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CHAPTER 8

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

REVIEW AND PREVIEW TO CHAPTER 8

Laws of Exponents

Rational powers are defined by

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m \quad (m, n \in I, n > 0)$$

If $a > 0$, $b > 0$, and $r, s \in Q$, then

$$\boxed{\begin{aligned} a^r \times a^s &= a^{r+s} \\ \frac{a^r}{a^s} &= a^{r-s} \\ (a^r)^s &= a^{rs} \\ (ab)^r &= a^r b^r \\ \left(\frac{a}{b}\right)^r &= \frac{a^r}{b^r} \\ a^0 &= 1 \\ a^{-r} &= \frac{1}{a^r} \end{aligned}}$$

EXERCISE 1

1. Evaluate.
   (a) $(-3)^5$
   (b) 4^{-3}
   (c) $2^{-3}5^4$
   (d) $3^{-2} - (1.7)^0$
   (e) $36^{\frac{1}{2}}$
   (f) $(-64)^{\frac{1}{3}}$
   (g) $125^{\frac{2}{3}}$
   (h) $9^{-\frac{7}{2}}$

2. Write each number as a power of 2.

(a) 128 (b) $2^6 \times 8^4$

(c) $(2^9)^4$ (d) $\frac{1}{4}$

(e) $\dfrac{2^{3.1}}{2^{4.6}}$ (f) $\sqrt{2}$

(g) $4\sqrt{2}$ (h) 1

3. Simplify and leave your answer with only positive exponents.

(a) $(12x^2y^4)\left(\frac{1}{2}x^5y\right)$ (b) $(2s^3t^{-1})\left(\frac{1}{4}s^6\right)(16t^4)$

(c) $\dfrac{x^9(2x)^4}{x^3}$ (d) $\dfrac{a^{-3}b^4}{a^{-5}b^5}$

(e) $(rs)^3(2s)^{-2}(4r)^4$ (f) $(2u^2v^3)^3(3u^3v)^{-2}$

(g) $\dfrac{(x^2y^3)^4(xy^4)^{-3}}{x^2y}$ (h) $\left(\dfrac{c^4d^3}{cd^2}\right)\left(\dfrac{d^2}{c^3}\right)^3$

(i) $\dfrac{a^{-1}+b^{-1}}{(a+b)^{-1}}$ (j) $\dfrac{(y^{10}z^{-5})^{\frac{1}{5}}}{(y^{-2}z^3)^{\frac{1}{3}}}$

(k) $\dfrac{(9st)^{\frac{3}{2}}}{(27s^3t^{-4})^{\frac{2}{3}}}$ (l) $\left(\dfrac{a^2b^{-3}}{x^{-1}y^2}\right)^3\left(\dfrac{x^{-2}b^{-1}}{a^{\frac{3}{2}}y^{\frac{1}{3}}}\right)$

Logarithms

The expression $\log_b x$ is read "the logarithm to the base b of x." It is defined as follows.

$$\boxed{\log_b x = y \quad \text{if and only if} \quad b^y = x} \qquad (1)$$

In words, this says that

$$\boxed{\log_b x \text{ is the exponent to which the base } b \text{ must be raised to give } x}$$

In using (1) to switch back and forth between the logarithmic form $\log_b x = y$ and the exponential form $b^y = x$, it is helpful to notice that in both cases the base is the same:

$$\underset{\substack{\uparrow \\ \text{base}}}{\log_b} x = \overset{\substack{\text{exponent} \\ \downarrow}}{y} \quad \Leftrightarrow \quad \underset{\substack{\uparrow \\ \text{base}}}{b}^{\overset{\substack{\text{exponent} \\ \downarrow}}{y}} = x$$

Example 1 Express in exponential form.

(a) $\log_2\left(\frac{1}{2}\right) = -1$ (b) $\log_{10} 100\,000 = 5$ (c) $\log_3 z = t$

Solution (a) $\log_2\left(\frac{1}{2}\right) = -1$ (b) $\log_{10} 100\,000 = 5$ (c) $\log_3 z = t$

$2^{-1} = \frac{1}{2}$ $\qquad 10^5 = 100\,000$ $\qquad 3^t = z$

Example 2 Express in logarithmic form.

(a) $1000 = 10^3$ (b) $2^{-3} = \frac{1}{8}$ (c) $s = 5^r$

Solution (a) $10^3 = 1000$ (b) $2^{-3} = \frac{1}{8}$ (c) $5^r = s$

$\log_{10} 1000 = 3$ $\qquad \log_2\left(\frac{1}{8}\right) = -3$ $\qquad \log_5 s = r$

Example 3 Evaluate. (a) $\log_3 81$ (b) $\log_{16} 4$ (c) $\log_{10} 0.0001$

Solution (a) $\log_3 81 = 4$ because $3^4 = 81$

(b) $\log_{16} 4 = \frac{1}{2}$ because $16^{\frac{1}{2}} = 4$

(c) $\log_{10} 0.0001 = -4$ because $10^{-4} = 0.0001$

Example 4 Solve for x.

