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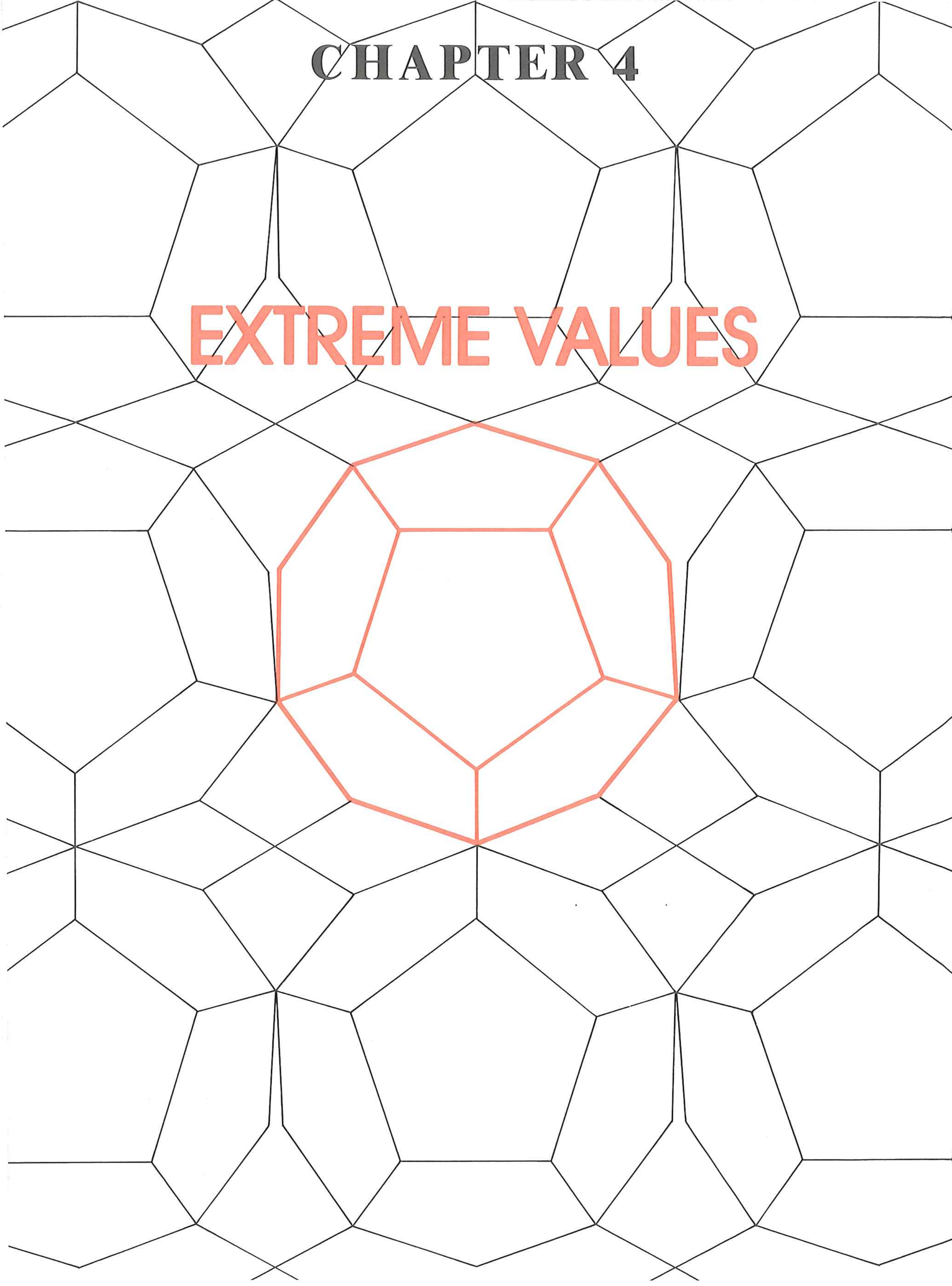
CHAPTER 4

EXTREME VALUES



REVIEW AND PREVIEW TO CHAPTER 4

Intervals

There are certain sets of real numbers, called **intervals**, that occur frequently in calculus and correspond geometrically to line segments. For example, if $a < b$, the **open interval** from a to b consists of all numbers between a and b and is denoted by the symbol (a, b). Using set-builder notation, we can write

$$(a, b) = \{x \mid a < x < b\}$$

Notice that the endpoints of the interval—namely, a and b—are excluded. This is indicated by the parentheses ( ) and the open circles in the figure. The **closed interval** from a to b is the set

$$[a, b] = \{x \mid a \leqslant x \leqslant b\}$$

Here, the endpoints of the interval are included. This is indicated by the square brackets [ ] and the solid circles in the figure.

The open interval (a, b)

The closed interval $[a, b]$

We also need to consider infinite intervals such as

$$(a, \infty) = \{x \mid x > a\}$$

This does not mean that ∞ (''infinity'') is a number. The notation (a, ∞) stands for the set of all numbers that are greater than a, so the symbol ∞ simply indicates that the interval extends indefinitely far in the positive direction.

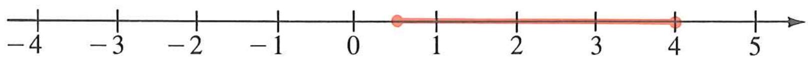
The following table lists the nine possible types of intervals. When these intervals are discussed, it will always be assumed that $a < b$.

Notation	Set Description	Picture
(a, b)	$\{x \mid a < x < b\}$	
$[a, b]$	$\{x \mid a \leqslant x \leqslant b\}$	
$[a, b)$	$\{x \mid a \leqslant x < b\}$	
$(a, b]$	$\{x \mid a < x \leqslant b\}$	
(a, ∞)	$\{x \mid x > a\}$	
$[a, \infty)$	$\{x \mid x \geqslant a\}$	
$(-\infty, b)$	$\{x \mid x < b\}$	
$(-\infty, b]$	$\{x \mid x \leqslant b\}$	
$(-\infty, \infty)$	R (set of all real numbers)	

Example Express the following intervals in terms of inequalities and graph the intervals:

(a) $\left[\frac{1}{2}, 4\right]$, (b) $[-2, 1)$, (c) $(-4, \infty)$.

Solution (a) $\left[\frac{1}{2}, 4\right] = \{x \mid \frac{1}{2} \leqslant x \leqslant 4\}$



(b) $[-2, 1) = \{x \mid -2 \leqslant x < 1\}$

(c) $(-4, \infty) = \{x \mid x > -4\}$

EXERCISE 1

1. Express the interval in terms of inequalities and graph the interval.
   (a) $(-2, 6)$ (b) $[-3, -2)$
   (c) $(1, 4]$ (d) $[-2, 1.5]$
   (e) $[3, \infty)$ (f) $(-\infty, 2)$
   (g) $(-\infty, 1]$ (h) $(-\frac{3}{2}, \infty)$
2. Express the inequality in interval notation and graph the corresponding interval.
   (a) $x < 2$ (b) $0 < x < 3$
   (c) $-1 \leqslant x < 2$ (d) $x > 1$
   (e) $-1 \leqslant x \leqslant 3$ (f) $x \leqslant -1$

Inequalities

Rules for Inequalities

1. If $a < b$, then $a + c < b + c$.
2. If $a < b$ and $c < d$, then $a + c < b + d$.
3. If $a < b$ and $c > 0$, then $ac < bc$.
4. If $a < b$ and $c < 0$, then $ac > bc$.
5. If $0 < a < b$, then $\dfrac{1}{a} > \dfrac{1}{b}$.

These rules also apply to any of the other order relations $>$, $\leqslant$, and $\geqslant$. Rule 1 says that we can add (or subtract) any number to (or from) both sides of an inequality and Rule 2 says that two inequalities can be added. However, we have to be careful with multiplication. Rule 3 says that we can multiply (or divide) both sides of an inequality by a *positive* number, but Rule 4 says that *if we multiply both sides of an inequality by a negative number, then we reverse the direction of the inequality*. For example, if we take the inequality $3 < 5$ and multiply by 2, we get $6 < 10$, but if we multiply by -2, we get $-6 > -10$. Finally, Rule 5 says that if we take reciprocals, then we reverse the direction of an inequality (provided the numbers are positive).

Example 1 Solve the inequality $1 + x < 6x - 4$.

Solution

$$
\begin{aligned}
1 + x &< 6x - 4 \\
5 + x &< 6x && \text{(Rule 1, } c = 4) \\
5 &< 5x && \text{(Rule 1, } c = -x) \\
1 &< x && \text{(Rule 3, } c = \tfrac{1}{5})
\end{aligned}
$$

In interval notation, the solution of the inequality is $(1, \infty)$.

Example 2 Solve $x^2 + 2x > 0$.

Solution 1 For a quadratic inequality we first factor:

$$x^2 + 2x = x(x + 2)$$

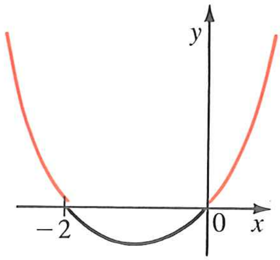
The product of two factors is positive when both factors are positive or both are negative, so there are two cases to consider.

TAKE CASES

Case 1: *both positive*

$$
\begin{aligned}
x > 0 \quad &\text{and} \quad x + 2 > 0 \\
x > 0 \quad &\text{and} \quad x > -2
\end{aligned}
$$

The intersection of these two intervals is $x > 0$.



$y = x(x+2)$
$y > 0$ when
$x < -2$ or $x > 0$

Case 2: *both negative.*

$$x < 0 \quad \text{and} \quad x + 2 < 0$$
$$x < 0 \quad \text{and} \quad x < -2$$

The intersection of these two intervals is $x < -2$.

Thus, the solution set of the given inequality is

$$\{x \mid x > 0 \text{ or } x < -2\} = (-\infty, -2) \cup (0, \infty)$$

Solution 2 We know that the corresponding equation $x(x+2) = 0$ has the solutions 0 and -2. These numbers divide the number line into three intervals

$$(\infty, -2) \qquad (-2, 0) \qquad (0, \infty)$$

On each of these intervals we determine the signs of the factors. For instance,

$$x < -2 \;\Rightarrow\; x + 2 < 0$$

Then we record these signs in the following chart and deduce the sign of the product.

Interval	x	$x+2$	$x(x+2)$
$x < -2$	$-$	$-$	$+$
$-2 < x < 0$	$-$	$+$	$-$
$x > 0$	$+$	$+$	$+$

We read from the chart that $x(x+2)$ is positive when $x < -2$ or $x > 0$. Thus, the solution of the given inequality is

$$\{x \mid x > 0 \text{ or } x < -2\} = (-\infty, -2) \cup (0, \infty)$$

When more than two factors are involved in an inequality, the method of Solution 1 of Example 2 is cumbersome, so we use the chart method.

Example 3 (a) Solve the inequality $(x+1)(x-2)(x-3) < 0$.
(b) Solve the inequality $(x+1)(x-2)(x-3) > 0$.

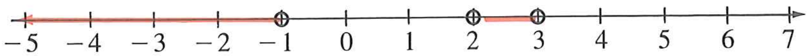
Solution We combine parts (a) and (b) by making a chart as in Solution 2 of Example 2:

Interval	$x+1$	$x-2$	$x-3$	$(x+1)(x-2)(x-3)$
$x < -1$	$-$	$-$	$-$	$-$
$-1 < x < 2$	$+$	$-$	$-$	$+$
$2 < x < 3$	$+$	$+$	$-$	$-$
$x > 3$	$+$	$+$	$+$	$+$

We read from the chart that $(x + 1)(x - 2)(x - 3)$ is negative when $x < -1$ or $2 < x < 3$. Thus, the solution of the inequality $(x + 1)(x - 2)(x - 3) < 0$ is

$$\{x \mid x < -1 \text{ or } 2 < x < 3\} = (-\infty, -1) \cup (2, 3)$$

and is shown in the following figure.



Similarly, the solution of the inequality $(x + 1)(x - 2)(x - 3) > 0$ is

$$\{x \mid -1 < x < 2 \text{ or } x > 3\} = (-1, 2) \cup (3, \infty)$$

Another method for obtaining the information in the charts in Examples 2 and 3 is to use test values. For instance in Example 3, if we use the test value $x = 1$ for the interval $(-1, 2)$, then substitution in $(x + 1)(x - 2)(x - 3)$ gives

$$(2)(-1)(-2) = 4$$

or, schematically,

$$(+)(-)(-) = +$$

The function $f(x) = (x + 1)(x - 2)(x - 3)$ does not change sign within any of the intervals, so we conclude that it is positive on the interval $(-1, 2)$.

