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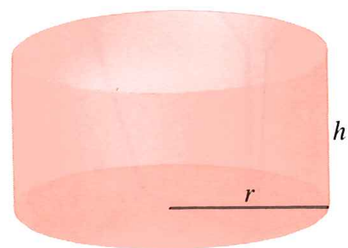
CHAPTER 11

INTEGRALS

REVIEW AND PREVIEW TO CHAPTER 11

Volume of a Cylinder



$$V = \pi r^2 h$$

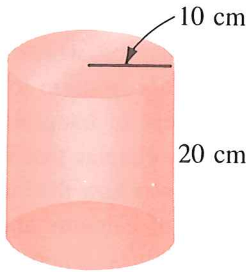
where r = radius of base

h = height

EXERCISE 1

1. Find the volume of each solid.

(a)



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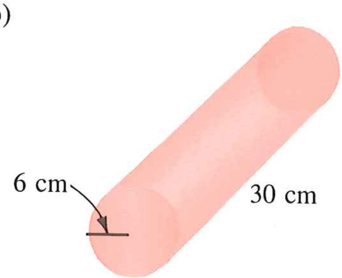

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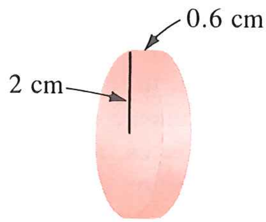

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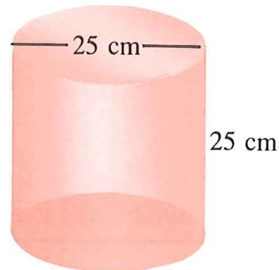

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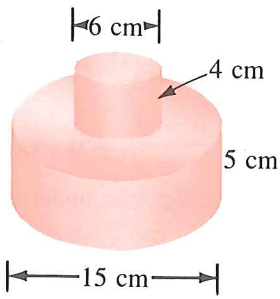

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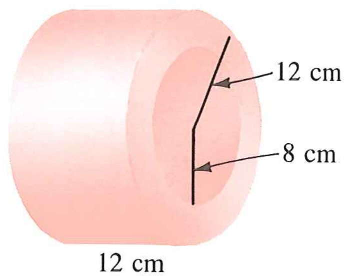


INTRODUCTION

There are two main branches of calculus, differential calculus and integral calculus. The first nine chapters of this book were concerned with differential calculus, whose central idea is that of a derivative and which arose from the solution to the tangent problem.

The main concept in integral calculus is the definite integral, and we will see in this chapter that it arises from the solution of the area problem. We discover how the Fundamental Theorem of Calculus links the two branches of calculus and enables us to compute integrals in terms of antiderivatives. Then we study a number of techniques for calculating integrals and use them to find volumes.

Chapters 10 and 11 of this book constitute an introduction to integral calculus and we apply it to find areas between curves and volumes of solids of revolution. In further courses you will see that integrals can also be used to compute lengths of curves and areas of surfaces in geometry; work, forces, and centres of mass in physics; and other quantities in probability, chemistry, biology, and economics.

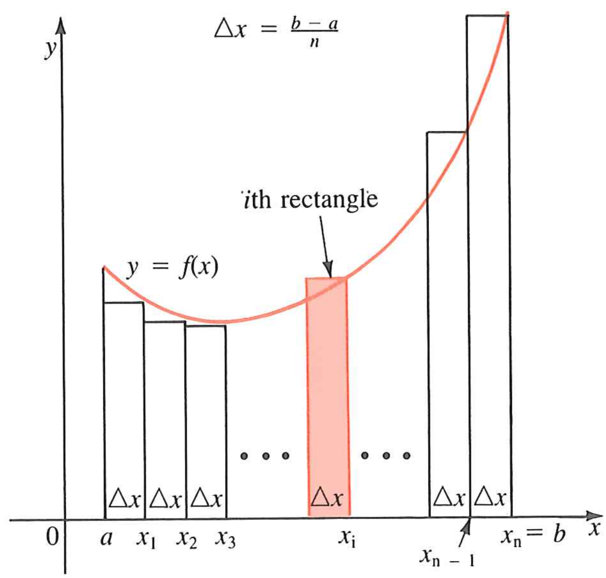
11.1 THE DEFINITE INTEGRAL

In Section 10.4 we saw that a limit of the form

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \tag{1}$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$

arises in finding the area under the curve $y = f(x)$ from a to b.



In Section 11.7 we will see that similar limits occur in calculating volumes. In fact, it turns out that limits of the form (1) also arise in computing lengths of curves, areas of surfaces, and physical quantities such as work and force. Since this type of limit occurs so frequently, we give it a special name and notation.

Let f be a continuous function defined on an interval $[a, b]$. The **definite integral of *f* from *a* to *b*** is

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $\quad \Delta x = \dfrac{b-a}{n} \quad$ and $\quad x_i = a + i\Delta x$

The symbol $\int$ was introduced by Leibniz and is called an **integral sign**. It is an elongated S and he chose it because an integral is a limit of sums. In the notation $\int_a^b f(x)dx$, $f(x)$ is called the **integrand** and a and b are called the **limits of integration**; a is the **lower limit** and b is the **upper limit**. You should regard $\int_a^b f(x)dx$ as a single symbol; the significance of dx will be explained in Section 11.3. The procedure of calculating an integral is called **integration**.

If the integrand is a positive function, then the integral represents an area. In fact, from Section 10.4 we know the following.

For the special case where $f(x) \geqslant 0$,

$$\int_a^b f(x)dx = \text{the area under the graph of } f \text{ from } a \text{ to } b$$

In general, however, *an integral need not represent an area*. This is the case in the following example.

Example 1 Evaluate $\displaystyle\int_0^5 (3x - x^2)dx$.

Solution We apply the definition of a definite integral with integrand $f(x) = 3x - x^2$, lower limit $a = 0$, and upper limit $b = 5$. Thus

$$\Delta x = \frac{b-a}{n} = \frac{5-0}{n} = \frac{5}{n}$$

and $$x_i = a + i\Delta x = 0 + i\left(\frac{5}{n}\right) = \frac{5i}{n}$$

Therefore we have

$$
\begin{aligned}
\int_0^5 (3x - x^2)dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{5i}{n}\right)\frac{5}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left[3\left(\frac{5i}{n}\right) - \left(\frac{5i}{n}\right)^2\right]\frac{5}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{75i}{n^2} - \frac{125i^2}{n^3}\right) \\
&= \lim_{n\to\infty} \left(\frac{75}{n^2}\sum_{i=1}^{n} i - \frac{125}{n^3}\sum_{i=1}^{n} i^2\right) \\
&= \lim_{n\to\infty} \left[\left(\frac{75}{n^2}\right)\frac{n(n+1)}{2} - \left(\frac{125}{n^3}\right)\frac{n(n+1)(2n+1)}{6}\right] \\
&= \lim_{n\to\infty} \left[\frac{75}{2}\left(1 + \frac{1}{n}\right) - \frac{125}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)\right] \\
&= \frac{75}{2}(1) - \frac{125}{6}(1)(2) \\
&= -\frac{25}{6}
\end{aligned}
$$

Notice that the value of the integral in Example 1 is a negative number. This is not surprising because the function $f(x) = 3x - x^2$ takes on both positive and negative values for $0 \leq x \leq 5$, and so the integral does not represent an area. In the following example we compute the value of the approximating sum for $n = 10$ and show that it is negative. This will enable us to see what the integral does represent.

Example 2 If $f(x) = 3x - x^2$, $a = 0$, and $b = 5$, find the value of the sum

$$\sum_{i=1}^{10} f(x_i)\,\Delta x$$

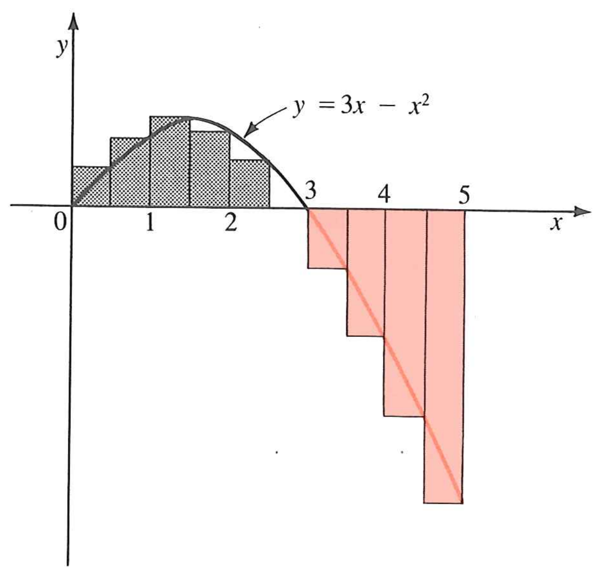
Solution With $n = 10$ we have

$$\Delta x = \frac{b - a}{n} = \frac{5 - 0}{10} = \frac{1}{2}$$

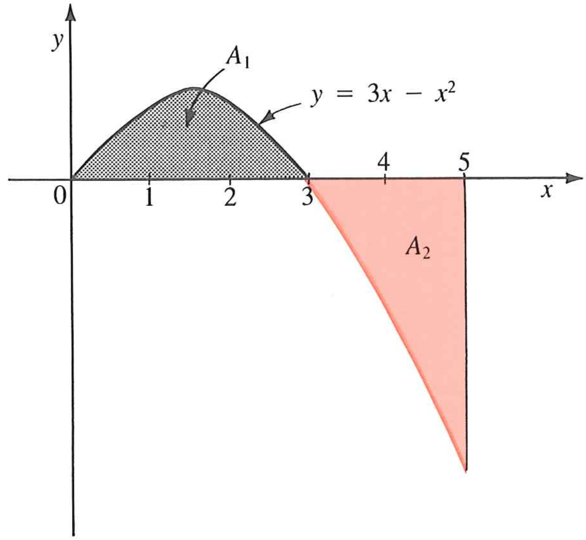
Therefore

$$
\begin{aligned}
\sum_{i=1}^{10} f(x_i)\Delta x &= \tfrac{1}{2}\sum_{i=1}^{10} f(x_i) \\
&= \tfrac{1}{2}[f(0.5) + f(1) + f(1.5) + f(2) + f(2.5) + f(3) \\
&\quad + f(3.5) + f(4) + f(4.5) + f(5)] \\
&= \tfrac{1}{2}[1.25 + 2 + 2.25 + 2 + 1.25 + 0 - 1.75 \\
&\quad - 4 - 6.75 - 10] \\
&= -6.875
\end{aligned}
$$

The following diagram illustrates Example 2. Notice that the function $f(x) = 3x - x^2$ is positive for $0 < x < 3$ and negative for $3 < x \leq 5$. As in Section 10.4, the areas of the grey rectangles represent the first five terms in the sum. The areas of the red rectangles are the negatives of the last four terms in the sum. Thus the total sum represents the areas of the rectangles that lie above the x-axis minus the areas of the rectangles below the x-axis.



When we take the limit of such sums we get the value of the integral in Example 1. This suggests that the integral can be interpreted as the area of the grey region minus the area of the red region in the following diagram.

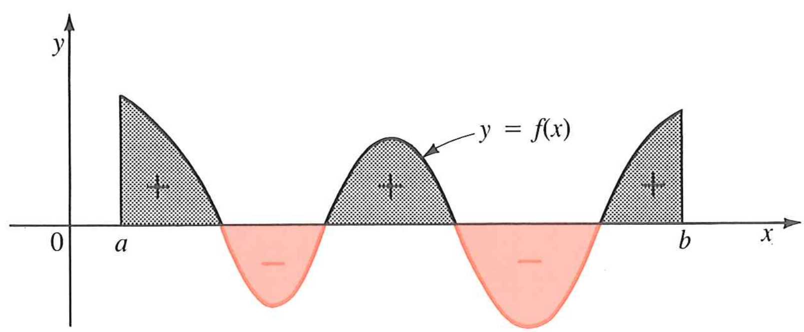


$$\int_0^5 (3x - x^2)dx = A_1 - A_2$$

In general, if a function f takes on both positive and negative values, then we can interpret $\int_a^b f(x)dx$ as a difference of areas:

$$\int_a^b f(x)dx = A_1 - A_2$$

where A_1 = area of region above the x-axis and below the graph of f
A_2 = area of the region below the x-axis and above the graph of f



Example 3 Evaluate $\int_{-1}^{2} x^3\,dx$ and interpret the value as a difference of areas.

Solution We have $a = -1$, $b = 2$, and $f(x) = x^3$, so

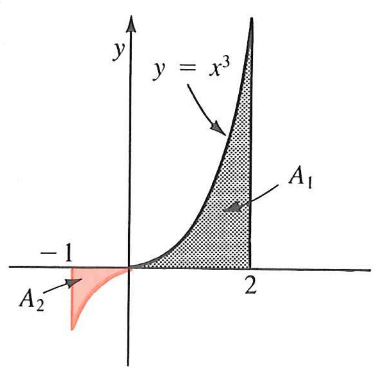
$$\Delta x = \frac{b-a}{n} = \frac{2-(-1)}{n} = \frac{3}{n}$$

$$x_i = a + i\Delta x = -1 + \frac{3i}{n}$$

and

$$\begin{aligned}
\int_{-1}^{2} x^3 dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(-1 + \frac{3i}{n}\right)\frac{3}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(-1 + \frac{3i}{n}\right)^3 \frac{3}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(-1 + \frac{9i}{n} - \frac{27i^2}{n^2} + \frac{27i^3}{n^3}\right)\frac{3}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(-\frac{3}{n} + \frac{27i}{n^2} - \frac{81i^2}{n^3} + \frac{81i^3}{n^4}\right) \\
&= \lim_{n\to\infty} \left(-\frac{3}{n}\sum_{i=1}^{n} 1 + \frac{27}{n^2}\sum_{i=1}^{n} i - \frac{81}{n^3}\sum_{i=1}^{n} i^2 + \frac{81}{n^4}\sum_{i=1}^{n} i^3\right) \\
&= \lim_{n\to\infty} \left[-\frac{3}{n}n + \frac{27}{n^2}\frac{n(n+1)}{2} - \frac{81}{n^3}\frac{n(n+1)(2n+1)}{6} + \frac{81}{n^4}\frac{n^2(n+1)^2}{4}\right] \\
&= \lim_{n\to\infty} \left[-3 + \frac{27}{2}\left(1 + \frac{1}{n}\right) - \frac{81}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) + \frac{81}{4}\left(1 + \frac{1}{n}\right)^2\right] \\
&= -3 + \frac{27}{2}(1) - \frac{81}{6}(1)(2) + \frac{81}{4}(1) \\
&= 3.75
\end{aligned}$$

Since $x^3 \geqslant 0$ for $0 \leqslant x \leqslant 2$ and $x^3 \leqslant 0$ for $-1 \leqslant x \leqslant 0$, the integral can be interpreted as $A_1 - A_2$, where A_1 and A_2 are the areas shown in the diagram.



