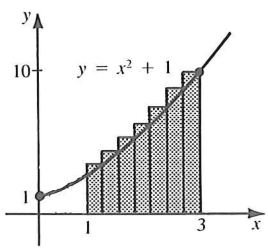
1. (a) 24 (b) 28



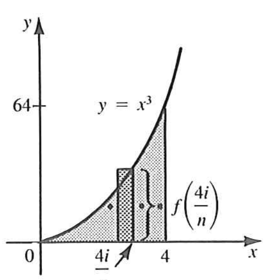


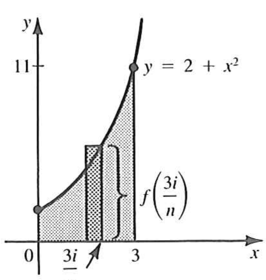
2. (a) $\frac{32}{3}$ (b) 11.48



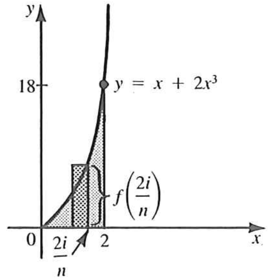


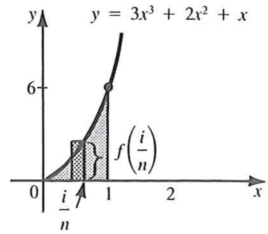
3. (a) 64 (b) 15



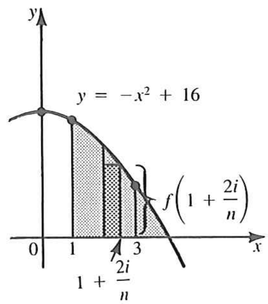


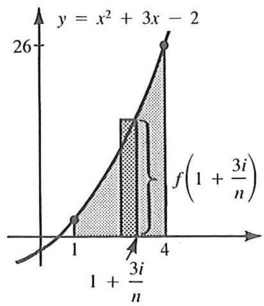
(c) 10 (d) $\frac{23}{12}$



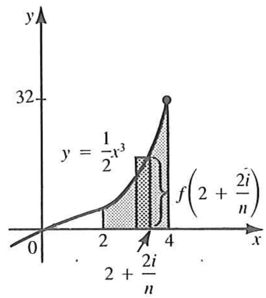


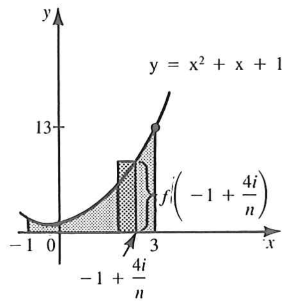
4. (a) $\frac{70}{3}$ (b) $\frac{75}{2}$



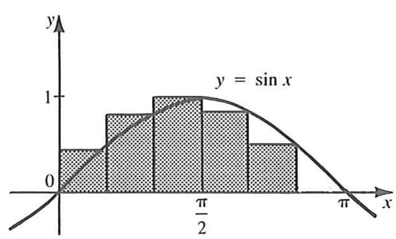


(c) 30 (d) $\frac{52}{3}$

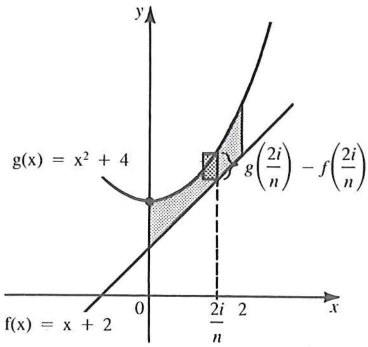




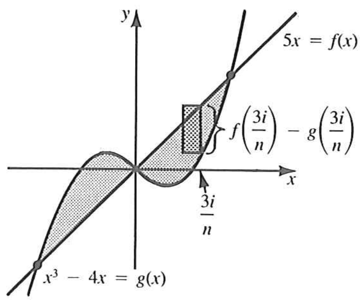
5. $\frac{\pi}{6}(1 + \sqrt{3})$



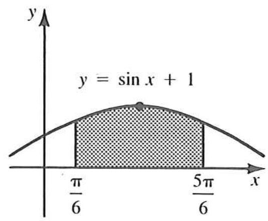
6. (a) $\frac{14}{3}$

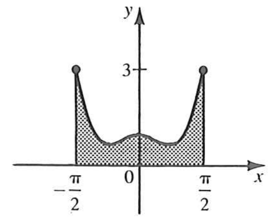


(b) $\frac{81}{2}$

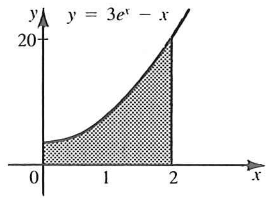


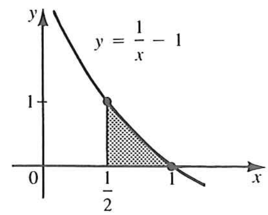
(c) $\frac{2\pi}{3} + \sqrt{3}$ (d) $\frac{\pi^3}{12} + 2\sqrt{2}$