EXERCISE 2

1. Solve the inequality.

(a) $3x + 7 > 0$
(b) $18 - 4x < 0$
(c) $17 - 2x \geqslant 13$
(d) $2x + 1 < 5x - 11$
(e) $2x - 1 < 19$
(f) $x^2 - 7x + 6 > 0$
(g) $12 - x - x^2 > 0$
(h) $x^2 < 3x$
(i) $x^2 - 9 > 0$
(j) $x^2 \leqslant 5$
(k) $(x + 1)(2x + 1)(x - 6) > 0$
(l) $x^3 + 3x^2 - 10x < 0$
(m) $x^3 + 3x^2 - 4 < 0$
(n) $x^3 + 2x^2 - 9x - 18 > 0$
(o) $x^3 - 8 \geqslant 0$
(p) $x^9 + x > 0$

2. Solve the inequality.

(a) $\dfrac{2x + 1}{x^2 + 1} > 0$
(b) $\dfrac{x + 2}{x - 3} > 0$
(c) $\dfrac{x^2 + x}{(x - 1)^3} < 0$
(d) $\dfrac{5x}{(x^2 - 1)^2} < 0$

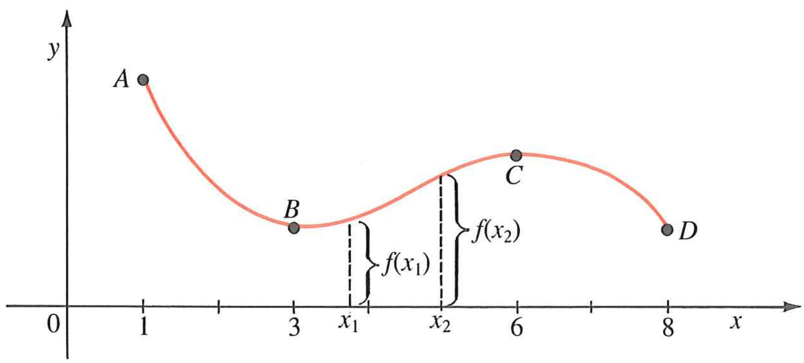
INTRODUCTION

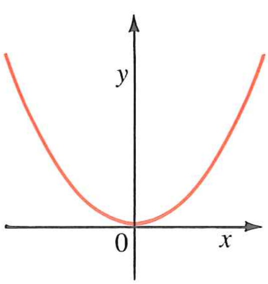
In this chapter, we use our knowledge of derivatives to find maximum and minimum values of functions. Then we apply this skill to practical problems that require us to maximize an area or a volume or to minimize a cost or, in general, to find the best possible outcome of a situation.

4.1 INCREASING AND DECREASING FUNCTIONS

Interval notation was introduced in the Review and Preview to this chapter.

One of the most useful things to know about a function is where its graph rises and where it falls. The graph of f, shown in the figure below, falls from A to B, rises from B to C, and falls again from C to D. We say that f is *decreasing* on the interval $(1, 3)$, *increasing* on $(3, 6)$, and decreasing on $(6, 8)$. Notice that for any two numbers x_1 and x_2 between 3 and 6 with $x_1 < x_2$, we have $f(x_1) < f(x_2)$.





In general, a function f is called **increasing on an interval I** if

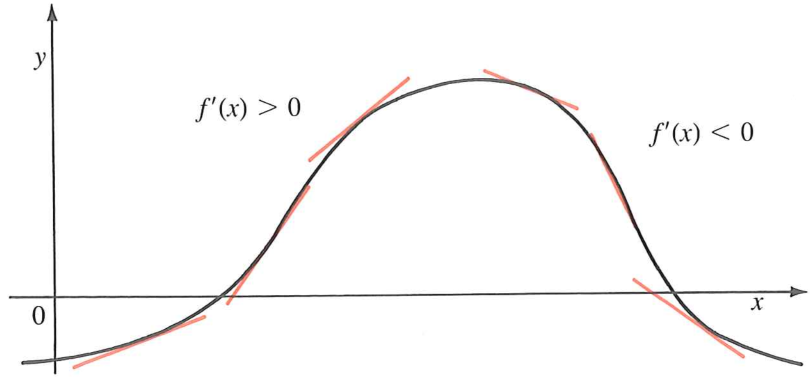
$$f(x_1) < f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in I}$$

It is called **decreasing on I** if

$$f(x_1) > f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in I}$$

For instance, you can see from the graph of the function $f(x) = x^2$ that f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

To see how the derivative of a function can tell us where a function is increasing or decreasing, look at the following diagram. When $f'(x) > 0$, the tangents have positive slope; that is, the tangents slope upward to the right. Thus, it appears that a positive derivative indicates an increasing function. When $f'(x) < 0$, the tangents slope downward to the right and f is decreasing. These facts are proved in more advanced courses.



Test for Increasing or Decreasing Functions

1. If $f'(x) > 0$ for all x in an interval I, then f is increasing on I.
2. If $f'(x) < 0$ for all x in I, then f is decreasing on I.

Example 1 Find the intervals on which the function $f(x) = 1 - 5x + 4x^2$ is increasing and decreasing.

Solution First we find the derivative:

$$f'(x) = -5 + 8x$$

The function f will be increasing when

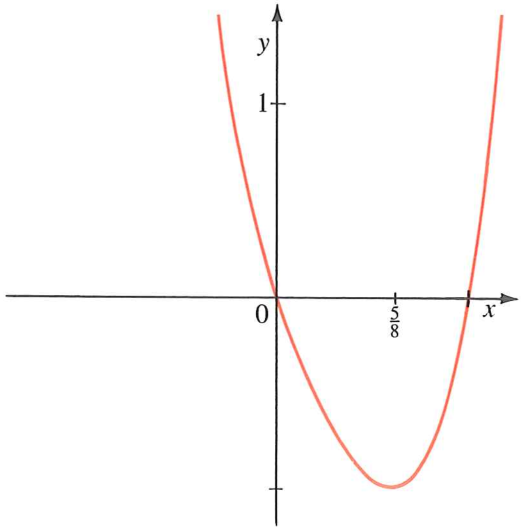
$$\begin{aligned} -5 + 8x &> 0 \\ 8x &> 5 \\ x &> \tfrac{5}{8} \end{aligned}$$

Thus, f is increasing on the interval $\left(\frac{5}{8}, \infty\right)$. Similarly,

$$-5 + 8x < 0 \quad \text{when} \quad x < \tfrac{5}{8}$$

So f is decreasing on the interval $\left(-\infty, \frac{5}{8}\right)$.

Since the function in Example 1 is quadratic, its graph is a parabola and is shown in the figure.



Example 2 Where is the function $y = x^3 + 6x^2 + 9x + 2$ increasing?

Solution We compute y' and factor it:

$$\begin{aligned} y' &= 3x^2 + 12x + 9 \\ &= 3(x^2 + 4x + 3) \\ &= 3(x + 1)(x + 3) \end{aligned}$$

The function will be increasing when $y' > 0$, so we have to solve the quadratic inequality

$$(x + 1)(x + 3) > 0$$

We observe that the product is 0 when $x = -1$ or -3. These numbers divide the line into three intervals $(-\infty, -3)$, $(-3, -1)$, and $(-1, \infty)$, on each of which the product keeps a constant sign as in the following chart. The last column of the chart gives the conclusion based on the Test for Increasing and Decreasing Functions.

Interval	$x + 1$	$x + 3$	$f'(x)$	f
$x < -3$	$-$	$-$	$+$	increasing on $(-\infty, -3)$
$-3 < x < -1$	$-$	$+$	$-$	decreasing on $(-3, -1)$
$x > -1$	$+$	$+$	$+$	increasing on $(-1, \infty)$

Another method of solving the quadratic inequality in Example 2 is to take cases as in Solution 1 of Example 2 in the Review and Preview to this chapter. A third method would be to graph the parabola $y = (x + 1)(x + 3)$ and observe that it lies above the x-axis when $x < -3$ or $x > -1$. For more complicated functions, however, the chart method is usually simplest, as in the following example.

Example 3 Find the intervals of increase and decrease for the function $g(x) = x^4 - 4x^3 - 8x^2 - 1$.

Solution

$$\begin{aligned} g'(x) &= 4x^3 - 12x^2 - 16x \\ &= 4x(x^2 - 3x - 4) \\ &= 4x(x + 1)(x - 4) \end{aligned}$$

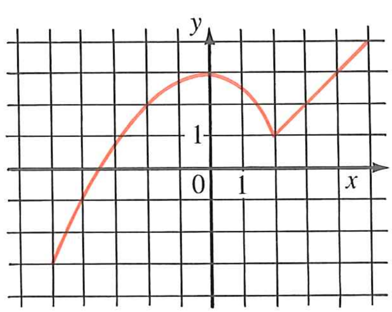
This expression is 0 when $x = 0, -1$, and 4. As in Example 2, we indicate the signs of the factors and the conclusion about g in a chart.

Interval	$4x$	$x + 1$	$x - 4$	$g'(x)$	g
$x < -1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$-$	$+$	$-$	$+$	increasing on $(-1, 0)$
$0 < x < 4$	$+$	$+$	$-$	$-$	decreasing on $(0, 4)$
$x > 4$	$+$	$+$	$+$	$+$	increasing on $(4, \infty)$

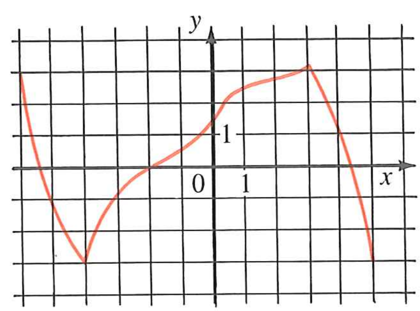
EXERCISE 4.1

A **1.** State the intervals of increase or decrease for the functions whose graphs are given.

(a)



(b)



B **2.** Find the intervals on which the following functions are increasing.

(a) $f(x) = 12 + x - x^2$ (b) $f(x) = x^4$

(c) $g(x) = x^3 - 3x + 2$ (d) $g(x) = 2x^3 - 3x^2$

(e) $y = 3x^4 + 4x^3 - 12x^2 + 7$ (f) $y = x^5 + 8x^3 + x$

3. Find the intervals on which the following functions are decreasing.

(a) $f(x) = x^2 + x^3$

(b) $g(x) = 2x^3 - 3x^2 - 36x + 62$

(c) $h(x) = (1 - x^2)^2$

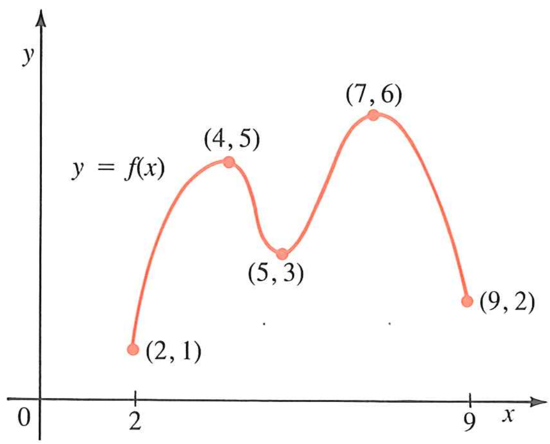
(d) $F(x) = 4x + x^4$

4. Find the intervals of increase and decrease for the following functions.
(a) $f(x) = 3x^2 - 18x + 1$
(b) $f(x) = 2x^3 - 9x^2 - 60x + 82$
(c) $g(x) = x^4 - 2x^2 + 16$
(d) $g(x) = 3x^4 - 16x^3 + 6x^2 + 72x + 8$
(e) $h(x) = x^3(x - 1)^4$
(f) $h(x) = \dfrac{x - 1}{x + 1}$
(g) $y = x\sqrt{4 - x}$
(h) $y = (x^2 - 9)^{\frac{2}{3}}$

C 5. Where is the function $y = 12x^5 + 15x^4 - 20x^3 + 27$ decreasing?

4.2 MAXIMUM AND MINIMUM VALUES

The graph of a function f is shown. Notice that the highest point on the graph is $(7, 6)$ and so the largest value taken on by the function is $f(7) = 6$. We say that f has an *absolute maximum* at 7 and the maximum value is $f(7) = 6$. The lowest point on the graph is $(2, 1)$, so the smallest value of the function is $f(2) = 1$. We say that f has an *absolute minimum* at 2 and the minimum value is $f(2) = 1$.

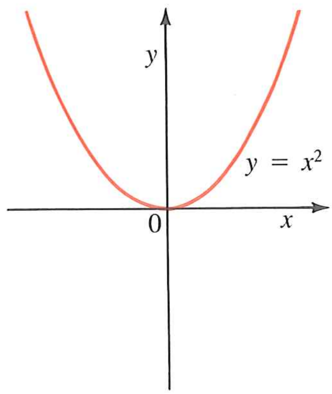


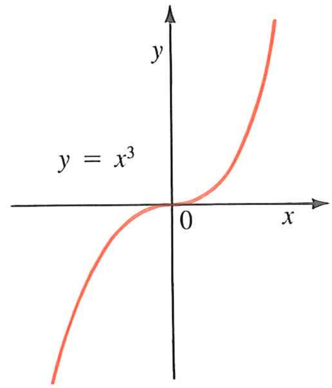
In general, a function f has an **absolute maximum** at c if $f(c) \geqslant f(x)$ for all x in the domain of f, and the number $f(c)$ is called the **maximum value** of f. The function has an **absolute minimum** at c if $f(c) \leqslant f(x)$ for all x in the domain, and the number $f(c)$ is called the **minimum value** of f. The **extreme values** of f are the maximum and minimum values.

