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CHAPTER 5

CURVE SKETCHING

REVIEW AND PREVIEW TO CHAPTER 5

Intercepts

To find the x-intercepts of $y = f(x)$, set $y = 0$ and solve for x.
To find the y-intercept of $y = f(x)$, set $x = 0$; the y-intercept is $f(0)$.

EXERCISE 1

1. Find the intercepts of the following curves.

(a) $y = 25 - 4x^2$

(b) $y = 2x^2 - x - 1$

(c) $y = \dfrac{x^2 + 2x - 3}{x^2 + 1}$

(d) $y = x^2 + x + 1$

(e) $y = 3x^2 + 4x - 6$

(f) $y = x^3 - 3x$

(g) $y = x^3 - x^2 - x + 1$

(h) $y = 2x^3 - 9x^2 - 18x$

(i) $y = x^3 + 8$

(j) $y = x^4 - 16$

2. Find the intercepts of the curve $y = 9x - x^3$ and use them, together with the methods of Chapter 4, to sketch the curve.

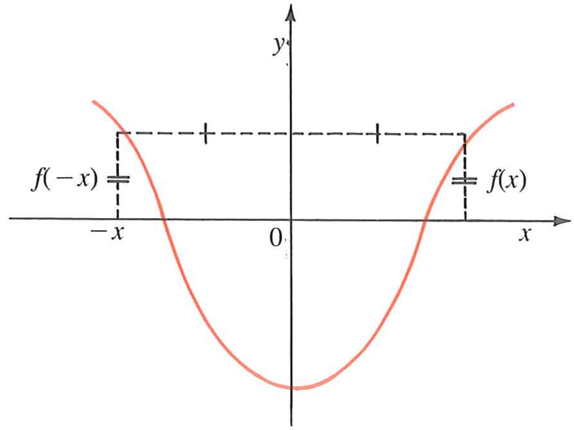
3. Use Newton's method to find the x-intercepts of the curve $y = x^3 - 3x + 1$ correct to two decimal places.

Symmetry

An **even function** satisfies

$$f(-x) = f(x)$$

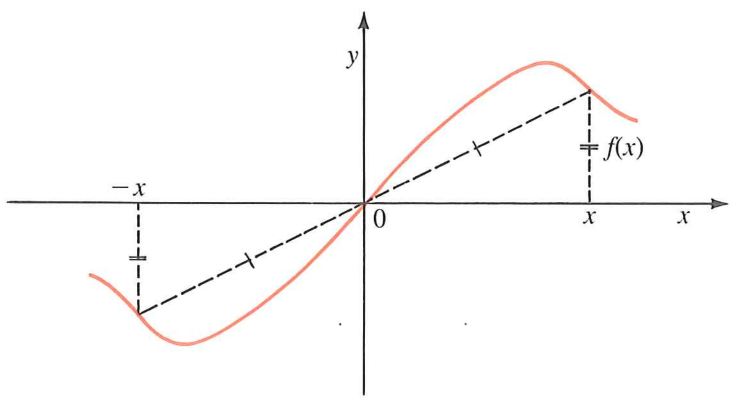
for all x in its domain. Thus, a function is even if it is unchanged when x is replaced by $-x$. The graph of an even function is symmetric about the y-axis.



An **odd function** satisfies

$$f(-x) = -f(x)$$

for all x in its domain. The graph of an odd function is symmetric about the origin.



Symmetry is used to reduce the amount of work in graphing. If we have graphed an even function for $x \geqslant 0$, we just reflect in the y-axis to get the entire graph. For an odd function we just rotate through 180° about the origin.

Example Determine whether each function is even, or odd, or neither.

(a) $f(x) = x^6$ (b) $g(x) = x^3 + \dfrac{1}{x}$

Solution (a) $f(-x) = (-x)^6$
$= (-1)^6 x^6$
$= x^6$
$= f(x)$

Thus f is even.

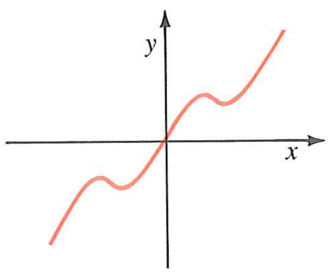
(b) $g(-x) = (-x)^3 + \dfrac{1}{-x}$
$= -x^3 - \dfrac{1}{x}$
$= -\left(x^3 + \dfrac{1}{x}\right)$
$= -g(x)$

Thus g is odd.

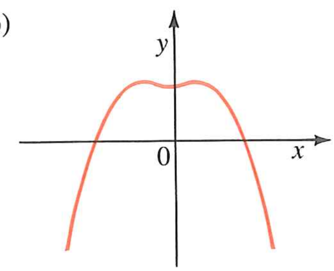
EXERCISE 2

1. State whether the functions of the following graphs are even, odd, or neither.

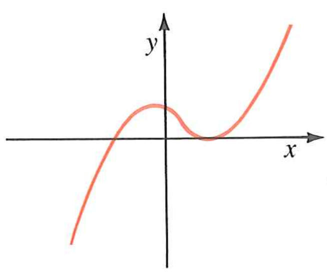
(a)



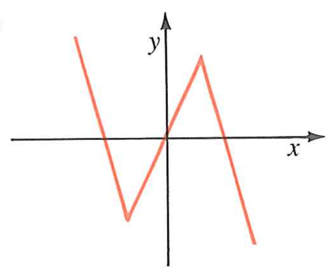
(b)



(c)



(d)



2. Determine whether the function is even, odd, or neither.

(a) $f(x) = x^2$

(b) $f(x) = x^3$

(c) $g(x) = x^2 + x^3$

(d) $g(x) = \dfrac{2}{x^4 + 1}$

(e) $h(x) = (x + x^5)^3$

(f) $h(x) = x^6(1 + x - x^2)$

(g) $y = |x|$

(h) $y = \dfrac{x^3}{x^4 + x^2 + 1}$

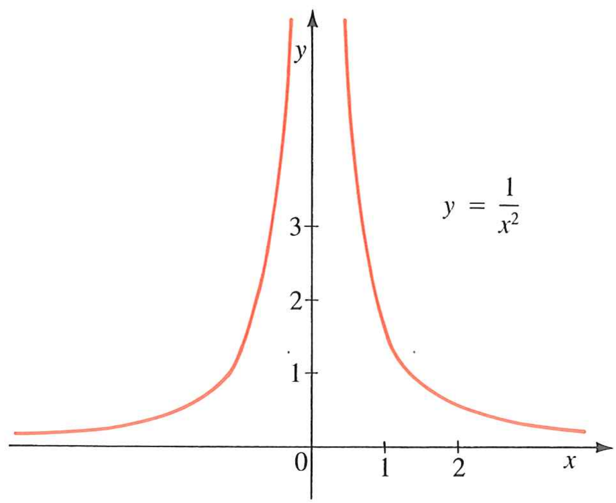
INTRODUCTION

In this chapter we look at further aspects of curves—vertical asymptotes, horizontal asymptotes, concavity, and inflection points. Then we use them, together with intervals of increase and decrease and maximum and minimum values, to develop a procedure for curve sketching.

5.1 VERTICAL ASYMPTOTES

Let us examine the behaviour of the function $f(x) = \frac{1}{x^2}$ for x close to 0.

x	$f(x) = \frac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10 000
± 0.001	1 000 000



The values in the table and the graph show that the closer we take x to 0, the larger $\frac{1}{x^2}$ becomes. In fact, it appears that by taking x close enough to 0, we can make $f(x)$ as large as we like. We indicate this type of behaviour by writing

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

and we say that the line $x = 0$ is a *vertical asymptote* of $y = \dfrac{1}{x^2}$.

Notice that f is not defined at $x = 0$, so f has a discontinuity at 0 (see Section 1.3). This type of discontinuity is called an **infinite discontinuity**.

In general, if $f(x)$ is defined on both sides of the number a, we write symbolically

$$\lim_{x \to a} f(x) = \infty$$

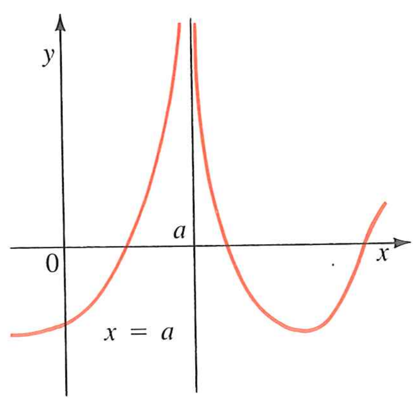
if the values of $f(x)$ can be made arbitrarily large (as large as we like) by taking x sufficiently close to a, but not equal to a. This type of limit is called an **infinite limit**. Roughly speaking, the values of $f(x)$ become larger and larger (or "increase without bound") as x gets closer and closer to a.

The symbol ∞ is not a number, but the expression

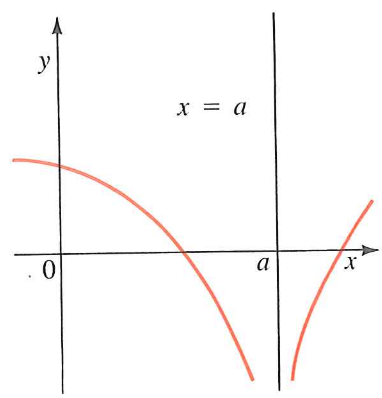
$$\lim_{x \to a} f(x) = \infty$$

is usually read as

"the limit of $f(x)$, as x approaches a, is infinity"



$$\lim_{x \to a} f(x) = \infty$$



$$\lim_{x \to a} f(x) = -\infty$$

These figures illustrate the definition of an infinite limit and a similar type of limit, denoted by

$$\lim_{x \to a} f(x) = -\infty$$

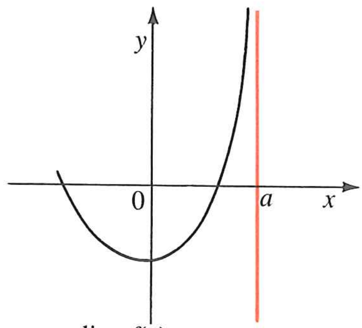
for functions that become large negatives as x becomes close to a.

Similar definitions can be given for the **one-sided infinite limits**

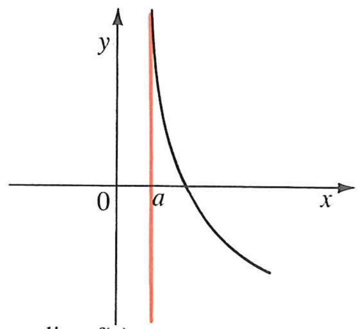
$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

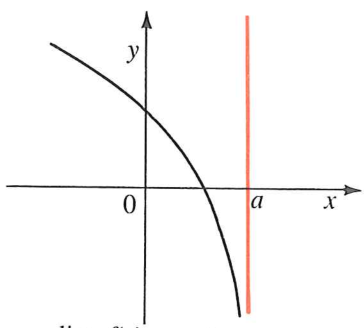
remembering that "$x \to a^-$" means that x approaches a from the left and "$x \to a^+$" means that x approaches a from the right. If any of these four statements is true, we say that $x = a$ is a **vertical asymptote**. Illustrations are shown in the following figures.



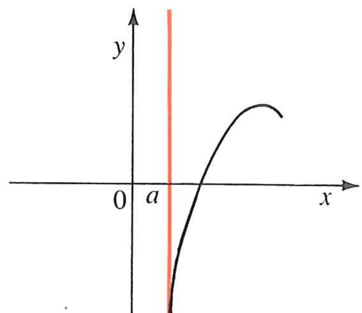
$\lim_{x \to a^-} f(x) = \infty$



$\lim_{x \to a^+} f(x) = \infty$



$\lim_{x \to a^-} f(x) = -\infty$



$\lim_{x \to a^+} f(x) = -\infty$

Example 1 Find $\lim_{x \to 0^+} \frac{1}{x}$ and $\lim_{x \to 0^-} \frac{1}{x}$.

Solution If x is close to 0, but positive, then $\frac{1}{x}$ is a large positive number. For instance,

$$\frac{1}{0.01} = 100 \qquad \frac{1}{0.000\,1} = 10\,000 \qquad \frac{1}{0.000\,001} = 1\,000\,000$$

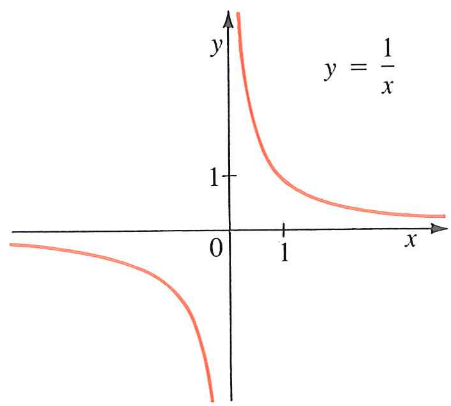
As x approaches 0 from the right, $\frac{1}{x}$ becomes increasingly large. Therefore,

$$\lim_{x\to 0^+} \frac{1}{x} = \infty$$

If x is close to 0, but negative, then $\frac{1}{x}$ is a large negative number. Thus,

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty$$

These limits can also be seen from the graph of the hyperbola $y = \frac{1}{x}$. The line $x = 0$ (the y-axis) is a vertical asymptote.



Example 2 Find $\lim_{x\to 6} \left[2 - \frac{5}{(x-6)^2}\right]$.

Solution If x is close to 6 (on either side of 6), $(x - 6)^2$ is a small positive number, and so

$$-\frac{5}{(x-6)^2}$$

is a large negative number. Therefore,

$$\lim_{x\to 6} \left[2 - \frac{5}{(x-6)^2}\right] = -\infty$$

To find the vertical asymptotes of a rational function, we find the values of x where the denominator is zero and compute the limits of the function from the right and left.

Example 3 (a) Find the vertical asymptotes of the function $y = \dfrac{x}{x^2 - x - 6}$.

(b) Sketch the graph near the asymptotes.

Solution (a) First we factor the denominator:

$$y = \frac{x}{(x-3)(x+2)}$$

Since the denominator is 0 when $x = 3$ or -2, the lines $x = 3$ and $x = -2$ are candidates for vertical asymptotes.

If x is close to 3, but $x > 3$, the denominator is close to 0 and $y > 0$ (since $x > 0$, $x - 3 > 0$, and $x + 2 > 0$). Symbolically, we could write

$$\frac{x}{(x-3)(x+2)} = \frac{[\text{positive}]}{[\text{small positive}][\text{positive}]} = \text{large positive}$$

Therefore,

$$\lim_{x\to 3^+} \frac{x}{(x-3)(x+2)} = \infty$$

If x is close to 3, but $x < 3$, we have

$$\frac{x}{(x-3)(x+2)} = \frac{[\text{positive}]}{[\text{small negative}][\text{positive}]} = \text{large negative}$$

and so $\displaystyle\lim_{x\to 3^-} \frac{x}{(x-3)(x+2)} = -\infty$

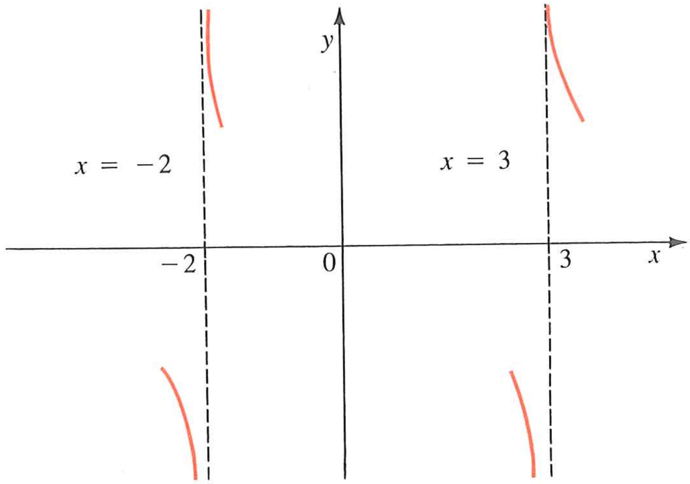
Similar reasoning gives

$$\lim_{x\to -2^+} \frac{x}{(x-3)(x+2)} = \infty$$

and $\displaystyle\lim_{x\to -2^-} \frac{x}{(x-3)(x+2)} = -\infty$

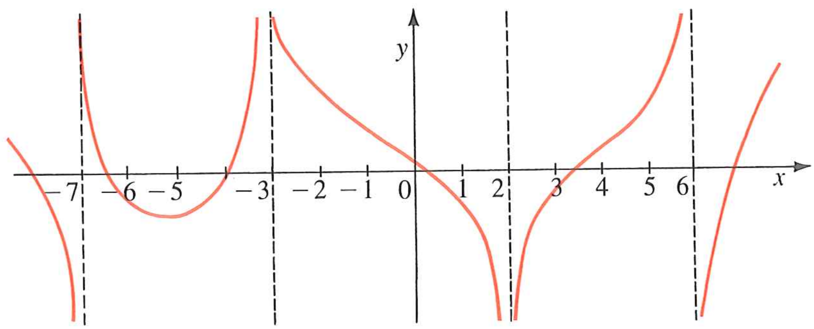
The vertical asymptotes are $x = 3$ and $x = -2$.