(a) $\log_2(25 - x) = 3$ (b) $3^{x+2} = 7$

Solution (a)

$$\begin{aligned} \log_2(25 - x) &= 3 \\ 2^3 &= 25 - x \\ 8 &= 25 - x \\ x &= 17 \end{aligned}$$

(b)

$$\begin{aligned} 3^{x+2} &= 7 \\ \log_3 7 &= x + 2 \\ x &= \log_3 7 - 2 \end{aligned}$$

EXERCISE 2

1. Express each equation in exponential form.
   (a) $\log_2 64 = 6$ (b) $\log_5 1 = 0$
   (c) $\log_{10} 0.01 = -2$ (d) $\log_8 4 = \frac{2}{3}$
   (e) $\log_8 512 = 3$ (f) $\log_2\left(\frac{1}{16}\right) = -4$
   (g) $\log_a b = c$ (h) $\log_r v = w$
2. Express each equation in logarithmic form.
   (a) $2^3 = 8$ (b) $10^5 = 100\,000$
   (c) $10^{-4} = 0.0001$ (d) $81^{\frac{1}{2}} = 9$
   (e) $4^{-\frac{3}{2}} = 0.125$ (f) $6^{-1} = \frac{1}{6}$
   (g) $r^s = t$ (h) $10^m = n$

3. Evaluate.

(a) $\log_6 6^4$
(b) $\log_2 32$
(c) $\log_4 64$
(d) $\log_8 8^{17}$
(e) $\log_9 9$
(f) $\log_6 1$
(g) $\log_3\left(\frac{1}{27}\right)$
(h) $\log_4 8$
(i) $\log_8 0.25$
(j) $\log_9 \sqrt{3}$

4. Solve each equation for x.

(a) $\log_2 x = 10$
(b) $\log_5 x = 4$
(c) $\log_{10}(3x + 5) = 2$
(d) $\log_3(2 - x) = 3$
(e) $2^{1-x} = 3$
(f) $3^{2x-1} = 5$
(g) $\log_2(\log_3 x) = 4$
(h) $10^{5^x} = 3$

Laws of Logarithms

Suppose that $x > 0$, $y > 0$, and r is any rational number. Then

1. $\log_b(xy) = \log_b x + \log_b y$
2. $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$
3. $\log_b(x^r) = r \log_b x$

Example 1 Use the Laws of Logarithms to rewrite the following.

(a) $\log_2(6x)$ (b) $\log_5 x^3y^6$ (c) $\log_{10} \dfrac{ab}{\sqrt[3]{c}}$

Solution (a) $\log_2(6x) = \log_2 6 + \log_2 x$

(b) $\log_5(x^3y^6) = \log_5 x^3 + \log_5 y^6 = 3 \log_5 x + 6 \log_5 y$

(c)
$$\log_{10} \frac{ab}{\sqrt[3]{c}} = \log_{10} ab - \log_{10} \sqrt[3]{c}$$
$$= \log_{10} a + \log_{10} b - \log_{10} c^{\frac{1}{3}}$$
$$= \log_{10} a + \log_{10} b - \tfrac{1}{3} \log_{10} c$$

Example 2 Express $3 \log_2 s + \frac{1}{2} \log_2 t - 4 \log_2(t^2 + 1)$ as a single logarithm.

Solution
$$3 \log_2 s + \tfrac{1}{2} \log_2 t - 4 \log_2(t^2 + 1) = \log_2 s^3 + \log_2 t^{\frac{1}{2}} - \log_2(t^2 + 1)^4$$
$$= \log_2(s^3 t^{\frac{1}{2}}) - \log_2(t^2 + 1)^4$$
$$= \log_2\left(\frac{s^3 \sqrt{t}}{(t^2 + 1)^4}\right)$$

EXERCISE 3

1. Use the Laws of Logarithms to rewrite each expression in a form with no logarithms of products, quotients, or powers.

(a) $\log_2 x(x - 1)$
(b) $\log_5\left(\frac{x}{2}\right)$
(c) $\log_2(AB^2)$
(d) $\log_6 \sqrt[4]{17}$
(e) $\log_3(x\sqrt{y})$
(f) $\log_2(xy)^{10}$
(g) $\log_5 \sqrt[3]{x^2 + 1}$
(h) $\log_b \frac{x^2}{yz^3}$
(i) $\log_{10} \frac{x^3y^4}{z^6}$
(j) $\log_{10} \frac{a^2}{b^4 \sqrt{c}}$