Notice that if we restrict our attention to the interval $(2, 5)$, then the largest value is $f(4) = 5$. For that reason, we say that $f(4) = 5$ is a *local maximum value* of f. Likewise $f(5) = 3$ is called a *local minimum value* because it is the smallest value of f if we consider values of x that are near 5 [for instance, values of x in the interval $(4, 7)$].

In general, a function f has a **local maximum** (also called a **relative maximum**) at c if $f(c) \geqslant f(x)$ when x is close to c (on both sides of c). Similarly, f has a **local minimum** (or **relative minimum**) at c if $f(c) \leqslant f(x)$ when x is close to c.

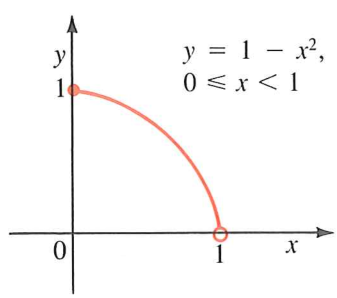
Example 1 If $f(x) = x^2$, then $f(x) \geqslant f(0)$ for all x since $x^2 \geqslant 0$. So $f(0) = 0$ is the absolute (and local) minimum value. But this function has no maximum value.

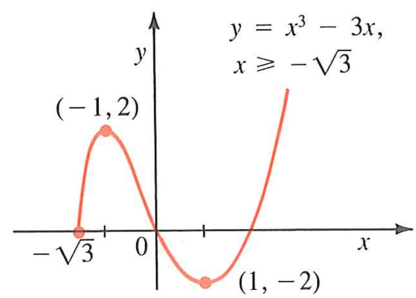




Example 2 We see from the graph of the function $f(x) = x^3$ that it has no absolute, or local, maximum or minimum value.

Example 3 The function $f(x) = 1 - x^2$, $0 \leqslant x < 1$, has maximum value $f(0) = 1$, but it has no minimum value. It takes on values very close to 0, but it never actually attains the value 0.





Example 4 The graph of the function

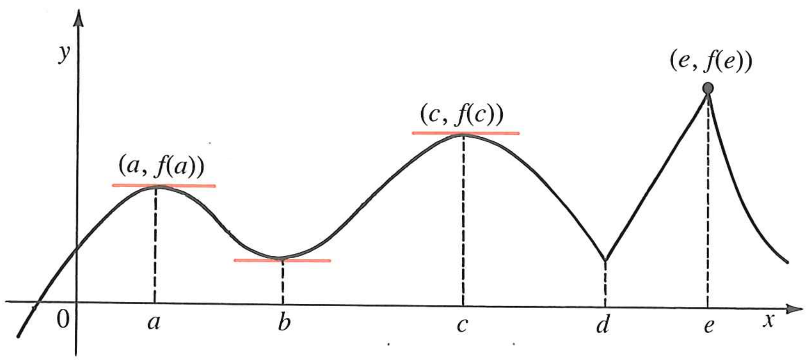
$$f(x) = x^3 - 3x, \qquad x \geq -\sqrt{3}$$

shows the following:

$f(-1) = 2$ is a local maximum
$f(1) = -2$ is a local and absolute minimum
f has no absolute maximum

We see from these examples that a function need not possess maximum and minimum values. But it can be proved that any continuous function defined on a closed interval $[a, b]$ has both an absolute maximum value and an absolute minimum value.

If we do not already know what the graph of a function looks like, how can we find the maximum and minimum values? Looking at the graph of f in the figure, we see that f has local maximum values when $x = a$, c, and e. It appears that the tangent lines at $(a, f(a))$ and $(c, f(c))$ are horizontal and so the slopes are $f'(a) = 0$ and $f'(c) = 0$. At $(e, f(e))$ there is no tangent line (since there is a corner) and so $f'(e)$ does not exist; that is, f is not differentiable at e.



Similarly, it appears that f has a local minimum value at b, where $f'(b) = 0$, and at d, where $f'(d)$ does not exist. These facts can be proved generally and are known as Fermat's Theorem after the French mathematician Pierre Fermat (1601–1665).

See the biography of Fermat at the end of this chapter

Fermat's Theorem

If f has a local maximum or minimum at c, then either $f'(c) = 0$ or $f'(c)$ does not exist.

We must be careful when using Fermat's Theorem. If $f'(c) = 0$, it does not automatically follow that f has a local maximum or minimum at c. For instance, the function $f(x) = x^3$ considered in Example 2 has no maximum and yet $f'(x) = 3x^2$, so $f'(0) = 0$. The significance of the derivative being 0 is just that the tangent is horizontal.

Fermat's Theorem does say that we should at least start looking for extreme values at the numbers c for which $f'(c) = 0$ or $f'(c)$ does not exist. These numbers are called *critical numbers*.

> A **critical number** of a function is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Example 5 Find the critical numbers of the following functions.
(a) $f(x) = x^3 + 6x^2 + 9x + 2$ (b) $f(x) = |x|$

Solution (a) First we find the derivative and factor it:

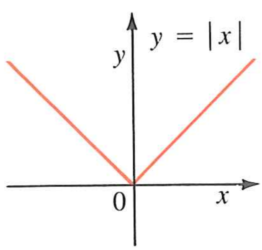
$$\begin{aligned} f'(x) &= 3x^2 + 12x + 9 \\ &= 3(x^2 + 4x + 3) \\ &= 3(x+1)(x+3) \end{aligned}$$

The derivative exists for all values of x, so the only critical numbers occur when $f'(x) = 0$, that is, when

$$\begin{aligned} &3(x+1)(x+3) = 0 \\ &x = -1 \quad \text{or} \quad x = -3 \end{aligned}$$

Therefore the critical numbers are -1 and -3.

(b) We know from Example 7 in Section 2.1 that the absolute value function is not differentiable at 0; that is, $f'(0)$ does not exist. Therefore 0 is a critical number.



Since

$$f(x) = |x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

we have

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Thus, $f'(x)$ is never 0. The only critical number is 0.

The absolute maximum or minimum of a continuous function on a closed interval is either a local maximum or minimum, in which case it occurs at a critical number, or it occurs at an endpoint of the interval. Therefore, we have the following.

Procedure for Finding the Absolute Maximum and Minimum Values of a Continuous Function on a Closed Interval $[a, b]$.

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints; that is, evaluate $f(a)$ and $f(b)$.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example 6 Find the absolute maximum and minimum values of the function

$$f(x) = x^3 + 6x^2 + 9x + 2, \quad -3.5 \leqslant x \leqslant 1$$

Solution From Example 5 we know that the critical numbers of f are -1 and -3. (Notice that each of these critical numbers lies in the interval $[-3.5, 1]$.) The values of f at these numbers are

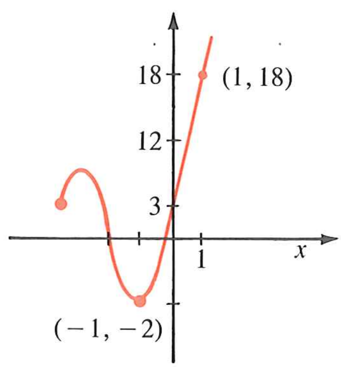
$$f(-1) = -2 \qquad f(-3) = 2$$

The values of f at the endpoints of the interval are

$$f(-3.5) = 1.125 \qquad f(1) = 18$$

Comparing these four numbers, we see that the absolute maximum value is $f(1) = 18$ and the absolute minimum value is $f(-1) = -2$.

Note that in this example the absolute maximum occurs at an endpoint, whereas the absolute minimum occurs at a critical number. (See the following graph of f).



$f(x) = x^3 + 6x^2 + 9x + 2,$
$-3.5 \leqslant x \leqslant 1$

Example 7 Find the absolute maximum and minimum values of the function

$$g(x) = x^{\frac{2}{3}}(5 + x), \qquad -5 \leqslant x \leqslant 1$$

Solution We could differentiate this function using the Product Rule, but it is perhaps simpler to rewrite the function first.

$$g(x) = x^{\frac{2}{3}}(5 + x) = 5x^{\frac{2}{3}} + x^{\frac{5}{3}}$$
$$g'(x) = \tfrac{10}{3}x^{-\frac{1}{3}} + \tfrac{5}{3}x^{\frac{2}{3}} = \frac{10 + 5x}{3x^{\frac{1}{3}}} = \frac{5(2 + x)}{3\sqrt[3]{x}}$$

This expression shows that $g'(x) = 0$ when $2 + x = 0$, that is, $x = -2$, and $g'(x)$ does not exist when $x = 0$. So the critical numbers are -2 and 0, and

$$g(0) = 0$$
$$g(-2) = (-2)^{\frac{2}{3}}(3) = 3(4^{\frac{1}{3}}) \doteq 4.8$$

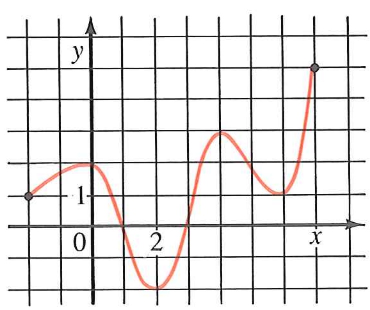
At the endpoints of the given interval $[-5, 1]$ we have

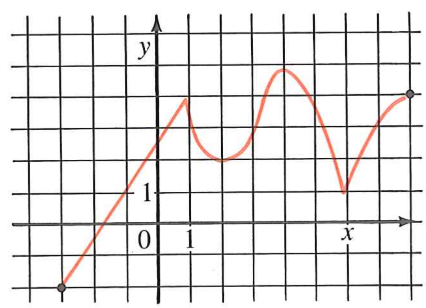
$$g(-5) = 0 \qquad g(1) = 6$$

We compare these values and see that the absolute maximum value is $g(1) = 6$ and the absolute minimum is $g(0) = g(-5) = 0$.

EXERCISE 4.2

A **1.** For the functions whose graphs are given, state
(a) the absolute maximum value,
(b) the absolute minimum value,
(c) the local maximum values,
(d) the local minimum values.

(i)


(ii)


B **2.** Sketch the graph of each function and use it to state the absolute and local maximum and minimum values of the function.

(a) $f(x) = 3x - 1,\ x > -1$ (b) $g(x) = 3x - 1,\ x \geqslant -1$
(c) $f(x) = x^2 + 1$ (d) $y = x^2 + 1,\ -1 < x < 2$
(e) $y = x^2 + 1,\ -1 \leqslant x \leqslant 2$ (f) $y = 2 - x^3$
(g) $y = |x - 2| - 1$ (h) $f(x) = \begin{cases} 2 - x & \text{if } x < 1 \\ x & \text{if } x \geqslant 1 \end{cases}$

3. Find the critical numbers of the given functions.

(a) $f(x) = 17 - 6x + 12x^2$ (b) $f(x) = x^3 - 3x + 2$
(c) $g(x) = x^4 - 4x^3 - 8x^2 - 1$
(d) $g(x) = 3x^4 - 16x^3 + 6x^2 + 72x + 8$
(e) $y = 2x^3 + 3x^2 - 6x + 3$ (f) $y = x^3 + x^2 + x + 1$
(g) $y = |x + 6|$ (h) $y = \sqrt[3]{x}$
(i) $y = x - \sqrt{x}$ (j) $y = x\sqrt{x - 1}$
(k) $y = \dfrac{t}{t + 1}$ (l) $y = \dfrac{t}{t^2 + 1}$

4. Find the absolute maximum value and absolute minimum value of the function.

(a) $f(x) = 2x^2 - 8x + 1,\ 0 \leqslant x \leqslant 3$
(b) $f(x) = 3 + 2(x + 1)^2,\ -3 \leqslant x \leqslant 2$
(c) $f(x) = 2x^3 - 3x^2,\ -2 \leqslant x \leqslant 2$
(d) $f(x) = 2x^3 - 3x^2 - 36x + 62,\ -3 \leqslant x \leqslant 4$
(e) $f(x) = x^4 - 2x^2 + 16,\ -3 \leqslant x \leqslant 2$
(f) $f(x) = x^5 + 3x^3 + x,\ -1 \leqslant x \leqslant 2$
(g) $g(x) = x^2 + \dfrac{16}{x},\ 1 \leqslant x \leqslant 4$
(h) $f(x) = 3x^{\frac{2}{3}} - 2x,\ 1 \leqslant x \leqslant 3$
(i) $f(x) = (x^2 - 9)^{\frac{2}{3}},\ -6 \leqslant x \leqslant 6$
(j) $f(x) = |2x - 1| - 1,\ 0 \leqslant x \leqslant 2$

C **5.** Show that the function $y = x^{21} + x^{11} + 13x$ does not have a local maximum or a local minimum.