(b) Using the information from part (a), we can sketch the part of the graph that lies near the asymptotes. We will be able to complete the picture with the information from the next section.



EXERCISE 5.1

A **1.** The graph of f is given.



(a) State the equations of the vertical asymptotes.

(b) State the following.

(i) $\lim\limits_{x \to -7^-} f(x)$ (ii) $\lim\limits_{x \to -7^+} f(x)$

(iii) $\lim\limits_{x \to -3} f(x)$ (iv) $\lim\limits_{x \to 2} f(x)$

(v) $\lim\limits_{x \to 6^-} f(x)$ (vi) $\lim\limits_{x \to 6^+} f(x)$

B **2.** Find each limit.

(a) $\lim\limits_{x \to 8} \dfrac{1}{(x - 8)^2}$ (b) $\lim\limits_{x \to 1^-} \dfrac{3}{x - 1}$

(c) $\lim\limits_{x \to 1^+} \dfrac{3}{x - 1}$ (d) $\lim\limits_{x \to -1} \dfrac{-2}{(x + 1)^2}$

(e) $\lim_{x \to 2^+} \frac{x-4}{x-2}$ (f) $\lim_{x \to 2^-} \frac{x-4}{x-2}$

(g) $\lim_{x \to -4} \left[1 + \frac{2x}{(x+4)^6}\right]$ (h) $\lim_{x \to 3^+} \left[x + \frac{2-x}{x-3}\right]$

(i) $\lim_{x \to -2^+} \frac{x}{x^2-4}$ (j) $\lim_{x \to -2^-} \frac{x}{x^2-4}$

(k) $\lim_{x \to 9^+} \frac{5-x}{\sqrt{x-9}}$ (l) $\lim_{x \to -3^+} \frac{10}{x^2-x-12}$

3. Find the vertical asymptotes and sketch the graph near the asymptotes.

(a) $y = \frac{2}{x+1}$ (b) $y = \frac{3}{(x-6)^2}$

(c) $y = \frac{x}{(x+2)^2}$ (d) $y = \frac{1}{x^2-1}$

(e) $y = \frac{x}{x^2-1}$ (f) $y = \frac{6x^3}{x^2+4x+3}$

(g) $y = \frac{1}{x^2(x+1)}$ (h) $y = \frac{1}{x^4-4x^2}$

C **4.** Find $\lim_{x \to 0^+} \left(\frac{5}{x} - \frac{2}{x^2}\right)$.

5. How small do we have to take x so that $\frac{1}{x^4} > 100\,000\,000$?

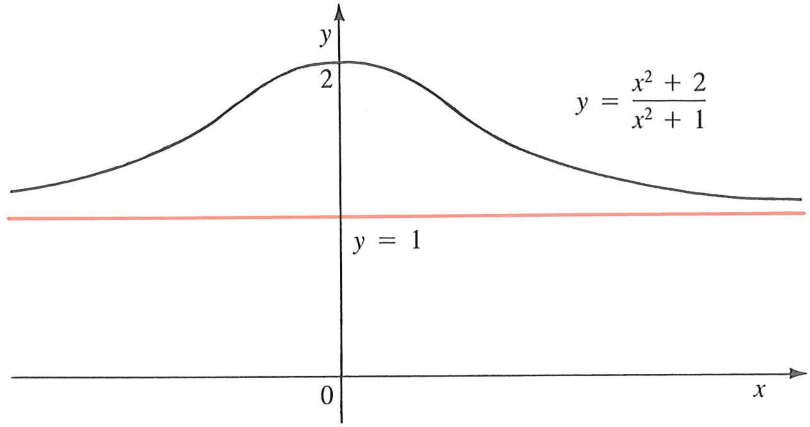
5.2 HORIZONTAL ASYMPTOTES

The table gives the values, correct to six decimal places, of the function

$$f(x) = \frac{x^2+2}{x^2+1}$$

for large values of x.

x	$f(x) = \frac{x^2+2}{x^2+1}$
0	2.000 000
±1	1.500 000
±2	1.200 000
±3	1.100 000
±4	1.058 824
±5	1.038 462
±10	1.009 901
±50	1.000 400
±100	1.000 100
±1000	1.000 001



From the table and the graph we see that the values of $f(x)$ become closer and closer to 1 as x grows larger and larger. In fact, it appears that by taking x large enough, we can make $f(x)$ as close to 1 as we like. We indicate this type of behaviour by writing

$$\lim_{x \to \infty} \frac{x^2 + 2}{x^2 + 1} = 1$$

and we say that the line $y = 1$ is a *horizontal asymptote*.

Similarly, we see that, for large negative values of x, the values of $f(x)$ are close to 1. By letting x decrease through negative values without bound, we can make $f(x)$ as close as we like to 1. We express this by writing

$$\lim_{x \to -\infty} \frac{x^2 + 2}{x^2 + 1} = 1$$

In general, we define a **limit at infinity** by writing

$$\lim_{x \to \infty} f(x) = L$$

if the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large. In other words, the values of $f(x)$ become closer and closer to L as x gets larger and larger. (Notice the similarity to the limit of a sequence in Section 1.6.)

Again the symbol ∞ is not a number, but the expression

$$\lim_{x \to \infty} f(x) = L$$

is read as

"the limit of $f(x)$, as x approaches infinity, is L"

Similarly, the notation

$$\lim_{x \to -\infty} f(x) = L$$

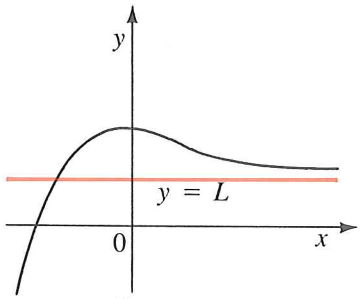
means that the values of $f(x)$ approach L as x becomes large negative.

The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

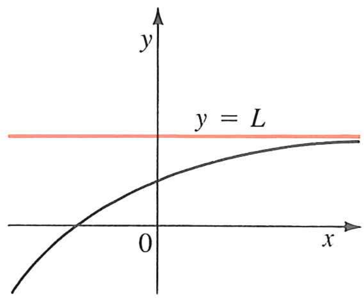
$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L$$

For instance, the line $y = 1$ is a horizontal asymptote of the function f considered at the beginning of this section.

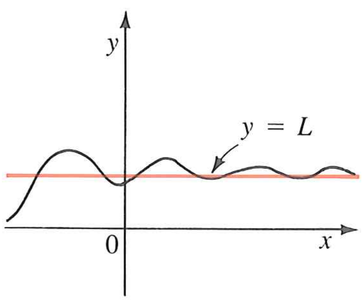
The figures show that there are many ways for a curve to approach a horizontal asymptote.



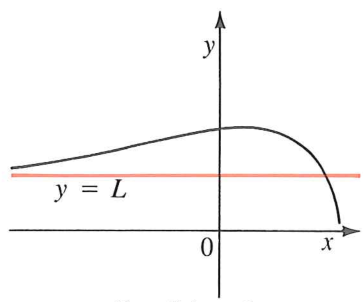
$\lim_{x \to \infty} f(x) = L$



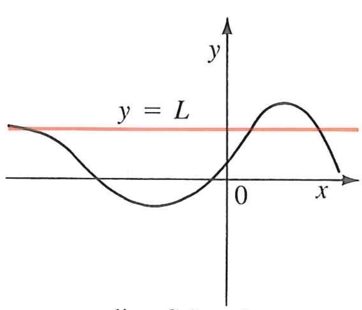
$\lim_{x \to \infty} f(x) = L$



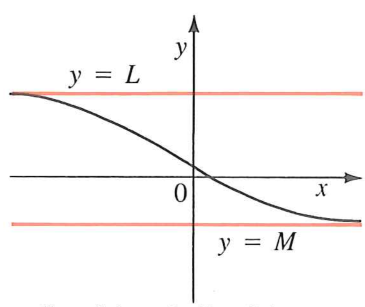
$\lim_{x \to \infty} f(x) = L$



$\lim_{x \to -\infty} f(x) = L$



$\lim_{x \to -\infty} f(x) = L$



$\lim_{x \to -\infty} f(x) = L, \lim_{x \to \infty} f(x) = M$

Example 1 Find $\lim_{x \to \infty} \frac{1}{x}$ and $\lim_{x \to -\infty} \frac{1}{x}$.

Solution When x is large, its reciprocal $\frac{1}{x}$ is small. For instance,

$$\frac{1}{100} = 0.01 \qquad \frac{1}{10\,000} = 0.0001 \qquad \frac{1}{1\,000\,000} = 0.000\,001$$

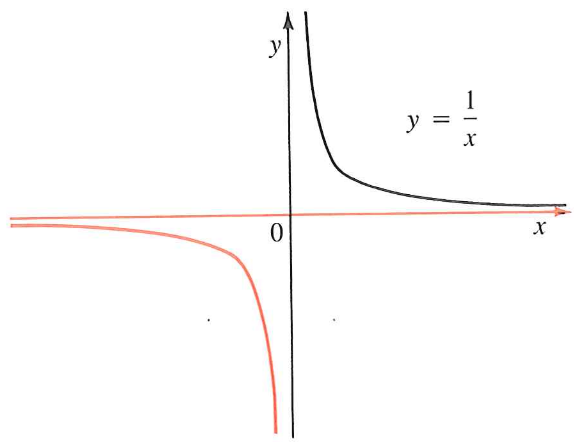
In fact, by taking x large enough, we can make $\frac{1}{x}$ as close to 0 as we like. Therefore,

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Likewise, when x is large negative, $\frac{1}{x}$ is small negative, so

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

Thus, the line $y = 0$ (the x-axis) is a horizontal asymptote of the curve $y = \frac{1}{x}$, as illustrated by the figure.



The seven properties of limits that were stated in Section 1.2 are also valid for finite limits at infinity. In particular, if we combine the results of Example 1 with Properties 6 and 7, we get the following rule.

If r is a positive rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

If r is a positive rational number and x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

In the following example, we use this rule to compute a more complicated limit.

Example 2 Evaluate $\displaystyle\lim_{x \to \infty} \frac{4x^2 - x + 2}{6x^2 + 5x + 1}$.

Solution To evaluate the limit at infinity of a rational function, we first divide the numerator and denominator by the highest power of x that occurs. (We can assume that $x \neq 0$ because we are interested only in large values of x.) In this case the highest power of x is x^2, so we proceed as follows:

$$\begin{aligned}
\lim_{x \to \infty} \frac{4x^2 - x + 2}{6x^2 + 5x + 1} &= \lim_{x \to \infty} \frac{\dfrac{4x^2 - x + 2}{x^2}}{\dfrac{6x^2 + 5x + 1}{x^2}} \\
&= \lim_{x \to \infty} \frac{4 - \dfrac{1}{x} + \dfrac{2}{x^2}}{6 + \dfrac{5}{x} + \dfrac{1}{x^2}} \\
&= \frac{\displaystyle\lim_{x \to \infty} 4 - \lim_{x \to \infty} \frac{1}{x} + 2 \lim_{x \to \infty} \left(\frac{1}{x^2}\right)}{\displaystyle\lim_{x \to \infty} 6 + 5 \lim_{x \to \infty} \frac{1}{x} + \lim_{x \to \infty} \left(\frac{1}{x^2}\right)} \\
&= \frac{4 - 0 + 2(0)}{6 + 5(0) + 0} \\
&= \frac{4}{6} \\
&= \frac{2}{3}
\end{aligned}$$

Example 3 Find the horizontal and vertical asymptotes of the function

$$y = \frac{x+1}{x-2}$$

and sketch its graph.

Solution We find the horizontal asymptotes by computing the limit at infinity. The first step is to divide the numerator and denominator by x.

$$\begin{aligned}
\lim_{x\to\infty} \frac{x+1}{x-2} &= \lim_{x\to\infty} \frac{1+\dfrac{1}{x}}{1-\dfrac{2}{x}} \\
&= \frac{\lim\limits_{x\to\infty} 1 + \lim\limits_{x\to\infty} \dfrac{1}{x}}{\lim\limits_{x\to\infty} 1 - 2\lim\limits_{x\to\infty} \dfrac{1}{x}} \\
&= \frac{1+0}{1-2(0)} \\
&= 1
\end{aligned}$$

A similar calculation shows that

$$\lim_{x\to-\infty} \frac{x+1}{x-2} = 1$$

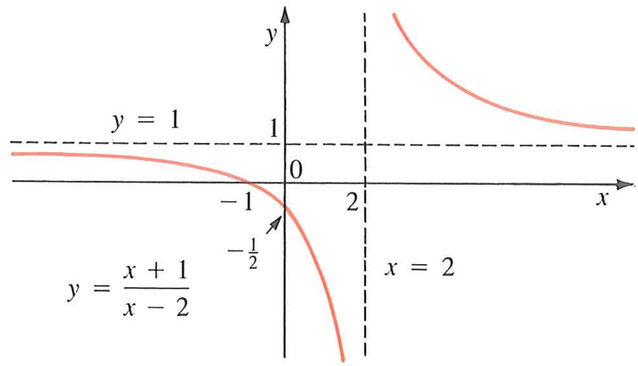
and so the horizontal asymptote is $y = 1$.

The denominator is 0 when $x = 2$, so we find the following limits by noting that $y > 0$ for $x > 2$ and $y < 0$ when $x < 2$ and x is close to 2.

$$\lim_{x\to2^+} \frac{x+1}{x-2} = \infty \qquad \lim_{x\to2^-} \frac{x+1}{x-2} = -\infty$$

Thus, the line $x = 2$ is the vertical asymptote.

We sketch the asymptotes as broken lines. Then we plot the x-intercept, -1, and the y-intercept, $-\frac{1}{2}$, and use the information from the limits to sketch the graph of the given function. [It could be verified using the derivative that the function is decreasing on each of the intervals $(-\infty, 2)$ and $(2, \infty)$.]



Example 4 Find the horizontal and vertical asymptotes of the function

$$y = \frac{x}{x^2 - x - 6}$$

and sketch the graph.

Solution As in Example 2, we divide numerator and denominator by x^2:

$$\lim_{x\to\infty} \frac{x}{x^2 - x - 6} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1 - \frac{1}{x} - \frac{6}{x^2}}$$

$$= \frac{\lim_{x\to\infty} \frac{1}{x}}{\lim_{x\to\infty} 1 - \lim_{x\to\infty} \frac{1}{x} - 6\lim_{x\to\infty} \frac{1}{x^2}}$$

$$= \frac{0}{1 - 0 - 6(0)}$$

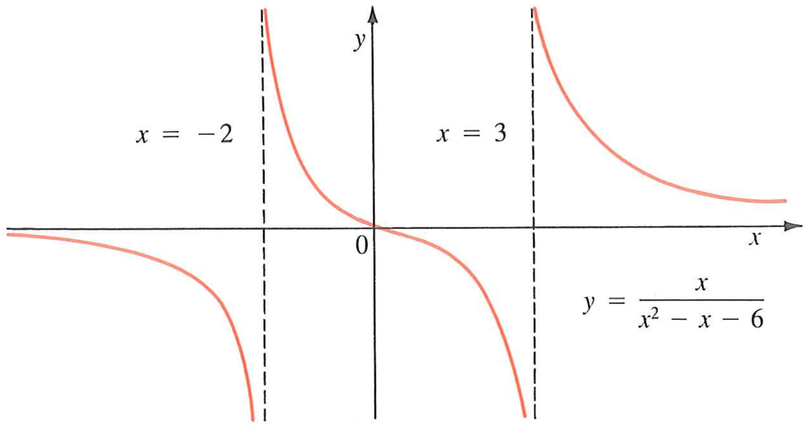
$$= 0$$

A similar calculation shows that

$$\lim_{x\to-\infty} \frac{x}{x^2 - x - 6} = 0$$

Therefore the line $y = 0$ (the x-axis) is a horizontal asymptote.

In Example 3 in Section 5.1 we found that the vertical asymptotes are $x = 3$ and $x = -2$, and we sketched the graph near these asymptotes. We now combine that information with the intercepts (both 0) and limits at infinity to sketch the complete graph. [Calculation of the derivative would confirm that $y' < 0$, so y is decreasing on the intervals $(-\infty, -2)$, $(-2, 3)$, and $(3, \infty)$.]



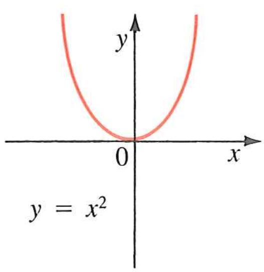
Infinite Limits at Infinity

We use the notation

$$\lim_{x \to \infty} f(x) = \infty$$

to indicate that the values of $f(x)$ become large as x becomes large. There are similar meanings for the following symbols:

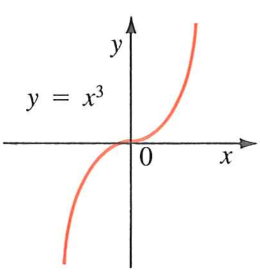
$$\lim_{x \to \infty} f(x) = -\infty \qquad \lim_{x \to -\infty} f(x) = \infty \qquad \lim_{x \to -\infty} f(x) = -\infty$$



For example, we can see from the graphs of $y = x^2$ and $y = x^3$ that

$$\lim_{x \to -\infty} x^2 = \infty \qquad \lim_{x \to \infty} x^2 = \infty$$

$$\lim_{x \to -\infty} x^3 = -\infty \qquad \lim_{x \to \infty} x^3 = \infty$$



For infinite limits, not all of the properties of limits hold, as we see in the next example.