EXERCISE 11.1

B **1.** (a) Evaluate the integral $\int_0^4 (x^2 - 2x)dx$.

(b) Find the value of the approximating sum

$$\sum_{i=1}^{8} f(x_i)\Delta x$$

where $f(x) = x^2 - 2x$, $a = 0$, and $b = 4$.

(c) Draw a diagram showing the approximating rectangles (like the diagram following Example 2).

(d) Interpret the integral in part (a) as a difference of areas and illustrate with a diagram.

2. Evaluate each integral.

(a) $\int_{-2}^{3} (1 - 4x)dx$ (b) $\int_0^1 (1 + 4x - 6x^2)dx$

(c) $\int_0^1 x^3\, dx$ (d) $\int_1^4 (x^2 - 6)dx$

3. Evaluate each integral and interpret the value as a difference of areas.

(a) $\int_0^3 (1 - x^2)dx$ (b) $\int_3^5 (2x - 7)dx$

4. Evaluate $\int_a^b x^2\, dx$.

5. (a) Use the methods of Chapter 5 to sketch the curve $y = x^3 - 4x$.

(b) Evaluate the integral $\int_{-2}^{2} (x^3 - 4x)dx$.

(c) Give a geometric explanation for the value of the integral in part (b).

11.2 THE FUNDAMENTAL THEOREM OF CALCULUS

In Section 11.1 we used the definition of an integral to calculate integrals as limits of sums. This procedure is sometimes long and difficult and so we now give a much easier method, based on the Fundamental Theorem of Calculus.

For the special case where $f(x) \geqslant 0$, we know that the definite integral $\int_a^b f(x)dx$ represents the area under the curve $y = f(x)$ from a to b. From Section 10.1 we know that this area is equal to $F(b) - F(a)$, where F is any antiderivative of f. Therefore, in this case, we have

$$\int_a^b f(x)dx = F(b) - F(a) \qquad \text{where } F' = f$$

It turns out that this equation is true even when f is not a positive function.

Fundamental Theorem of Calculus

If f is continuous on the interval $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f.

Example 1 Evaluate $\int_{-1}^{2} x^3\, dx$.

Solution The function $f(x) = x^3$ is a polynomial, so it is continuous on the interval $[-1, 2]$. An antiderivative of f is $F(x) = \frac{1}{4}x^4$, so the Fundamental Theorem of Calculus gives

$$\begin{aligned}\int_{-1}^{2} x^3\, dx &= F(2) - F(-1)\\ &= \tfrac{1}{4}(2)^4 - \tfrac{1}{4}(-1)^4\\ &= \tfrac{15}{4}\end{aligned}$$

If you compare this solution with Example 3 in Section 11.1, you will see that the Fundamental Theorem of Calculus gives a much simpler solution.

We often use the notation

$$F(b) - F(a) = F(x)\Big]_a^b \quad \text{or} \quad F(x)\Big|_a^b$$

and so the conclusion of the Fundamental Theorem of Calculus can be written as

$$\int_a^b f(x)dx = F(x)\Big]_a^b \quad \text{where } F' = f$$

Thus the solution of Example 1 could be streamlined as follows:

$$\int_{-1}^{2} x^3\, dx = \frac{x^4}{4}\Big]_{-1}^{2} = \frac{2^4 - (-1)^4}{4} = \frac{15}{4}$$

Example 2 Find $\int_{\pi}^{2\pi} \sin x \, dx$.

Solution An antiderivative of $\sin x$ is $-\cos x$, so

$$\begin{aligned}\int_{\pi}^{2\pi} \sin x \, dx &= -\cos x \Big]_{\pi}^{2\pi} \\ &= (-\cos 2\pi) - (-\cos \pi) \\ &= -1 - 1 \\ &= -2\end{aligned}$$

In order to make effective use of the Fundamental Theorem of Calculus, we must have a supply of antiderivatives of functions and we list a number of them in the following table. We use the traditional notation $\int f(x)dx$ for an antiderivative of f. (This notation is used because of the relation between integrals and antiderivatives given by the Fundamental Theorem.) The antiderivative $\int f(x)dx$ is often called an **indefinite integral of f**.

Table of Indefinite Integrals (Antiderivatives)

$\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + C \quad (n \neq -1)$	$\int \dfrac{1}{x} \, dx = \ln \lvert x \rvert + C$
$\int \sin x \, dx = -\cos x + C$	$\int \cos x \, dx = \sin x + C$
$\int \sec^2 x \, dx = \tan x + C$	$\int \csc^2 x \, dx = -\cot x + C$
$\int \sec x \tan x \, dx = \sec x + C$	$\int \csc x \cot x \, dx = -\csc x + C$
$\int e^x \, dx = e^x + C$	$\int a^x \, dx = \dfrac{a^x}{\ln a} + C \quad (a \neq 1)$
$\int \dfrac{1}{x^2+1} \, dx = \tan^{-1} x + C$	$\int \dfrac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C$

Any formula in this table can be verified by differentiating the function on the right side. For instance,

$$\int \sec^2 x \, dx = \tan x \quad \text{because} \quad \frac{d}{dx} \tan x = \sec^2 x$$

We should distinguish carefully between definite and indefinite integrals. A definite integral $\int_a^b f(x)dx$ is a number, whereas an indefinite integral $\int f(x)dx$ is a function. The Fundamental Theorem of Calculus gives the connection between them:

$$\int_a^b f(x)dx = \int f(x)dx \Big]_a^b$$

In finding antiderivatives of more complicated functions, we use the following rules:

Properties of Integrals

$$\int cf(x)dx = c\int f(x)dx \quad (c \text{ a constant})$$

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

These can be verified by differentiating both sides of each equation.

In view of the Fundamental Theorem of Calculus, it follows that the definite integral has similar properties:

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx \quad (c \text{ a constant})$$

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

Example 3 Find the general indefinite integral $\int (6x^2 + \csc^2 x)dx$.

Solution From the table, we have

$$\begin{aligned}\int (6x^2 + \csc^2 x)dx &= 6\int x^2\,dx + \int \csc^2 x\,dx\\ &= 6\left(\frac{x^3}{3}\right) + (-\cot x) + C\\ &= 2x^3 - \cot x + C\end{aligned}$$

Example 4 Find $\int_2^5 (2x^3 - 3x^2 + 7x + 2)dx$.

Solution

$$\begin{aligned}\int_2^5 (2x^3 - 3x^2 + 7x + 2)dx &= 2\left(\frac{x^4}{4}\right) - 3\left(\frac{x^3}{3}\right) + 7\left(\frac{x^2}{2}\right) + 2x\Big]_2^5\\ &= \tfrac{1}{2}x^4 - x^3 + \tfrac{7}{2}x^2 + 2x\Big]_2^5\\ &= \left[\tfrac{1}{2}(5)^4 - 5^3 + \tfrac{7}{2}(5)^2 + 2(5)\right]\\ &\qquad - \left[\tfrac{1}{2}(2)^4 - 2^3 + \tfrac{7}{2}(2)^2 + 2(2)\right]\\ &= 267\end{aligned}$$

Example 5 Evaluate $\int_1^8 \frac{1}{\sqrt[3]{x^2}}\,dx$.

Solution

$$\begin{aligned}\int_1^8 \frac{1}{\sqrt[3]{x^2}}\,dx &= \int_1^8 x^{-\frac{2}{3}}\,dx \\ &= \left.\frac{x^{-\frac{2}{3}+1}}{-\frac{2}{3}+1}\right|_1^8 \\ &= \left.3x^{\frac{1}{3}}\right|_1^8 \\ &= 3(2) - 3(1) \\ &= 3\end{aligned}$$

Example 6 Find $\int_1^4 \frac{t^2+\sqrt{t}-2}{t}\,dt$.

Solution

$$\begin{aligned}\int_1^4 \frac{t^2+\sqrt{t}-2}{t}\,dt &= \int_1^4 \left(t + \frac{1}{\sqrt{t}} - \frac{2}{t}\right)dt \\ &= \int_1^4 \left(t + t^{-\frac{1}{2}} - \frac{2}{t}\right)dt \\ &= \left[\frac{t^2}{2} + 2t^{\frac{1}{2}} - 2\ln t\right]_1^4 \\ &= (8 + 4 - 2\ln 4) - \left(\tfrac{1}{2} + 2 - 2\ln 1\right) \\ &= 9.5 - 2\ln 4\end{aligned}$$

$\ln|t| = \ln t$ since $t > 0$

Example 7 Find $\int_0^1 \frac{1}{x^2+1}\,dx$.

Solution

$$\begin{aligned}\int_0^1 \frac{1}{x^2+1}\,dx &= \left.\tan^{-1} x\right|_0^1 \\ &= \tan^{-1} 1 - \tan^{-1} 0 \\ &= \frac{\pi}{4} - 0 \\ &= \frac{\pi}{4}\end{aligned}$$

The Fundamental Theorem of Calculus says that if f is continuous, then

$$\int_a^b f(x)dx = F(b) - F(a) \quad \text{where } F' = f$$

Thus, putting $f = F'$ in the left side, we can rewrite the equation as

$$\int_a^b F'(x)dx = F(b) - F(a)$$

This version says that if we take a function F, differentiate it, and then integrate the result, we arrive at an expression that involves the original function F. Thus the Fundamental Theorem says that differentiation and integration are inverse processes. This inverse relationship, first noticed by Isaac Barrow, was used by Newton and Leibniz to make calculus a powerful method for solving problems in mathematics and science.

EXERCISE 11.2

1. Evaluate the following definite integrals.

(a) $\int_{-6}^{7} 2\, dx$ (b) $\int_{-1}^{5} (6x - 7)dx$

(c) $\int_{1}^{2} (5 + 4x - 6x^2)dx$ (d) $\int_{0}^{1} (t^2 + 6t - 1)dt$

(e) $\int_{-1}^{2} (x^3 - x^2 + 4x)dx$ (f) $\int_{0}^{1} (x^{99} + 1)dx$

(g) $\int_{2}^{3} \frac{1}{t^2}\, dt$ (h) $\int_{1}^{4} (x - \sqrt{x})dx$

(i) $\int_{0}^{1} \sqrt[4]{x^5}\, dx$ (j) $\int_{1}^{8} \frac{2}{\sqrt[3]{x}}\, dx$

(k) $\int_{1}^{2} \frac{x^3 + x^2 + 1}{x^3}\, dx$ (l) $\int_{1}^{4} \left(\frac{\sqrt{x} + 1}{x}\right)dx$

(m) $\int_{0}^{64} \sqrt{y}(1 + \sqrt[3]{y})dy$ (n) $\int_{0}^{\frac{\pi}{2}} (8x + \cos x)dx$

(o) $\int_{0}^{\frac{\pi}{6}} (\sec x \tan x)dx$ (p) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (3 \sin \theta - \sec^2 \theta)d\theta$

2. Find the following general indefinite integrals.

(a) $\int (x^5 - 2x^3 + 4)dx$ (b) $\int x^2\sqrt{x}\, dx$

(c) $\int \left(t + \frac{2}{t}\right)dt$ (d) $\int (1 + \sqrt{x})^2 dx$

(e) $\int \frac{x - 5}{\sqrt[4]{x}}\, dx$ (f) $\int (\cos \theta + \sin \theta)d\theta$

(g) $\int (5x^4 - 2 \csc x \cot x)dx$ (h) $\int (2 \csc^2 x + 1)dx$

3. Evaluate each integral.

(a) $\int_0^1 e^x\,dx$ (b) $\int_{-1}^1 2^x\,dx$

(c) $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx$ (d) $\int_1^{\sqrt{3}} \frac{12}{1+x^2}\,dx$

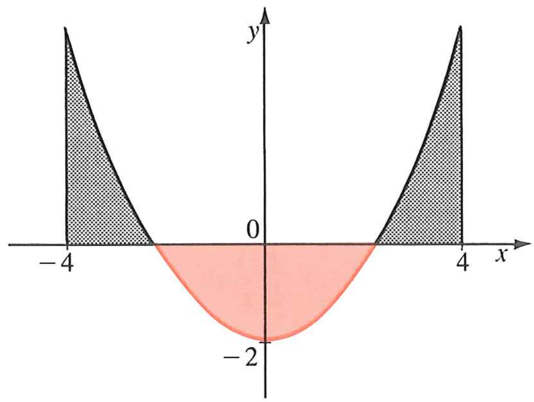
(e) $\int_{-1}^1 \left(x+1+\frac{3}{x^2+1}\right)dx$ (f) $\int_{-\pi}^0 (2e^x+\sin x)dx$

4. What is wrong with the following calculation?

$$\int_{-2}^1 \frac{1}{x^4}\,dx = \frac{x^{-3}}{-3}\Bigg]_{-2}^1 = -\frac{1}{3x^3}\Bigg]_{-2}^1 = -\frac{1}{3} - \frac{1}{24} = -\frac{9}{24}$$

PROBLEMS PLUS

Given that the area above the x-axis is equal to the area below the x-axis, find the equation of the parabola.