6. Find the value of k if the function $y = x^2 + kx + 72$ has a local minimum at $x = 4$.

7. Find the values of a and b if the function $y = 2x^3 + ax^2 + bx + 36$ has a local maximum when $x = -4$ and a local minimum when $x = 5$.

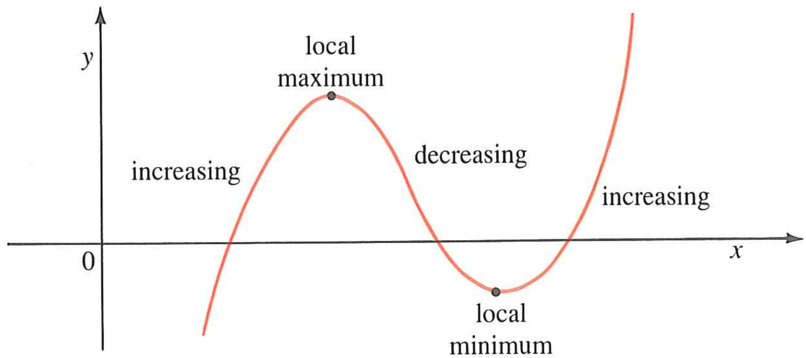
8. (a) Use Newton's method to find the critical numbers of the function $f(x) = 2x^5 - 5x^2 - 20x + 12$ correct to three decimal places.

(b) Find the absolute minimum value of the function $f(x) = 2x^5 - 5x^2 - 20x + 12,\ -1 \leqslant x \leqslant 2$, correct to two decimal places.

4.3 THE FIRST DERIVATIVE TEST

If f has a local maximum or minimum at c, then c must be a critical number of f (Fermat's Theorem), but not all critical numbers give rise to a maximum or minimum. For instance, recall that 0 is a critical number of the function $y = x^3$ but this function has no maximum or minimum. Therefore we need a test that will tell us whether or not a function has a maximum or minimum at a critical number.

One way of resolving the question is suggested by the graph in the figure below. If f is increasing just to the left of a critical number c and decreasing just to the right of c, then f has a local maximum at c. Similarly, if f is decreasing just to the left of c and increasing just to the right, then f has a local minimum at c.



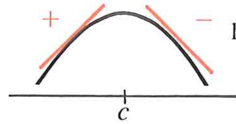
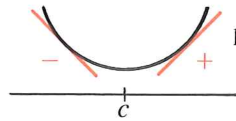
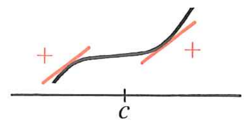
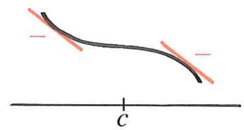
We know from Section 4.1 that f is increasing when $f'(x) > 0$ and decreasing when $f'(x) < 0$. Therefore we have the following test.

First Derivative Test

Let c be a critical number of a continuous function f.

1. If $f'(x)$ changes from positive to negative at c, then f has a local maximum at c.
2. If $f'(x)$ changes from negative to positive at c, then f has a local minimum at c.
3. If $f'(x)$ does not change sign at c, then f has no maximum or minimum at c.

The sketches in the following table illustrate how the First Derivative Test works.

Sign of $f'(x)$ to the left of c	Sign of $f'(x)$ to the right of c	Graph	$f(c)$
+	−		local maximum
−	+		local minimum
+	+		neither
−	−		neither

Example 1 Find the local maximum and minimum values of $f(x) = x^3 - 3x + 1$.

Solution First we find the critical numbers. The derivative is

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1)$$

Since $f'(x) = 0$ when $x^2 = 1$, the critical numbers are $x = \pm 1$.

Next we analyze the sign of the derivative. To see where it is positive or negative, we solve the following inequalities.

$\sqrt{x^2} = |x|$

$$\begin{aligned} f'(x) &> 0 \\ x^2 - 1 &> 0 \\ x^2 &> 1 \\ |x| &> 1 \\ x < -1 &\text{ or } x > 1 \end{aligned} \qquad\qquad \begin{aligned} f'(x) &< 0 \\ x^2 - 1 &< 0 \\ x^2 &< 1 \\ |x| &< 1 \\ -1 < x &< 1 \end{aligned}$$

Therefore $f'(x)$ changes sign from positive to negative at -1 and from negative to positive at 1. By the First Derivative Test it follows that

$$\begin{aligned} f(-1) &= 3 \qquad \text{is a local maximum} \\ f(1) &= -1 \qquad \text{is a local minimum} \end{aligned}$$

Another method of solving Example 1 is to factor $x^2 - 1$ as $(x - 1)(x + 1)$ and use a chart.

Example 2 Find the local maximum and minimum values of $g(x) = x^4 - 4x^3 - 8x^2 - 1$ and use this information to sketch the graph of g.

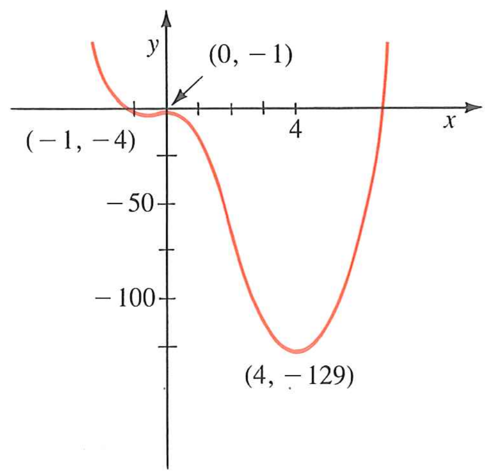
Solution In Example 3 in Section 4.1 we found that $g'(x) = 4x(x + 1)(x - 4)$, so the critical numbers are 0, -1, and 4. We reproduce the chart that gave the sign of $g'(x)$:

Interval	$4x$	$x + 1$	$x - 4$	$g'(x)$	g
$x < -1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$-$	$+$	$-$	$+$	increasing on $(-1, 0)$
$0 < x < 4$	$+$	$+$	$-$	$-$	decreasing on $(0, 4)$
$x > 4$	$+$	$+$	$+$	$+$	increasing on $(4, \infty)$

From the chart we see immediately that $g'(x)$ changes from negative to positive at -1, from positive to negative at 0, and from negative to positive at 4. Therefore, by the First Derivative Test,

$$\begin{aligned} g(-1) &= -4 && \text{is a local minimum} \\ g(0) &= -1 && \text{is a local maximum} \\ g(4) &= -129 && \text{is a local minimum} \end{aligned}$$

Using this information we sketch the graph of g.



Example 3 Find the critical numbers, intervals of increase and decrease, and local maximum and minimum values of the function $f(x) = 2x - 3x^{\frac{2}{3}}$.

Solution The derivative is

$$f'(x) = 2 - 2x^{-\frac{1}{3}} = \frac{2(\sqrt[3]{x} - 1)}{\sqrt[3]{x}}$$

which is not defined when $x = 0$. (But note that $f(x)$ is defined everywhere.) Also $f'(x) = 0$ when $x = 1$. So the critical numbers are 0 and 1.

The intervals of increase and decrease are obtained in the following chart.

Interval	$\sqrt[3]{x}$	$\sqrt[3]{x} - 1$	$f'(x)$	f
$x < 0$	$-$	$-$	$+$	increasing on $(-\infty, 0)$
$0 < x < 1$	$+$	$-$	$-$	decreasing on $(0, 1)$
$x > 1$	$+$	$+$	$+$	increasing on $(1, \infty)$

From the chart we see that the derivative changes from positive to negative at 0 and from negative to positive at 1. Thus, by the First Derivative Test,

$$f(0) = 0 \quad \text{is a local maximum}$$
$$f(1) = -1 \quad \text{is a local minimum}$$

In certain circumstances, the First Derivative Test can be used to find an *absolute* maximum or minimum.

First Derivative Test for Absolute Extreme Values

Let c be a critical number of a continuous function f defined on an interval.

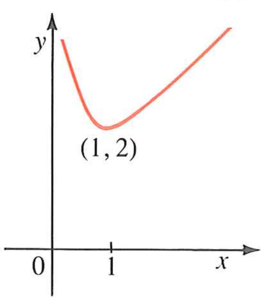
1. If $f'(x)$ is positive for all $x < c$ and $f'(x)$ is negative for all $x > c$, then $f(c)$ is the absolute maximum value.
2. If $f'(x)$ is negative for all $x < c$ and $f'(x)$ is positive for all $x > c$, then $f(c)$ is the absolute minimum value.

Example 4 Find the absolute minimum value of the function

$$f(x) = x + \frac{1}{x}, \quad x > 0.$$

Solution The derivative is

$$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$$



Thus $f'(x) = 0$ when $x^2 = 1$, that is, $x = 1$ (since $x > 0$). Also $f'(x) > 0$ when $x^2 > 1$, that is, $x > 1$. Similarly, $f'(x) < 0$ when $0 < x < 1$.

Thus, by the First Derivative Test, the absolute minimum value of f is $f(1) = 2$. The graph of f illustrates this fact.

EXERCISE 4.3

B **1.** Find the local maximum and minimum values of f.

(a) $f(x) = 3x^2 - 4x + 13$ (b) $f(x) = x^3 - 12x - 5$

(c) $f(x) = 2 + 5x - x^5$ (d) $f(x) = x^4 - x^3$

2. Find the critical numbers, intervals of increase and decrease, and local maximum values of the function. Then use this information to sketch the graph of f.

(a) $f(x) = 2 + 6x - 6x^2$

(b) $f(x) = x^3 - 9x^2 + 24x - 10$

(c) $g(x) = 1 + 3x^2 - 2x^3$

(d) $g(x) = 3x^4 - 16x^3 + 18x^2 + 1$

(e) $h(x) = x^4 - 8x^2 + 6$

(f) $h(x) = 3x^5 - 5x^3$

3. Find the local maximum and minimum values of f.

(a) $f(x) = 2x^{\frac{2}{3}}(3 - 4x^{\frac{1}{3}})$ (b) $f(x) = \dfrac{x^2}{x^2 - 1}$

(c) $f(x) = x\sqrt{4 - x}$ (d) $f(x) = x\sqrt{1 - x^2}$

4. Find the absolute maximum or minimum value of the function.

(a) $f(x) = 27 + x - x^2$ (b) $f(x) = 3 - \dfrac{1}{\sqrt{x^2 + 1}}$

(c) $g(x) = \dfrac{x^2 - 1}{x^2 + 1}$ (d) $g(x) = \dfrac{x^2 - x + 1}{x^2 + 1}, x \geqslant 0$

C **5.** Sketch the graph of a function f that satisfies all of the following conditions.

(a) $f(2) = 3, f(5) = 6$

(b) $f'(2) = f'(5) = 0$

(c) $f'(x) \geqslant 0$ for $x < 5$

(d) $f'(x) < 0$ for $x > 5$

6. Find the local maximum and minimum values of the function f defined by

$$f(x) = \begin{cases} -x & \text{if } x < 0 \\ 2x^3 - 15x^2 + 36x & \text{if } 0 \leqslant x \leqslant 4 \\ 216 - x & \text{if } x > 4 \end{cases}$$

PROBLEMS PLUS

Find the absolute maximum value of the function

$$f(x) = \frac{1}{1 + |x|} + \frac{1}{1 + |x - 2|}$$

4.4 APPLIED MAXIMUM AND MINIMUM PROBLEMS

One of the most important applications of derivatives occurs in the solution of ''optimization'' problems, in which a quantity must be maximized or minimized. In this section and the next, we solve such problems as maximizing areas, volumes, and profits, and minimizing distances, times, and costs.

In solving these problems, the first step is to express the problem in mathematical language by setting up the function that is to be maximized or minimized. Then we use the methods of this chapter to find the extreme value.

Example 1 Find two positive numbers whose product is 10 000 and whose sum is a minimum.

Solution Let x be the first number and y the second number. We wish to minimize the sum

INTRODUCE NOTATION

$$S = x + y$$

but we first express S in terms of just one variable. To eliminate one of the variables we use the given condition that the product of the numbers is 10 000:

$$xy = 10\,000$$

Solving for y, we get

$$y = \frac{10\,000}{x}$$

and, substituting into the equation for S, we have

$$S = x + \frac{10\,000}{x}, \quad x > 0$$

$$\frac{dS}{dx} = 1 - \frac{10\,000}{x^2} = \frac{x^2 - 10\,000}{x^2}$$

To find the critical numbers, we equate the derivative to 0:

$$\frac{x^2 - 10\,000}{x^2} = 0$$

$$x^2 = 10\,000$$

$$x = 100 \qquad \text{(since } x > 0\text{)}$$

Now we verify that $x = 100$ gives a minimum value for S. Since

$$\frac{dS}{dx} < 0 \text{ for } 0 < x < 100 \quad \text{and} \quad \frac{dS}{dx} > 0 \text{ for } x > 100$$

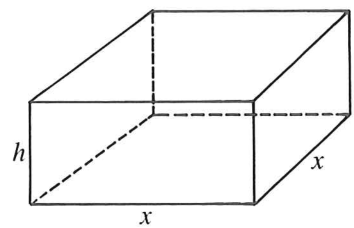
S has an absolute minimum at $x = 100$ by the First Derivative Test for Absolute Extreme Values.