Example 5 Find $\lim_{x\to\infty} (x^4 - x)$.

Solution It would be wrong to write this as a difference of limits:

$$\lim_{x\to\infty} (x^4 - x) \neq \lim_{x\to\infty} x^4 - \lim_{x\to\infty} x$$

because ∞ is not a number ($\infty - \infty$ cannot be defined).

But we can write

$$\lim_{x\to\infty} (x^4 - x) = \lim_{x\to\infty} x(x^3 - 1) = \infty$$

since x and $x^3 - 1$ both become large and therefore their product becomes large.

Using limits at infinity, together with intercepts, we can get a rough idea of the graph of a polynomial.

Example 6 Sketch the graph of $y = (x - 3)^2(x + 2)(1 - x)$ by finding its intercepts and its limits as $x \to \infty$ and $x \to -\infty$.

Solution The y-intercept is

$$f(0) = (-3)^2(2)(1) = 18$$

To find the x-intercepts, we set

$$y = (x - 3)^2(x + 2)(1 - x) = 0$$

and find that $x = 3, -2$, or 1. Note that since $(x - 3)^2$ is positive, the function does not change sign at 3. Therefore the graph does not cross the x-axis at 3.

When x is large and positive we have

$$\begin{aligned}(x - 3)^2(x + 2)(1 - x) &= [\text{large positive}][\text{large positive}][\text{large negative}]\\ &= \text{large negative}\end{aligned}$$

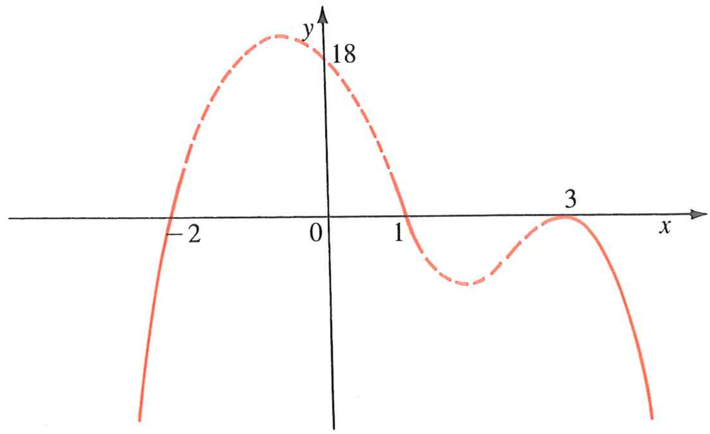
and so $$\lim_{x\to\infty} (x - 3)^2(x + 2)(1 - x) = -\infty$$

When x is large and negative we have

$$\begin{aligned}(x - 3)^2(x + 2)(1 - x) &= [\text{large positive}][\text{large negative}][\text{large positive}]\\ &= \text{large negative}\end{aligned}$$

and so $$\lim_{x\to-\infty} (x - 3)^2(x + 2)(1 - x) = -\infty$$

Combining this information, we give a rough sketch of the graph.

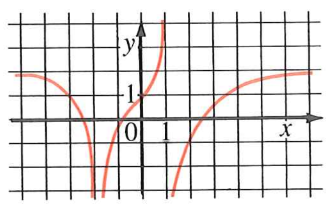


The use of derivatives would enable us to complete the picture by giving the precise location of the maximum and minimum points.

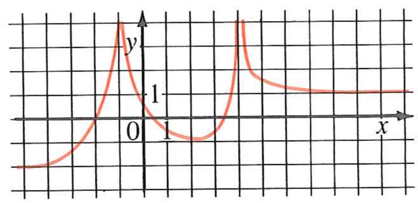
EXERCISE 5.2

A **1.** State the equations of the horizontal and vertical asymptotes.

(a)



(b)



B **2.** Find the limit.

(a) $\lim_{x\to\infty} \frac{6}{\sqrt{x}}$

(b) $\lim_{x\to-\infty} 3x^{-5}$

(c) $\lim_{x\to\infty} \frac{2x+1}{x-3}$

(d) $\lim_{x\to-\infty} \frac{2x+1}{x-3}$

(e) $\lim_{x\to\infty} \frac{1-x}{3+5x}$

(f) $\lim_{x\to-\infty} \frac{x^2-x+1}{x^2+3x-2}$

(g) $\lim_{x\to\infty} \frac{x+3}{x^2-5x+7}$

(h) $\lim_{x\to\infty} \frac{x^2-1}{(x+3)(2x+4)}$

(i) $\displaystyle\lim_{x\to-\infty} \frac{3x^3 + x^2 - 5}{x^3 - 4x + 1}$ (j) $\displaystyle\lim_{x\to-\infty} \frac{12x^2 - 2x + 1}{3x^4 - 14x^2 + x - 3}$

3. Find the horizontal asymptotes of each curve.

(a) $y = \dfrac{2x - 3}{5 - 4x}$ (b) $y = \dfrac{x}{x^2 + 1}$

(c) $y = \dfrac{x^3 + 1}{x^3 - 1}$ (d) $y = 1 - \dfrac{x}{x^2 - 2}$

4. Find the horizontal and vertical asymptotes. Use them, together with intercepts, to sketch the graph.

(a) $y = \dfrac{2}{x + 1}$ (b) $y = \dfrac{x}{x + 1}$

(c) $y = \dfrac{4x + 5}{3 - 2x}$ (d) $y = \dfrac{1}{x^2 - 1}$

(e) $y = \dfrac{x}{x^2 - 1}$ (f) $y = \dfrac{2x^2}{x^2 + 3x - 4}$

(g) $y = \dfrac{x}{(x + 2)^2}$ (h) $y = \dfrac{x^2}{(x + 2)^2}$

5. Find the limit.

(a) $\displaystyle\lim_{x\to\infty} \sqrt{x}$ (b) $\displaystyle\lim_{x\to-\infty} x^5$

(c) $\displaystyle\lim_{x\to\infty} (x^3 - x^2)$ (d) $\displaystyle\lim_{x\to-\infty} (x^3 - x^2)$

(e) $\displaystyle\lim_{x\to\infty} x^2(2x + 1)(x - 2)$ (f) $\displaystyle\lim_{x\to\infty} (x + 2)^4(3 - x)$

6. Find the limits as $x \to \infty$ and $x \to -\infty$. Use this information, together with intercepts, to give a rough sketch of the graph.

(a) $y = (x + 1)(x - 2)(3 - x)$ (b) $y = x^2(x - 2)(2x + 5)$

(c) $y = (1 - x)^2(2 - x)(5 - x)$ (d) $y = (x + 1)^3(x - 2)^4$

7. Find the horizontal asymptotes of $y = \dfrac{x}{|x| + 1}$.

C **8.** Find $\displaystyle\lim_{x\to\infty} \frac{\sqrt{4x^2 + 1}}{2x - 3}$. [*Hint:* Divide numerator and denominator by x.]

9. Find $\displaystyle\lim_{x\to-\infty} \frac{3x}{\sqrt{x^2 + 6}}$. [*Hint:* Note that $\sqrt{x^2} = -x$ when $x < 0$.]

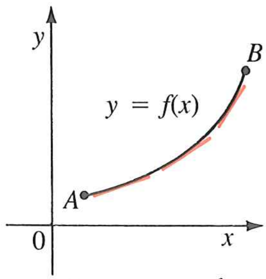
10. Find $\displaystyle\lim_{x\to\infty} (\sqrt{x^2 + 5x + 1} - x)$. [*Hint:* Rationalize.]

11. Find $\displaystyle\lim_{x\to-\infty} \frac{x^{10} + 6x^6 - 3}{x^5 + 2x}$. [*Hint*: Divide numerator and denominator by x^5.]

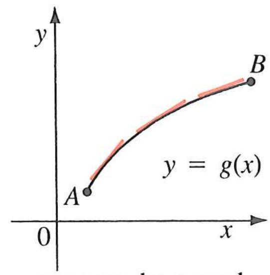
12. How large do we have to take x so that $\dfrac{1}{x^2} < 0.000\,001$?

5.3 CONCAVITY AND POINTS OF INFLECTION

The graphs of the functions f and g, shown in the diagrams, each connect point A to point B, but they bend in different directions. The graph of f lies above its tangent lines and is called *concave upward*; the graph of g lies below its tangent lines and is called *concave downward*.



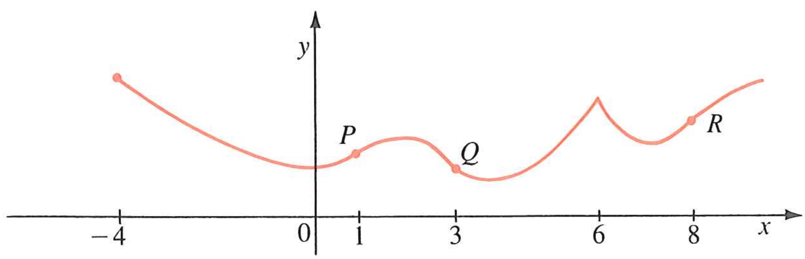
concave upward



concave downward

In general, the graph of f is called **concave upward** on an interval I if it lies above all of its tangents on I. It is called **concave downward** on I if it lies below all of these tangents.

For instance, the function whose graph is shown is concave upward (abbreviated *CU*) on the intervals $(-4, 1)$, $(3, 6)$, and $(6, 8)$, and concave downward (*CD*) on $(1, 3)$ and $(8, \infty)$.

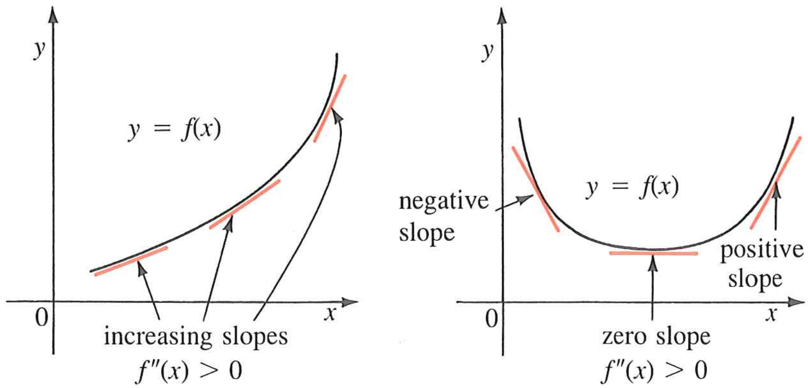


A point P on a curve is called a **point of inflection** if the curve changes from concave upward to concave downward or from concave downward to concave upward at P. For instance, the graph just considered has three points of inflection, namely, P, Q, and R. Notice that if a curve has a tangent line at a point of inflection, then the curve crosses its tangent line there.

To see how the sign of the second derivative affects the direction of concavity, we consider a function f with positive second derivative, that is, $f''(x) > 0$ on an interval I. Thus,

$$\frac{d}{dx}f'(x) > 0 \qquad \text{for } x \in I$$

We know that if a function has a positive derivative, then it is increasing, so $f'(x)$ is increasing on I. This says that the slopes of the tangent lines increase from left to right. From the figures, we see that the graph is concave upward.



Similarly, if $f''(x) < 0$, then $f'(x)$ is decreasing, so the slopes of the tangent lines decrease. You should draw the corresponding pictures, which suggest that f is concave downward.

Test For Concavity

If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.
If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

It follows from the Test for Concavity that there will be a point of inflection at any point where the second derivative changes sign.

Example 1 (a) Determine where the curve $y = x^3 - 3x^2 + 4x - 5$ is concave upward and where it is concave downward.
(b) Find the points of inflection.
(c) Use this information to sketch the curve.

Solution (a) If $f(x) = x^3 - 3x^2 + 4x - 5$
then $f'(x) = 3x^2 - 6x + 4$
and $f''(x) = 6x - 6 = 6(x - 1)$

The curve is concave upward when $f''(x) > 0$, that is,

$$\begin{aligned} 6(x - 1) &> 0 \\ x - 1 &> 0 \\ x &> 1 \end{aligned}$$

Also, the curve is concave downward when $f''(x) < 0$, that is, $x < 1$. Thus, the curve is concave upward on $(1, \infty)$ and concave downward on $(-\infty, 1)$.

(b) The curve changes from concave downward to concave upward when $x = 1$, so the point $(1, -3)$ is a point of inflection.

(c) We note that there is no critical number because the discriminant of the equation

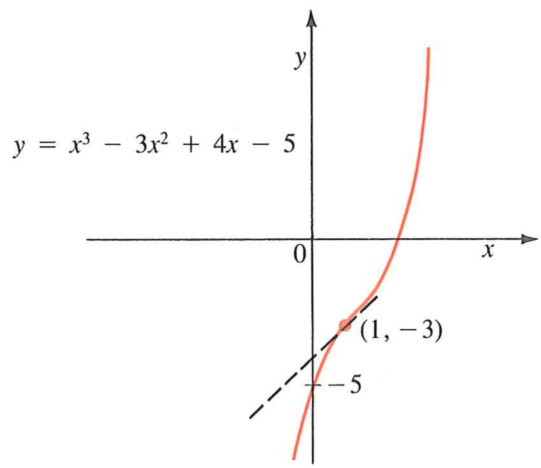
$$f'(x) = 3x^2 - 6x + 4 = 0$$

is

$$b^2 - 4ac = 36 - 48 < 0$$

Thus $f'(x) > 0$, so the function is always increasing and there is no maximum or minimum. This information, together with parts (a) and (b) and the y-intercept, allows us to sketch the curve. You can see that the information regarding concavity is very helpful in sketching this particular curve.

The slope of the tangent at the inflection point is $f'(1) = 1$.



Example 2 Discuss the curve $y = \dfrac{x}{x^2 + 1}$ with respect to concavity and points of inflection.

Solution If

$$f(x) = \frac{x}{x^2 + 1}$$

then

$$f'(x) = \frac{x^2 + 1 - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$$

and

$$f''(x) = \frac{(1 + x^2)^2(-2x) - (1 - x^2)2(x^2 + 1)(2x)}{(x^2 + 1)^4} = \frac{2x(x^2 - 3)}{(x^2 + 1)^3}$$

Noting that the denominator of $f''(x)$ is always positive and the numerator is 0 when $x = 0$ or $x = \pm\sqrt{3}$, we analyze the concavity of f in the following chart.

Interval	$2x$	$x^2 - 3$	$f''(x)$	f
$x < -\sqrt{3}$	$-$	$+$	$-$	CD on $(-\infty, -\sqrt{3})$
$-\sqrt{3} < x < 0$	$-$	$-$	$+$	CU on $(-\sqrt{3}, 0)$
$0 < x < \sqrt{3}$	$+$	$-$	$-$	CD on $(0, \sqrt{3})$
$x > \sqrt{3}$	$+$	$+$	$+$	CU on $(\sqrt{3}, \infty)$

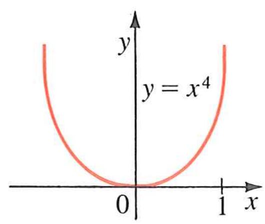
From the chart we also see that the direction of concavity changes when $x = -\sqrt{3}$, 0, and $\sqrt{3}$. Therefore there are three points of inflection:

$$\left(-\sqrt{3}, -\frac{\sqrt{3}}{4}\right) \qquad (0, 0) \qquad \left(\sqrt{3}, \frac{\sqrt{3}}{4}\right)$$

The following example shows that inflection points cannot be located simply by setting $f''(x) = 0$. There must be a change in the direction of concavity.

Example 3 Show that the function $f(x) = x^4$ satisfies $f''(0) = 0$ but has no inflection point.

Solution Since $f(x) = x^4$, we have $f'(x) = 4x^3$ and $f''(x) = 12x^2$, so $f''(0) = 0$. But $12x^2 > 0$ for both $x < 0$ and $x > 0$, so the direction of concavity does not change and there is no inflection point (see the diagram).



Example 4 For the function $f(x) = x^{\frac{1}{3}}(x + 3)^{\frac{2}{3}}$,

(a) find the intervals of increase and decrease,
(b) find the local maximum and minimum values,
(c) find the intervals of concavity,
(d) find the points of inflection,
(e) sketch the graph of f.

Solution (a)

$$f'(x) = \tfrac{1}{3}x^{-\frac{2}{3}}(x + 3)^{\frac{2}{3}} + x^{\frac{1}{3}}\left(\tfrac{2}{3}\right)(x + 3)^{-\frac{1}{3}}$$

$$= \frac{x + 3 + 2x}{3x^{\frac{2}{3}}(x + 3)^{\frac{1}{3}}} = \frac{x + 1}{x^{\frac{2}{3}}(x + 3)^{\frac{1}{3}}}$$

Therefore, $f'(x) = 0$ when $x = -1$ and $f'(x)$ does not exist when $x = 0$ or -3. So the critical numbers are -3, -1, and 0. We set up a chart accordingly.