11.3 THE SUBSTITUTION RULE

If we use the Chain Rule to differentiate $y = \sqrt{x^2+1}$, we get

$$\frac{d}{dx}\sqrt{x^2+1} = \tfrac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x) = \frac{x}{\sqrt{x^2+1}}$$

Therefore in terms of indefinite integrals, we have

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \sqrt{x^2+1} + C$$

Many other integrals can also be evaluated by reversing the Chain Rule, so first let us recall the general version of the Chain Rule:

$$\frac{d}{dx}[F(g(x))] = F'(g(x))\, g'(x)$$

Turning this around and stating it in terms of integrals, we get

$$\int F'(g(x))g'(x)dx = F(g(x)) + C$$

Let us change from the variable x to the variable $u = g(x)$. Then we have

$$\int F'(g(x))g'(x)dx = F(g(x)) + C = F(u) + C = \int F'(u)du$$

If we now write $F' = f$, we get

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Thus we have the following rule for evaluating an indefinite integral when we make a *substitution* or *change of variable* $u = g(x)$.

Substitution Rule for Indefinite Integrals

If $u = g(x)$, then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Example 1 Find $\int (x^2 - 5)^8\, 2x\, dx$.

Solution 1 We let

$$u = g(x) = x^2 - 5 \quad \text{and} \quad f(u) = u^8$$

Then $\quad g'(x) = 2x \quad$ and $\quad f(g(x)) = (x^2 - 5)^8$

so we have

$$\begin{aligned}\int (x^2 - 5)^8\, 2x\, dx &= \int f(g(x))g'(x)dx \\ &= \int f(u)du \qquad \text{(Substitution Rule)} \\ &= \int u^8\, du \\ &= \frac{u^9}{9} + C \\ &= \frac{(x^2 - 5)^9}{9} + C\end{aligned}$$

The Substitution Rule can be remembered most easily through the idea of a *differential*. If $u = g(x)$ is a differentiable function, we define the **differential** dx to be an independent variable; that is, dx can be given the value of any real number. Then the **differential** du is defined in terms of dx by the equation

$$du = g'(x)dx$$

For example, if $u = 8x^4$, then $du = 32x^3\,dx$.

If we now regard the dx and the du after integral signs as differentials, we have

$$\int f(g(x))g'(x)dx = \int f(u)du$$

which is the Substitution Rule. This means that in using the Substitution Rule, all we have to do is treat dx and du after integral signs as differentials. For instance, we could solve Example 1 as follows:

Solution 2 Let $u = x^2 - 5$. Then $du = 2x\,dx$, so

$$\int (x^2 - 5)^8\, 2x\,dx = \int u^8\,du = \frac{u^9}{9} + C = \frac{(x^2-5)^9}{9} + C$$

Example 2 Evaluate $\displaystyle\int \frac{x^2}{\sqrt{1-x^3}}\,dx$.

Solution Let $u = 1 - x^3$. Then $du = -3x^2\,dx$. We notice that $x^2\,dx$ occurs in the integrand, so we solve for it: $x^2\,dx = -\frac{1}{3}\,du$. Thus

$$\begin{aligned}\int \frac{x^2}{\sqrt{1-x^3}}\,dx &= \int \frac{-\frac{1}{3}\,du}{\sqrt{u}} \\ &= -\tfrac{1}{3}\int u^{-\frac{1}{2}}\,du \\ &= \left(-\tfrac{1}{3}\right)2u^{\frac{1}{2}} + C \\ &= -\tfrac{2}{3}(1-x^3)^{\frac{1}{2}} + C\end{aligned}$$

In using the Substitution Rule, the idea is to replace a complicated integral by a simpler integral by changing to a new variable u. For instance, in Example 1 we started with the integral $\int (x^2 - 5)^8\, 2x\,dx$ and obtained the simpler integral $\int u^8\,du$.

In thinking of an appropriate substitution, *we try to choose u to be some function in the integrand whose differential also occurs (except perhaps for a constant factor)*. In Example 1, we chose $u = x^2 - 5$ because $du = 2x\,dx$ occurs. In Example 2, we chose $u = 1 - x^3$ because $du = -3x^2\,dx$ is a constant multiple of $x^2\,dx$.

Example 3 Find $\displaystyle\int \frac{\ln x}{x}\,dx$.

Solution We let $u = \ln x$ because its differential is $du = \frac{1}{x}\,dx$. Thus

$$\begin{aligned}\int \frac{\ln x}{x}\,dx &= \int u\,du\\ &= \tfrac{1}{2}u^2 + C\\ &= \tfrac{1}{2}(\ln x)^2 + C\end{aligned}$$

Example 4 Find $\int \sin 4x\,dx$.

Solution Here we let $u = 4x$. Then $du = 4\,dx$, so $dx = \frac{1}{4}\,du$. Therefore,

$$\begin{aligned}\int \sin 4x\,dx &= \int (\sin u)\left(\tfrac{1}{4}\,du\right)\\ &= \tfrac{1}{4}\int \sin u\,du\\ &= \tfrac{1}{4}(-\cos u) + C\\ &= -\tfrac{1}{4}\cos 4x + C\end{aligned}$$

Example 5 Find $\int (2 + \sin x)^{10}\cos x\,dx$.

Solution Observe that if we let $u = 2 + \sin x$, then $du = \cos x\,dx$. So,

$$\begin{aligned}\int (2 + \sin x)^{10}\cos x\,dx &= \int u^{10}\,du\\ &= \frac{u^{11}}{11} + C\\ &= \tfrac{1}{11}(2 + \sin x)^{11} + C\end{aligned}$$

When performing a definite integration using substitution, we have to change the limits of integration so that they are the appropriate values of u.

Substitution Rule for Definite Integrals

If $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Example 6 Evaluate $\int_1^9 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$.

Solution Let $u = \sqrt{x}$. Then

$$du = \tfrac{1}{2}x^{-\frac{1}{2}}\,dx = \frac{1}{2\sqrt{x}}\,dx$$

So $\qquad \dfrac{1}{\sqrt{x}}\,dx = 2\,du$

Now we find the new limits of integration as follows.

When $x = 1$, $u = \sqrt{1} = 1$.
When $x = 9$, $u = \sqrt{9} = 3$.

Thus,
$$\int_1^9 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int_1^3 e^u(2\,du)$$
$$= 2\int_1^3 e^u\,du$$
$$= 2e^u\Big]_1^3$$
$$= 2(e^3 - e)$$

Example 7 Find the area under the curve $y = \dfrac{1}{2x + 1}$ from 0 to 1.

Solution Since the given function is positive for $0 \leqslant x \leqslant 1$, the area under its graph is equal to the integral:

$$A = \int_0^1 \frac{1}{2x + 1}\,dx$$

Let $u = 2x + 1$. Then $du = 2\,dx$, so $dx = \frac{1}{2}\,du$. We find the new limits of integration.

When $x = 0$, $u = 2(0) + 1 = 1$.
When $x = 1$, $u = 2(1) + 1 = 3$.

Therefore
$$A = \int_0^1 \frac{1}{2x + 1}\,dx$$
$$= \int_1^3 \frac{\frac{1}{2}\,du}{u}$$
$$= \frac{1}{2}\int_1^3 \frac{1}{u}\,du$$
$$= \frac{1}{2}\ln|u|\Big]_1^3$$
$$= \frac{1}{2}(\ln 3 - \ln 1)$$
$$= \frac{1}{2}\ln 3$$

EXERCISE 11.3

A **1.** Suggest an appropriate substitution for each integral.

(a) $\int \sin(x^2)\,2x\,dx$ (b) $\int \frac{(\ln x)^2}{x}\,dx$

(c) $\int \cos 5x\,dx$ (d) $\int \sqrt{\sin x}\cos x\,dx$

B **2.** Evaluate each integral by making the given substitution.

(a) $\int x(1 - x^2)^{10}\,dx,\ u = 1 - x^2$

(b) $\int e^{5x}\,dx,\ u = 5x$

(c) $\int \sqrt{x - 1}\,dx,\ u = x - 1$

(d) $\int \frac{x + 1}{x^2 + 2x - 6}\,dx,\ u = x^2 + 2x - 6$

3. Evaluate the following indefinite integrals.

(a) $\int x(x^2 + 4)^8\,dx$ (b) $\int x^2\sqrt{x^3 + 2}\,dx$

(c) $\int (x + 6)^{10}\,dx$ (d) $\int \frac{1}{(3x - 1)^2}\,dx$

(e) $\int \sec^2 3x\,dx$ (f) $\int (1 + 2x^4)x^3\,dx$

(g) $\int \sin^2 x\cos x\,dx$ (h) $\int \frac{\sqrt{\ln x}}{x}\,dx$

(i) $\int t^2e^{t^3}\,dt$ (j) $\int \frac{1}{1 - x}\,dx$

(k) $\int \frac{3x^2 - 2}{(x^3 - 2x + 1)^3}\,dx$ (l) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

(m) $\int e^{3-x}\,dx$ (n) $\int e^{\cos x}\sin x\,dx$

(o) $\int \sqrt{1 + \tan x}\sec^2 x\,dx$ (p) $\int x\sin(x^2)\,dx$

(q) $\int \sin x\sin(\cos x)dx$ (r) $\int \frac{\tan^{-1} x}{1 + x^2}\,dx$

4. Evaluate the following definite integrals.

(a) $\int_0^1 e^{2x+1}\,dx$ (b) $\int_0^2 \frac{1}{(1 + 5x)^4}\,dx$

(c) $\int_0^2 x\sqrt{4 - x^2}\,dx$ (d) $\int_0^1 \sin \pi t\,dt$

(e) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos\theta}{\sin^3\theta}\,d\theta$ (f) $\int_0^1 x^4(x^5 + 1)^5\,dx$

(g) $\int_{\frac{1}{2}}^{1} \frac{\left(1 + \frac{1}{x}\right)^5}{x^2}\,dx$ (h) $\int_1^2 (x + 1)e^{3x^2+6x-4}\,dx$

5. Find (a) $\int \tan x \, dx$ (b) $\int \cot x \, dx$
6. Find the area under the curve $y = \sqrt{4x + 1}$ from 0 to 10.
7. Find the area under the curve $y = \cos\left(\dfrac{x}{2}\right)$, $0 \leqslant x \leqslant \pi$.
8. Find the area bounded by the curves $y = e^{-x}$, $y = e^{2x}$, and $x = 1$.

C 9. Evaluate

(a) $\displaystyle\int \frac{1}{x + \sqrt{x}} \, dx$ (b) $\displaystyle\int \frac{x + 1}{x + 2} \, dx$

11.4 INTEGRATION BY PARTS

For every differentiation rule there is a corresponding integration rule. Just as we reversed the Chain Rule to get the Substitution Rule, we can reverse the Product Rule to get the rule for integration by parts.

The Product Rule states that if f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$$

In terms of integrals, this becomes

$$\int [f(x)g'(x) + f'(x)g(x)]dx = f(x)g(x)$$

or

$$\int f(x)g'(x)dx + \int f'(x)g(x)dx = f(x)g(x)$$

We can rewrite this equation as follows.

Integration by Parts

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx \qquad (1)$$

Example 1 Find $\int xe^x \, dx$.

Solution Suppose we choose

$$f(x) = x \quad \text{and} \quad g'(x) = e^x$$

Then

$$f'(x) = 1 \qquad g(x) = e^x$$

(For g we can choose *any* antiderivative of g'.) Then Formula 1 gives

$$\begin{aligned}\int xe^x \, dx &= f(x)g(x) - \int f'(x)g(x)dx \\ &= xe^x - \int e^x \, dx \\ &= xe^x - e^x + C\end{aligned}$$

In integration by parts the aim is to get a simpler integral than the one we started with. Thus, in Example 1 we started with $\int xe^x\,dx$ and expressed it in terms of the simpler integral $\int e^x\,dx$. If we had chosen $f(x) = e^x$ and $g'(x) = x$, then $f'(x) = e^x$ and $g(x) = \frac{1}{2}x^2$, so integration by parts gives

$$\int xe^x\,dx = e^x\left(\tfrac{1}{2}x^2\right) - \tfrac{1}{2}\int x^2e^x\,dx$$

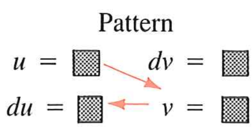
But $\int x^2e^x\,dx$ is a more difficult integral than the one we started with. In general, *we try to choose f to be a function that becomes simpler when differentiated, as long as $g'(x)$ can be readily integrated to give $g(x)$.*

Formula 1 is often stated in differential notation. Let $u = f(x)$ and $v = g(x)$. Then $du = f'(x)dx$ and $dv = g'(x)dx$. By the Substitution Rule, Formula 1 can be written as follows.

Integration by Parts in Differential Notation

$$\int u\,dv = uv - \int v\,du \qquad ②$$

Using Formula 2, we could rewrite the solution to Example 1 as follows:



Let $\quad u = x \qquad\qquad dv = e^x\,dx$

Then $\quad du = dx \qquad\qquad v = e^x$

So,
$$\int \underbrace{x}_{u}\,\underbrace{e^x\,dx}_{dv} = \underbrace{xe^x}_{uv} - \int \underbrace{e^x}_{v}\,\underbrace{dx}_{du} = xe^x - e^x + C$$

Example 2 Find $\int x\cos 3x\,dx$.

Solution Let

$u = x \qquad\qquad dv = \cos 3x\,dx$

Then $\quad du = dx \qquad\qquad v = \frac{1}{3}\sin 3x$

Thus,
$$\int x\cos 3x\,dx = x\left(\tfrac{1}{3}\sin 3x\right) - \tfrac{1}{3}\int \sin 3x\,dx$$
$$= \tfrac{1}{3}x\sin 3x + \tfrac{1}{9}\cos 3x + C$$

Example 3 Evaluate $\int x^2\sin 3x\,dx$.