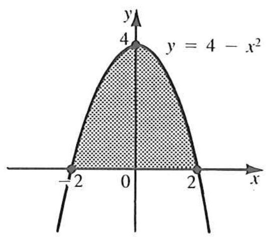


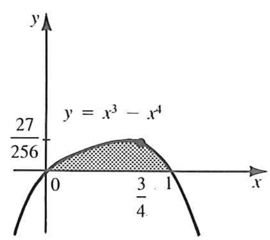
(e) $3e^2 - 5$ (f) $\ln 2 - \frac{1}{2}$



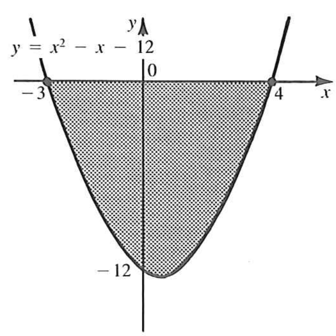


2. (a) $\frac{32}{3}$ (b) $\frac{1}{20}$





(c) $\frac{343}{6}$



EXERCISE 10.5

1. (a) (i) 6.521 610 (ii) 6.322 986
(b) $e^2 - 1 \doteq 6.389\,056$

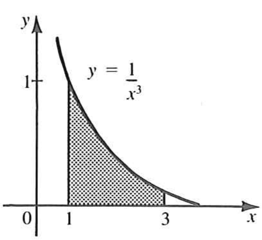
2. (a) (i) 1.657 458 (ii) 1.732 039
(b) $\cos \frac{\pi}{4} - \cos \pi \doteq 1.707\,107$

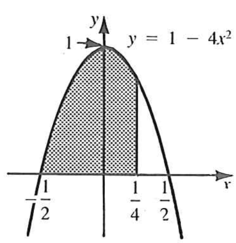
3. 40.862 771 **4.** 0.343 793 **5.** 1.116 667

6. 0.329 675

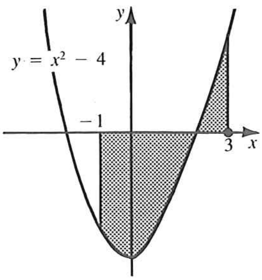
EXERCISE 10.6

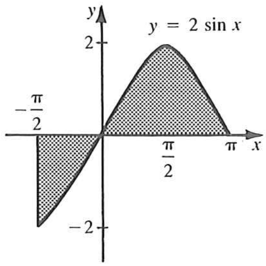
1. (a) $\frac{4}{9}$ (b) $\frac{9}{16}$



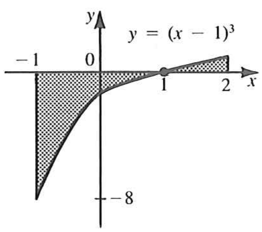


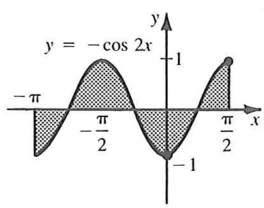
3. (a) $\frac{34}{3}$ (b) 6



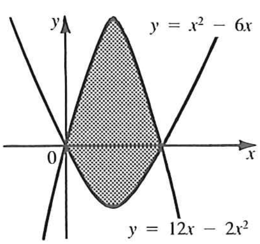


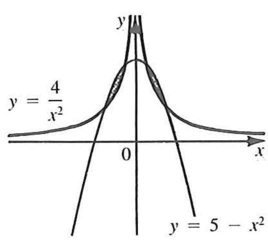
(c) $\frac{17}{4}$ (d) 3



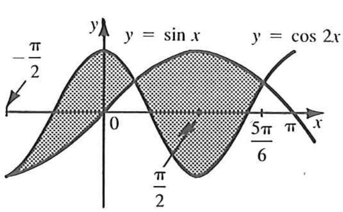


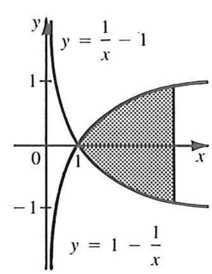
4. (a) 108 (b) $\frac{4}{3}$



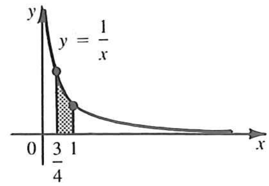


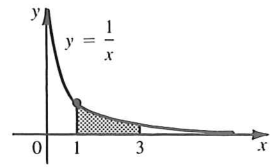
(c) $\dfrac{9\sqrt{3}}{4}$ (d) $6 - \ln 16$