Interval	$x + 1$	$x^{\frac{2}{3}}$	$(x+3)^{\frac{1}{3}}$	$f'(x)$	f
$x < -3$	$-$	$+$	$-$	$+$	increasing on $(-\infty, -3)$
$-3 < x < -1$	$-$	$+$	$+$	$-$	decreasing on $(-3, -1)$
$-1 < x < 0$	$+$	$+$	$+$	$+$	increasing on $(-1, 0)$
$x > 0$	$+$	$+$	$+$	$+$	increasing on $(0, \infty)$

(b) From part (a) and the First Derivative Test, $f(-3) = 0$ is a local maximum and $f(-1) = -4^{\frac{1}{3}} \doteq -1.6$ is a local minimum.

(c)
$$f''(x) = \frac{x^{\frac{2}{3}}(x+3)^{\frac{1}{3}} - (x+1)\left[\frac{2}{3}x^{-\frac{1}{3}}(x+3)^{\frac{1}{3}} + x^{\frac{2}{3}}\left(\frac{1}{3}\right)(x+3)^{-\frac{2}{3}}\right]}{x^{\frac{4}{3}}(x+3)^{\frac{2}{3}}}$$

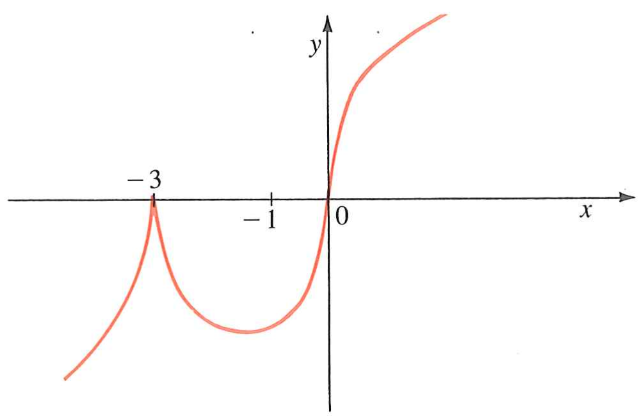
Multiplying numerator and denominator by $x^{\frac{1}{3}}(x+3)^{\frac{2}{3}}$, we get

$$f''(x) = \frac{x(x+3) - (x+1)\left[\frac{2}{3}(x+3) + \frac{1}{3}x\right]}{x^{\frac{5}{3}}(x+3)^{\frac{4}{3}}} = \frac{-2}{x^{\frac{5}{3}}(x+3)^{\frac{4}{3}}}$$

Since $(x+3)^{\frac{4}{3}} \geqslant 0$ for all x (it is a fourth power), we see that $f''(x) > 0$ when $x < 0$ $(x \neq -3)$ and $f''(x) < 0$ when $x > 0$. Thus, f is concave upward on $(-\infty, -3)$ and $(-3, 0)$ and concave downward on $(0, \infty)$.

(d) Since the curve changes from concave upward to concave downward at 0, $(0, 0)$ is an inflection point.

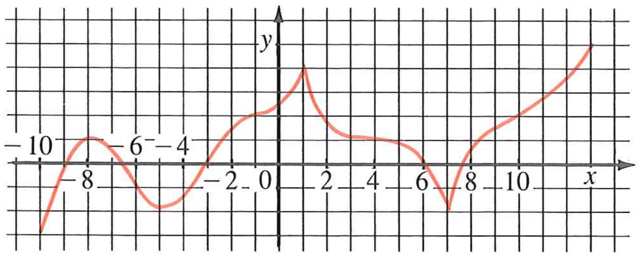
(e) In sketching the curve, notice the shape near the intercepts. Although f is concave upward on $(-\infty, -3)$ and $(-3, 0)$, it is not concave upward on $(-\infty, 0)$. The fact that $f'(x)$ does not exist at -3 and 0 is reflected in the almost vertical shape of the curve near these points.



EXERCISE 5.3

A **1.** (a) State the intervals on which f is concave upward or concave downward.

(b) State the coordinates of the points of inflection.



B **2.** Find the intervals on which the curve is concave upward or concave downward and state the points of inflection.

(a) $y = 2 + 5x - 12x^2$
(b) $y = 6x^2 - 12x + 1$
(c) $y = 16 + 4x + x^2 - x^3$
(d) $y = 2x^3 + 24x^2 - 5x - 21$
(e) $y = x^4 - 2x^3 + x - 2$
(f) $y = x^4 - 24x^2 + x - 1$
(g) $y = \dfrac{1}{x - 1}$
(h) $y = \dfrac{x - 2}{5 - x}$
(i) $y = \dfrac{1}{x^2 + 1}$
(j) $y = \dfrac{1 - x^2}{x^3}$
(k) $y = x^{\frac{2}{3}}(5 + x)$
(l) $y = \dfrac{x^2}{\sqrt{x + 1}}$

3. For each of the following functions,

(a) find the intervals of increase or decrease,
(b) find the local maximum and minimum values,
(c) find the intervals of concavity,
(d) find the points of inflection,
(e) sketch the curve.

(i) $y = 4 - 13x - 6x^2 - x^3$
(ii) $y = x^4 - 8x^2$
(iii) $y = x\sqrt{x^2 + 4}$
(iv) $y = 3x^{\frac{2}{3}} - 2x$

C **4.** For what values of the constants c and d is $(4, -7)$ a point of inflection of the cubic curve $y = x^3 + cx^2 + x + d$?

5. Show that the function $f(x) = x|x|$ has an inflection point at $(0, 0)$, but $f''(0)$ does not exist.

6. Sketch the graph of a continuous function that satisfies all of the following conditions.

(a) $f(0) = f(3) = 0, f(-1) = f(1) = -2$
(b) $f'(-1) = f'(1) = 0$
(c) $f'(x) < 0$ for $x < -1$ and for $0 < x < 1$, $f'(x) > 0$ for $-1 < x < 0$ and for $x > 1$

(d) $f''(x) > 0$ for $x < 3$ $(x \neq 0)$, $f''(x) < 0$ for $x > 3$
(e) $\lim_{x\to\infty} f(x) = 1$, $\lim_{x\to-\infty} f(x) = \infty$

7. Sketch the graph of a continuous function that satisfies all of the following conditions.
(a) $f'(x) > 0$ for $0 < x < 1$, $f'(x) < 0$ for $x > 1$
(b) $f''(x) < 0$ for $0 < x < 2$, $f''(x) > 0$ for $x > 2$
(c) $\lim_{x\to\infty} f(x) = 0$
(d) $f(-x) = -f(x)$ for all x

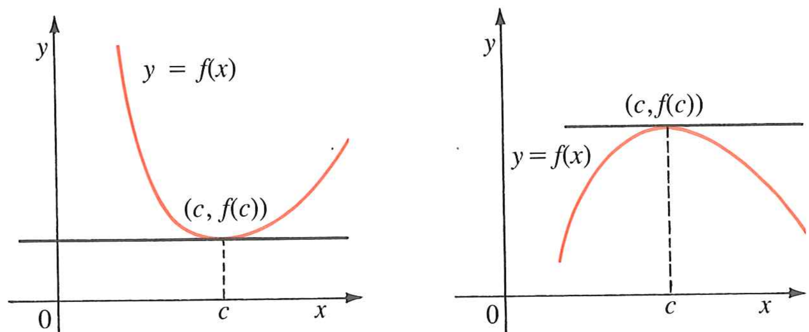
8. Use Newton's method to find the coordinates of the inflection point of the curve $y = x^5 + 2x^3 + 6x^2 - 5x + 4$ correct to three decimal places.

9. Suppose that f is positive, concave upward, and $f''(x)$ exists on an interval I. Show that the function $g(x) = [f(x)]^2$ is also concave upward on I.

5.4 THE SECOND DERIVATIVE TEST

Another application of the second derivative in curve-sketching occurs in locating the local maximum and minimum values of a function f. We assume that $f''(x)$ exists and is continuous throughout the domain of f.

The figure shows the graph of a function f with $f''(c) > 0$ and $f'(c) = 0$. Since $f''(c) > 0$, the graph of f is concave upward near c and therefore lies above its tangent at $(c, f(c))$. But since $f'(c) = 0$, this tangent is horizontal. Therefore, f has a local minimum at c.



$f'(c) = 0$, $f''(c) > 0$ $\qquad$ $f'(c) = 0$, $f''(c) < 0$

Similarly, if $f'(c) = 0$ and $f''(c) < 0$, then the graph of f is concave downward near c and therefore lies below its horizontal tangent at $(c, f(c))$. Thus, f has a local maximum at c.

Second Derivative Test

If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Example 1 Find the local maximum and minimum values of $f(x) = x^3 - 12x + 5$.

Solution First we find the critical numbers.

$$\begin{aligned} f'(x) &= 3x^2 - 12 = 3(x^2 - 4) = 0 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

To apply the Second Derivative Test we find the second derivative:

$$f''(x) = 6x$$

Since $f'(2) = 0$ and $f''(2) = 12 > 0$,

$$f(2) = -11 \text{ is a local minimum}$$

Since $f'(-2) = 0$ and $f''(-2) = -12 < 0$,

$$f(-2) = 21 \text{ is a local maximum}$$

Example 2 Find the maximum and minimum values of $y = x^4 - 8x^3$. Use these, together with concavity and points of inflection, to sketch the curve.

Solution If $f(x) = x^4 - 8x^3$, then

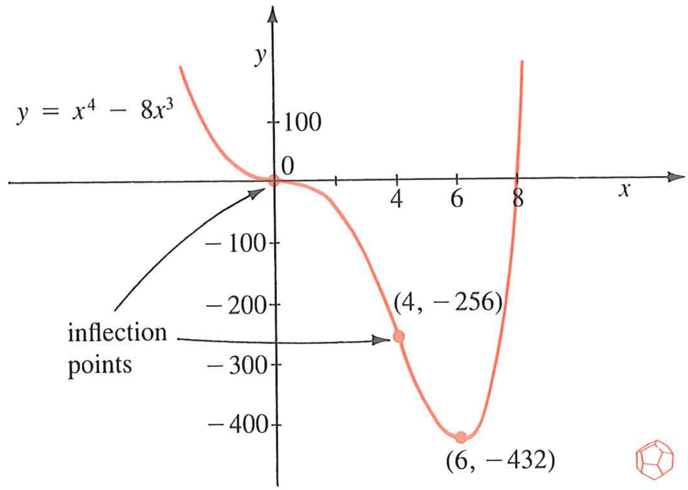
$$\begin{aligned} f'(x) &= 4x^3 - 24x^2 = 4x^2(x - 6) \\ f''(x) &= 12x^2 - 48x = 12x(x - 4) \end{aligned}$$

To find the critical numbers we set $f'(x) = 0$ and obtain $x = 0$ and $x = 6$. Then to use the Second Derivative Test we evaluate f'' at these numbers:

$$f''(0) = 0 \qquad f''(6) = 144$$

Since $f'(6) = 0$ and $f''(6) > 0$, $f(6) = -432$ is a local minimum. Since $f''(0) = 0$, the Second Derivative Test gives no information about the critical number 0. But since the first derivative does not change sign at 0 (it is negative on both sides of 0), the First Derivative Test tells us that f has no maximum or minimum at 0.

Since $f''(x) = 12x(x - 4)$, we have $f''(x) > 0$ for $x < 0$ or $x > 4$ and $f''(x) < 0$ for $0 < x < 4$. So f is concave upward on $(-\infty, 0)$ and $(4, \infty)$ and concave downward on $(0, 4)$. The inflection points are $(0, 0)$ and $(4, -256)$.



Note: Example 2 illustrates the fact that the Second Derivative Test gives no information when $f''(c) = 0$. It also fails when $f''(c)$ does not exist. For instance, in Example 4 in Section 5.3 the function has a local maximum value when $x = -3$, but $f''(-3)$ does not exist and so the Second Derivative Test does not apply. In such cases we must use the First Derivative Test. In fact, the First Derivative Test has the added advantage that we need not calculate the second derivative.

EXERCISE 5.4

B **1.** Use the Second Derivative Test to find the local maximum and minimum values of each function, wherever possible.

(a) $f(x) = 3x^2 - 4x + 13$
(b) $f(x) = 2 + 6x - 6x^2$
(c) $g(x) = 2x^3 - 48x - 17$
(d) $g(x) = 1 + 3x^2 - 2x^3$
(e) $h(x) = x^3 - 9x^2 + 24x - 10$
(f) $h(x) = x^4 - x^3$
(g) $F(x) = 3x^4 - 16x^3 + 18x^2 + 1$
(h) $F(x) = 2 + 5x - x^5$
(i) $G(x) = (1 - 3x^2 + x^3)^5$
(j) $G(x) = x^2 + \dfrac{16}{x}$

2. Use any method to find the local maximum and minimum values of each function.

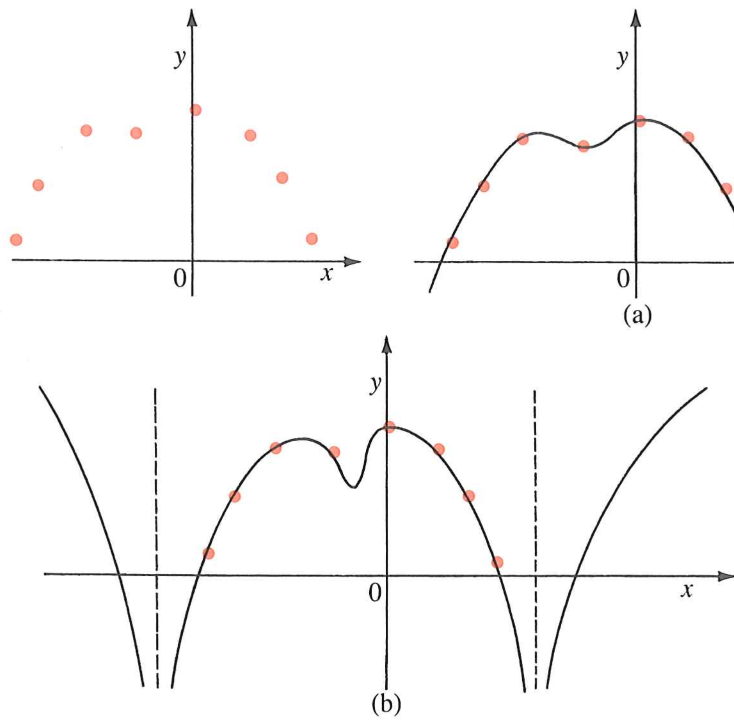
(a) $f(x) = x^4 - 6x^2 + 10$
(b) $f(x) = x\sqrt{x - 1}$
(c) $g(x) = \dfrac{x}{x^2 + 9}$
(d) $g(x) = \dfrac{x}{(2x - 3)^2}$
(e) $f(t) = \dfrac{t^2}{2t + 5}$
(f) $f(t) = t + 3t^{\frac{2}{3}}$

3. Find the local maximum and minimum values of each function. Use this information, together with concavity, to sketch the curve.

(a) $y = x - x^3$
(b) $y = x^4 - 3x^3 + 3x^2 - x + 1$
(c) $y = 3x^5 - 25x^3 + 60x$
(d) $y = x\sqrt{10 + x}$

5.5 A PROCEDURE FOR CURVE SKETCHING

You may have wondered why we bother using calculus to sketch curves when we could easily use a calculator to plot points and join them with a smooth curve. The danger in this approach can be seen from the points plotted in the figures below. We might join the points to produce the curve shown in (a), but the correct graph might be as in (b).



If we just plot points, we don't know when to stop. (How far should we go to the left or right?) Furthermore, it would be easy to miss such essential features as maximum or minimum values. But if we use calculus, we can be sure that all the important aspects of the curve are illustrated.

You are encouraged to use the sophisticated function-plotting software that is available. Even here, calculus is useful in choosing significant domains.

The procedure we use for sketching a curve $y = f(x)$ is to assemble the following information.

A. ***Domain.*** The first step is to find the domain of the function. (See the Review and Preview to Chapter 2.)

B. ***Intercepts.*** Next we find the x-intercepts and the y-intercept. (See the Review and Preview to this chapter.)

C. ***Symmetry.*** If $f(-x) = f(x)$, then f is even and its graph is symmetric about the x-axis. If $f(-x) = -f(x)$, then f is odd and its graph is symmetric about the origin. (See the Review and Preview to this chapter.)

D. ***Asymptotes.***

(i) *Vertical Asymptotes.* The vertical asymptotes of a rational function can be found by equating the denominator to 0 after dividing out any common factors. If $x = a$ is a vertical asymptote, then the limits

$$\lim_{x\to a^-} f(x) \quad \text{and} \quad \lim_{x\to a^+} f(x)$$

should be identified as either ∞ or $-\infty$. (See Section 5.1.)