Solution Let

$u = x^2 \qquad\qquad dv = \sin 3x\,dx$

Then $\quad du = 2x\,dx \qquad\qquad v = -\frac{1}{3}\cos 3x$

So,
$$\int x^2\sin 3x\,dx = x^2\left(-\tfrac{1}{3}\cos 3x\right) - \int\left(-\tfrac{1}{3}\cos 3x\right)(2x)dx$$
$$= -\tfrac{1}{3}x^2\cos 3x + \tfrac{2}{3}\int x\cos 3x\,dx$$

The integral we have obtained, $\int x \cos 3x\, dx$, was evaluated in Example 2. Using the result of that example, we get

$$\begin{aligned}\int x^2 \sin 3x\, dx &= -\tfrac{1}{3}x^2 \cos 3x + \tfrac{2}{3}\int x \cos 3x\, dx \\ &= -\tfrac{1}{3}x^2 \cos 3x + \tfrac{2}{3}\left(\tfrac{1}{3}x \sin 3x + \tfrac{1}{9}\cos 3x + C\right) \\ &= -\tfrac{1}{3}x^2 \cos 3x + \tfrac{2}{9}x \sin 3x + \tfrac{2}{27}\cos 3x + \tfrac{2}{3}C\end{aligned}$$

But $\frac{2}{3}C$ is a constant; let us call it K. Then

$$\int x^2 \sin 3x\, dx = -\tfrac{1}{3}x^2 \cos 3x + \tfrac{2}{9}x \sin 3x + \tfrac{2}{27}\cos 3x + K$$

Example 4 Find $\displaystyle\int x^2 \ln x\, dx$.

Solution In this example it is not appropriate to let $u = x^2$ because we don't have a formula for the integral of $\ln x$. So we let

$$u = \ln x \qquad dv = x^2\, dx$$
$$du = \frac{1}{x}\, dx \qquad v = \tfrac{1}{3}x^3$$

Then,
$$\begin{aligned}\int x^2 \ln x\, dx &= (\ln x)\left(\tfrac{1}{3}x^3\right) - \int \left(\tfrac{1}{3}x^3\right)\frac{1}{x}\, dx \\ &= \tfrac{1}{3}x^3 \ln x - \tfrac{1}{3}\int x^2\, dx \\ &= \tfrac{1}{3}x^3 \ln x - \tfrac{1}{9}x^3 + C\end{aligned}$$

When using integration by parts to find a definite integral, we evaluate both sides of the equation between the appropriate limits. For instance, Formula 1 becomes the following.

Definite Integration by Parts

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)dx$$

Example 5 Evaluate $\displaystyle\int_1^e \ln x\, dx$.

Solution Here there is not much choice:

$$u = \ln x \qquad dv = dx$$
$$du = \frac{1}{x}\, dx \qquad v = x$$

$$\begin{aligned}\int_1^e \ln x\,dx &= x\ln x\Big]_1^e - \int_1^e (x)\frac{1}{x}\,dx\\ &= e\ln e - 1\ln 1 - \int_1^e 1\,dx\\ &= e(1) - 1(0) - x\Big]_1^e\\ &= e - (e-1)\\ &= 1\end{aligned}$$

EXERCISE 11.4

B **1.** Evaluate the following indefinite integrals.

(a) $\int x\cos x\,dx$ (b) $\int xe^{2x}\,dx$

(c) $\int x\ln x\,dx$ (d) $\int t\sec^2 t\,dt$

(e) $\int x^2e^x\,dx$ (f) $\int (3x-5)e^{-4x}\,dx$

(g) $\int \tan^{-1} x\,dx$ (h) $\int \dfrac{xe^x}{(x+1)^2}\,dx$

2. Evaluate the following definite integrals.

(a) $\int_0^{\pi} x\sin x\,dx$ (b) $\int_0^1 xe^{-x}\,dx$

(c) $\int_1^2 x^4\ln x\,dx$ (d) $\int_0^{2\pi} x^2\cos x\,dx$

3. (a) Show that if $n \geqslant 1$, then

$$\int_0^1 x^ne^x\,dx = e - n\int_0^1 x^{n-1}e^x\,dx$$

(b) Use the formula in part (a) repeatedly to evaluate the integral $\int_0^1 x^3e^x\,dx$.

4. Evaluate the integral $\int e^{\sqrt{x}}\,dx$ by first making the substitution $t = \sqrt{x}$ and then integrating by parts.

5. Find the area under the curve $y = xe^{-3x}$ from 0 to 2.

6. Find the area of the region bounded by the curves $y = \ln x$, $y = 0$, and $x = 5$.

C **7.** Evaluate $\int e^x \sin x\,dx$. [*Hint:* Integrate by parts twice.]

11.5 TRIGONOMETRIC SUBSTITUTION

In this section we learn how to integrate certain combinations of trigonometric functions. Then we apply this knowledge to integrate functions containing radicals such as $\sqrt{1 - x^2}$. This will enable us to find the area enclosed by an ellipse.

Example 1 Evaluate $\int_0^{\frac{\pi}{2}} \cos^3 x \sin^4 x \, dx$.

Solution We write

$$\cos^3 x = (\cos^2 x)(\cos x) = (1 - \sin^2 x)\cos x$$

Now the integrand is expressed in terms of $\sin x$, except for the extra factor of $\cos x$:

$$\begin{aligned} \cos^3 x \sin^4 x &= (1 - \sin^2 x)\cos x \sin^4 x \\ &= (\sin^4 x - \sin^6 x)\cos x \end{aligned}$$

This is useful because if we make the substitution $u = \sin x$, then $du = \cos x \, dx$. We change the limits of integration as follows:

When $x = 0$, $u = \sin 0 = 0$.

When $x = \frac{\pi}{2}$, $u = \sin \frac{\pi}{2} = 1$.

Thus,
$$\begin{aligned} \int_0^{\frac{\pi}{2}} \cos^3 x \sin^4 x \, dx &= \int_0^{\frac{\pi}{2}} (\sin^4 x - \sin^6 x)\cos x \, dx \\ &= \int_0^1 (u^4 - u^6) du \\ &= \left[\frac{u^5}{5} - \frac{u^7}{7}\right]_0^1 \\ &= \tfrac{1}{5} - \tfrac{1}{7} \\ &= \tfrac{2}{35} \end{aligned}$$

m or n odd

In general, we can integrate a product of the form $\sin^m x \cos^n x$ if either m or n is an odd positive integer. For instance, if m is odd, we change all but one sine into cosines (using $\sin^2 \theta = 1 - \cos^2 \theta$) and make the substitution $u = \cos \theta$.

An example where both m and n are even follows.

Example 2 Find $\int \sin^2 x \, dx$.

Solution Recall that $\sin^2 x$ occurs in one of the double angle formulas for cosine:

$$\cos 2x = 1 - 2 \sin^2 x$$

Solving for $\sin^2 x$, we have

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x)$$

Thus,
$$\begin{aligned}\int \sin^2 x\,dx &= \int \tfrac{1}{2}(1 - \cos 2x)dx \\ &= \tfrac{1}{2}\int (1 - \cos 2x)dx \\ &= \tfrac{1}{2}\left(x - \tfrac{1}{2}\sin 2x\right) + C \\ &= \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x + C\end{aligned}$$

m and *n* even

In general, we can integrate $\sin^m x \cos^n x$, where both m and n are even, by using the formulas

$$\boxed{\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)}$$

Functions that involve an expression of the form $\sqrt{a^2 - x^2}$ can be integrated by making the *trigonometric substitution* $x = a \sin \theta$, where $-\dfrac{\pi}{2} \leqslant \theta \leqslant \dfrac{\pi}{2}$. Use of the identity $1 - \sin^2\theta = \cos^2\theta$ will convert such an integral into an integral involving $\sin\theta$ and $\cos\theta$.

Example 3 Evaluate $\displaystyle\int \frac{1}{x^2\sqrt{1 - x^2}}\,dx$.

Solution Let $x = \sin\theta$, $-\dfrac{\pi}{2} \leqslant \theta \leqslant \dfrac{\pi}{2}$. Then $dx = \cos\theta\,d\theta$ and

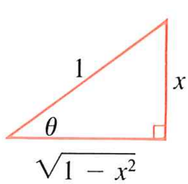
$$\sqrt{1 - x^2} = \sqrt{1 - \sin^2\theta} = \sqrt{\cos^2\theta}$$

But $\cos\theta \geqslant 0$ for $-\dfrac{\pi}{2} \leqslant \theta \leqslant \dfrac{\pi}{2}$, and so

$$\sqrt{1 - x^2} = \cos\theta$$

Thus, the Substitution Rule gives

$$\begin{aligned}\int \frac{1}{x^2\sqrt{1 - x^2}}\,dx &= \int \frac{\cos\theta}{\sin^2\theta\cos\theta}\,d\theta \\ &= \int \frac{1}{\sin^2\theta}\,d\theta \\ &= \int \csc^2\theta\,d\theta \\ &= -\cot\theta + C\end{aligned}$$



Since this is an indefinite integral, we must return to the original variable x. This can be done by looking at the diagram. Since $\sin\theta = x$, we label the sides as shown and see that

$$\cot\theta = \frac{\sqrt{1 - x^2}}{x}$$

Thus $\displaystyle\int \frac{1}{x^2\sqrt{1-x^2}}\,dx = -\frac{\sqrt{1-x^2}}{x} + C$

We could have used the substitution $x = \cos\theta$ in Example 3 but with the restriction $0 \leqslant \theta \leqslant \pi$ since this ensures that $\sin\theta \geqslant 0$.

Example 4 Find the area enclosed by the ellipse

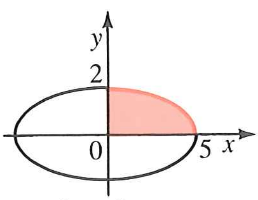
$$\frac{x^2}{25} + \frac{y^2}{4} = 1$$

Solution Solving the equation of the ellipse for y, we get

$$\frac{y^2}{4} = 1 - \frac{x^2}{25} = \frac{25 - x^2}{25}$$

$$y^2 = \tfrac{4}{25}(25 - x^2)$$

$$y = \pm\tfrac{2}{5}\sqrt{25 - x^2}$$



$\frac{x^2}{25} + \frac{y^2}{4} = 1$

The ellipse is symmetric with respect to both axes, so the total area A is four times the area in the first quadrant. The part of the ellipse in the first quadrant is given by

$$y = \tfrac{2}{5}\sqrt{25 - x^2},\ 0 \leqslant x \leqslant 5$$

So $\displaystyle \tfrac{1}{4}A = \int_0^5 \tfrac{2}{5}\sqrt{25 - x^2}\,dx$

$$A = \tfrac{8}{5}\int_0^5 \sqrt{25 - x^2}\,dx$$

To evaluate this integral we substitute $x = 5\sin\theta$. Then $dx = 5\cos\theta\,d\theta$. Also,

$$\sqrt{25 - x^2} = \sqrt{25 - 25\sin^2\theta} = 5\sqrt{1 - \sin^2\theta} = 5\cos\theta$$

We change the limits of integration as follows.

When $x = 0$, $\sin\theta = 0$, so $\theta = 0$.

When $x = 5$, $\sin\theta = 1$, so $\theta = \dfrac{\pi}{2}$.

$$
\begin{aligned}
\text{Thus}\quad A &= \tfrac{8}{5}\int_0^5 \sqrt{25 - x^2}\,dx \\
&= \tfrac{8}{5}\int_0^{\frac{\pi}{2}} (5\cos\theta)5\cos\theta\,d\theta \\
&= 40\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta \\
&= 40\int_0^{\frac{\pi}{2}} \tfrac{1}{2}(1 + \cos 2\theta)d\theta \\
&= 20\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}} \\
&= 20\left[\left(\frac{\pi}{2} + \tfrac{1}{2}\sin\pi\right) - (0 + 0)\right] \\
&= 10\pi
\end{aligned}
$$

EXERCISE 11.5

B **1.** Evaluate the following integrals.

(a) $\int \cos^3 x\,dx$ (b) $\int \sin^3 x\cos^2 x\,dx$

(c) $\int \sin^2 x\cos^2 x\,dx$ (d) $\int_0^{\frac{\pi}{2}} \sin^5 x\,dx$

(e) $\int_0^{\frac{\pi}{2}} \cos^5 x\sin^4 x\,dx$ (f) $\int_0^{\frac{\pi}{2}} \sin^4 x\,dx$

2. Evaluate each integral.

(a) $\int x^3\sqrt{1 - x^2}\,dx$ (b) $\int_0^2 x^2\sqrt{4 - x^2}\,dx$

3. (a) Use the trigonometric substitution $x = \tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, to show that

$$\int_0^1 \frac{x^3}{\sqrt{x^2 + 1}}\,dx = \int_0^{\frac{\pi}{4}} \tan^3\theta\sec\theta\,d\theta$$

(b) Evaluate $\int_0^{\frac{\pi}{4}} \tan^3 \theta \sec \theta \, d\theta$ by using the identity $\tan^2 \theta = \sec^2 \theta - 1$ and making the substitution $u = \sec \theta$.

4. Find the area enclosed by the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$.

5. Evaluate the integral $\int x\sqrt{1 - x^2}\, dx$ by two methods:
(a) Using the substitution $x = \sin \theta$
(b) Using the substitution $u = 1 - x^2$

6. Show that the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is πab.

11.6 PARTIAL FRACTIONS

We have learned how to add fractions using a common denominator.

$$\begin{aligned}\frac{2}{x-1} + \frac{3}{x+2} &= \frac{2(x+2) + 3(x-1)}{(x-1)(x+2)}\\ &= \frac{5x+1}{(x-1)(x+2)}\end{aligned}$$

A technique of integration is to reverse this process and resolve a fraction into a sum of fractions with simpler denominators, which are easier to integrate.

$$\begin{aligned}\int \frac{5x+1}{(x-1)(x+2)}\, dx &= \int\left(\frac{2}{x-1} + \frac{3}{x+2}\right)dx\\ &= \int \frac{2}{x-1}\, dx + \int \frac{3}{x+2}\, dx\\ &= 2 \ln|x-1| + 3 \ln|x+2| + C\end{aligned}$$

The method of reversing this process is illustrated in Example 1.

Example 1 Express $\dfrac{5x+1}{(x-1)(x+2)}$ as the sum of two fractions.

Solution Let
$$\frac{5x+1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

Multiplying both sides by the common denominator gives us

$$\begin{aligned}5x + 1 &= A(x+2) + B(x-1)\\ &= Ax + 2A + Bx - B\\ &= (A+B)x + (2A - B)\end{aligned}$$

This is an identity if and only if

$$A + B = 5$$

and $\quad 2A - B = 1$

Adding the two equations, we get

$$3A = 6$$
$$A = 2$$

Substituting $A = 2$ in the first equation, we have $B = 3$.