When $x = 100$, we have $y = \dfrac{10\,000}{100} = 100$. Therefore the two numbers are both 100.

Example 2 If 2700 cm^2 of material is available to make a box with a square base and open top, find the largest possible volume of the box.

Solution

DRAW A DIAGRAM



INTRODUCE NOTATION

Let x be the length of the base of the box and h its height, in centimetres. The quantity to be maximized is the volume of the box:

$$V = x^2h$$

We eliminate h by finding a relationship between x and h. To do this, we use the fact that the area of the available material is 2700 cm^2. Since the box is open at the top, its surface area is given by

$$\begin{aligned}(\text{area of base}) + (\text{area of four sides}) &= 2700\\ x^2 + 4xh &= 2700\end{aligned}$$

Solving for h, we get

$$h = \frac{2700 - x^2}{4x}$$

This allows us to express V as a function of x:

$$V = x^2h = x^2\left(\frac{2700 - x^2}{4x}\right) = x\left(\frac{2700 - x^2}{4}\right)$$

$$V = 675x - \tfrac{1}{4}x^3$$

It is important to identify the domain of this function. Since x represents the base, we have $x \geqslant 0$. Also $h \geqslant 0$, so

$$\begin{aligned}2700 - x^2 &\geqslant 0\\ x^2 &\leqslant 2700\\ x &\leqslant \sqrt{2700} = 30\sqrt{3}\end{aligned}$$

Thus, the domain of the function V is given by $0 \leqslant x \leqslant 30\sqrt{3}$.

Now we differentiate:

$$V' = 675 - \tfrac{3}{4}x^2$$

The critical numbers occur when $V' = 0$:

$$\begin{aligned} 675 - \tfrac{3}{4}x^2 &= 0 \\ \tfrac{3}{4}x^2 &= 675 \\ x^2 &= \tfrac{4}{3}(675) = 900 \\ x &= 30 \end{aligned}$$

To find the absolute maximum of V on the interval $[0, 30\sqrt{3}]$, we evaluate V at the critical number and the endpoints:

$$\begin{aligned} V(30) &= (675)(30) - \tfrac{1}{4}(30)^3 = 13\ 500 \\ V(0) &= 0 \\ V(30\sqrt{3}) &= (675)(30\sqrt{3}) - \tfrac{1}{4}(30\sqrt{3})^3 = 0 \end{aligned}$$

Thus, the absolute maximum volume occurs when $x = 30$ cm and the maximum volume is 13 500 cm^3.

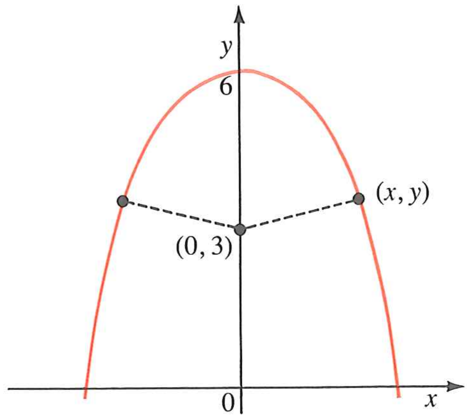
We summarize the steps in solving applied maximum and minimum problems by adapting the problem-solving principles stated in the Review and Preview to Chapter 3 and by keeping in mind the procedures used in solving Examples 1 and 2.

1. ***Understand the problem.*** The first step is to read the problem carefully until it is clearly understood. Ask yourself: What is the unknown? What are the given quantities? What are the given conditions?
2. ***Draw a diagram.*** In most problems it is useful to draw a diagram and identify the given and required quantities on the diagram.
3. ***Introduce notation.*** Assign a symbol to the quantity that is to be maximized or minimized (let us call it Q for now). Also select symbols (a, b, c, ..., x, y) for other unknown quantities and label the diagram with these symbols. It may help to use initials as suggestive variables—for example, A for area, h for height, t for time.
4. Express Q in terms of some of the other symbols from Step 3.
5. If Q has been expressed as a function of more than one variable in Step 4, use the given information to find relationships (in the form of equations) among these variables. Then use these equations to eliminate all but one of the variables in the expression for Q. Thus, Q will be given as a function of *one* variable x—say, $Q = f(x)$. Write the domain of this function.
6. Use the methods of Section 4.2 and 4.3 to find the *absolute* maximum or minimum value of f.

Example 3 Find the points on the parabola $y = 6 - x^2$ that are closest to the point $(0, 3)$.

Solution From the sketch it appears that there are two points at a minimum distance from $(0, 3)$.

DRAW A DIAGRAM



The distance d from the point $(0, 3)$ to the point (x, y) is

$$d = \sqrt{x^2 + (y - 3)^2}$$

But if (x, y) lies on the parabola, then $y = 6 - x^2$. Substituting, we get

$$d = \sqrt{x^2 + (6 - x^2 - 3)^2} = \sqrt{x^2 + (3 - x^2)^2}$$

Instead of minimizing d, we minimize the simpler expression for d^2. (Note that d is smallest when d^2 is smallest; see Question 9 in Exercise 4.4.) Therefore we need to find the critical numbers of the function

$$f(x) = x^2 + (3 - x^2)^2 = x^4 - 5x^2 + 9$$

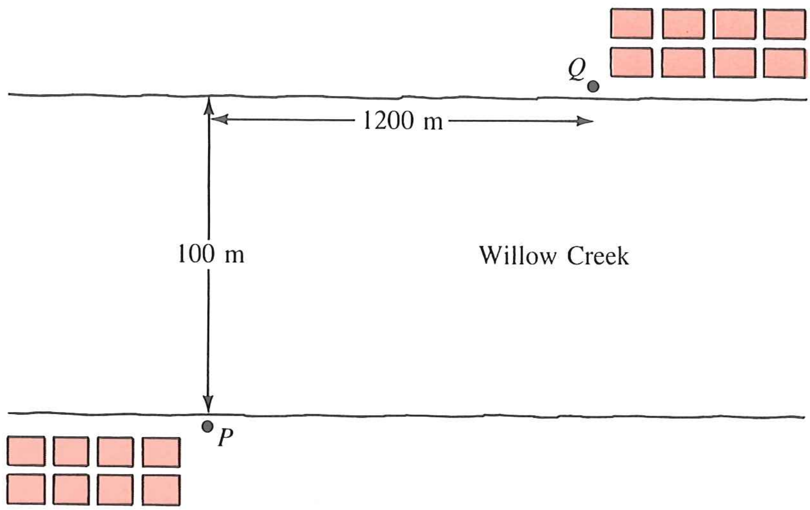
Differentiation gives $f'(x) = 4x^3 - 10x$, so the critical numbers occur when

$$4x^3 - 10x = 0$$
$$2x(2x^2 - 5) = 0$$
$$x = 0, \pm\sqrt{2.5}$$

The First Derivative Test shows that $x = 0$ gives a local maximum, whereas $x = \sqrt{2.5}$ and $x = -\sqrt{2.5}$ minimize $f(x)$ and, therefore, the distance d.

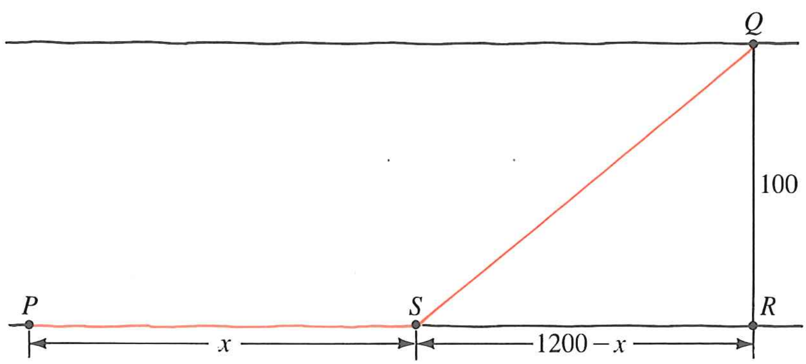
When $x = \pm\sqrt{2.5}$, we have $y = 6 - 2.5 = 3.5$, so the points on the parabola closest to $(0, 3)$ are $(\sqrt{2.5}, 3.5)$ and $(-\sqrt{2.5}, 3.5)$.

Example 4 A cable television company is laying cable in an area with underground utilities. Two subdivisions are located on opposite sides of Willow Creek, which is 100 m wide. The company has to connect points P and Q with cable, where Q is on the north bank 1200 m east of P. It costs \$40/m to lay cable underground and \$80/m to lay cable underwater. What is the least expensive way to lay the cable?



Solution The company can lay the cable on the south shore to the point R directly opposite Q, or it can proceed east a distance x metres to a point S before laying underwater cable from S to Q.

DRAW A DIAGRAM



The cost of the underground portion is

$$40x$$

Since $\qquad SQ = \sqrt{100^2 + (1200 - x)^2}$

the cost of the underwater portion is

$$80SQ = 80\sqrt{100^2 + (1200 - x)^2}$$

So the total cost is

$$C(x) = 40x + 80\sqrt{100^2 + (1200 - x)^2},\ 0 \leqslant x \leqslant 1200$$

and

$$C'(x) = 40 + \frac{40(2)(1200 - x)(-1)}{\sqrt{100^2 + (1200 - x)^2}}$$

Thus, $C'(x) = 0$ when $40 + \dfrac{40(2)(1200 - x)(-1)}{\sqrt{100^2 + (1200 - x)^2}} = 0$

$$\begin{aligned} 1 - \frac{2(1200 - x)}{\sqrt{100^2 + (1200 - x)^2}} &= 0 \\ 2(1200 - x) &= \sqrt{100^2 + (1200 - x)^2} \\ 4(1200 - x)^2 &= 100^2 + (1200 - x)^2 \\ 3(1200 - x)^2 &= 100^2 \\ x - 1200 &= \pm\frac{100}{\sqrt{3}} \end{aligned}$$

$$x = 1200 \pm \frac{100}{\sqrt{3}} \doteq 1142 \text{ or } 1258$$

Notice that 1258 lies outside the domain of C. So we evaluate C at the critical number 1142 and at the endpoints of the interval $[0, 1200]$:

$$\begin{aligned} C(1142) &= \$54\,928 \\ C(0) &= \$96\,333 \\ C(1200) &= \$56\,000 \end{aligned}$$

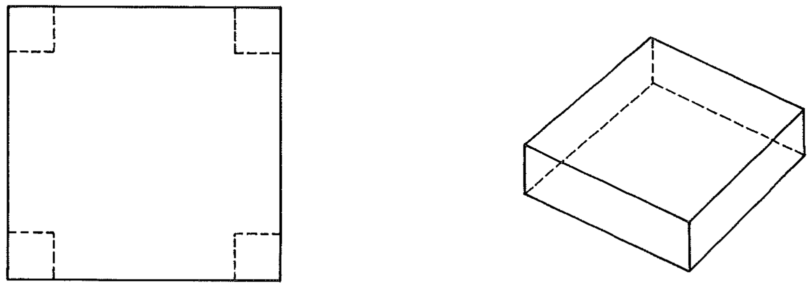
The cheapest method is to lay cable underground to a point about 1142 m east of P and then to lay the remaining cable underwater.

EXERCISE 4.4

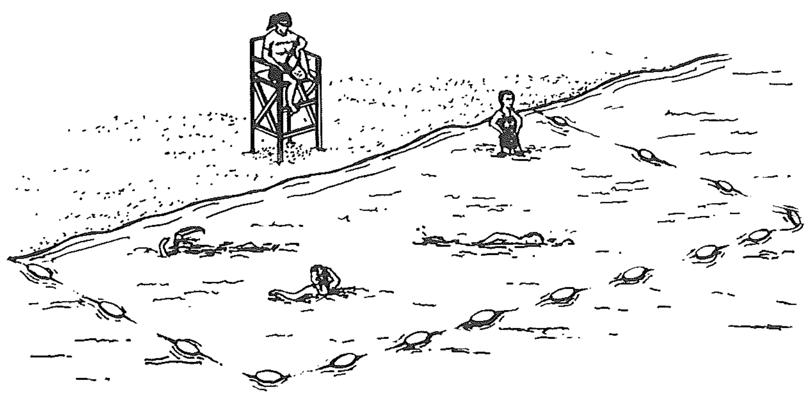
1. Find two numbers whose difference is 150 and whose product is a minimum.
2. Find two positive numbers with product 200 such that the sum of one number and twice the second number is as small as possible.
3. A rectangle has a perimeter of 100 cm. What length and width should it have so that its area is a maximum?
4. Show that a rectangle with given area has minimum perimeter when it is a square.