(ii) *Horizontal Asymptotes.* Recall from Section 5.2 that the line $y = L$ is a horizontal asymptote of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L$$

E. ***Intervals of Increase or Decrease.*** We calculate $f'(x)$ and use the Test for Increasing or Decreasing Functions. (See Section 4.3.)

F. ***Local Maximum and Minimum Values.*** We find the critical numbers of f and use the First Derivative Test. (See Section 4.3.) It is also possible to use the Second Derivative Test. (See Section 5.3.)

G. ***Concavity and Points of Inflection.*** We calculate $f''(x)$ and use the Test for Concavity. (See Section 5.3.) Inflection points occur where the direction of concavity changes.

H. ***Sketch of the Curve.*** We draw the asymptotes as broken lines and sketch the portions of the curve near the asymptotes. We plot the intercepts, maximum and minimum points, and inflection points. Then we draw the curve so that it passes through these points, rising and falling according to E, with concavity according to G, and joining with the parts near the asymptotes.

Example 1 Discuss the curve $y = 3x^5 - 5x^3$ under the headings A–H.

Solution Let $f(x) = 3x^5 - 5x^3$.

A. ***Domain.*** The domain is R.

B. ***Intercepts.*** The y-intercept is 0. The x-intercepts occur when $3x^5 - 5x^3 = x^3(3x^2 - 5) = 0$, so they are 0 and $\pm\sqrt{\frac{5}{3}}$.

C. ***Symmetry.*** Since

$$f(-x) = 3(-x)^5 - 5(-x)^3 = -3x^5 + 5x^3 = -f(x)$$

f is an odd function. The curve is symmetric about the origin.

D. ***Asymptotes.*** A polynomial has no asymptote, but it is still useful to note that

$$\lim_{x\to\infty} (3x^5 - 5x^3) = \infty \quad \text{and} \quad \lim_{x\to-\infty} (3x^5 - 5x^3) = -\infty$$

E. ***Intervals of Increase or Decrease.***

$$f'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1)$$

Thus $f'(x) > 0$ when $x^2 > 1$, that is, when $x > 1$ or $x < -1$, so f is increasing on $(-\infty, -1)$ and $(1, \infty)$. Also, $f'(x) < 0$ when $-1 < x < 1$ $(x \neq 0)$, so f is decreasing on $(-1, 0)$ and $(0, 1)$.

F. ***Local Maximum and Minimum Values.*** By part E and the First Derivative Test, $f(-1) = 2$ is a local maximum and $f(1) = -2$ is a local minimum.

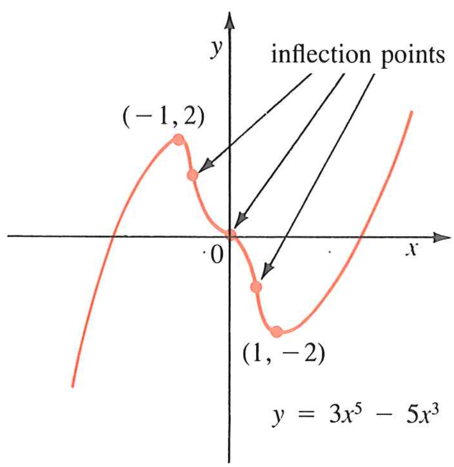
G. ***Concavity and Points of Inflection.***

$$f''(x) = 60x^3 - 30x = 30x(2x^2 - 1)$$

Interval	x	$2x^2 - 1$	$f''(x)$	f
$x < -\frac{1}{\sqrt{2}}$	$-$	$+$	$-$	CD on $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$
$-\frac{1}{\sqrt{2}} < x < 0$	$-$	$-$	$+$	CU on $\left(-\frac{1}{\sqrt{2}}, 0\right)$
$0 < x < \frac{1}{\sqrt{2}}$	$+$	$-$	$-$	CD on $\left(0, \frac{1}{\sqrt{2}}\right)$
$x > \frac{1}{\sqrt{2}}$	$+$	$+$	$+$	CU on $\left(\frac{1}{\sqrt{2}}, \infty\right)$

The inflection points occur when $x = 0, \pm\frac{1}{\sqrt{2}}$.

H. ***Sketch of the Curve.*** Using the information in A–G, we sketch the curve. Notice that we need only sketch it for $x \geq 0$ and use symmetry.



Example 2 Sketch the graph of the function $f(x) = \dfrac{1}{1 + x^2}$.

Solution. A. ***Domain.*** The domain is R.

B. ***Intercepts.*** There is no x-intercept. The y-intercept is $f(0) = 1$.

C. ***Symmetry.*** Since

$$f(-x) = \frac{1}{1 + (-x)^2} = \frac{1}{1 + x^2} = f(x)$$

the function is even and its graph is symmetric about the y-axis.

D. ***Asymptotes.*** The denominator is never 0, so there is no vertical asymptote. To find any horizontal asymptote we compute as follows:

$$\lim_{x \to \infty} \frac{1}{1 + x^2} = \lim_{x \to \infty} \frac{\dfrac{1}{x^2}}{\dfrac{1}{x^2} + 1} = \frac{0}{0 + 1} = 0$$

Similarly, $\displaystyle\lim_{x \to -\infty} \frac{1}{1 + x^2} = 0$

So the line $y = 0$ (the x-axis) is a horizontal asymptote.

Charts are possible but not necessary in E and G

E. ***Intervals of Increase or Decrease.***

$$f'(x) = -\frac{2x}{(1 + x^2)^2}$$

Since the denominator is positive, $f'(x) > 0$ when $x < 0$ and $f'(x) < 0$ when $x > 0$. Therefore f is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

F. ***Local Maximum and Minimum Values.*** By the First Derivative Test, $f(0) = 1$ is a local (and absolute) maximum.

G. ***Concavity and Points of Inflection.***

$$f''(x) = -\frac{2(1 + x^2)^2 - (2x)2(1 + x^2)(2x)}{(1 + x^2)^4} = \frac{2(3x^2 - 1)}{(1 + x^2)^3}$$

Thus, $f''(x) > 0$ when

$$3x^2 > 1$$

$$x^2 > \tfrac{1}{3}$$

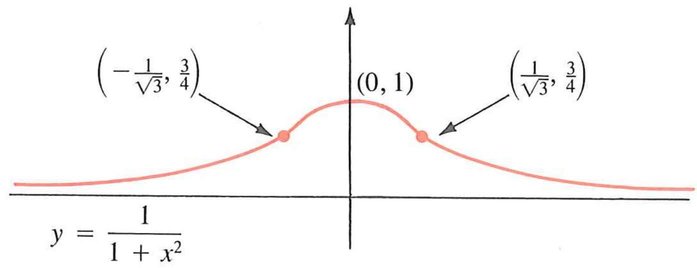
$$|x| > \frac{1}{\sqrt{3}}$$

$$x > \frac{1}{\sqrt{3}} \quad \text{or} \quad x < -\frac{1}{\sqrt{3}}$$

Thus f is concave upward on $\left(-\infty, -\dfrac{1}{\sqrt{3}}\right)$ and $\left(\dfrac{1}{\sqrt{3}}, \infty\right)$. It is concave downward on $\left(-\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$. The inflection points are

$$\left(-\frac{1}{\sqrt{3}}, \frac{3}{4}\right) \quad \text{and} \quad \left(\frac{1}{\sqrt{3}}, \frac{3}{4}\right)$$

H. ***Sketch of the Curve.***



Example 3 Sketch the graph of the function $f(x) = \dfrac{x^2}{1 - x^2}$.

Solution A. ***Domain.*** The domain is $\{x \mid x^2 \neq 1\} = \{x \mid x \neq \pm 1\}$.

B. ***Intercepts.*** The intercepts are both 0.

C. ***Symmetry.*** $f(-x) = f(x)$, so f is even and the curve is symmetric about the y-axis.

D. ***Asymptotes.***

(i) Noting that $f(x) > 0$ for $-1 < x < 1$ and $f(x) < 0$ for $x > 1$ and $x < -1$, we have

$$\lim_{x \to 1^-} \frac{x^2}{1 - x^2} = \infty \qquad \lim_{x \to 1^+} \frac{x^2}{1 - x^2} = -\infty$$

$$\lim_{x \to -1^-} \frac{x^2}{1 - x^2} = -\infty \qquad \lim_{x \to -1^+} \frac{x^2}{1 - x^2} = \infty$$

So $x = 1$ and $x = -1$ are the vertical asymptotes.

(ii) $$\lim_{x \to \pm\infty} \frac{x^2}{1 - x^2} = \lim_{x \to \pm\infty} \frac{1}{\dfrac{1}{x^2} - 1} = \frac{1}{0 - 1} = -1$$

So $y = -1$ is the horizontal asymptote.

E. ***Intervals of Increase or Decrease.***

$$f'(x) = \frac{(1 - x^2)(2x) - x^2(-2x)}{(1 - x^2)^2} = \frac{2x}{(1 - x^2)^2}$$

Thus $f'(x) > 0$ for $x > 0$ $(x \neq 1)$ and $f'(x) < 0$ for $x < 0$ $(x \neq -1)$. The function is increasing on $(0, 1)$ and $(1, \infty)$ and decreasing on $(-\infty, -1)$ and $(-1, 0)$.

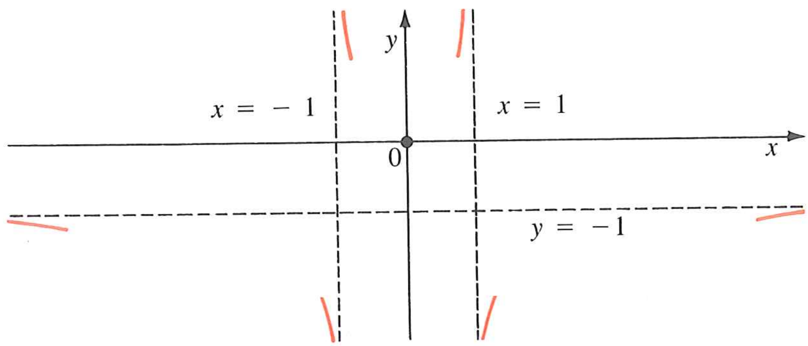
F. ***Local Maximum and Minimum Values.*** By the First Derivative Test, $f(0) = 0$ is a local minimum.

G. ***Concavity and Points of Inflection.***

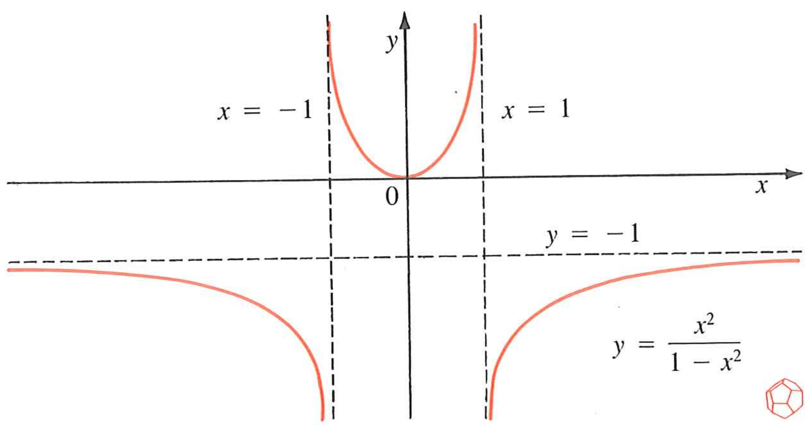
$$f''(x) = \frac{(1 - x^2)^2(2) - (2x)2(1 - x^2)(-2x)}{(1 - x^2)^4} = \frac{2 + 6x^2}{(1 - x^2)^3}$$

So $f''(x) > 0$ when $x^2 < 1$ and $f''(x) < 0$ when $x^2 > 1$. Thus, f is concave upward on $(-1, 1)$ and concave downward on $(-\infty, -1)$ and $(1, \infty)$. There is no inflection point because the numbers -1 and 1 are not in the domain.

H. ***Sketch of the Curve.*** We first sketch the asymptotes and the nearby parts of the curve.



Then we use the remaining information to complete the picture.



Example 4 Sketch the curve $y = x\sqrt{2 - x}$.

Solution Let $f(x) = x\sqrt{2 - x}$.

A. ***Domain.*** The domain is

$$\{x \mid 2 - x \geqslant 0\} = \{x \mid x \leqslant 2\} = (-\infty, 2]$$

B. ***Intercepts.*** The y-intercept is $f(0) = 0$. The x-intercepts occur when $y = 0$, so they are 0 and 2.

C. ***Symmetry.*** There is no symmetry.

D. ***Asymptotes.*** There is no asymptote, but

$$\lim_{x \to -\infty} x\sqrt{2 - x} = -\infty$$

E. ***Intervals of Increase or Decrease.***

$$\begin{aligned} f'(x) &= \sqrt{2 - x} + x \frac{-1}{2\sqrt{2 - x}} \\ &= \frac{2(2 - x) - x}{2\sqrt{2 - x}} \\ &= \frac{4 - 3x}{2\sqrt{2 - x}} \end{aligned}$$

So $f'(x) > 0$ when

$$4 - 3x > 0 \quad \Leftrightarrow \quad 3x < 4 \quad \Leftrightarrow \quad x < \tfrac{4}{3}$$

and f is increasing on $\left(-\infty, \frac{4}{3}\right)$. Also, $f'(x) < 0$ when $x > \frac{4}{3}$, so f is decreasing on $\left(\frac{4}{3}, 2\right)$. (Remember that the domain is $(-\infty, 2)$.)

F. ***Local Maximum and Minimum Values.*** By the First Derivative Test,

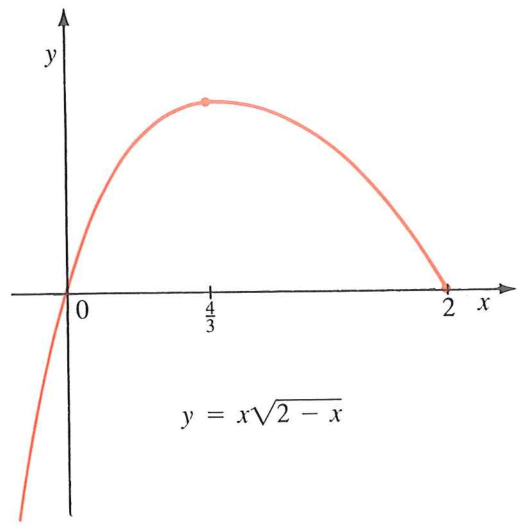
$$f\left(\tfrac{4}{3}\right) = \frac{4\sqrt{2}}{3\sqrt{3}} \text{ is a local (and absolute) maximum}$$

G. ***Concavity and Points of Inflection.***

$$\begin{aligned} f''(x) &= \frac{(2\sqrt{2 - x})(-3) - (4 - 3x)\left(-\dfrac{1}{\sqrt{2 - x}}\right)}{4(2 - x)} \\ &= \frac{-6(2 - x) + 4 - 3x}{4(2 - x)^{\frac{3}{2}}} \\ &= \frac{3x - 8}{4(2 - x)^{\frac{3}{2}}} \end{aligned}$$

Note that the domain of f is $(-\infty, 2]$ and $f''(x) < 0$ on this domain. (The denominator is positive since it is a power of a square root.) Thus, f is concave downward on $(-\infty, 2)$.

H. ***Sketch of the Curve.***



EXERCISE 5.5

B

Discuss the curve in each question under the headings A. Domain; B. Intercepts; C. Symmetry; D. Asymptotes; E. Intervals of Increase or Decrease; F. Local Maximum and Minimum Values; G. Concavity and Points of Inflection; and H. Sketch of the Curve.

1. $y = 3x^5 - 10x^3 + 45x$

2. $y = (x^2 - 1)^3$

3. $y = \dfrac{x - 4}{x + 4}$

4. $y = \dfrac{x^2}{x^2 + 3}$

5. $y = \dfrac{x}{x^2 - 1}$

6. $y = \dfrac{x}{(x - 1)^2}$

7. $y = \dfrac{1}{x^3 - x}$

8. $y = \dfrac{x^2 - 1}{x^3}$

9. $y = x\sqrt{1 - x^2}$

10. $y = \dfrac{x}{\sqrt{x^2 - 4}}$

11. $y = \dfrac{\sqrt{x}}{\sqrt{x} + 1}$

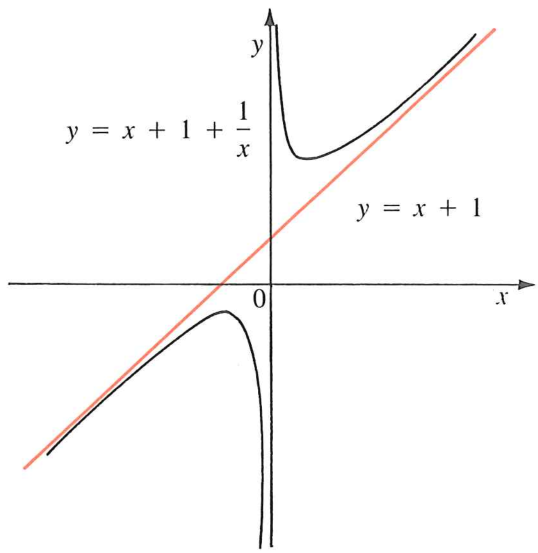
12. $y = x - \sqrt[3]{x}$

**5.6 SLANT ASYMPTOTES

Consider the function

$$f(x) = x + 1 + \frac{1}{x}$$

For large values of x, $\frac{1}{x}$ is small and so the values of $f(x)$ are close to $x + 1$. This means that the graph of f is close to the graph of the line $y = x + 1$. This line is called a *slant asymptote* or an *oblique asymptote* of the graph of f.