2. Evaluate.

(a) $\log_5 \sqrt{125}$
(b) $\log_2 112 - \log_2 7$
(c) $\log_{10} 2 + \log_{10} 5$
(d) $\log_{10} \sqrt{0.1}$
(e) $\log_4 192 - \log_4 3$
(f) $\log_{12} 9 + \log_{12} 16$

3. Rewrite each expression as a single logarithm.

(a) $\log_{10} 12 + \frac{1}{2} \log_{10} 7 - \log_{10} 2$
(b) $\log_2 A + \log_2 B - 2 \log_2 C$
(c) $\log_5(x^2 - 1) - \log_5(x - 1)$
(d) $4 \log_2 x - \frac{1}{3} \log_2(x^2 + 1) + \log_2(x - 1)$
(e) $\frac{1}{2}[\log_5 x + 2 \log_5 y - 3 \log_5 z]$
(f) $\log_a b + c \log_a d - r \log_a s$

INTRODUCTION

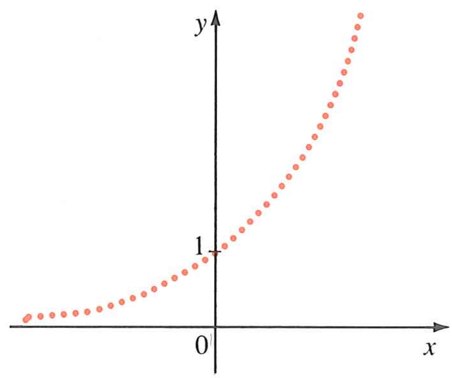
Some of the most important applications of calculus require the exponential function and its inverse function, the logarithmic function. In this chapter we learn how to differentiate these functions so that we can apply them to solve growth and decay problems.

8.1 EXPONENTIAL FUNCTIONS

An **exponential function** is a function of the form

$$f(x) = b^x$$

where the base b is a positive constant. We have defined b^x if x is a rational number, but what is meant by an irrational power such as $2^{\sqrt{3}}$? To help us answer this question we first look at the graph of the function $y = 2^x$, where x is rational. A representation of this graph is shown.



$y = 2^x$, $x \in Q$

We want to enlarge the domain of $y = 2^x$ to include both rational and irrational numbers. We want to fill in the holes in the graph by defining $f(x) = 2^x$, where $x \in R$, so that f is a continuous, increasing function.

In particular, since

$$1.7 < \sqrt{3} < 1.8$$

we must have

$$2^{1.7} < 2^{\sqrt{3}} < 2^{1.8}$$

Similarly, using better approximations for $\sqrt{3}$, we obtain better approximations for $2^{\sqrt{3}}$:

$$
\begin{array}{rcl}
1.73 < \sqrt{3} < 1.74 & \text{so} & 2^{1.73} < 2^{\sqrt{3}} < 2^{1.74} \\
1.732 < \sqrt{3} < 1.733 & \text{so} & 2^{1.732} < 2^{\sqrt{3}} < 2^{1.733} \\
1.732\,0 < \sqrt{3} < 1.732\,1 & \text{so} & 2^{1.732\,0} < 2^{\sqrt{3}} < 2^{1.732\,1} \\
1.732\,05 < \sqrt{3} < 1.732\,06 & \text{so} & 2^{1.732\,05} < 2^{\sqrt{3}} < 2^{1.732\,06}
\end{array}
$$

It can be shown that there is exactly one number that is greater than all of the numbers

$$2^{1.7},\ 2^{1.73},\ 2^{1.732},\ 2^{1.732\,0},\ 2^{1.732\,05},\ \ldots$$

and less than all of the numbers

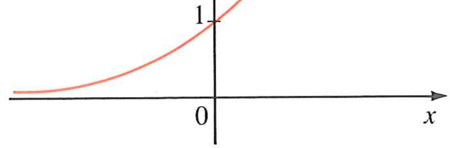
$$2^{1.8},\ 2^{1.74},\ 2^{1.733},\ 2^{1.732\,1},\ 2^{1.732\,06},\ \ldots$$

We define $2^{\sqrt{3}}$ to be this number. Using the above approximation process we can compute it correct to 6 decimal places:

$$2^{\sqrt{3}} \doteq 3.321\,997$$

In general, if x is any irrational number we can define 2^x (or b^x, if $b > 0$) in a similar manner as a limit of approximations. It can be proved that the Laws of Exponents are still true when the exponents are real numbers.

The graph of $f(x) = 2^x$, where $x \in R$, is shown. It is a continuous, increasing function.