Therefore, $\quad \dfrac{5x + 1}{(x - 1)(x + 2)} = \dfrac{2}{x - 1} + \dfrac{3}{x + 2}$

This technique of expressing a single fraction as a sum of two or more fractions is called **the method of partial fractions**. The method involves four cases that are distinguished by the type of factors that occur in the denominator. Example 1 demonstrated the technique when the denominator is *the product of distinct linear factors*.

In Example 2 we examine the case when the denominator is *the product of linear factors, some of which are repeated*.

Example 2 (a) Resolve $\dfrac{6x + 7}{(x + 2)^2}$ into partial fractions.

(b) Evaluate $\displaystyle\int \frac{6x + 7}{(x + 2)^2}\, dx$.

Solution (a) Let $\quad \dfrac{6x + 7}{(x + 2)^2} = \dfrac{A}{x + 2} + \dfrac{B}{(x + 2)^2}$

Multiplying both sides by the common denominator gives us

$$\begin{aligned} 6x + 7 &= A(x + 2) + B \\ &= Ax + 2A + B \end{aligned}$$

Comparing coefficients, we get

$$A = 6$$

and $\quad 2A + B = 7$

So $\quad B = -5$

Therefore, $\quad \dfrac{6x + 7}{(x + 2)^2} = \dfrac{6}{x + 2} + \dfrac{-5}{(x + 2)^2}$

Let $u = x + 2$

$$\int \frac{dx}{(x+2)^2} = \int u^{-2}\,du = -u^{-1} + C = -\frac{1}{x+2} + C$$

(b)
$$\int \frac{6x+7}{(x+2)^2}\,dx = \int\left(\frac{6}{x+2} + \frac{-5}{(x+2)^2}\right)dx$$
$$= 6\int \frac{dx}{x+2} - 5\int \frac{dx}{(x+2)^2}$$
$$= 6\ln|x+2| + \frac{5}{x+2} + C$$

If we wished to express $\dfrac{x^3 - x + 2}{x^2(x+2)^3}$ as a sum of partial fractions, the initial step would be to let

$$\frac{x^3 - x + 2}{x^2(x+2)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2} + \frac{D}{(x+2)^2} + \frac{E}{(x+2)^3}$$

However, we confine our work to simpler examples.

The third case has *non-repeated irreducible quadratic factors* in the denominator.

Example 3 (a) Resolve $\dfrac{-2x+4}{(x^2+1)(x-1)^2}$ into partial fractions.

(b) Evaluate $\displaystyle\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx$.

Solution (a) Let $\dfrac{-2x+4}{(x^2+1)(x-1)^2} = \dfrac{Ax+B}{x^2+1} + \dfrac{C}{x-1} + \dfrac{D}{(x-1)^2}$

$x^2 + 1$ is irreducible since it cannot be factored.

Notice that when the denominator is an irreducible quadratic the numerator is a function of the form $Ax + B$.

Multiplying both sides by the common denominator gives us

$$-2x + 4 = (Ax + B)(x-1)^2 + C(x^2+1)(x-1) + D(x^2+1)$$
$$= Ax^3 - 2Ax^2 + Ax + Bx^2 - 2Bx + B + Cx^3 - Cx^2 + Cx - C + Dx^2 + D$$
$$= (A + C)x^3 + (-2A + B - C + D)x^2 + (A - 2B + C)x + (B - C + D)$$

Comparing coefficients we get

$$0 = A + C \quad ①$$
$$0 = -2A + B - C + D \quad ②$$
$$-2 = A - 2B + C \quad ③$$
$$4 = B - C + D \quad ④$$

Subtracting ④ from ②

$$-4 = -2A$$
$$A = 2$$

Substituting $A = 2$ in ①

$$0 = 2 + C$$
$$C = -2$$

Substituting $A = 2$ and $C = -2$ in ③

$$-2 = 2 - 2B - 2$$
$$B = 1$$

Substituting $B = 1$ and $C = -2$ in ④

$$4 = 1 + 2 + D$$
$$D = 1$$

Therefore,

$$\begin{aligned}\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} &= \frac{2x + 1}{x^2 + 1} + \frac{-2}{x - 1} + \frac{1}{(x - 1)^2} \\ &= \frac{2x}{x^2 + 1} + \frac{1}{x^2 + 1} - \frac{2}{x - 1} + \frac{1}{(x - 1)^2}\end{aligned}$$

(b)

$$\begin{aligned}\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2}\,dx &= \int \frac{2x}{x^2 + 1}\,dx + \int \frac{1}{x^2 + 1}\,dx - \int \frac{2}{x - 1}\,dx + \int \frac{1}{(x - 1)^2}\,dx \\ &= \ln(x^2 + 1) + \tan^{-1} x - 2\ln|x - 1| - \frac{1}{x - 1} + C\end{aligned}$$

Let $u = x^2 + 1$
$du = 2x\,dx$

$$\begin{aligned}&\int \frac{2x}{x^2 + 1}\,dx \\ &= \int \frac{du}{u} \\ &= \ln|u| + C \\ &= \ln(x^2 + 1) + C\end{aligned}$$

A fourth case involves *repeated irreducible quadratic factors* in the denominator. We will not be dealing with these, but the initial step of one such type is presented.

$$\frac{x^2 - x + 1}{(x + 1)(x^2 + 1)^3} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2} + \frac{Fx + G}{(x^2 + 1)^3}$$

In each of our examples, the degree of the numerator has been less than the degree of the denominator. Thus we have been dealing with **proper rational functions**. A theorem in algebra guarantees that every proper rational function can be resolved into partial fractions.

If the degree of the numerator is the same as or greater than that of the denominator, we begin by dividing the numerator by the denominator and proceed from there.

Example 4 Evaluate $\displaystyle\int_3^4 \frac{x + 4}{x - 2}\,dx$.

Solution Since the degree of the numerator is the same as the degree of the denominator, we divide and get

$$\begin{array}{r} 1 \\ x-2\overline{)x+4} \\ \underline{x-2} \\ 6 \end{array}$$

$$\frac{x+4}{x-2} = 1 + \frac{6}{x-2}$$

Therefore,
$$\int_3^4 \frac{x+4}{x-2}\,dx = \int_3^4 \left(1 + \frac{6}{x-2}\right)dx$$
$$= \Big[x + 6\ln|x-2|\Big]_3^4$$
$$= 4 + 6\ln 2 - (3 + 6\ln 1)$$
$$= 1 + 6\ln 2$$

Example 5 Evaluate $\int \frac{5x^3 - x^2}{x^2 - 1}\,dx$.

Solution Since the degree of the numerator exceeds the degree of the denominator, we divide and get

$$\begin{array}{r} 5x - 1 \\ x^2-1\overline{)5x^3 - x^2} \\ \underline{5x^3 - 5x} \\ -x^2 + 5x \\ \underline{-x^2 + 1} \\ 5x - 1 \end{array}$$

$$\frac{5x^3 - x^2}{x^2 - 1} = 5x - 1 + \frac{5x-1}{x^2-1}$$

We resolve $\frac{5x-1}{x^2-1}$ into partial fractions.

Let $\frac{5x-1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1}$

Multiplying both sides by the common denominator gives us

$$\begin{aligned} 5x - 1 &= A(x-1) + B(x+1) \\ &= Ax - A + Bx + B \\ &= (A+B)x + (-A+B) \end{aligned}$$

Comparing coefficients, we get

$$A + B = 5$$

and
$$-A + B = -1$$

Adding the equations

$$\begin{aligned} 2B &= 4 \\ B &= 2 \end{aligned}$$

Substituting $B = 2$ in $A + B = 5$, we get $A = 3$.

Therefore, $\frac{5x^3 - x^2}{x^2 - 1} = 5x - 1 + \frac{3}{x+1} + \frac{2}{x-1}$

Now,
$$\int \frac{5x^3 - x^2}{x^2-1}\,dx = 5\int x\,dx - \int dx + \int \frac{3}{x+1}\,dx + \int \frac{2}{x-1}\,dx$$
$$= \tfrac{5}{2}x^2 - x + 3\ln|x+1| + 2\ln|x-1| + C$$

EXERCISE 11.6

A

State the form of the partial fraction decomposition of the given function. Do not determine numerical values.

1. $\dfrac{1}{(x+2)(x-3)}$

2. $\dfrac{x+3}{(x+2)(x+5)^2}$

3. $\dfrac{x^2+x+1}{(x-1)(x+1)^2(x-2)^3}$

4. $\dfrac{5x}{(x^2+x+1)(x-7)}$

5. $\dfrac{2-3x}{(x+5)(x^2+4)(x^2+2x+6)}$

6. $\dfrac{2x-1}{x^2-16}$

7. $\dfrac{x^2+1}{(x^2+7x+12)}$

8. $\dfrac{x^3-2x^2+2}{(x-5)^3(x^2+5x+10)^2}$

B

Evaluate.

9. $\displaystyle\int \frac{dx}{x^2-1}$

10. $\displaystyle\int_2^5 \frac{2t+3}{t-1}\,dt$

11. $\displaystyle\int \frac{x^3-3x^2+x}{x^2-3x+2}\,dx$

12. $\displaystyle\int \frac{t^3}{t^2+7t+12}\,dt$

13. $\displaystyle\int \frac{x+4}{2x-x^2-x^3}\,dx$

14. $\displaystyle\int \frac{dx}{(x-1)^2(x+1)}$

15. $\displaystyle\int \frac{x^3-1}{x^3+3x^2}\,dx$

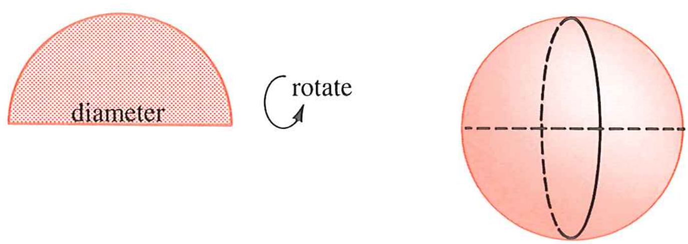
16. $\displaystyle\int_1^4 \frac{4}{x^3+4x}\,dx$

17. $\displaystyle\int \frac{1-3x}{x^3-1}\,dx$

18. $\displaystyle\int \frac{4x^2+5x+4}{(x^2+1)(x^2+2x+2)}\,dx$

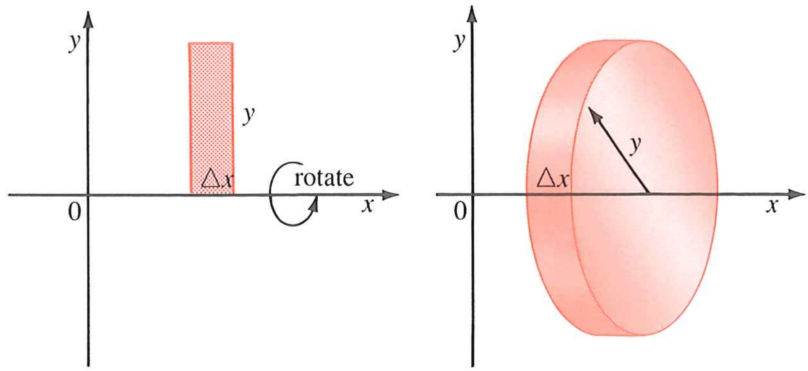
11.7 VOLUMES OF REVOLUTION

Rotating a plane region about a line produces a solid. For example, a semicircular region rotated about its diameter is a sphere.

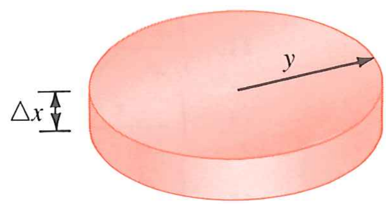


In this section we rotate a plane region about the x-axis and find the volume of the solid that is produced.

Example 1 Find the volume of the solid that is produced when the rectangle in the diagram is rotated about the x-axis.



Solution The solid produced is a cylinder with height Δx and radius y.

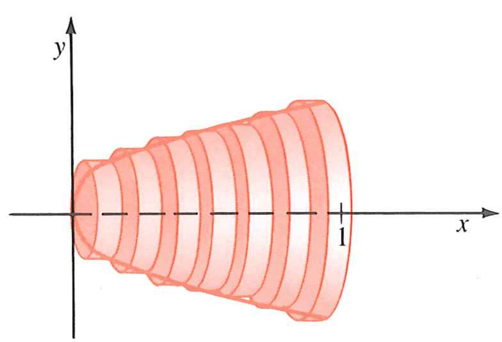


The volume V, of a cylinder with height h, and radius r, is

$$V = \pi r^2 h$$

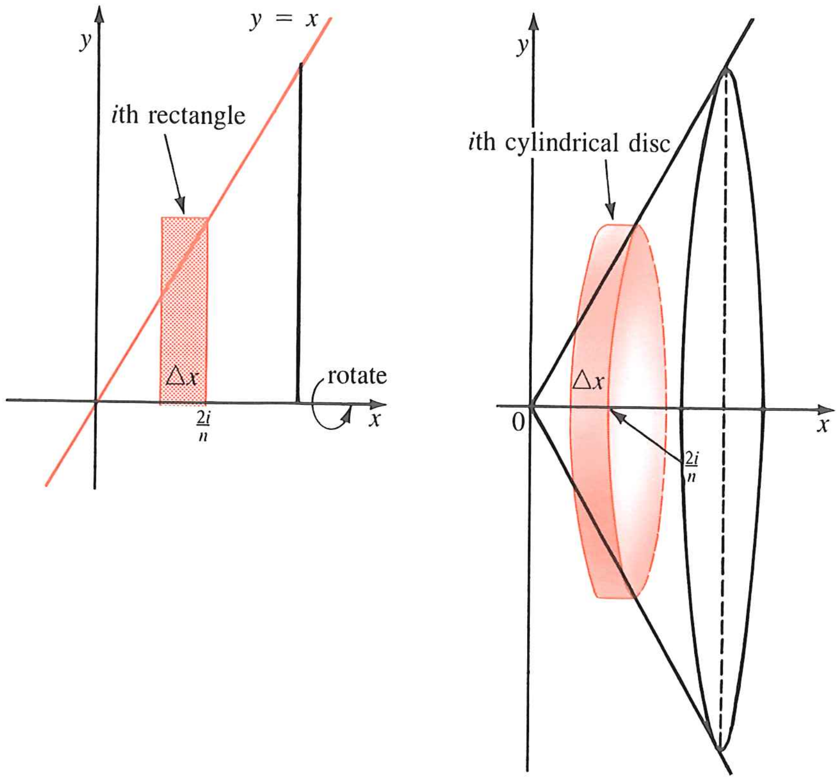
Therefore, $\quad V = \pi y^2 \Delta x$

Our method of calculating the volume of a solid that has been generated by rotating a plane region about the x-axis parallels the method that we used to calculate the area of a region in Section 10.4. We subdivide the solid into slices of equal thickness and approximate the volume of each slice by the volume of a cylindrical disc. Each cylindrical disc is generated by rotating a rectangle about the x-axis.