In terms of limits, we have

$$\begin{aligned}\lim_{x\to\infty} [f(x) - (x + 1)] &= \lim_{x\to\infty} \left[\left(x + 1 + \frac{1}{x}\right) - (x + 1)\right] \\ &= \lim_{x\to\infty} \frac{1}{x} \\ &= 0\end{aligned}$$

In general, the line $y = mx + b$ is a **slant asymptote** if the vertical distance between the curve $y = f(x)$ and the line approaches 0 as x gets large (or large negative). We can write this condition as

$$\lim_{x\to\infty} [f(x) - (mx + b)] = 0$$

or $$\lim_{x\to-\infty} [f(x) - (mx + b)] = 0$$

For rational functions, slant asymptotes occur when the degree of the numerator is one more than the degree of the denominator. For instance, another way of writing

$$f(x) = x + 1 + \frac{1}{x} \quad \text{is} \quad f(x) = \frac{x^2 + x + 1}{x}$$

In general, for any rational function:

> The equation of a slant asymptote can be found by division.

Example 1 Find the slant asymptote of the curve $y = \dfrac{2x^3 - 3x^2 + x - 3}{x^2 + 1}$.

Solution First we note that there will be a slant asymptote because the degree of the numerator is one more than the degree of the denominator. Long division gives

$$\begin{array}{r|l} & 2x - 3 \\ \hline x^2 + 1 & 2x^3 - 3x^2 + x - 3 \\ & \underline{2x^3 + 2x} \\ & - 3x^2 - x - 3 \\ & \underline{- 3x^2 - 3} \\ & - x \end{array}$$

and so

$$f(x) = \frac{2x^3 - 3x^2 + x - 3}{x^2 + 1} = 2x - 3 - \frac{x}{x^2 + 1}$$

Therefore

$$\begin{aligned} \lim_{x \to \infty} [f(x) - (2x - 3)] &= \lim_{x \to \infty} \left[2x - 3 - \frac{x}{x^2 + 1} - (2x - 3)\right] \\ &= \lim_{x \to \infty} \left[-\frac{x}{x^2 + 1}\right] \\ &= \lim_{x \to \infty} \frac{-\dfrac{1}{x}}{1 + \dfrac{1}{x^2}} \\ &= \frac{0}{1 + 0} \\ &= 0 \end{aligned}$$

Thus the line $y = 2x - 3$ is the slant asymptote.

Example 2 Find the slant asymptote of the curve $y = \dfrac{1 + x - x^2}{x - 1}$, then use it to help sketch the curve.

Solution Long division gives

$$\begin{array}{r} -x \\ x - 1 \overline{\smash{)} -x^2 + x + 1} \\ \underline{-x^2 + x} \\ 1 \end{array}$$

$$f(x) = \frac{1 + x - x^2}{x - 1} = -x + \frac{1}{x - 1}$$

Therefore, $\lim\limits_{x \to \infty} [f(x) - (-x)] = \lim\limits_{x \to \infty} \dfrac{1}{x - 1} = 0$

and so the slant asymptote is $y = -x$.

We analyze the other aspects of the curve using the headings of Section 5.5.

A. ***Domain.*** The domain is $\{x \mid x \neq 1\}$.

B. ***Intercepts.*** The y-intercept is $f(0) = -1$. The x-intercepts occur when

$$x^2 - x - 1 = 0$$

$$x = \frac{1 \pm \sqrt{5}}{2}$$

C. ***Symmetry.*** There is no symmetry.

D. ***Asymptotes.*** $\lim\limits_{x \to 1^+} \left(-x + \dfrac{1}{x - 1}\right) = \infty$

$$\lim_{x \to 1^-} \left(-x + \frac{1}{x - 1}\right) = -\infty$$

So $x = 1$ is a vertical asymptote. Since there is a slant asymptote, there cannot be a horizontal asymptote.

E. ***Intervals of Increase or Decrease.*** $f'(x) = -1 - \dfrac{1}{(x - 1)^2}$

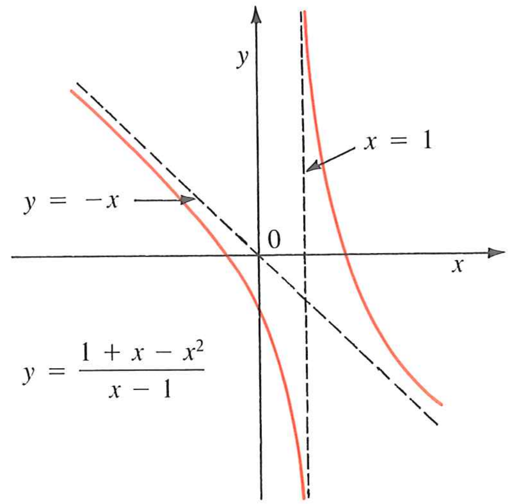
Thus $f'(x) < 0$ for all $x \neq 1$, so f is decreasing on $(-\infty, 1)$ and $(1, \infty)$.

F. ***Local Maximum and Minimum Values.*** There is no maximum or minimum.

G. ***Concavity and Points of Inflection.*** $f''(x) = \dfrac{2}{(x - 1)^3}$

So $f''(x) > 0$ for $x > 1$ and $f''(x) < 0$ for $x < 1$. Thus, f is concave upward on $(1, \infty)$ and concave downward on $(-\infty, 1)$. There is no inflection point.

H. ***Sketch of the Curve.***



EXERCISE 5.6

B **1.** Find the equation of the slant asymptote.

(a) $y = \dfrac{2x - x^2 - 1}{x}$ (b) $y = \dfrac{x^3 - 1}{x^2}$

(c) $y = \dfrac{3x^2 + 4x + 2}{x + 1}$ (d) $y = \dfrac{4x^2}{2x + 1}$

(e) $y = \dfrac{x^3 + 4x^2 + 5x + 16}{x^2 + 4}$ (f) $y = \dfrac{x + x^2 - x^4}{x^3 - 1}$

2. Find the slant asymptote of the curve. Thus, use it to help graph the curve.

(a) $y = \dfrac{x^2 + 9}{x}$ (b) $y = \dfrac{x^2 - 2x - 1}{x}$

(c) $y = \dfrac{x^3}{x^2 - 1}$ (d) $y = \dfrac{(x - 1)^3}{x^2}$

C **3.** Let $$f(x) = \frac{x^3 + 1}{x}$$

Show that
$$\lim_{x \to \infty} [f(x) - x^2] = 0$$

This shows that the graph of f approaches the graph of $y = x^2$, and we say that the graph of f is *asymptotic* to the parabola $y = x^2$. Use this fact to help sketch the graph of f.

5.7 REVIEW EXERCISE

1. Find the limit.

(a) $\lim_{x\to 4^+} \frac{2}{4-x}$ (b) $\lim_{x\to 4} \frac{6}{(x-4)^2}$

(c) $\lim_{x\to -1^-} \frac{1}{(x+1)^3}$ (d) $\lim_{x\to 5^-} \frac{x+3}{x^2-4x-5}$

(e) $\lim_{x\to -2} \frac{x}{(x+2)^2}$ (f) $\lim_{x\to \infty} \frac{6-x}{6+5x}$

(g) $\lim_{x\to -\infty} \frac{x}{x^3-1}$ (h) $\lim_{x\to \infty} \frac{4x^2-3x+5}{2x^2+5x-4}$

(i) $\lim_{x\to \infty} (x^4-2x^2)$ (j) $\lim_{x\to \infty} (\sqrt{x^2+x}-\sqrt{x^2-x})$

2. Find the vertical and horizontal asymptotes.

(a) $y = \frac{6x-1}{1-2x}$ (b) $y = \frac{1}{x^2+6x+9}$

(c) $y = \frac{x}{2x^2-5x-3}$ (d) $y = \frac{x^3}{x^3-1}$

3. Find the intervals on which the curve is concave upward or concave downward and state the points of inflection.

(a) $y = 5x^3 + 12x^2 - 3x + 2$

(b) $y = x^4 - x^3 - 3x^2 + x - 12$

(c) $y = \frac{x^2}{x^2+4}$ (d) $y = x + \frac{1}{\sqrt{x}}$

4. Find the local maximum and minimum values of each function.

(a) $f(x) = x^2 - x^3$ (b) $f(x) = 2x^3 + 15x^2 - 36x$

(c) $g(x) = \frac{x^2}{x-1}$ (d) $g(x) = x + \sqrt{1-x}$

5. Discuss each curve under the headings A–H given in Section 5.5.

(a) $y = x^3 - 6x^2 + 9x$ (b) $y = x^3 - x^4$

(c) $y = \frac{2}{4+x}$ (d) $y = \frac{1-x^2}{1+x^2}$

(e) $y = \frac{1+x^2}{1-x^2}$ (f) $y = x^{\frac{1}{3}}(x-4)^{\frac{2}{3}}$

6. Sketch the graph of a function f that satisfies all the following conditions.

(a) $f(0) = 0, f'(0) = 1$

(b) $f''(x) > 0$ for $x < 0$, $f''(x) < 0$ for $x > 0$

(c) $\lim_{x\to \infty} f(x) = 2$, $\lim_{x\to -\infty} f(x) = -2$

7. Sketch the graph of a function g that satisfies the following conditions.
   (a) $g(0) = 0$
   (b) $g''(x) > 0$ for $x \neq 0$
   (c) $\lim_{x \to 0^-} g'(x) = \infty$, $\lim_{x \to 0^+} g'(x) = -\infty$
   (d) $\lim_{x \to -\infty} g(x) = -\infty$, $\lim_{x \to \infty} g(x) = \infty$

PROBLEMS PLUS

Sketch the curve $y = x + \sqrt{|x|}$

5.8 CHAPTER 5 TEST

1. Find the following limits.
   (a) $\lim_{x\to\infty} \dfrac{6x^3 - 3x + 1}{2x^3 + x^2 - 5}$
   (b) $\lim_{x\to -2+} \dfrac{x - 1}{x^2 - 4}$
2. Find the vertical and horizontal asymptotes of the curve
$$y = \frac{1 - 2x}{3x + 5}.$$
3. (a) Find the intervals on which the curve $y = \dfrac{x}{(x + 1)^2}$ is concave upward or concave downward.
   (b) Find any points of inflection.
4. For the curve $y = 2 - 12x + 9x^2 - 2x^3$,
   (a) find the intervals of increase and decrease,
   (b) find the local maximum and minimum values,
   (c) find the intervals of concavity,
   (d) find any inflection points,
   (e) sketch the curve.
5. Discuss the curve $y = \dfrac{x}{x^2 - 9}$ under the following headings.
   (a) Domain
   (b) Intercepts
   (c) Symmetry
   (d) Asymptotes
   (e) Intervals of Increase or Decrease
   (f) Local Maximum and Minimum Values
   (g) Concavity and Points of Inflection
   (h) Sketch of the Curve.

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ANSWERS

CHAPTER 5 CURVE SKETCHING

REVIEW AND PREVIEW TO CHAPTER 5

EXERCISE 1

1. (a) x-intercepts $\pm\frac{5}{2}$, y-intercept 25

(b) x-intercepts $-\frac{1}{2}$, 1, y-intercept -1

(c) x-intercepts 1, -3, y-intercept -3

(d) no x-intercept, y-intercept 1

(e) x-intercepts $\dfrac{-2 \pm \sqrt{22}}{3}$, y-intercept -6

(f) x-intercepts 0, $\pm\sqrt{3}$, y-intercept 0

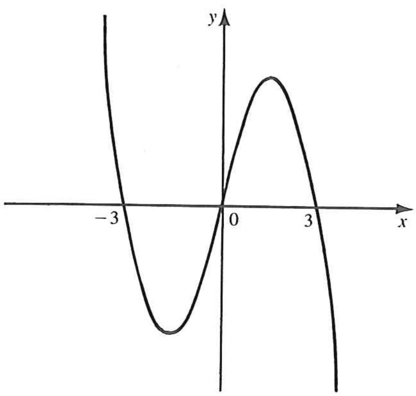
(g) x-intercepts ± 1, y-intercept 1

(h) x-intercepts 0, $-\frac{3}{2}$, 6, y-intercept 0

(i) x-intercept -2, y-intercept 8

(j) x-intercepts ± 2, y-intercept -16

2. x-intercepts 0, ± 3, y-intercept 0



3. 1.53, 0.35, -1.88

EXERCISE 2

1. (a) odd (b) even (c) neither (d) odd

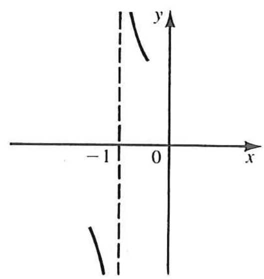
2. (a) even (b) odd (c) neither (d) even (e) odd (f) neither (g) even (h) odd

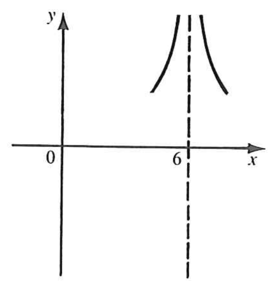
EXERCISE 5.1

1. (a) $x = -7$, $x = -3$, $x = 2$, $x = 6$
(b) (i) $-\infty$ (ii) ∞ (iii) ∞ (iv) $-\infty$
(v) ∞ (vi) $-\infty$

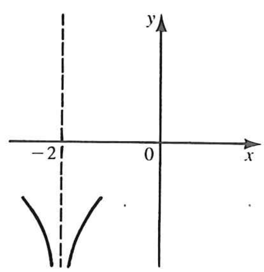
2. (a) ∞ (b) $-\infty$ (c) ∞ (d) $-\infty$ (e) $-\infty$
(f) ∞ (g) $-\infty$ (h) $-\infty$ (i) ∞ (j) $-\infty$
(k) $-\infty$ (l) $-\infty$

3. (a) $x = -1$ (b) $x = 6$

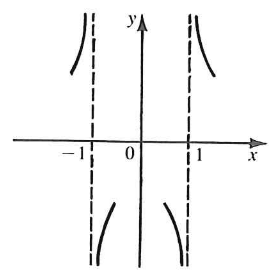




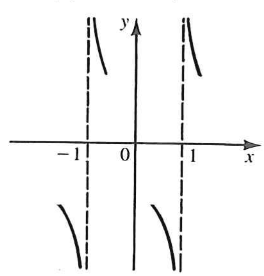
(c) $x = -2$



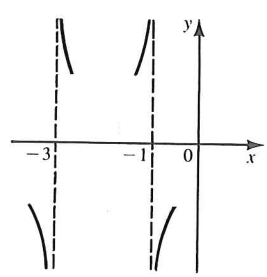
(d) $x = -1$, $x = 1$



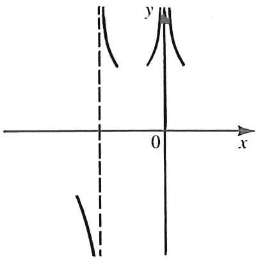
(e) $x = -1$, $x = 1$

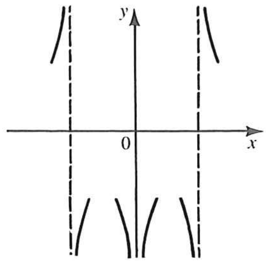


(f) $x = -3$, $x = -1$



(g) $x = -1, x = 0$ (h) $x = -2, x = 0,$ $x = 2$





4. $-\infty$ **5.** $|x| < \frac{1}{100}$

EXERCISE 5.2

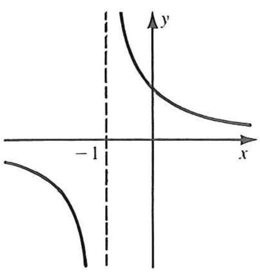
Abbreviations: VA, vertical asymptote; HA, horizontal asymptote

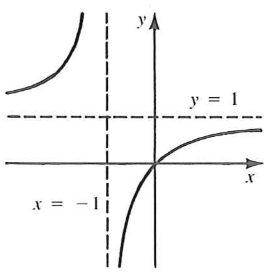
1. (a) HA: $y = 2$, VA: $x = -2, x = 1$
(b) HA: $y = -2, y = 1$; VA: $x = -1, x = 4$

2. (a) 0 (b) 0 (c) 2 (d) 2 (e) $-\frac{1}{5}$ (f) 1
(g) 0 (h) $\frac{1}{2}$ (i) 3 (j) 0