5. (a) (b)





6. (a) ln 3 (b) trapezoid with vertices $(4, 0)$, $(12, 0)$ $\left(12, \frac{1}{12}\right)$, $\left(4, \frac{1}{4}\right)$ and area $\doteq 1.333\,333$

8. (a) $\frac{4}{3}$ (b) 18 (c) $393\frac{1}{3}$

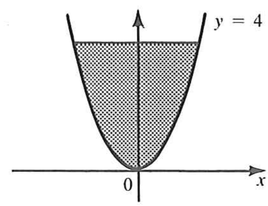
9. (a) 0.789 219 (b) 2.866 105

10. 37.326 155

11. 2.166 253 if the height is determined by the midpoint of the base

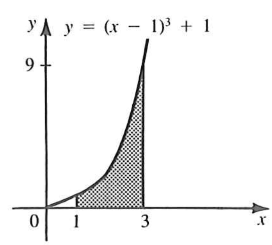
12. (a) $\frac{1}{6}$ (b) $\frac{1}{4}$

13. (a) $\frac{32}{3}$ (b) $2\sqrt[3]{2}$

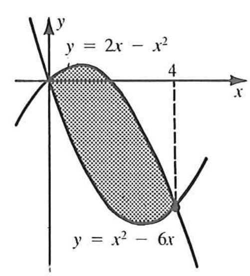


10.7 CHAPTER 10 TEST

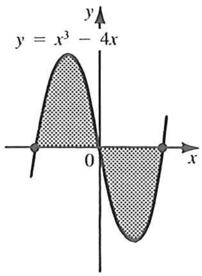
1. 6



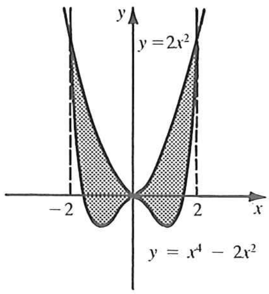
2. $\frac{64}{3}$



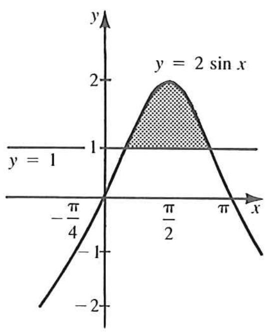
3. 8



4. $\frac{128}{15}$



5. $4\sqrt{3} - \sqrt{2} - 1 - \dfrac{5\pi}{12}$



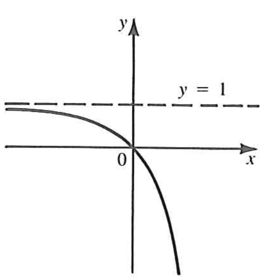
6. (a) ln 2.5 (b) the area of the trapezoid with vertices (2, 0), (5, 0), $\left(5, \frac{1}{5}\right)$, $\left(2, \frac{1}{2}\right)$ is 1.05

7. 18 **8.** 0.537 607 **9.** 77.177 154

10. $\frac{95}{6}$

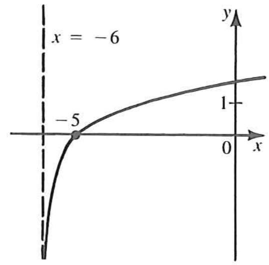
CUMULATIVE REVIEW FOR CHAPTERS 8 TO 10

1. (a)



(b) R, $(-\infty, 1)$, $y = 1$

2. (a)



(b) $(-6, \infty)$, R, $x = -6$

3. (a) 1 (b) $-\infty$ **4.** (a) 2 (b) 9

5. (a) $\frac{1}{2}(\ln 20 - 1) \doteq 0.997\ 866$

(b) $1 - \dfrac{1}{e^2} \doteq 0.864\ 665$

6. (a) $y' = -(3 + 4x)e^{3-4x}$

(b) $y' = \dfrac{2x^2 - (x^2 + 1)\ln(x^2 + 1)}{x^2(x^2 + 1)}$

(c) $y' = \dfrac{2xe^{x^2}}{1 + e^{x^2}}$ (d) $y' = -\dfrac{10^{-\sqrt{x}} \ln 10}{2\sqrt{x}}$