Example 2 Find the volume of the solid that is generated when the region under $y = 2x$ from 0 to 2 is rotated about the x-axis.

Solution Subdivide the region into n slices of equal thickness $\frac{2}{n}$.



The height of a cylindrical disc is the width of a slice. The ith cylindrical disc has height $\frac{2}{n}$ and base radius $f\left(\frac{2i}{n}\right)$ determined by the right-hand endpoint of the ith slice.

The volume of the ith cylindrical disc is

$$\begin{aligned}\pi y^2 \Delta x &= \pi\left(\frac{2i}{n}\right)^2\left(\frac{2}{n}\right)\\ &= \frac{8}{n^3}\pi i^2\end{aligned}$$

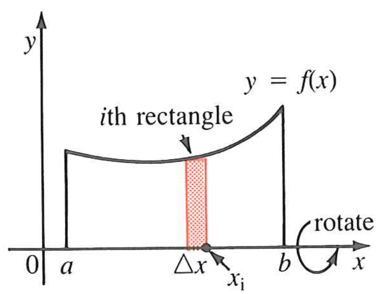
The sum of the volumes of the n cylindrical discs is

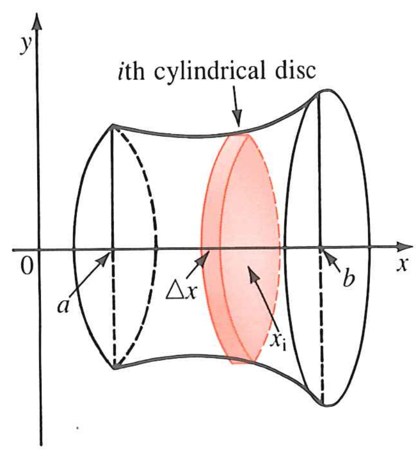
$$\sum_{i=1}^{n} \frac{8}{n^3}\pi i^2 = \frac{8\pi}{n^3}\sum_{i=1}^{n} i^2$$
$$= \frac{8\pi}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$
$$= \frac{4\pi}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

In order to find the volume of the solid, we take the limit as n approaches infinity:

$$V = \lim_{n\to\infty} \frac{4\pi}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$
$$= \tfrac{4}{3}\pi(1)(2)$$
$$= \tfrac{8}{3}\pi$$

We develop a formula by calculating the volume of the solid that is generated when the region under $y = f(x)$ from a to b is rotated about the x-axis.





The solid is subdivided into n slices of equal thickness and the volume of each slice is approximated by the volume of a cylindrical disc.

The height of each cylindrical disc is

$$\frac{b-a}{n} = \Delta x$$

The right-hand endpoint of the ith slice is

$$x_i = a + i\Delta x$$

The volume of the ith cylindrical disc is

$$\pi y^2 \Delta x = \pi[f(x_i)]^2 \Delta x$$

The sum of the volumes of the n cylindrical discs is

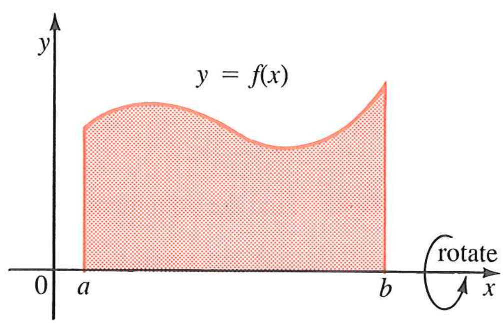
$$\sum_{i=1}^{n} \pi[f(x_i)]^2 \Delta x$$

and the volume of the region is

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} \pi[f(x_i)]^2 \Delta x = \int_a^b \pi[f(x)]^2 dx$$

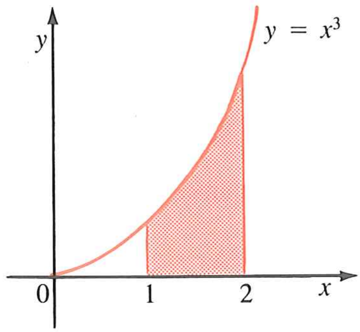
Volume of Revolution

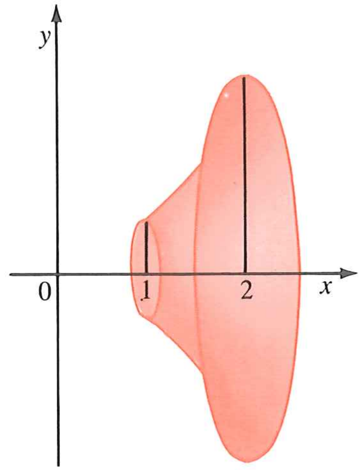
$$V = \pi \int_a^b [f(x)]^2 dx$$



Example 3 Find the volume of the solid that is generated when the region under $y = x^3$ from 1 to 2 is rotated about the x-axis. Include a sketch of the region that is to be rotated and a sketch of the solid.

Solution First we sketch the region and the solid.





Since $\quad V = \pi\displaystyle\int_a^b [f(x)]^2 dx$

therefore, $\quad V = \pi\displaystyle\int_1^2 x^6\, dx$

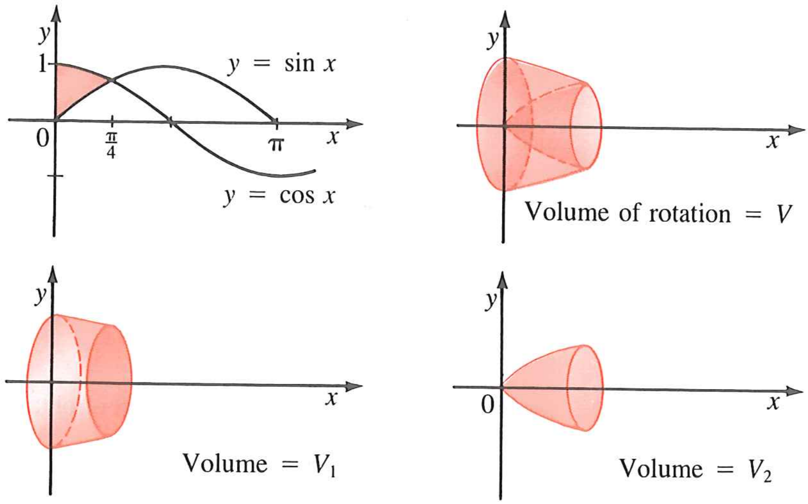
$$= \pi\left[\frac{x^7}{7}\right]_1^2$$

$$= \frac{128}{7}\pi - \frac{1}{7}\pi$$

$$= \frac{127}{7}\pi$$

Example 4 Find the volume of the solid that is generated when the region between $y = \sin x$ and $y = \cos x$ from 0 to $\frac{\pi}{4}$ is rotated about the x-axis.

Solution First we sketch the curve and note that $y = \cos x$ is above $y = \sin x$ in the required region. If V_1 is the volume of the solid generated when the region under $y = \cos x$ from 0 to $\frac{\pi}{4}$ is rotated about the x-axis and V_2 is the volume of the solid generated when the region under $y = \sin x$ from 0 to $\frac{\pi}{4}$ is rotated about the x-axis, then the required volume is $V_1 - V_2$.



$$\begin{aligned}
V &= V_1 - V_2 \\
&= \pi \int_0^{\frac{\pi}{4}} \cos^2 x \, dx - \pi \int_0^{\frac{\pi}{4}} \sin^2 x \, dx \\
&= \pi \int_0^{\frac{\pi}{4}} (\cos^2 x - \sin^2 x) dx \\
&= \pi \int_0^{\frac{\pi}{4}} \cos 2x \, dx \\
&= \pi \left[\frac{\sin 2x}{2} \right]_0^{\frac{\pi}{4}} \\
&= \frac{\pi}{2}\left(\sin \frac{\pi}{2} - \sin 0 \right) \\
&= \frac{\pi}{2}(1 - 0) \\
&= \frac{\pi}{2}
\end{aligned}$$

EXERCISE 11.7

B **1.** Find the volume of the solid obtained when the given region is rotated about the x-axis. Include a sketch of the region and the solid.

(a) Under $y = x^2 + 1$ from 1 to 3

(b) Between $y = 2x - x^2$ and the x-axis

(c) Under $y = \dfrac{1}{x}$ from 1 to 4

(d) Between $y = x^2$ and $y^2 = x$

2. Find the volume of the solid obtained when the given region is rotated about the x-axis
   (a) Under $y = \sin^{\frac{3}{2}} x$ from 0 to π
   (b) Bounded by $y = 4 - x^2$ and $y = 3x$ and $x = 0$
   (c) Under $y = \sec x$ from 0 to $\dfrac{\pi}{4}$
   (d) Under $y = 2e^{2x}$ from $\ln 1$ to $\ln 3$
3. Find the volume of the sphere obtained when $x^2 + y^2 = 25$, $y \geqslant 0$, is rotated about the x-axis.
4. Find the volume of the ellipsoid obtained when $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ is rotated about the x-axis.
5. Use integration to find the volume of a sphere with radius r.
6. Use integration to find the volume of a cone of height h and base radius r.
7. Find the volume of the solid obtained when the region under $y = \sqrt{x \ln x}$ from 1 to 2 is rotated about the x-axis.
8. Find the volume of the solid obtained when the region under $y = \dfrac{x}{x+1}$ from 0 to 1 is rotated about the x-axis.

C 9. Find the volume of the ellipsoid generated by rotating $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $y \geqslant 0$, about the x-axis. Use the result to find the volume generated in Question 4.

10. Find the volume of the solid obtained by rotating the region bounded by $y = 2x - x^2$, $y = 2x$, and $x = 2$ about the line $y = -1$.
11. Find the volume of the solid obtained when the region in the first quadrant bounded by $y = x^2$, $4y = x^2$, and $y = 1$ is rotated about the y-axis.

11.8 REVIEW EXERCISE

1. Evaluate each integral directly from the definition as a limit of sums.

(a) $\int_{-1}^{4}(3x + 2)dx$ (b) $\int_{0}^{1}(x^3 - 2x^2)dx$

2. (a) Evaluate the integral $\int_{-3}^{0}(x^2 + 2x)dx$ as a limit of sums.

(b) Evaluate the integral in part (a) using the Fundamental Theorem of Calculus.

(c) Interpret the integral in part (a) as a difference of areas.

3. Evaluate the following indefinite integrals.

(a) $\int(x^4 - 12x^3 + 6x)dx$ (b) $\int\sqrt{x}(1 - x + 3x^2)dx$

(c) $\int(2x + \sec x \tan x)dx$ (d) $\int\frac{x + 2}{\sqrt[3]{x}}dx$

(e) $\int\frac{\sin x + x \cos x}{x \sin x}dx$ (f) $\int\frac{x}{\sqrt{4 + x^2}}dx$

(g) $\int e^x\sqrt{1 + e^x}\,dx$ (h) $\int\sec 4x \tan 4x\,dx$

(i) $\int\sqrt{x}\ln x\,dx$ (j) $\int\frac{1}{x \ln x}dx$

(k) $\int\frac{dx}{x - x^2}$ (l) $\int\frac{x + 4}{x^3 + 3x^2 - 10x}dx$

(m) $\int xe^{-3x}\,dx$ (n) $\int\sin^3 x\,dx$

(o) $\int\frac{x + 4}{(x + 1)^2}dx$ (p) $\int\frac{dx}{x(x^2 + x + 1)}$

(q) $\int\frac{x^3 + 4x^2}{x^2 + 4x + 3}dx$ (r) $\int\frac{x^3}{x^3 + 4x}dx$

(s) $\int\frac{\sin^{-1}x}{\sqrt{1 - x^2}}dx$ (t) $\int\frac{1}{(9 - x^2)^{\frac{3}{2}}}dx$

(u) $\int\frac{e^x}{1 - e^x}dx$ (v) $\int\frac{e^x}{e^{2x} + 3e^x + 2}dx$

4. Evaluate the following definite integrals.

(a) $\int_{-1}^{1} (1 + 4x - x^2)dx$ (b) $\int_{0}^{1} e^{-3x}\, dx$

(c) $\int_{1}^{3} \frac{1 + 3x}{x^2}\, dx$ (d) $\int_{0}^{4} \frac{3}{2x + 1}\, dx$

(e) $\int_{0}^{\frac{\pi}{2}} \cos^3 x \sin^2 x\, dx$ (f) $\int_{0}^{\frac{\pi}{2}} x^2 \sin x\, dx$

(g) $\int_{0}^{1} \frac{x}{x + 1}\, dx$ (h) $\int_{0}^{3} \sqrt{9 - x^2}\, dx$

5. Find the area of the region bounded by the curves $y = \sin x$ and $y = \sin^2 x$ from $x = 0$ to $x = \pi$.

6. Find the volume of the solid obtained when the given region is rotated about the x-axis. Include a sketch of the region.

(a) Under $y = \sqrt{x}$ from 0 to 4

(b) Under $y = \sqrt{\sin x \cos x}$ from 0 to $\frac{\pi}{2}$

(c) Under $y = \frac{1}{\sqrt{x}}$ from 1 to 2

(d) Under $y = e^{-x}$ from 0 to $\ln 2$

7. Find the volume of the solid obtained when the given region is rotated about the x-axis. Include a sketch of the region.