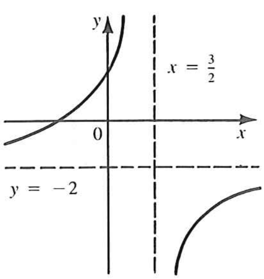
3. (a) $y = -\frac{1}{2}$ (b) $y = 0$ (c) $y = 1$ (d) $y = 1$

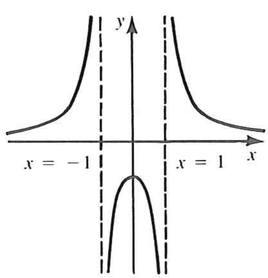
4. (a) HA: $y = 0$, VA: $x = -1$ (b) HA: $y = 1$, VA: $x = -1$



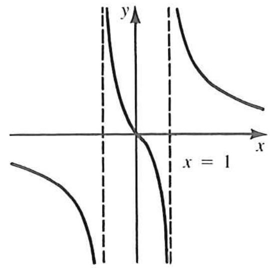


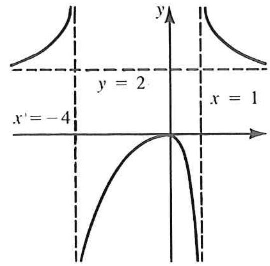
(c) HA: $y = -2$, VA: $x = \frac{3}{2}$
(d) HA: $y = 0$, VA: $x = -1, x = 1$



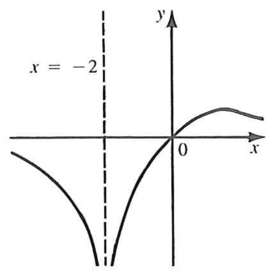


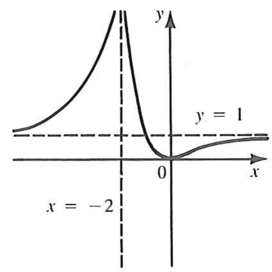
(e) HA: $y = 0$, VA: $x = -1, x = 1$
(f) HA: $y = 2$, VA: $x = -4, x = 1$





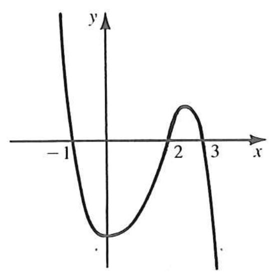
(g) HA: $y = 0$, VA: $x = -2$ (h) HA: $y = 1$, VA: $x = -2$

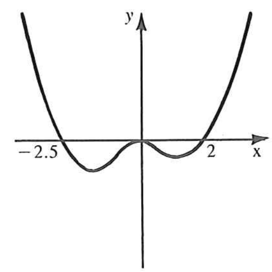




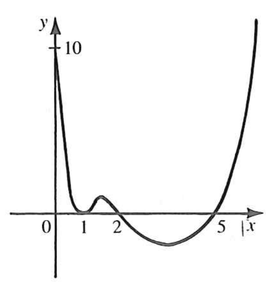
5. (a) ∞ (b) $-\infty$ (c) ∞ (d) $-\infty$ (e) ∞ (f) $-\infty$

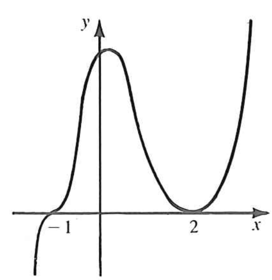
6. (a) $-\infty, \infty$ (b) ∞, ∞





(c) ∞, ∞ (d) $\infty, -\infty$





7. $y = 1, y = -1$ **8.** 1 **9.** -3 **10.** $\frac{5}{2}$

11. $-\infty$ **12.** $x > 1000$

EXERCISE 5.3

Abbreviations: CU, concave upward; CD, concave downward; IP, inflection point

1. CU on $(-6, -3), (-1, 1), (1, 3), (10, 13)$, CD on $(-10, -6), (-3, -1), (3, 7), (7, 10)$, IP $(-6, -1), (-3, 0), (-1, 2), (3, 1), (10, 2)$

2. (*a*) CD on $(-\infty, \infty)$ (b) CU on $(-\infty, \infty)$

(c) CU on $\left(-\infty, \frac{1}{3}\right)$, CD on $\left(\frac{1}{3}, \infty\right)$, IP $\left(\frac{1}{3}, \frac{470}{27}\right)$

(d) CU on $(-4, \infty)$, CD on $(-\infty, -4)$, IP $(-4, 255)$

(e) CU on $(-\infty, 0), (1, \infty)$, CD on $(0, 1)$, IP $(0, -2), (1, -2)$

(f) CU on $(-\infty, -2), (2, \infty)$, CD on $(-2, 2)$, IP $(-2, -83), (2, -79)$

(g) CU on $(1, \infty)$, CD on $(-\infty, 1)$

(h) CU on $(-\infty, 5)$, CD on $(5, \infty)$ (i) CU on $\left(-\infty, -\frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{3}}, \infty\right)$, CD on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, IP $\left(-\frac{1}{\sqrt{3}}, \frac{3}{4}\right), \left(\frac{1}{\sqrt{3}}, \frac{3}{4}\right)$ (j) CU on $(-\infty, -\sqrt{6}), (0, \sqrt{6})$, CD on $(-\sqrt{6}, 0), (\sqrt{6}, \infty)$, IP $\left(-\sqrt{6}, \frac{5}{(\sqrt{6})^3}\right), \left[\sqrt{6}, -\frac{5}{(\sqrt{6})^3}\right)$ (k) CU on $(1, \infty)$, CD on $(-\infty, 1)$, IP $(1, 6)$ (l) CU on $(-1, \infty)$

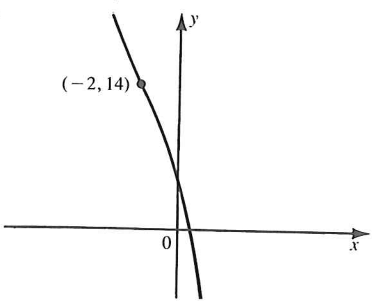
3. (i) (a) decreasing $(-\infty, \infty)$

(b) none

(c) CU on $(-\infty, -2)$, CD on $(-2, \infty)$

(d) IP $(-2, 14)$

(e)



(ii) (a) increasing on $(-2, 0), (2, \infty)$ decreasing on $(-\infty, -2), (0, 2)$

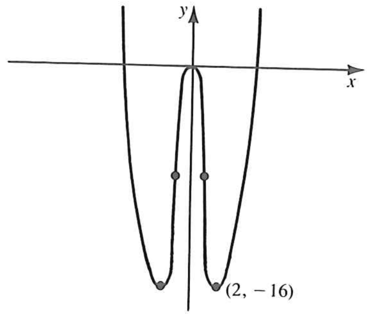
(b) local maximum $f(0) = 0$

local minima $f(2) = f(-2) = -16$

(c) CU on $\left(-\infty, -\frac{2}{\sqrt{3}}\right)$ and $\left(\frac{2}{\sqrt{3}}, \infty\right)$ CD on $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$

(d) IP $\left(-\frac{2}{\sqrt{3}}, -\frac{80}{9}\right)$ and $\left(\frac{2}{\sqrt{3}}, -\frac{80}{9}\right)$

(e)



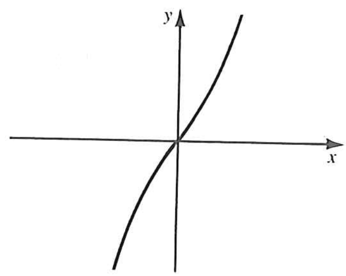
(iii) (a) increasing on $(-\infty, \infty)$

(b) none

(c) CU on $(0, \infty)$, CD on $(-\infty, 0)$

(d) IP $(0, 0)$

(e)



(iv) (a) increasing on $(0, 1)$

decreasing on $(-\infty, 0), (1, \infty)$

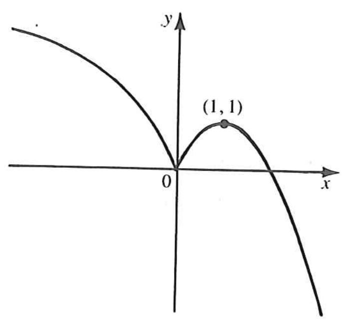
(b) local minimum $f(0) = 0$

local maximum $f(1) = 1$

(c) CD on $(-\infty, 0), (0, \infty)$

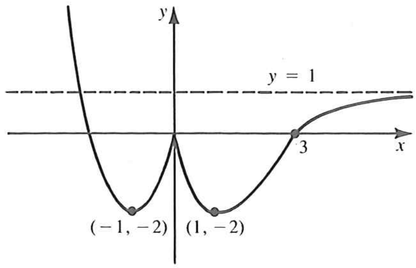
(d) none

(e)

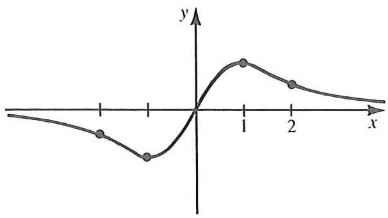


4. $c = -12, d = 117$

6.



7.



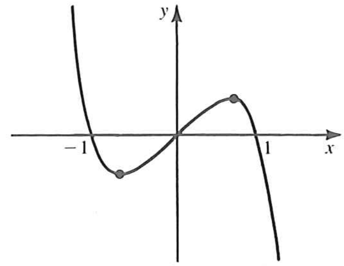
8. $(-0.614, 8.782)$

EXERCISE 5.4

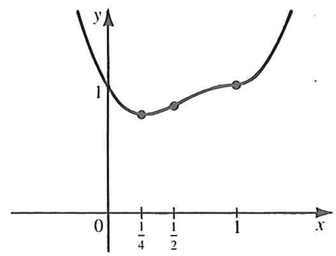
1. (a) local minimum $f\left(\frac{2}{3}\right) = \frac{35}{3}$ (b) local maximum $f\left(\frac{1}{2}\right) = \frac{7}{2}$ (c) local maximum $g(-2\sqrt{2}) = 64\sqrt{2} - 17$, local minimum $g(2\sqrt{2}) = -64\sqrt{2} - 17$ (d) local maximum $g(1) = 2$, local minimum $g(0) = 1$ (e) local maximum $h(2) = 10$, local minimum $h(4) = 6$ (f) local minimum $h\left(\frac{3}{4}\right) = -\frac{27}{256}$, no information on $h(0)$ (g) local maximum $F(1) = 6$, local minima $F(0) = 1$, $F(3) = -26$
(h) local maximum $F(1) = 6$, local minimum $F(-1) = -2$ (i) local maximum $G(0) = 1$, local minimum $G(2) = -243$, no information at other critical numbers (j) local minimum $G(2) = 12$

2. (a) local maximum $f(0) = 10$, local minima $f(\sqrt{3}) = f(-\sqrt{3}) = 1$ (b) none
(c) local maximum $g(3) = \frac{1}{6}$,
local minimum $g(-3) = -\frac{1}{6}$
(d) local minimum $f\left(-\frac{3}{2}\right) = -\frac{1}{24}$
(e) local maximum $f(-5) = -5$, local minimum $f(0) = 0$
(f) local maximum $f(-8) = 4$

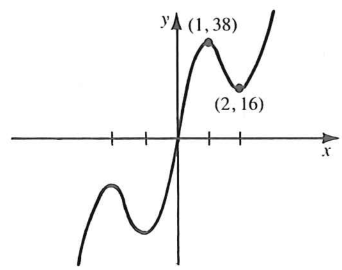
3. (a) local maximum $f\left(\frac{1}{\sqrt{3}}\right) = \frac{2\sqrt{3}}{9}$
local minimum $f\left(-\frac{1}{\sqrt{3}}\right) = -\frac{2\sqrt{3}}{9}$



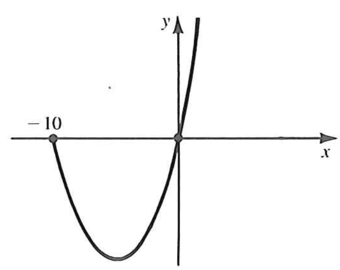
(b) local minimum $f\left(\frac{1}{4}\right) = \frac{229}{256}$



(c) local maxima $f(1) = 38$, $f(-2) = -16$
local minima $f(2) = 16$, $f(-1) = -38$



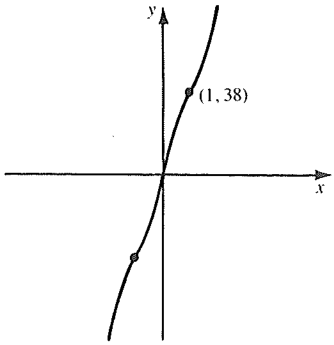
(d) local minimum $f\left(-\frac{20}{3}\right) = -\frac{20\sqrt{30}}{9}$



EXERCISE 5.5

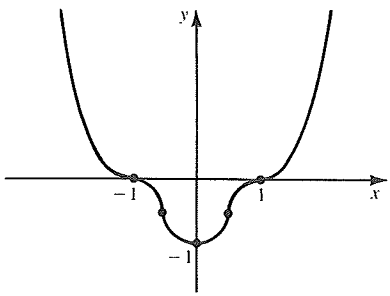
1. A. R B. x-intercept 0, y-intercept 0 C. about the origin D. none E. increasing on R
F. none G. CU on $(-1, 0)$, $(1, \infty)$, CD on $(-\infty, -1)$, $(0, 1)$, IP $(-1, -38)$, $(1, 38)$, $(0, 0)$

H.



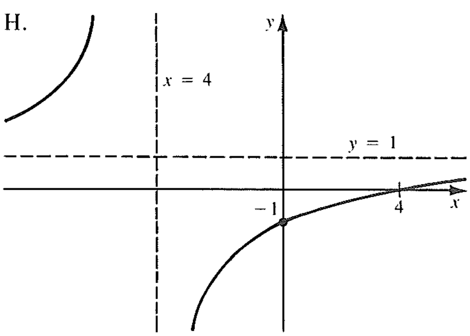
2. A. R B x-intercepts ± 1, y-intercept -1 C. about the y-axis D. none E. increasing on $(0, \infty)$, decreasing on $(-\infty, 0)$ F. local minimum $f(0) = -1$ G. CU on $(-\infty, -1)$, $\left(-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$, $(1, \infty)$, CD on $\left(-1, -\frac{1}{\sqrt{5}}\right)$, $\left(\frac{1}{\sqrt{5}}, 1\right)$, IP $(\pm 1, 0)$, $\left(\pm\frac{1}{\sqrt{5}}, -\frac{64}{125}\right)$

H.



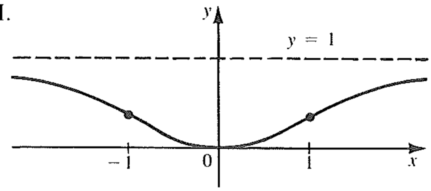
3. A. $(-\infty, -4) \cup (-4, \infty)$ B. y-intercept -1, x-intercept 4 C. none D. HA: $y = 1$, VA: $x = -4$ E. increasing on $(-\infty, -4)$, $(-4, \infty)$ F. none G. CU on $(-\infty, -4)$, CD on $(-4, \infty)$

H.



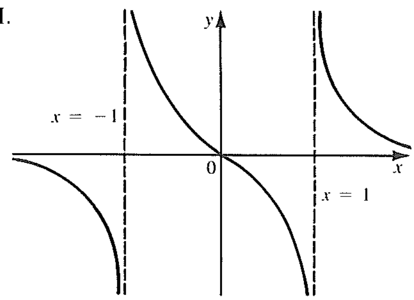
4. A. R B. x-intercept 0, y-intercept 0 C. about the y-axis D. HA: $y = 1$ E. increasing on $(0, \infty)$, decreasing on $(-\infty, 0)$ F. local minimum $f(0) = 0$ G. CU on $(-1, 1)$, CD on $(-\infty, -1)$, $(1, \infty)$, IP$\left(\pm 1, \frac{1}{4}\right)$

H.



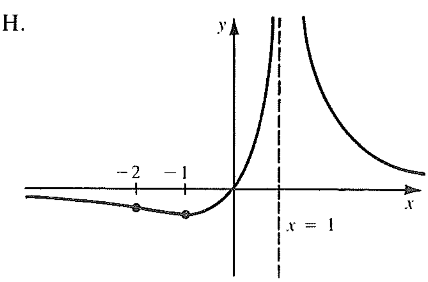
5. A. $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$ B. y-intercept 0, x-intercept 0 C. About the origin D. HA: $y = 0$, VA: $x = \pm 1$ E. decreasing on $(-\infty, -1)$, $(-1, 1)$, $(1, \infty)$ F. none G. CU on $(-1, 0)$, $(1, \infty)$, CD on $(-\infty, -1)$, $(0, 1)$, IP $(0, 0)$

H.