(e) $y' = \frac{1}{2}\left(\dfrac{1}{x} + \dfrac{3x^2}{1 - x^3}\right)$

(f) $y' = x^{\tan x}\left(\sec^2 x \ln x + \dfrac{\tan x}{x}\right)$

7. (a) $g'(x) = e^{f(x)}f'(x)$ (b) $h'(x) = f'(e^x)e^x$

8. $3x - y = 3(\ln 3 - 1)$

9. (a) increasing on $(-1, \infty)$, decreasing on $(-\infty, -1)$ (b) minimum $f(-1) = -\frac{1}{e}$

(c) CU on $(-2, \infty)$, CD on $(-\infty, -2)$, IP $(-2, -2e^{-2})$

10. (a) $F(x) = 3x^4 - 3x^3 + 4x^2 + 31x + C$

(b) $F(x) = -2 \cos 2x + \frac{5}{3} \sin(3x + 1) + C$

(c) $F(x) = -\frac{2}{3}e^{3x} - \frac{1}{12}e^{-4x} + C$

11. (a) $F(x) = \sqrt{2} \ln(x + 1) - \sqrt{3} \ln x + c$

(b) $F(x) = \frac{2}{3}(\sqrt{2} + \sqrt{5} + \sqrt{8})x^{\frac{3}{2}} + C$

12. (a) $F(x) = x^3 + x^2 - 9$

(b) $F(x) = 2\sqrt{x} - \frac{1}{2}x^2 + 5 - 2\sqrt{2}$

(c) $F(x) = \frac{3}{4}(e^{4x} - e^8) + 3$ **13.** 3.9 s

14. (a) $1200\left(\frac{10}{3}\right)^t$ (b) 44 444

(c) 53 510 bacteria/h (c) 1.76 h

15. (a) $(50)e^{-\frac{\ln 2}{19.7}t}$ (b) 0.73 g (c) 0.026 g/min (d) 1 h, 51 min

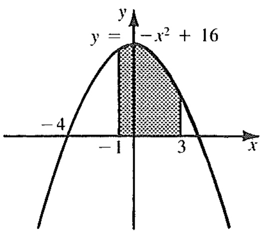
16. 11 min **17.** 4.53 kg

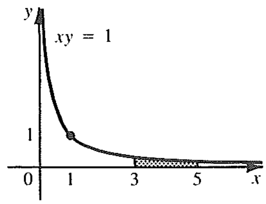
18. two years and eleven months

19. $s = 1.7 \cos 1.2t + 1.5 \sin 1.2t$

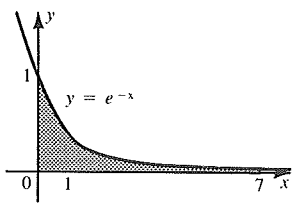
20. (a) $\frac{2\pi}{5}s$ (b) 0.75 m/s

21. (a) $\frac{164}{3}$ (b) $\ln \frac{5}{3}$

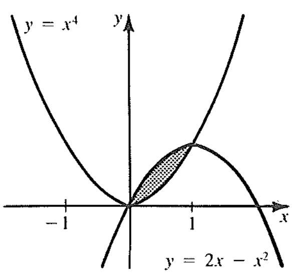


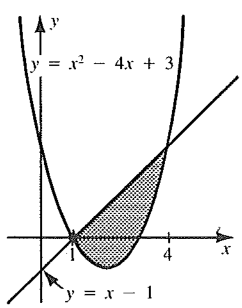


(c) $\frac{e^7 - 1}{e^7}$

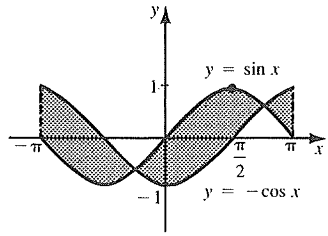


22. (a) $\frac{7}{15}$ (b) $\frac{9}{2}$

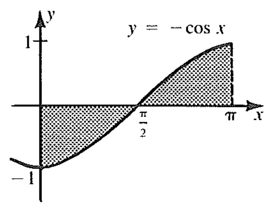


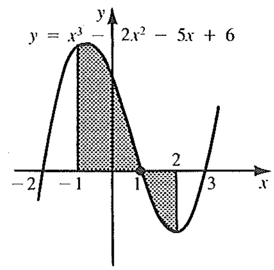


(c) $\frac{8}{\sqrt{2}}$



(d) 2 (e) $\frac{157}{12}$





23. (a) $\frac{1}{2}ma^2$ (b) $\frac{4}{3}$ **24.** 161.601 142

25. 1.860 176