(a) Between $y = 1$ and $y = 5 - x^2$

(b) Between $y^2 = 4x$ and $y = x$

(c) Between $y = \sin x$ and $y = -\cos x$ from $\frac{3\pi}{4}$ to π

11.9 CHAPTER 11 TEST

1. (a) State the definition of the definite integral $\int_a^b f(x)dx$.

(b) Use your definition in part (a) to evaluate $\int_0^3 (x^2 + 4x - 5)dx$.

(c) State the Fundamental Theorem of Calculus.

(d) Use the Fundamental Theorem of Calculus to evaluate the integral in part (b).

2. Evaluate the following definite integrals.

(a) $\int_0^{\frac{\pi}{4}} \sin 4x\, dx$

(b) $\int_1^2 \frac{1}{\sqrt[3]{x^4}}\, dx$

(c) $\int_0^{\frac{\pi}{2}} x \sin x\, dx$

(d) $\int_0^1 \frac{1}{(3x + 2)^2}\, dx$

3. Evaluate the following indefinite integrals.

(a) $\int x^2 e^{x^3}\, dx$

(b) $\int \cos^5 x\, dx$

(c) $\int \frac{\cos \sqrt{x}}{\sqrt{x}}\, dx$

(d) $\int \sin^2 x\, dx$

(e) $\int \frac{x^2 + 1}{(x - 1)(x - 2)(x - 3)}\, dx$

(f) $\int \frac{dx}{x^3 + 3x^2}$

(g) $\int \frac{x^2}{x^2 + 1}\, dx$

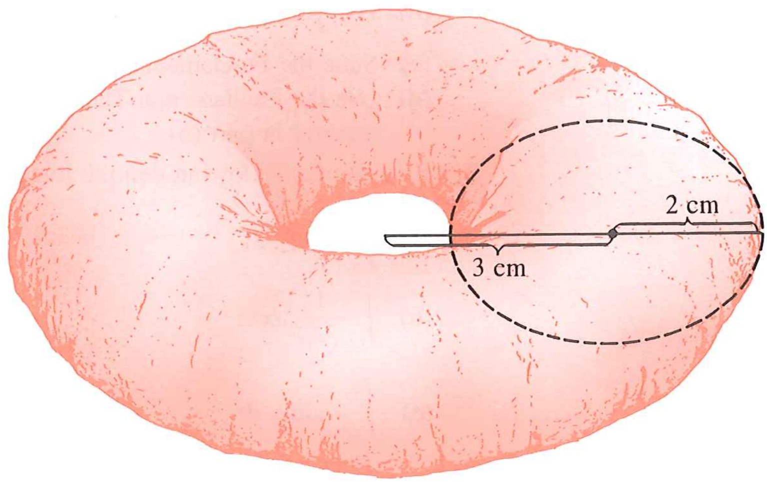
4. Find the volume of the solid obtained when the region under $y = e^{-2x}$ from $-\ln 2$ to $\ln 2$ is rotated about the x-axis.

5. (a) Find the area of the region between the curves $y^2 = 2x$ and $x = 2y$.

(b) Find the volume of the solid obtained when the region in part (a) is rotated about the x-axis.

PROBLEMS PLUS

Find the volume of the doughnut whose dimensions are shown.



APPENDIX

The following formula was used in the proof of the Power Rule.

$$\boxed{x^n - a^n = (x - a)(x^{n-1} + x^{n-2}a + \dots + xa^{n-2} + a^{n-1})}$$

For $n = 2$ or 3 this formula is just the formula for the difference of squares or difference of cubes:

$$x^2 - a^2 = (x - a)(x + a)$$
$$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$$

In general, we can prove it by multiplying out the right side:

$$\begin{aligned}(x - a)&(x^{n-1} + x^{n-2}a + \dots + xa^{n-2} + a^{n-1})\\ &= x^n + x^{n-1}a + x^{n-2}a^2 + \dots + x^2a^{n-2} + xa^{n-1}\\ &\qquad - x^{n-1}a - x^{n-2}a^2 - \dots - x^2a^{n-2} - xa^{n-1} - a^n\\ &= x^n - a^n\end{aligned}$$

A different proof of the Power Rule can be given using the Binomial Theorem:

$$(a + b)^n = a^n + na^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{r}a^{n-r}b^r + \dots + b^n$$
$$\text{where } \binom{n}{r} = \frac{n!}{r!(n - r)!}$$

(See Section 3.3 in *Finite Mathematics*.)

Alternative Proof of the Power Rule

If $f(x) = x^n$, then $f(x + h) = (x + h)^n$, which we can expand using the Binomial Theorem. Thus

$$\begin{aligned}\frac{f(x + h) - f(x)}{h} &= \frac{(x + h)^n - x^n}{h}\\ &= \frac{\left[x^n + nx^{n-1}h + \dfrac{n(n - 1)}{2}x^{n-2}h^2 + \dots + nxh^{n-1} + h^n\right] - x^n}{h}\\ &= \frac{nx^{n-1}h + \dfrac{n(n - 1)}{2}x^{n-2}h^2 + \dots + nxh^{n-1} + h^n}{h}\\ &= nx^{n-1} + \frac{n(n - 1)}{2}x^{n-2}h + \dots + nxh^{n-2} + h^{n-1}\end{aligned}$$

Therefore

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0}\left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \ldots + nxh^{n-2} + h^{n-1}\right] \\
&= nx^{n-1}
\end{aligned}
$$

because every term, except the first, has h as a factor and so approaches 0.

The following relationship between differentiable functions and continuous functions is needed in the proof of the Product Rule in Section 2.4.

> If f is differentiable at a, then f is continuous at a.

Proof

Since f if differentiable at a, we know that $f'(a)$ exists and we can use the formula

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

To verify that f is continuous at a, we have to prove that $\lim_{x\to a} f(x) = f(a)$ or, equivalently, that $\lim_{x\to a} [f(x) - f(a)] = 0$. For $x \neq a$, we can divide and multiply by the quantity $x - a$, so

$$
\begin{aligned}
\lim_{x\to a} [f(x) - f(a)] &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a}(x - a) \\
&= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \lim_{x\to a}(x - a) \\
&= f'(a)(0) \\
&= 0
\end{aligned}
$$

Therefore f is continuous at a.

The function $f(x) = |x|$ is continuous at 0, but not differentiable at 0 (see Example 7 in Section 2.1), so the converse of this theorem is false.

ANSWERS

CHAPTER 11 INTEGRALS

REVIEW AND PREVIEW TO CHAPTER 11

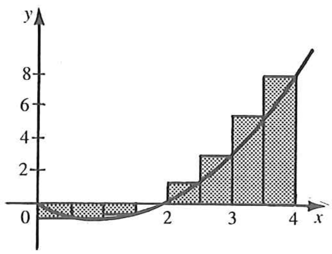
EXERCISE 1

1. (a) $2000\ \pi\ \text{cm}^3$ (b) $1080\ \pi\ \text{cm}^3$ (c) $2.4\ \pi\ \text{cm}^3$ (d) $3906.25\ \pi\ \text{cm}^3$ (e) $317.25\ \pi\ \text{cm}^3$ (f) $960\ \pi\ \text{cm}^3$

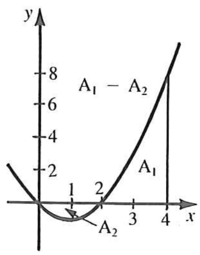
EXERCISE 11.1

1. (a) $\frac{16}{3}$ (b) 7.5

(c)

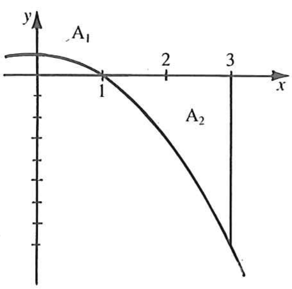


(d)

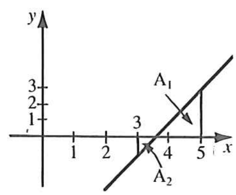


2. (a) -5 (b) 1 (c) $\frac{1}{4}$ (d) 3

3. (a) -6

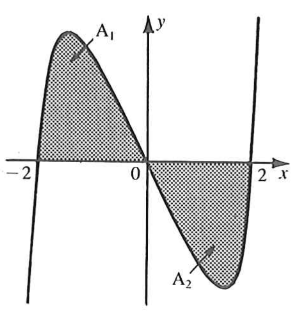


(b) 2



4. $\frac{1}{3}(b^3 - a^3)$

5. (a)



(b) 0 (c) $A_1 = A_2$

EXERCISE 11.2

1. (a) 26 (b) 30 (c) -3 (d) $\frac{7}{3}$ (e) $\frac{27}{4}$ (f) 1.01 (g) $\frac{1}{6}$ (h) $\frac{17}{6}$ (i) $\frac{4}{9}$ (j) 9 (k) $\frac{11}{8} + \ln 2$ (l) $2 + \ln 4$ (m) $\dfrac{48\,128}{33}$ (n) $\pi^2 + 1$ (o) $\dfrac{2}{\sqrt{3}} - 1$ (p) $\frac{1}{2}(3\sqrt{2} - 1) - \sqrt{3}$

2. (a) $\frac{1}{6}x^6 - \frac{1}{2}x^4 + 4x + C$ (b) $\frac{2}{7}x^{\frac{7}{2}} + C$
(c) $\frac{1}{2}t^2 + 2\ln|t| + C$
(d) $x + \frac{4}{3}x^{\frac{3}{2}} + \frac{1}{2}x^2 + C$ (e) $\frac{4}{7}x^{\frac{7}{4}} - \frac{20}{3}x^{\frac{3}{4}} + C$
(f) $\sin\theta - \cos\theta + C$ (g) $x^5 + 2\csc x + C$
(h) $x - 2\cot x + C$

3. (a) $e - 1$ (b) $\dfrac{3}{2\ln 2}$ (c) $\dfrac{\pi}{6}$ (d) π
(e) $2 + \frac{3}{2}\pi$ (f) $-2e^{-\pi}$

4. The function is not continuous on $[-2, 1]$, so the Fundamental Theorem does not apply.

EXERCISE 11.3

1. (a) $u = x^2$ (b) $u = \ln x$ (c) $u = 5x$ (d) $u = \sin x$

2. (a) $-\frac{1}{22}(1 - x^2)^{11} + C$ (b) $\frac{1}{5}e^{5x} + C$
(c) $\frac{2}{3}(x - 1)^{\frac{3}{2}} + C$ (d) $\frac{1}{2}\ln|x^2 + 2x - 6| + C$

3. (a) $\frac{1}{18}(x^2 + 4)^9 + C$ (b) $\frac{2}{9}(x^3 + 2)^{\frac{3}{2}} + C$
(c) $\frac{1}{11}(x + 6)^{11} + C$ (d) $-\dfrac{1}{3(3x - 1)} + C$
(e) $\frac{1}{3}\tan 3x + C$ (f) $\frac{1}{16}(1 + 2x^4)^2 + C$
(g) $\frac{1}{3}\sin^3 x + C$ (h) $\frac{2}{3}(\ln x)^{\frac{3}{2}} + C$
(i) $\frac{1}{3}e^{t^3} + C$ (j) $-\ln|1 - x| + C$
(k) $-\dfrac{1}{2(x^3 - 2x + 1)^2} + C$
(l) $-2\cos\sqrt{x} + C$ (m) $-e^{3-x} + C$
(n) $-e^{\cos x} + C$ (o) $\frac{2}{3}(1 + \tan x)^{\frac{3}{2}} + C$
(p) $-\frac{1}{2}\cos(x^2) + C$ (q) $\cos(\cos x) + C$
(r) $\frac{1}{2}(\tan^{-1} x)^2 + C$

4. (a) $\frac{1}{2}(e^3 - e)$ (b) $\frac{266}{3993}$ (c) $\frac{8}{3}$ (d) $\dfrac{2}{\pi}$ (e) $\frac{3}{2}$
(f) 2.1 (g) $\frac{665}{6}$ (h) $\frac{1}{6}(e^{20} - e^5)$

5. (a) $\ln|\sec x| + C$ (b) $\ln|\sin x| + C$

6. $\frac{1}{6}(41\sqrt{41} - 1)$ **7.** 2 **8.** $\frac{1}{2}(e^2 - 3) + \dfrac{1}{e}$

9. (a) $2\ln(\sqrt{x} + 1) + C$
(b) $x + 2 - \ln|x + 2| + C$

EXERCISE 11.4

1. (a) $x\sin x + \cos x + C$ (b) $\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$
(c) $\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$
(d) $t\tan t - \ln|\sec t| + C$
(e) $(x^2 - 2x + 2)e^x + C$
(f) $\left(\frac{17}{16} - \frac{3}{4}x\right)e^{-4x} + C$
(g) $x\tan^{-1}x - \frac{1}{2}\ln(x^2 + 1) + C$
(h) $\dfrac{e^x}{x + 1} + C$

2. (a) π (b) $1 - \dfrac{2}{e}$ (c) $\frac{32}{5}\ln 2 - \frac{31}{25}$ (d) 4π

3. (b) $6 - 2e$ **4.** $2e^{\sqrt{x}}(\sqrt{x} - 1) + C$

5. $\frac{1}{9}(1 - 7e^{-6})$ **6.** $5\ln 5 - 4$

7. $\frac{1}{2}e^x(\sin x - \cos x) + C$

EXERCISE 11.5

1. (a) $\sin x - \frac{1}{3}\sin^3 x + C$
(b) $\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$
(c) $\frac{1}{8}(x - \frac{1}{4}\sin 4x) + C$
(d) $\frac{8}{15}$ (e) $\frac{8}{315}$ (f) $\dfrac{3\pi}{16}$

2. (a) $\frac{1}{5}(1 - x^2)^{\frac{5}{2}} - \frac{1}{3}(1 - x^2)^{\frac{3}{2}} + C$ (b) π