5. A box with a square base and open top must have a volume of $4000\ \text{cm}^3$. Find the dimensions of the box that minimizes the amount of material used.

6. A box with an open top is to be constructed from a square piece of cardboard, 3 m wide, by cutting out a square from each of the four corners and bending up the sides, as indicated in the figure. Find the largest volume that such a box can have.



7. The lifeguard at a public beach has 400 m of rope available to lay out a rectangular restricted swimming area using the straight shoreline as one side of the rectangle.
   (a) If she wants to maximize the swimming area, what will the dimensions of the rectangle be?
   (b) To ensure the safety of swimmers, she decides that nobody should be more than 50 m from shore. What should the dimensions of the swimming area be with this added restriction?



8. A farmer wants to fence an area of $750\ 000\ \text{m}^2$ in a rectangular field and divide it in half with a fence parallel to one of the sides of the rectangle. How can this be done so as to minimize the cost of the fence?

9. Show that if the function $y = f(x)$ has a minimum at c, then the function $y = \sqrt{f(x)}$ also has a minimum at c.

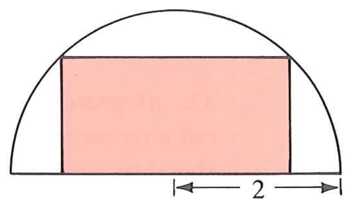
10. Find the point on the line $y = 5x + 4$ that is closest to the origin.

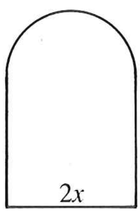
11. Find the point on the parabola $2y = x^2$ that is closest to the point $(-4, 1)$.

12. A can is to be made to hold a litre of oil. Find the radius of the can that will minimize the cost of the metal to make the can. (Take $1 \text{ L} = 1000 \text{ cm}^3$.)

13. A piece of wire 40 cm long is cut into two pieces. One piece is bent into the shape of a square and the other is bent into the shape of a circle. How should the wire be cut so that the total area enclosed is
(a) a maximum? (b) a minimum?

14. A rectangle is inscribed in a semicircle of radius 2 cm as shown. Find the largest area of such a rectangle.





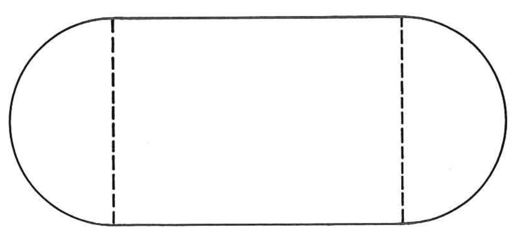
15. Solve the problem in Example 4 if it costs \$120/m to lay the cable underwater.

16. A Norman window has the shape of a rectangle capped by a semicircular region as shown. If the perimeter of the window is 8 m, find the width of the window that will admit the greatest amount of light.

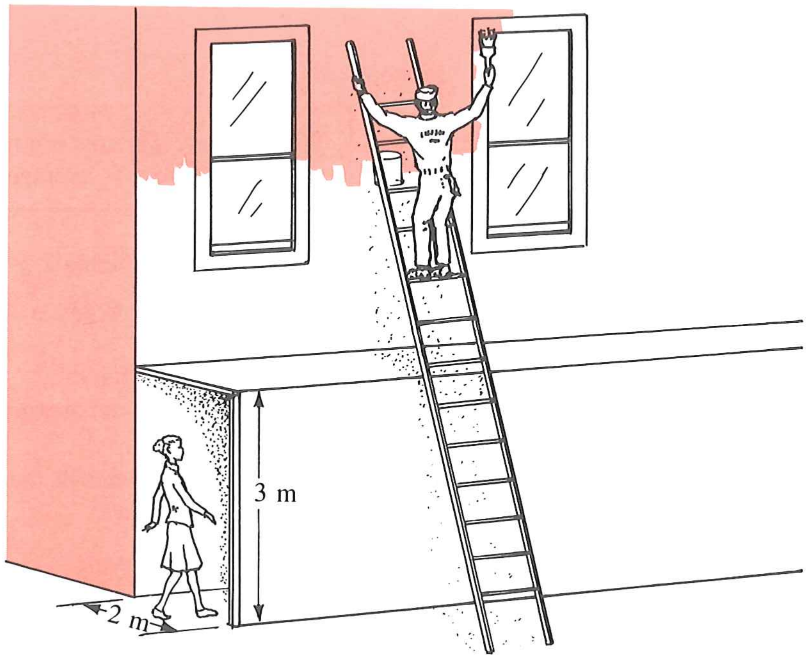
17. A boat leaves a dock at noon and heads west at a speed of 25 km/h. Another boat heads north at 20 km/h and reaches the same dock at 1:00 p.m. When were the boats closest to each other?

18. Find the largest possible volume of a right circular cylinder that is inscribed in a sphere of radius r.

19. A 1-km racetrack is to be built with two straight sides and semicircles at the ends (as in the figure). Find the dimensions of the track that encloses the maximum area.



20. Painters are painting the second floor exterior wall of a building that adjoins a busy sidewalk. A corridor 2 m wide and 3 m high is built to protect pedestrians as shown in the following figure. What is the length of the shortest ladder that will reach from the ground over the corridor to the wall of the building?



4.5 EXTREME VALUE PROBLEMS IN ECONOMICS

In this section we use the techniques of this chapter to solve problems that arise in business and economics: minimizing average costs and maximizing revenues or profits.

In Section 3.4 we introduced the idea of marginal cost. Recall that if it costs a company an amount $C(x)$ to produce x units of a commodity, then the function C is called a cost function. The marginal cost function is the rate of change of C with respect to x; that is, the derivative $C'(x)$ of the cost function.

The **average cost function**

$$c(x) = \frac{C(x)}{x} \tag{1}$$

is the cost per unit when x units are produced. We want to minimize the average cost and we do so by locating the critical number of c.

Using the Quotient Rule to differentiate Equation 1 and equating the derivative to zero, we have

$$c'(x) = \frac{xC'(x) - C(x)}{x^2} = 0$$

$$xC'(x) - C(x) = 0$$

$$C'(x) = \frac{C(x)}{x} = c(x)$$

In words, this says:

> When the average cost is a minimum
> marginal cost = average cost

Example 1 The cost, in dollars, of producing x 5-kg bags of flour is

$$C(x) = 140\,000 + 0.43x + 0.000\,001x^2$$

(See Example 1 in Section 3.4.)

(a) Find the average cost and marginal cost of producing 100 000 bags.

(b) At what production level will the average cost be smallest, and what is this average cost?

Solution (a) The average cost function is

$$\begin{aligned} c(x) &= \frac{C(x)}{x} \\ &= \frac{140\,000 + 0.43x + 0.000\,001x^2}{x} \\ &= \frac{140\,000}{x} + 0.43 + 0.000\,001x \end{aligned}$$

The average cost of producing 100 000 bags of flour is

$$\begin{aligned} c(100\,000) &= \frac{140\,000}{100\,000} + 0.43 + (0.000\,001)(100\,000) \\ &= 1.4 + 0.43 + 0.1 \\ &= \$1.93/\text{bag} \end{aligned}$$

The marginal cost function is

$$C'(x) = 0.43 + 0.000\,002x$$

When $x = 100\,000$, the marginal cost is

$$\begin{aligned} C'(100\,000) &= 0.43 + (0.000\,002)(100\,000) \\ &= 0.43 + 0.2 \\ &= \$0.63 \end{aligned}$$

(b) To minimize the average cost we could equate the marginal cost and the average cost, or we could simply differentiate $c(x)$:

$$c'(x) = \frac{-140\,000}{x^2} + 0.000\,001 = 0$$

$$\frac{140\,000}{x^2} = 0.000\,001$$

$$x^2 = \frac{140\,000}{0.000\,001} = 14 \times 10^{10}$$

$$x = \sqrt{14} \times 10^5 \doteq 3.74 \times 10^5$$

Since $c'(x) < 0$ for $x < \sqrt{14} \times 10^5$ and $c'(x) > 0$ for $x > \sqrt{14} \times 10^5$, this value gives an absolute minimum by the First Derivative Test.

Thus, the average cost will be smallest when the production level is about 374 000 bags and this minimum average cost is

$$c(\sqrt{14} \times 10^5) \doteq \$1.18/\text{bag}$$

Recall that the demand function $p(x)$ is the price per unit that a company can charge if it sells x units. The revenue function is

$$R(x) = xp(x)$$

and the marginal revenue function is its derivative, $R'(x)$. The profit is obtained by subtracting costs from revenue:

$$P(x) = R(x) - C(x)$$

To maximize profit we look for the critical numbers of P:

$$P'(x) = R'(x) - C'(x) = 0$$
$$R'(x) = C'(x)$$

> For maximum profit
> marginal revenue = marginal cost

Example 2 In Examples 2 and 3 in Section 3.4 we considered the case of Howard's Hamburgers with yearly demand function

$$p = \frac{800\,000 - x}{200\,000}$$

and cost function

$$C(x) = 125\,000 + 0.42x$$

What level of sales will maximize profits?

Solution The revenue function is

$$R(x) = xp(x) = \frac{1}{200\,000}(800\,000x - x^2)$$

so the marginal revenue is

$$R'(x) = \frac{1}{200\,000}(800\,000 - 2x) = \frac{1}{100\,000}(400\,000 - x)$$

The marginal cost is

$$C'(x) = 0.42$$

Profits are maximized when marginal revenue = marginal cost.

$$\begin{aligned} R'(x) &= C'(x) \\ \frac{1}{100\,000}(400\,000 - x) &= 0.42 \\ 400\,000 - x &= 42\,000 \\ x &= 358\,000 \end{aligned}$$

Sales of 358 000 will maximize profits.

Example 3 A store has been selling 200 compact disc players a week at \$350 each. A market survey indicates that for each \$10 rebate offered to the buyers, the number of sets sold will increase by 20 a week.

(a) Find the demand function and the revenue function.

(b) How large a rebate should the store offer to maximize its revenue?

Solution (a) If x is the number of CD players sold per week, then the weekly increase in sales is $x - 200$. For each increase of 20 players sold, the price is decreased by \$10. So for each additional player sold the decrease in price will be $\$\frac{1}{20}10$ and the demand function is

$$p(x) = 350 - \tfrac{10}{20}(x - 200) = 450 - \tfrac{1}{2}x$$

The revenue function is

$$R(x) = xp(x) = 450x - \tfrac{1}{2}x^2$$

(b) We find the critical numbers of R by differentiating:

$$R'(x) = 450 - x$$

Thus $R'(x) = 0$ when $x = 450$. (Or use marginal revenue = marginal cost.) This value of x gives an absolute maximum by the First Derivative Test (or by observing that the graph of R is a parabola that opens downward).

The corresponding price is

$$p(450) = 450 - \tfrac{1}{2}(450) = 225$$

and the rebate is

$$350 - 225 = 125$$

To maximize revenue the store should offer a rebate of \$125.

EXERCISE 4.5

B **1.** A company determines that the cost, in dollars, of producing x items is

$$C(x) = 280\,000 + 12.5x + 0.07x^2$$

(a) Find the average cost and marginal cost of producing 1000 items.

(b) At what production level will the average cost be least?

(c) What is the minimum average cost?

2. The cost, in dollars, for the production of x units of a commodity is

$$C(x) = 6400 + \frac{x}{10} + \frac{x^2}{1000}$$

(a) Find the average cost and marginal cost at a production level of 3000 units.

(b) Find the production level that will minimize the average cost.

(c) Find the smallest average cost.

3. The Bouchard Soup Company estimates that the cost, in dollars, of making x cans of pea soup is

$$C(x) = 48\,000 + 0.28x + 0.000\,01x^2$$

and the revenue is

$$R(x) = 0.68x - 0.000\,01\,x^2$$

In order to maximize profits how many cans of pea soup should be sold?

4. Sue's Submarines has found that the monthly demand for their submarines is given by

$$p = \frac{30\,000 - x}{10\,000}$$

and the cost of making x submarines is

$$C(x) = 6000 + 0.8x$$

What level of sales will maximize profits?

5. A baseball team plays in a stadium that holds 52 000 spectators. Average attendance at a game was 27 000 with tickets priced at \$10. When ticket prices were lowered to \$8, the average attendance rose to 33 000.

(a) Find the demand function, assuming that it is linear.
(b) How should the owners set ticket prices so as to maximize revenue?

6. A chain of stores has been selling a line of cameras for \$50 each and has been averaging sales of 8000 cameras a month. They decide to increase the price, but their market research indicates that for each \$1 increase in price, sales will fall by 100.
(a) Find the demand function.
(b) Find the price that will maximize their revenue.

7. The manager of a 120-unit apartment complex knows from experience that all units will be occupied if the rent is \$400 a month. A market survey suggests that, on the average, one additional unit will remain vacant for each \$10 increase in rent. What rent should she charge to maximize revenue?