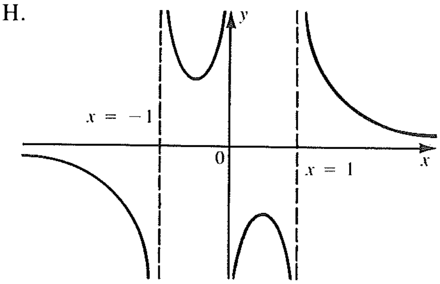
6. A. $(-\infty, 1) \cup (1, \infty)$ B. intercepts 0 C. none D. HA: $y = 0$, VA: $x = 1$ E. increasing on $(-1, 1)$, decreasing on $(-\infty, -1)$, $(1, \infty)$ F. $f(-1) = -\frac{1}{4}$ is a local minimum G. CU on $(-2, 1)$, $(1, \infty)$, CD on $(-\infty, -2)$, IP $\left(-2, -\frac{2}{9}\right)$

H.



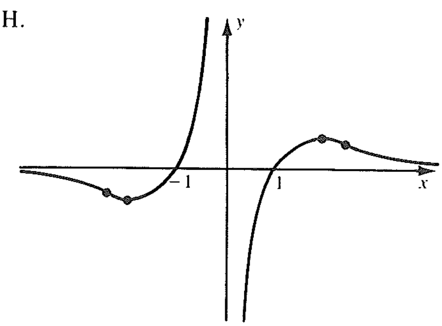
7. A. $(-\infty, -1) \cup (-1, 0) \cup (0, 1) \cup (1, \infty)$
B. none C. about the origin D. HA: $y = 0$, VA: $x = -1$, $x = 0$, $x = 1$ E. increases on $\left(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right)$, decreases on $\left(-\infty, -\sqrt{\frac{1}{3}}\right)$, $\left(\sqrt{\frac{1}{3}}, \infty\right)$
F. $f\left(-\sqrt{\frac{1}{3}}\right) = \dfrac{3\sqrt{3}}{2}$ is a local minimum,
$f\left(\sqrt{\frac{1}{3}}\right) = \dfrac{3\sqrt{3}}{2}$ is a local maximum.
G. CU on $(-1, 0)$, $(1, \infty)$,
CD on $(-\infty, -1)$, $(0, 1)$

H.



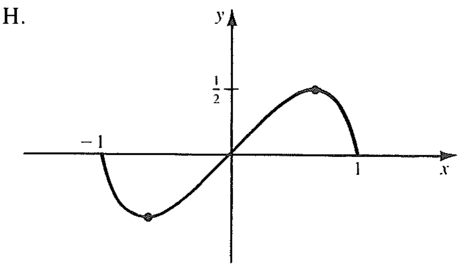
8. A. $(-\infty, 0) \cup (0, \infty)$ B. x-intercepts ± 1
C. about the origin D. HA: $y = 0$,
VA: $x = 0$ E. increasing on $(-\sqrt{3}, \sqrt{3})$,
decreasing on $(-\infty, -\sqrt{3})$, $(\sqrt{3}, \infty)$
F. $f(-\sqrt{3}) = -\dfrac{2\sqrt{3}}{9}$ is a local minimum,
$f(\sqrt{3}) = \dfrac{2\sqrt{3}}{9}$ is a local maximum G. CU on $(-\sqrt{6}, 0)$, $(\sqrt{6}, \infty)$, CD on $(-\infty, -\sqrt{6})$, $(0, \sqrt{6})$, IP $\left(-\sqrt{6}, -\dfrac{5}{6\sqrt{6}}\right)$, $\left(\sqrt{6}, \dfrac{5}{6\sqrt{6}}\right)$

H.



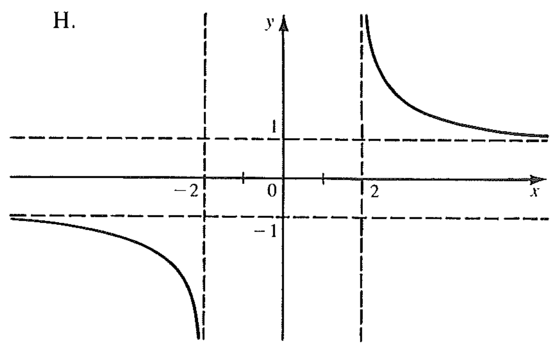
9. A. $[-1, 1]$ B. y-intercept 0, x-intercepts -1, 0, and 1 C. about the origin D. none
E. increasing on $\left(-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$, decreasing on $\left(-1, -\dfrac{1}{\sqrt{2}}\right)$, $\left(\dfrac{1}{\sqrt{2}}, 1\right)$
F. $f\left(-\dfrac{1}{\sqrt{2}}\right) = -\frac{1}{2}$ is a local minimum,
$f\left(\dfrac{1}{\sqrt{2}}\right) = \frac{1}{2}$ is a local maximum
G. CU on $(-1, 0)$, CD on $(0, 1)$, IP $(0, 0)$

H.



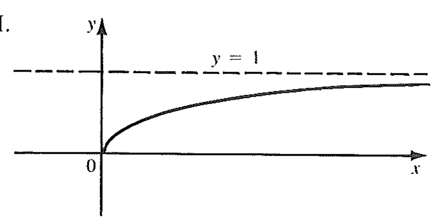
10. A. $(-\infty, -2) \cup (2, \infty)$ B. none C. about the origin D. HA: $y = \pm 1$, VA: $x = \pm 2$
E. decreasing on $(-\infty, -2)$, $(2, \infty)$ F. none
G. CD on $(-\infty, -2)$, CU on $(2, \infty)$

H.



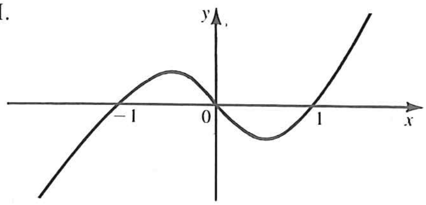
11. A. $[0, \infty)$ B. y-intercept 0, x-intercept 0
C. none D. HA: $y = 1$ E. increasing on $(0, \infty)$ F. none G. CD on $(0, \infty)$

H.



12. A. R B. y-intercept 0, x-intercepts $-1, 0, 1$ C. about the origin D. none E. increases on $\left(-\infty, -\frac{1}{3\sqrt{3}}\right)$, $\left(\frac{1}{3\sqrt{3}}, \infty\right)$, f decreases on $\left(-\frac{1}{3\sqrt{3}}, \frac{1}{3\sqrt{3}}\right)$, F. $f\left(\frac{1}{3\sqrt{3}}\right) = -\frac{2\sqrt{3}}{9}$ is a local minimum, $f\left(-3\frac{1}{\sqrt{3}}\right) = \frac{2\sqrt{3}}{9}$ is a local maximum G. CU on $(0, \infty)$, CD on $(-\infty, 0)$, IP $(0, 0)$

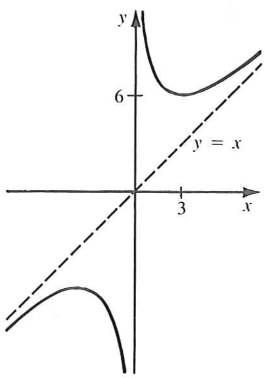
H.

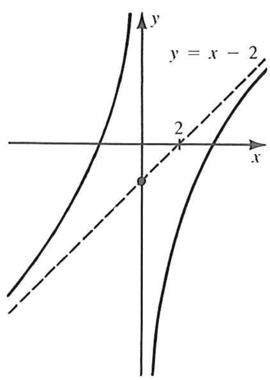


EXERCISE 5.6

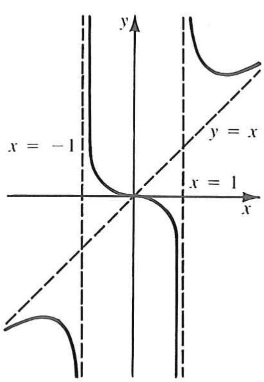
1. (a) $y = 2 - x$ (b) $y = x$ (c) $y = 3x + 1$ (d) $y = 2x - 1$ (e) $y = x + 4$ (f) $y = -x$

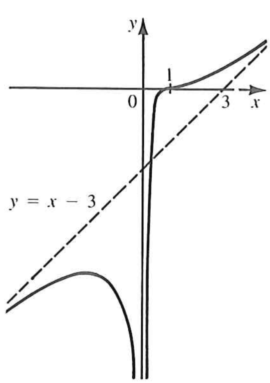
2. (a) $y = x$ (b) $y = x - 2$



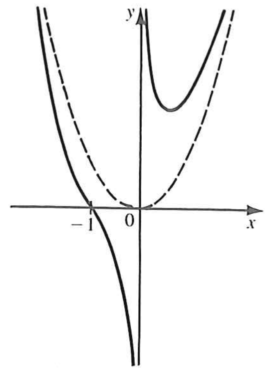


(c) $y = x$ (d) $y = x - 3$





3.



5.7 REVIEW EXERCISE

1. (a) $-\infty$ (b) ∞ (c) $-\infty$ (d) $-\infty$ (e) $-\infty$ (f) $-\frac{1}{5}$ (g) 0 (h) 2 (i) ∞ (j) 1

2. (a) VA: $x = \frac{1}{2}$, HA: $y = -3$
(b) VA: $x = -3$, HA: $y = 0$
(c) $x = -\frac{1}{2}$, $x = 3$, HA: $y = 0$
(d) VA: $x = 1$, HA: $y = 1$

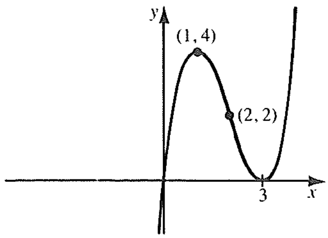
3. (a) CU on $\left(-\frac{4}{5}, \infty\right)$, CD on $\left(-\infty, -\frac{4}{5}\right)$, IP $\left(-\frac{4}{5}, \frac{238}{25}\right)$ (b) CU on $\left(-\infty, -\frac{1}{2}\right)$, $(1, \infty)$, CD on $\left(-\frac{1}{2}, 1\right)$, IP $\left(-\frac{1}{2}, -\frac{209}{16}\right)$ and $(1, -14)$

(c) CU on $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$, CD on $\left(-\infty, -\frac{2}{\sqrt{3}}\right)$, $\left(\frac{2}{\sqrt{3}}, \infty\right)$, IP $\left(\pm\frac{2}{\sqrt{3}}, \frac{1}{4}\right)$ (d) CU on $(0, \infty)$

4. (a) local maximum $f\left(\frac{2}{3}\right) = \frac{4}{27}$, local minimum $f(0) = 0$ (b) local maximum $f(-6) = 324$, local minimum $f(1) = -19$ (c) local maximum $g(0) = 0$, local minimum $g(2) = 4$ (d) local maximum $g\left(\frac{3}{4}\right) = \frac{5}{4}$

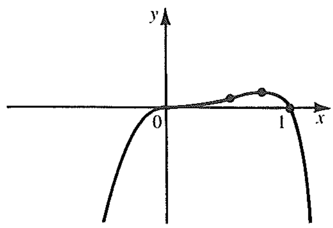
5. (a) A. R B. y-intercept 0, x-intercepts 0, 3 C. none D. none E. increasing on $(-\infty, 1)$, $(3, \infty)$, decreasing on $(1, 3)$ F. local maximum $f(1) = 4$, local minimum $f(3) = 0$ G. CU on $(2, \infty)$, CD on $(-\infty, 2)$, IP $(2, 2)$

H.



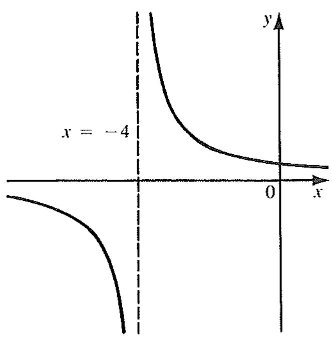
(b) A. R B. y-intercept 0, x-intercepts 0, 1
C. none D. none E. increasing on $\left(-\infty, \frac{3}{4}\right)$,
decreasing on $\left(\frac{3}{4}, \infty\right)$ F. local maximum
$f\left(\frac{3}{4}\right) = \frac{27}{256}$ G. CU on $\left(0, \frac{1}{2}\right)$, CD on $(-\infty, 0)$,
$\left(\frac{1}{2}, \infty\right)$, IP $(0, 0)$ and $\left(\frac{1}{2}, \frac{1}{16}\right)$

H.



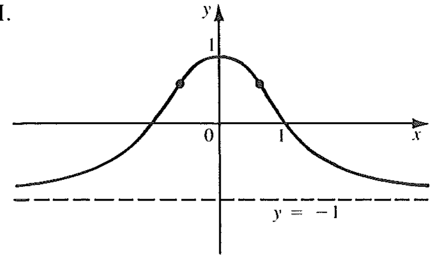
(c) A. $(-\infty, -4) \cup (-4, \infty)$ B. y-intercept $\frac{1}{2}$
C. none D. HA: $y = 0$, VA: $x = -4$
E. decreasing on $(-\infty, -4)$, $(-4, \infty)$ F. none
G. CU on $(-4, \infty)$, CD on $(-\infty, -4)$

H.



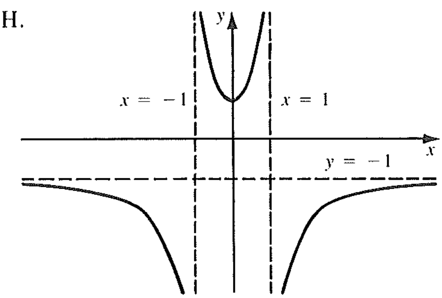
(d) A. R B. y-intercept 1, x-intercepts ± 1
C. about the y-axis D. HA: $y = -1$
E. increasing on $(-\infty, 0)$, decreasing on $(0, \infty)$
F. local maximum $f(0) = 1$
G. CU on $\left(-\infty, -\dfrac{1}{\sqrt{3}}\right)$, $\left(\dfrac{1}{\sqrt{3}}, \infty\right)$,
CD on $\left(-\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$, IP $\left(\pm\dfrac{1}{\sqrt{3}}, \frac{1}{2}\right)$

H.



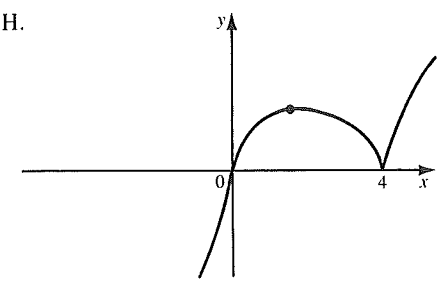
(e) A. $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
B. y-intercept 1 C. about the y-axis
D. HA: $y = -1$, VA: $x = -1$, $x = 1$
E. increasing on $(0, 1)$, $(1, \infty)$, decreasing on
$(-\infty, -1)$, $(-1, 0)$ F. local minimum $f(0) = 1$
G. CU on $(-1, 1)$, CD on $(-\infty, -1)$ and $(1, \infty)$

H.

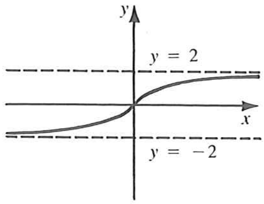


(f) A. R B. y-intercept 0, x-intercepts 0, 4
C. none D. none E. increasing on $\left(-\infty, \frac{4}{3}\right)$,
$(4, \infty)$, decreasing on $\left(\frac{4}{3}, 4\right)$ F. local minimum
$f(4) = 0$, local maximum $f\left(\frac{4}{3}\right) = \frac{4}{3}\sqrt[3]{4})$
G. CU on $(-\infty, 0)$, CD on $(0, 4)$, $(4, \infty)$,
IP $(0, 0)$

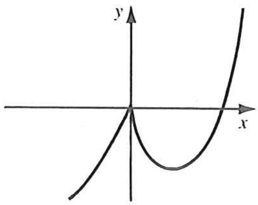
H.



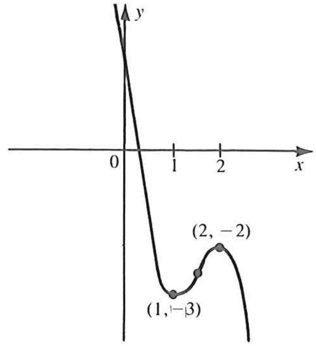
6.



7.



(e)



5.8 CHAPTER 5 TEST

1. (a) 3 (b) ∞ **2.** VA: $x = -\frac{5}{3}$, HA: $y = -\frac{2}{3}$

3. (a) CU on $(2, \infty)$, CD on $(-\infty, 2)$, IP $\left(2, \frac{2}{9}\right)$

4. (a) increasing on $(1, 2)$, decreasing on $(-\infty, 1)$, $(2, \infty)$ (b) local maximum $f(2) = -2$, local minimum $f(1) = -3$ (c) CU on $\left(-\infty, \frac{3}{2}\right)$, CD on $\left(\frac{3}{2}, \infty\right)$ (d) IP $\left(\frac{3}{2}, -\frac{5}{2}\right)$

5. (a) $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
(b) intercepts 0 (c) about the origin
(d) VA: $x = 3$, $x = -3$, HA: $y = 0$
(e) decreasing on $(-\infty, -3)$, $(-3, 3)$, $(3, \infty)$
(f) none (g) CU on $(-3, 0)$, $(3, \infty)$, CD on $(-\infty, -3)$, $(0, 3)$, IP $(0, 0)$