3. (b) $\frac{1}{3}(2 - \sqrt{2})$ **4.** 12π

5. $-\frac{1}{3}(1 - x^2)^{\frac{3}{2}} + C$

EXERCISE 11.6

1. $\dfrac{A}{x + 2} + \dfrac{B}{x - 3}$

2. $\dfrac{A}{x + 2} + \dfrac{B}{x + 5} + \dfrac{C}{(x + 5)^2}$

3. $\dfrac{A}{x - 1} + \dfrac{B}{x + 1} + \dfrac{C}{(x + 1)^2} + \dfrac{D}{x - 2} + \dfrac{E}{(x - 2)^2} + \dfrac{F}{(x - 2)^3}$

4. $\dfrac{Ax + B}{x^2 + x + 1} + \dfrac{C}{x - 7}$

5. $\dfrac{A}{x + 5} + \dfrac{Bx + C}{x^2 + 4} + \dfrac{Dx + E}{x^2 + 2x + 6}$

6. $\dfrac{A}{x - 4} + \dfrac{B}{x + 4}$

7. $1 - \left(\dfrac{A}{x + 3} + \dfrac{B}{x + 4}\right)$

8. $\dfrac{A}{x - 5} + \dfrac{B}{(x - 5)^2} + \dfrac{C}{(x - 5)^3} + \dfrac{Dx + E}{x^2 + 5x + 10} + \dfrac{Fx + G}{(x^2 + 5x + 10)^2}$

9. $\ln\sqrt{\dfrac{x - 1}{x + 1}} + C$ **10.** $6 + 5\ln 4$

11. $\frac{1}{2}x^2 - 2\ln|x - 2| + \ln|x - 1| + C$

12. $\frac{1}{2}t^2 - 27\ln|t + 3| + 64\ln|t + 4| + C$

13. $2\ln|x| - \frac{1}{3}\ln|2 + x| - \frac{5}{3}\ln|1 - x| + C$

14. $-\frac{1}{4}\ln|x - 1| - \dfrac{1}{2(x - 1)} + \frac{1}{4}\ln|x + 1| + K$

15. $x - \frac{1}{9}\ln|x| + \dfrac{1}{3x} + \frac{28}{9}\ln|x + 3| + K$

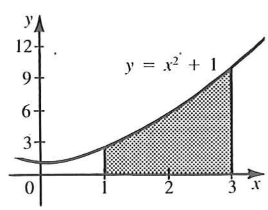
16. $\ln 2$

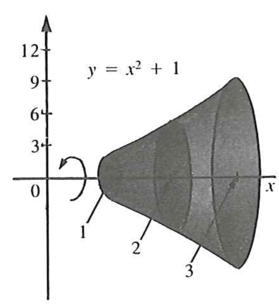
17. $-\frac{2}{3}\ln|x - 1| + \frac{1}{3}\ln|x^2 + x + 1| - \dfrac{4}{\sqrt{3}}\tan^{-1}\dfrac{2x + 1}{\sqrt{3}} + C$

18. $\frac{1}{2}\ln(x^2 + 1) + 2\tan^{-1}x - \frac{1}{2}\ln(x^2 + 2x + 2) + \tan^{-1}(x + 1) + K$

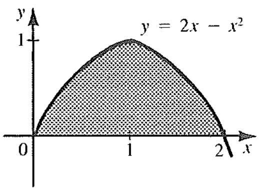
EXERCISE 11.7

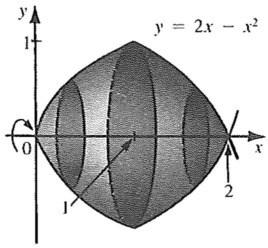
1. (a) $\frac{1016}{15}\pi$



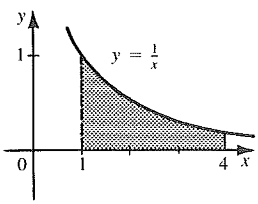


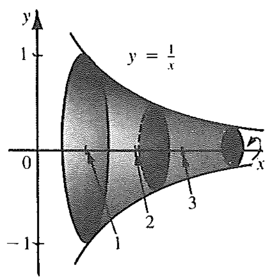
(b) $\frac{16}{15}\pi$



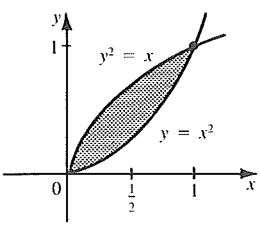


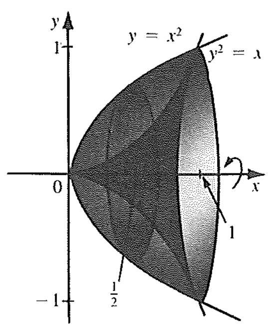
(c) $\frac{3}{4}\pi$



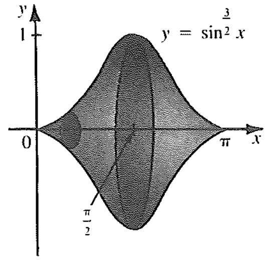


(d) $\frac{3}{10}\pi$

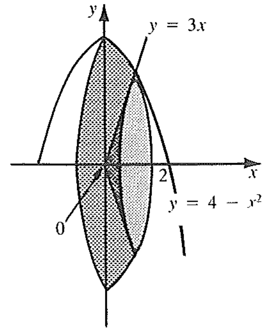




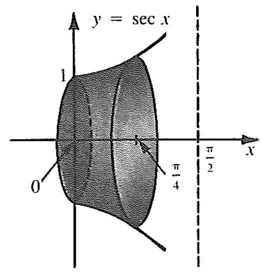
2. (a) $\frac{4}{3}\pi$



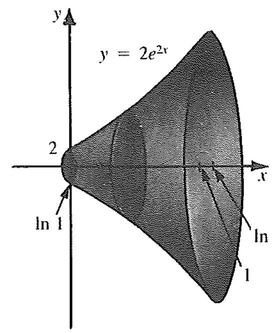
(b) $\frac{158}{15}\pi$



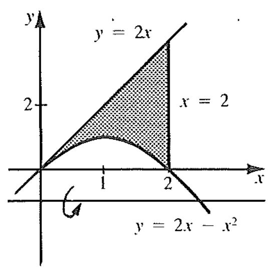
(c) π

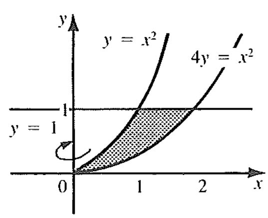


(d) 320π



3. $\frac{500}{3}\pi$ **4.** 48π **5.** $\frac{4}{3}\pi r^3$ **6.** $\frac{1}{3}\pi r^2 h$
7. $\left(2\ln 2 - \frac{3}{4}\right)\pi$ **8.** $\left(\frac{3}{2} - \ln 4\right)\pi$
9. $\frac{4}{3}\pi ab^2$ **10.** $\frac{224}{15}\pi$ **11.** $\frac{3}{2}\pi$





11.8 REVIEW EXERCISE

1. (a) 32.5 (b) $-\frac{5}{12}$ **2.** 0

3. (a) $\frac{1}{5}x^5 - 3x^4 + 3x^2 + C$

(b) $\frac{2}{3}x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}} + \frac{6}{7}x^{\frac{7}{2}} + C$

(c) $x^2 + \sec x + C$ (d) $\frac{3}{5}x^{\frac{5}{3}} + 3x^{\frac{2}{3}} + C$

(e) $\ln|x| + \ln|\sin x| + C$ (f) $\sqrt{4 + x^2} + C$

(g) $\frac{2}{3}(1 + e^x)^{\frac{3}{2}} + C$ (h) $\frac{1}{4}\sec 4x + C$

(i) $\frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{4}{9}x^{\frac{3}{2}} + C$ (j) $\ln|\ln x| + C$

(k) $\ln\left|\frac{x}{1 - x}\right| + C$ (l) $-\frac{2}{5}\ln|x| - \frac{1}{35}\ln$
$|x + 5| + \frac{3}{7}\ln|x - 2| + C$

(m) $-\frac{1}{9}e^{-3x}(3x + 1) + C$

(n) $-\cos x + \frac{1}{3}\cos^3 x + C$

(o) $\ln|x + 1| - \frac{3}{x + 1} + C$

(p) $\ln\frac{|x|}{\sqrt{x^2 + x + 1}} - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x + 1}{\sqrt{3}}\right) + C$

(q) $\frac{1}{2}x^2 + \frac{3}{2}\ln|x + 1| - \frac{9}{2}\ln|x + 3| + C$

(r) $x - 2\tan^{-1}\left(\frac{x}{2}\right) + C$ (s) $\frac{1}{2}(\sin^{-1}x)^2 + C$

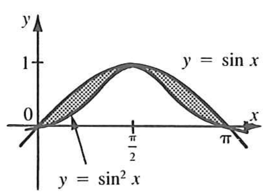
(t) $\frac{x}{9\sqrt{9 - x^2}} + C$ (u) $-\ln|1 - e^x| + C$

(v) $\ln(e^x + 1) - \ln(e^x + 2) + C$

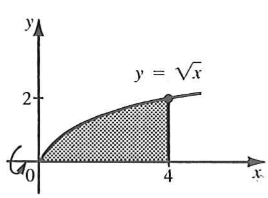
4. (a) $\frac{4}{3}$ (b) $\frac{1}{3}(1 - e^{-3})$ (c) $\frac{2}{3} + 3\ln 3$ (d) $3\ln 3$

(e) $\frac{2}{15}$ (f) $\pi - 2$ (g) $1 - \ln 2$ (h) $\frac{9\pi}{4}$

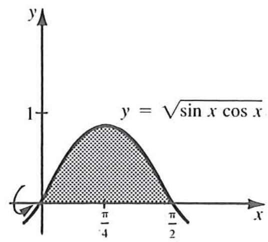
5. $2 - \frac{\pi}{2}$

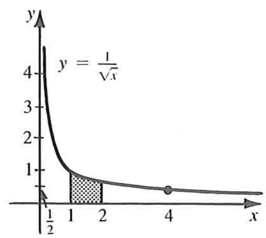


6. (a) 8π

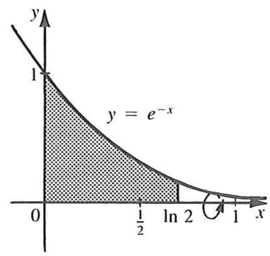


(b) $\frac{\pi}{2}$ (c) $\pi\ln 2$

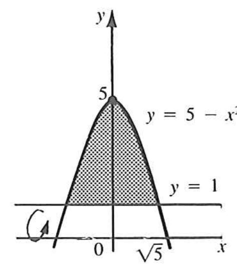




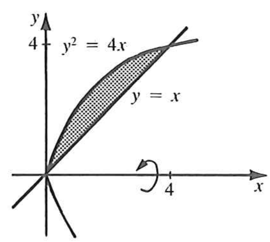
(d) $\frac{3}{8}\pi$

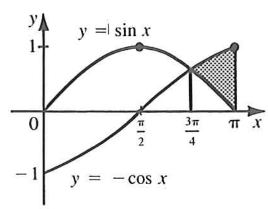


7. (a) $\frac{832}{15}\pi$



(b) $\frac{32}{3}\pi$ (c) $\frac{\pi}{2}$





11.9 CHAPTER 11 TEST

1. (b) 12 **2.** (a) $\frac{1}{2}$ (b) $3\left(1 - \frac{1}{\sqrt[3]{2}}\right)$

(c) 1 (d) $\frac{1}{10}$

3. (a) $\frac{1}{3}e^{x^3} + C$ (b) $\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$

(c) $2\sin\sqrt{x} + C$ (d) $\frac{1}{2}x - \frac{1}{4}\sin 2x + C$

(e) $\ln|x - 1| - 5\ln|x - 2| + 5\ln$
$|x - 3| + C$

(f) $-\frac{1}{9}\ln|x| - \frac{1}{3x} + \frac{1}{9}\ln|x + 3| + C$

(g) $x - \tan^{-1}x + C$ **4.** $\frac{255}{64}\pi$

5. (a) $\frac{16}{3}$ (b) $\frac{64}{3}\pi$

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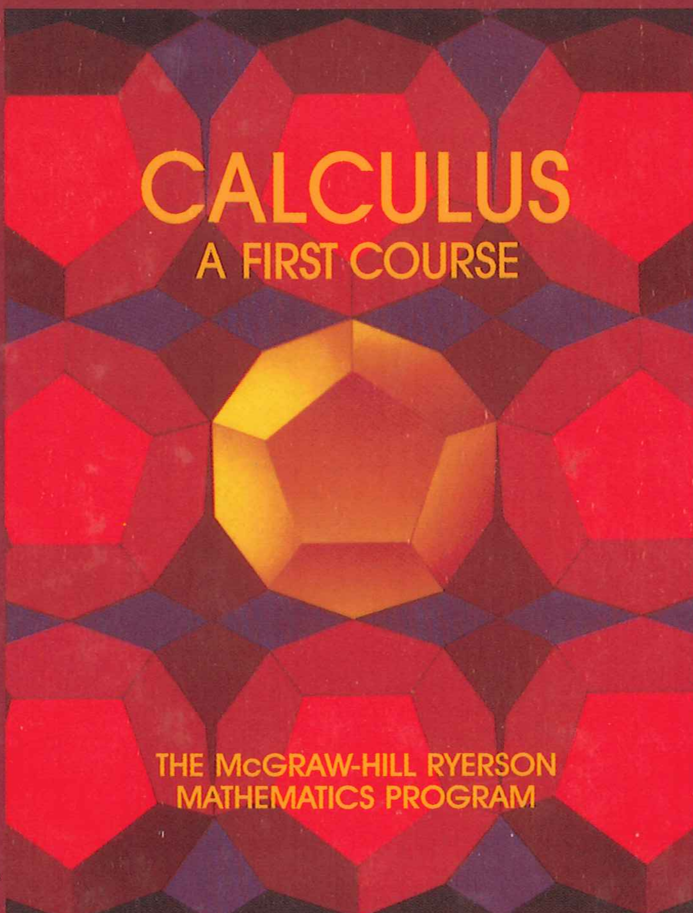

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