8. New Horizons Travel advertises a package plan for a Florida vacation. The fare for the flight is \$400/person plus \$8/person for each unsold seat on the plane. The plane holds 120 passengers and the flight will be cancelled if there are fewer than 50 passengers. What number of passengers will maximize revenue?

4.6 REVIEW EXERCISE

1. For each of the following functions find the critical numbers and the intervals of increase or decrease.
(a) $f(x) = x - x^3$ (b) $f(x) = x + x^3$
(c) $f(x) = 3x^4 - 8x^3 - 6x^2 + 24x + 3$
(d) $f(x) = \dfrac{2x + 1}{2x - 1}$
(e) $f(x) = \dfrac{x^2}{x + 1}$ (f) $f(x) = 3x^{\frac{5}{3}} - 15x^{\frac{2}{3}}$

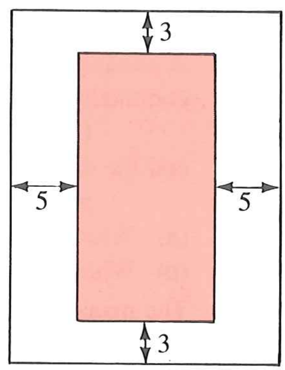
2. Find the absolute maximum value and absolute minimum value of each function.
(a) $f(x) = 4x^2 + 12x - 7, \ -2 \leqslant x \leqslant 1$
(b) $f(x) = x^3 - 27x + 32, \ -4 \leqslant x \leqslant 4$
(c) $g(x) = 3x^5 - 50x^3 + 135x, \ -2 \leqslant x \leqslant 4$
(d) $g(x) = \dfrac{1 + x}{1 - x}, \ 2 \leqslant x \leqslant 5$

3. Find the local maximum and minimum values of f.
(a) $f(x) = 7 + 72x + 3x^2 - 2x^3$
(b) $f(x) = x^4 - 72x^2 + 10$
(c) $f(x) = \sqrt{16 - x^2}$
(d) $f(x) = 12 - 2|x + 3|$

4. For the function $f(x) = x^4 - 8x^3 + 22x^2 - 24x$,
   (a) find the critical numbers,
   (b) find the intervals of increase or decrease,
   (c) find the local maximum and minimum values,
   (d) sketch the graph.

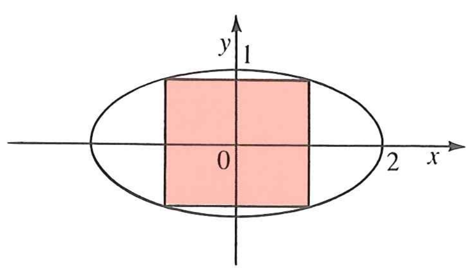
5. Find the absolute maximum value of the function
$$f(x) = \frac{1}{x^2 + x + 1}.$$

6. A printed page is to contain 60 cm^2 of printed material with clear margins of 5 cm on each side and 3 cm on the top and bottom. Find the minimum total area of the page.

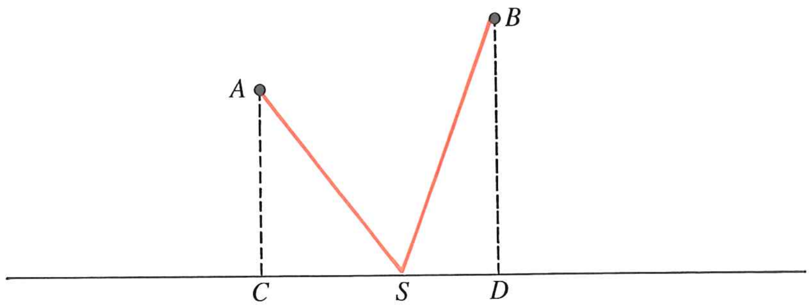


7. The illumination of an object by a light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. Two streetlights are 40 m apart and one is twice as strong as the other. Where is the darkest spot between the two lights?

8. Find the dimensions of the largest rectangle with sides parallel to the axes that can be inscribed in the ellipse $x^2 + 4y^2 = 4$ as shown in the figure.



9. Two towns A and B are 5 km and 7 km, respectively, from a railroad line. The points C and D nearest to A and B on the line are 6 km apart. Where should a station be located to minimize the length of a new road from A to S to B?



10. A company determines that the cost of making x units of a commodity is

$$C(x) = 480\,000 + 2.4x + 0.0008x^2$$

and the demand function is given by

$$p(x) = 4 - 0.001x$$

(a) What production level will minimize the average cost?

(b) What level of sales will maximize profits?

11. The manager of a 120-room resort hotel has found that, on the average, 50 rooms are booked when the price is \$100 per night and 80 rooms are booked when the price is \$80 per night.

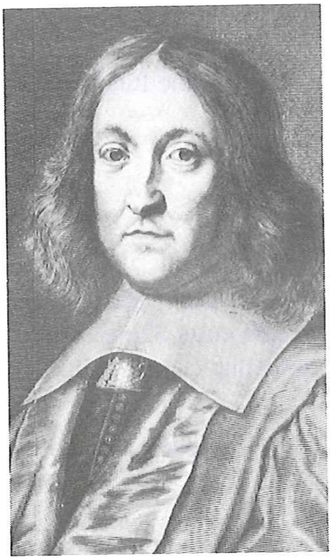
(a) Find the demand function, assuming that it is linear.

(b) What price should he charge to maximize revenue?

4.7 CHAPTER 4 TEST

1. Find the interval on which the function $f(x) = \dfrac{x}{x^2 + 1}$ is increasing.
2. Find the absolute maximum and minimum values of the function $f(x) = x^3 + 2x^2 + x - 1, \ -1 \leqslant x \leqslant 1$.
3. For the function $f(x) = x^4 - 8x^2 + 3$,
   (a) find the critical numbers,
   (b) find the intervals of increase and decrease,
   (c) find the local maximum and minimum values.
4. A box is to be built with a square base and an open top. Material for the base costs $\$4/\text{m}^2$, while material for the sides costs $\$2/\text{m}^2$. Find the dimensions of the box of maximum volume that can be built for \$1200.
5. A company estimates that the cost, in dollars, of producing x items is
   $$C(x) = 16\,000 + 22.5x + 0.004x^2$$
   At what production level will the average cost be lowest?
6. An apple orchard now has 80 trees planted per hectare and the average yield is 400 apples per tree. For each additional tree planted per hectare the average yield per tree is reduced by approximately four apples. How many trees per hectare will give the largest crop of apples?

FOUNDERS OF CALCULUS

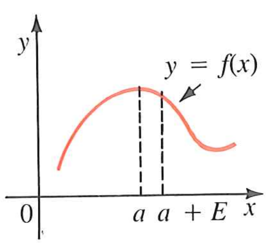


Pierre Fermat (1601–1665) was not a professional mathematician. He was a French lawyer and civil servant whose hobby was mathematics. Whenever he had some spare time, he discovered and proved mathematical theorems for pure enjoyment. But he found enough spare time to invent two of the most important areas of mathematics, analytic geometry and differential calculus, as well as to contribute to the revival of two other areas, number theory and probability theory. As an amateur, Fermat never published his discoveries, though his new results circulated in manuscript form through letters. Thus, although he invented analytic geometry at the same time as Descartes, he did not receive credit for it at the time.

Laplace called Fermat ''the true inventor of differential calculus'' in spite of the fact that Fermat did not formulate the general idea of a derivative. What Laplace meant was that Fermat's methods in solving maximum and minimum problems and tangent problems used ideas that Newton later employed in formulating derivatives.

For example, Fermat would have solved a maximum problem in the 1630s as follows. If $f(a)$ is a maximum value, then it seems intuitively clear from a picture that f changes very slowly near a.

Thus, if E is very small, then $f(a)$ and $f(a + E)$ are approximately equal:



$$f(a + E) \doteq f(a) \qquad \text{or} \qquad f(a + E) - f(a) \doteq 0$$

Dividing both sides by E,

$$\frac{f(a + E) - f(a)}{E} \doteq 0$$

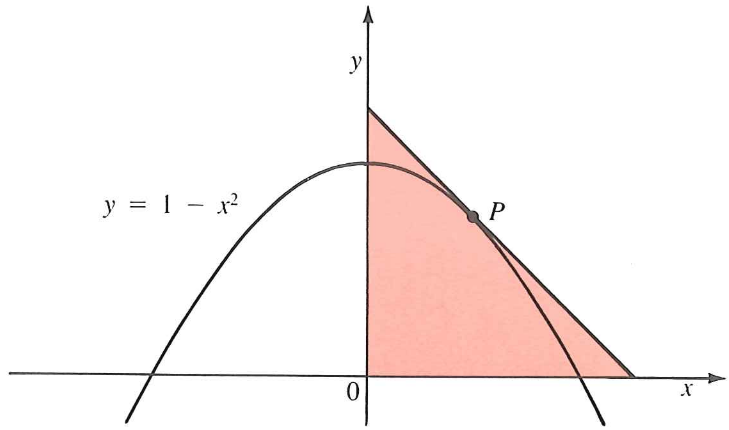
If f is a polynomial, we can carry out the division of E into $f(a + E) - f(a)$. Fermat then set $E = 0$ and solved the resulting equation for a. We see that, for polynomials, Fermat's method is equivalent to finding

$$\lim_{E\to 0} \frac{f(a + E) - f(a)}{E}$$

and equating it to 0. Fermat did not use limits, but essentially his method amounts to setting $f'(a) = 0$ as we would today.

PROBLEMS PLUS

Find the point P on the parabola $y = 1 - x^2$ at which the tangent line cuts from the first quadrant the triangle with the smallest area.



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ANSWERS

CHAPTER 4 EXTREME VALUES

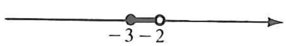
REVIEW AND PREVIEW TO CHAPTER 4

EXERCISE 1

1.

(a) $\{x \mid -2 < x < 6\}$

(b) $\{x \mid -3 \leq x < -2\}$

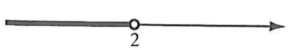


(c) $\{x \mid 1 < x \leq 4\}$

(d) $\{x \mid -2 \leq x \leq 1.5\}$

(e) $\{x \mid x \geq 3\}$

(f) $\{x \mid x < 2\}$

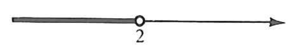


(g) $\{x \mid x \leq 1\}$

(h) $\left\{x \mid x > -\frac{3}{2}\right\}$

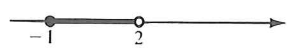
2.

(a) $(-\infty, 2)$



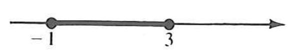
(b) $(0, 3)$

(c) $[-1, 2)$



(d) $(1, \infty)$

(e) $[-1, 3]$



(f) $(-\infty, -1]$

EXERCISE 2

1. (a) $\left(-\frac{7}{3}, \infty\right)$ (b) $\left(\frac{9}{2}, \infty\right)$ (c) $(-\infty, 2]$
(d) $(4, \infty)$ (e) $(-\infty, 10)$ (f) $(-\infty, 1) \cup (6, \infty)$
(g) $(-4, 3)$ (h) $(0, 3)$ (i) $(-\infty, -3) \cup (3, \infty)$

(j) $[-\sqrt{5},\sqrt{5}]$ (k) $\left(-1,-\frac{1}{2}\right) \cup (6,\infty)$

(l) $(-\infty,-5) \cup (0,2)$ (m) $(-\infty,1)$
(n) $(-3,-2) \cup (3,\infty)$ (o) $[2,\infty)$ (p) $(0,\infty)$

2. (a) $\left(-\frac{1}{2},\infty\right)$ (b) $(-\infty,-2) \cup (3,\infty)$

(c) $(-\infty,-1) \cup (0,1)$
(d) $(-\infty,-1) \cup (-1,0)$

EXERCISE 4.1

1. (a) increases on $(-5,0)$ and $(2,5)$, decreases on $(0,2)$ (b) increases on $(-4,3)$, decreases on $(-6,-4)$ and $(3,5)$

2. (a) $\left(-\infty,\frac{1}{2}\right)$ (b) $(0,\infty)$ (c) $(-\infty,-1)$, $(1,\infty)$

(d) $(-\infty,0)$, $(1,\infty)$ (e) $(-2,0)$, $(1,\infty)$
(f) $(-\infty,\infty)$

3. (a) $\left(-\frac{2}{3},0\right)$ (b) $(-2,3)$ (c) $(-\infty,-1)$, $(0,1)$

(d) $(-\infty,-1)$

4. (a) increases on $(3,\infty)$, decreases on $(-\infty,3)$
(b) increases on $(-\infty,-2)$, $(5,\infty)$, decreases on $(-2,5)$ (c) increases on $(-1,0)$, $(1,\infty)$, decreases on $(-\infty,-1)$, $(0,1)$
(d) increases on $(-1,2)$, $(3,\infty)$, decreases on $(-\infty,-1)$, $(2,3)$ (e) increases on $\left(-\infty,\frac{3}{7}\right)$, $(1,\infty)$, decreases on $\left(\frac{3}{7},1\right)$

(f) increases on $(-\infty,-1)$, $(-1,\infty)$
(g) increases on $\left(-\infty,\frac{8}{3}\right)$, decreases on $\left(\frac{8}{3},4\right)$

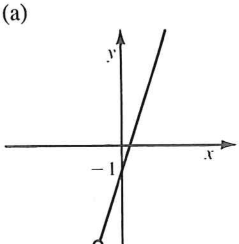
(h) increases on $(-3,0)$, $(3,\infty)$, decreases on $(-\infty,-3)$, $(0,3)$

5. $\left(\dfrac{-1-\sqrt{5}}{2},\dfrac{-1+\sqrt{5}}{2}\right)$

EXERCISE 4.2

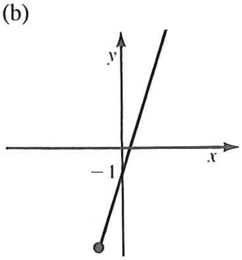
1 (i) (a) $f(7) = 5$ (b) $f(2) = -2$ (c) $f(0) = 2$, $f(4) = 3$ (d) $f(2) = -2$, $f(6) = 1$
(ii) (a) $f(4) = 5$ (b) $f(-3) = -2$
(c) $f(1) = 4$, $f(4) = 5$ (d) $f(2) = 2$, $f(6) = 1$

2. (a)



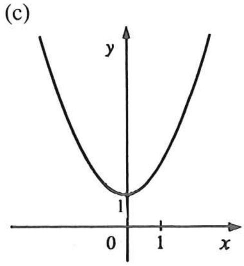
no maximum or minimum

(b)



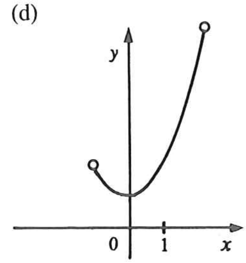
absolute minimum
$g(-1) = -4$, no maximum

(c)



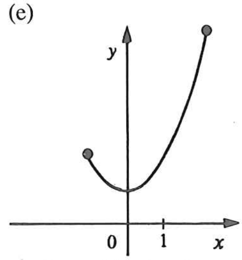
absolute and local minimum
$f(0) = 1$ no maximum

(d)



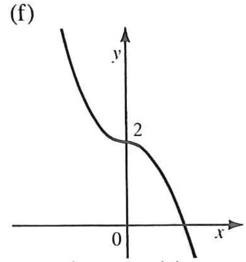
absolute and local minimum
$f(0) = 1$ no maximum

(e)



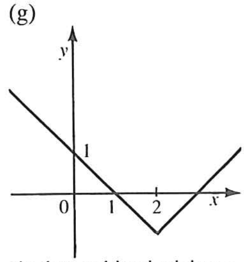
absolute and local minimum
$f(0) = 1$
absolute maximum $f(2) = 5$

(f)



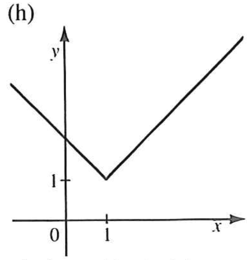
no maximum or minimum

(g)



absolute and local minimum
$f(2) = -1$ no maximum

(h)



absolute and local minimum
$f(1) = 1$, no maximum

3. (a) $\frac{1}{4}$ (b) ± 1 (c) $-1, 0, 4$ (d) $-1, 2, 3$

(e) $\frac{1}{2}(-1 \pm \sqrt{5})$ (f) none (g) -6 (h) 0

(i) $0, \frac{1}{4}$ (j) $1, \frac{2}{3}$ (k) -1 (l) ± 1

4. (a) $f(0) = 1$, $f(2) = -7$ (b) $f(2) = 21$, $f(-1) = 3$ (c) $f(2) = 4$, $f(-2) = -28$
(d) $f(-2) = 106$, $f(3) = -19$
(e) $f(-3) = 79$, $f(1) = f(-1) = 15$
(f) $f(2) = 58$, $f(-1) = -5$
(g) $g(4) = 20$, $g(2) = 12$
(h) $f(1) = 1$, $f(3) = 3\sqrt[3]{9} - 6$
(i) $f(\pm 6) = 9$, $f(\pm 3) = 0$
(j) $f(2) = 2$, $f\left(\frac{1}{2}\right) = -1$

6. -8 **7.** $a = -3$, $b = -120$

8. (a) -1.000, 1.353 (b) -15.14

EXERCISE 4.3

1. (a) local minimum $f\left(\frac{2}{3}\right) = \frac{35}{3}$

(b) local maximum $f(-2) = 11$, local minimum $f(2) = -21$

(c) local maximum $f(1) = 6$, local minimum $f(-1) = -2$

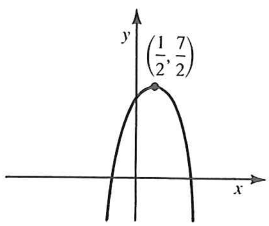
(d) local minimum $f\left(\frac{3}{4}\right) = -\frac{27}{256}$

2. (a) critical number $\frac{1}{2}$

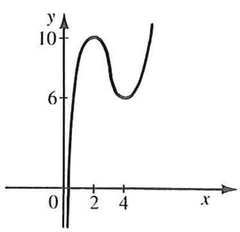
increases on $\left(-\infty, \frac{1}{2}\right)$

decreases on $\left(\frac{1}{2}, \infty\right)$

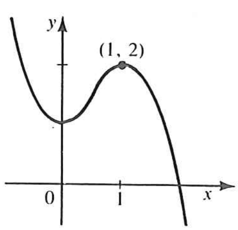
local maximum $f\left(\frac{1}{2}\right) = \frac{7}{2}$



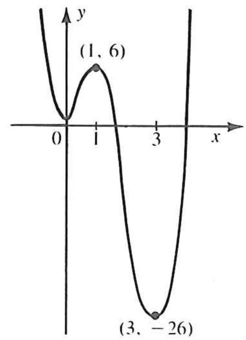
(b) critical numbers 2, 4
increases on $(-\infty, 2)$, $(4, \infty)$
decreases on $(2, 4)$
local maximum $f(2) = 10$
local minimum $f(4) = 6$



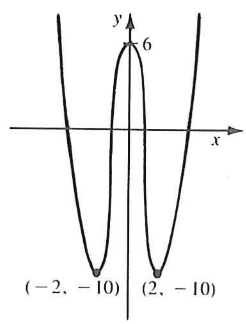
(c) critical numbers 0, 1
increases on $(0, 1)$
decreases on $(-\infty, 0)$, $(1, \infty)$
local maximum $g(1) = 2$
local minimum $g(0) = 1$



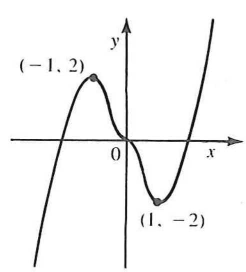
(d) critical numbers 0, 1, 3
increases on $(0, 1)$, $(3, \infty)$
decreases on $(-\infty, 0)$, $(1, 3)$
local maximum $g(1) = 6$
local minima $g(0) = 1$, $g(3) = -26$



(e) critical numbers -2, 0 2
increases on $(-2, 0)$, $(2, \infty)$
decreases on $(-\infty, -2)$, $(0, 2)$
local maximum $h(0) = 6$
local minima $h(\pm 2) = -10$



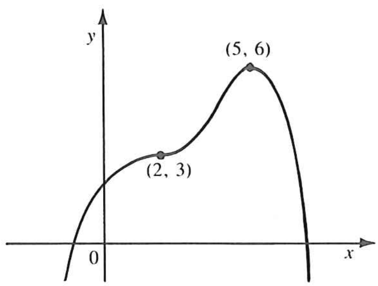
(f) critical numbers -1, 0, 1
increases on $(-\infty, -1)$, $(1, \infty)$
decreases on $(-1, 1)$
local maximum $h(-1) = 2$
local minimum $h(1) = -2$



3. (a) local maximum $f\left(\frac{1}{8}\right) = \frac{1}{2}$, local minimum $f(0) = 0$ (b) local maximum $f(0) = 0$
(c) local maximum $f\left(\frac{8}{3}\right) = \frac{16}{9}\sqrt{3}$
(d) local maximum $f\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$, local minimum $f\left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}$

4. (a) maximum $f\left(\frac{1}{2}\right) = \frac{109}{4}$
(b) minimum $f(0) = 2$
(c) minimum $g(0) = -1$
(d) minimum $g(1) = \frac{1}{2}$

5.



6. local maximum $f(2) = 28$, local minima $f(0) = 0$, $f(3) = 27$

EXERCISE 4.4

1. 75, -75 **2.** 20, 10 **3.** both 25 cm
5. 20 cm by 20 cm by 10 cm **6.** 2 m^3
7. (a) 100 m by 200 m (b) 50 m by 300 m
8. $500\sqrt{2}$ m by $750\sqrt{2}$ m **10.** $\left(-\frac{10}{13}, \frac{2}{13}\right)$
11. $(-2, 2)$ **12.** $\sqrt[3]{\frac{500}{\pi}} \doteq 5.4$ cm
13. (a) all wire for the circle
(b) $\frac{160}{4 + \pi} \doteq 22.4$ cm for the square
14. 4 cm^2
15. Lay cable underground to a point 1165 m east of P.
16. $\frac{16}{\pi + 4} \doteq 2.24$ m
17. $\frac{16}{41}$ h after noon (about 12:23 p.m.)
18. $\frac{4\sqrt{3}}{9}\pi r^3$
19. circular track, radius $\frac{1}{2\pi}$ km
20. about 7 m

EXERCISE 4.5

1. (a) \$362.50/item, \$152.50/item (b) 2000
(c) \$292.50/item
2. (a) \$5.23/unit, \$6.10/unit (b) 2530
(c) \$5.16/unit
3. 10 000 **4.** 11 000
5. (a) $p(x) = 19 - \frac{x}{3000}$ (b) \$9.50
6. (a) $p(x) = 130 - \frac{x}{100}$ (b) \$65
7. \$800 **8.** 85

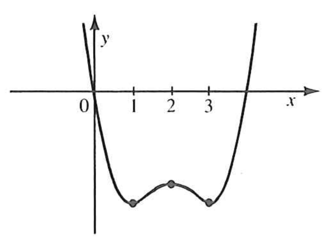
4.6 REVIEW EXERCISE

1. (a) critical numbers $\pm\frac{1}{\sqrt{3}}$, increases on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, decreases on $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$, $\left(\frac{1}{\sqrt{3}}, \infty\right)$ (b) no critical number, increases on $(-\infty, \infty)$ (c) critical numbers ± 1, 2, increases on $(-1, 1)$, $(2, \infty)$, decreases on $(-\infty, -1)$, $(1, 2)$ (d) no critical number, decreases on $\left(-\infty, \frac{1}{2}\right)$, $\left(\frac{1}{2}, \infty\right)$ (e) critical numbers -2, 0, increases on $(-\infty, -2)$, $(0, \infty)$, decreases on $(-2, -1)$, $(-1, 0)$ (f) critical numbers 0, 2, increases on $(-\infty, 0)$, $(2, \infty)$, decreases on $(0, 2)$

2. (a) $f(1) = 9$, $f\left(-\frac{3}{2}\right) = -16$ (b) $f(-3) = 86$, $f(3) = -22$ (c) $g(4) = 412$, $g(3) = -216$
(d) $g(5) = -\frac{3}{2}$, $g(2) = -3$

3. (a) local maximum $f(4) = 215$, local minimum $f(-3) = -128$ (b) local maximum $f(0) = 10$, local mimima, $f(\pm 6) = -1286$
(c) local maximum $f(0) = 4$
(d) local maximum $f(-3) = 12$

4. (a) 1, 2, 3 (b) increases on $(1, 2)$, $(3, \infty)$, decreases on $(-\infty, 1)$, $(2, 3)$
(c) local maximum $f(2) = -8$,
local minima $f(1) = f(3) = -9$
(d)



5. $f\left(-\frac{1}{2}\right) = \frac{4}{3}$ **6.** 240 cm^2

7. $\dfrac{40}{1 + \sqrt[3]{2}} \doteq 17.7$ m from dimmer light

8. $2\sqrt{2}$, $\sqrt{2}$ **9.** $\frac{21}{6}$ km from D

10. (a) 24 495 (b) 444

11. (a) $p(x) = \frac{1}{3}(400 - 2x)$ (b) \$66.67

4.7 CHAPTER 4 TEST

1. $(-1, 1)$ **2.** $f(1) = 3$, $f\left(-\frac{1}{3}\right) = \frac{31}{27}$

3. (a) $-2, 0, 2$ (b) increases on $(-2, 0)$, $(2, \infty)$, decreases on $(-\infty, -2)$, $(0, 2)$
(c) local maximum $f(0) = 3$, local minima $f(\pm 2) = -13$

4. A cube with side 10 m

5. 2000 **6.** 90