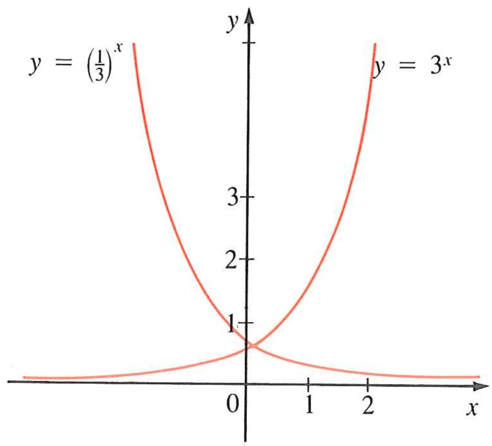
$y = 2^x,\ x \in R$

Example 1 Draw the graphs of the following functions.

(a) $f(x) = 3^x$ (b) $g(x) = \left(\frac{1}{3}\right)^x$

Solution We calculate values of $f(x)$ and $g(x)$, plot points, and join them to sketch the continuous graphs as shown.

x	$f(x) = 3^x$	$g(x) = \left(\frac{1}{3}\right)^x$
-3	$\frac{1}{27}$	27
-2	$\frac{1}{9}$	9
-1	$\frac{1}{3}$	3
0	1	1
1	3	$\frac{1}{3}$
2	9	$\frac{1}{9}$
3	27	$\frac{1}{27}$

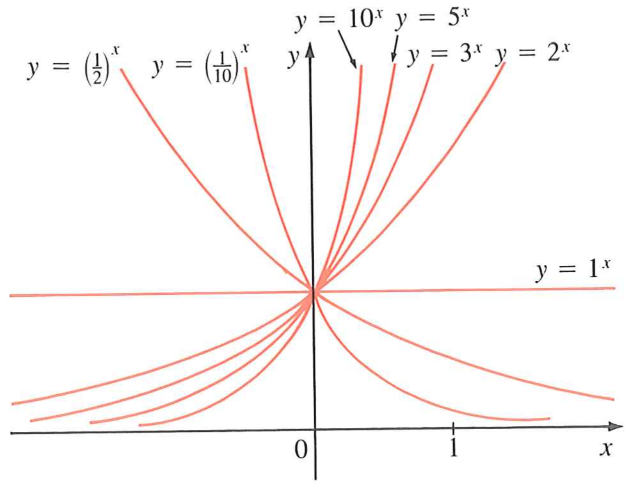


Notice that

$$g(x) = \left(\frac{1}{3}\right)^x = \frac{1}{3^x} = 3^{-x} = f(-x)$$

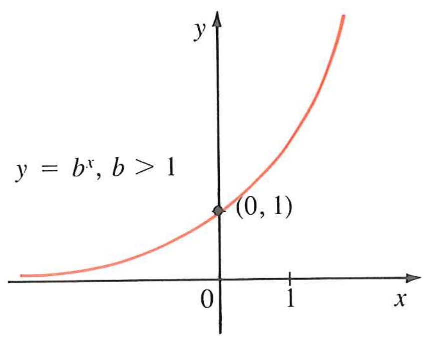
and so the graph of g could have been obtained from the graph of f by reflecting in the y-axis.

The graphs of the exponential functions $f(x) = b^x$ are shown for various values of the base b. Notice that all of these graphs pass through the same point $(0, 1)$ because $b^0 = 1$ for $b \neq 0$.



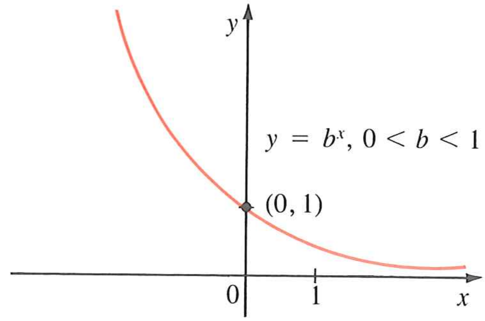
We see from these graphs that there are basically two kinds of exponential functions $y = b^x$, apart from the constant function $y = 1^x = 1$.

If $b > 1$, the function $y = b^x$ increases rapidly for x positive, and the larger the base the more rapid the increase. Notice that the graph approaches zero as x decreases through negative values, so the x-axis is a horizontal asymptote.



$$\boxed{\text{If } b > 1, \text{ then } \lim_{x \to -\infty} b^x = 0 \quad \text{and} \quad \lim_{x \to \infty} b^x = \infty.} \qquad (1)$$

If $0 < b < 1$, the exponential function $y = b^x$ is decreasing and approaches zero as x becomes large. Again the x-axis is a horizontal asymptote.



$$\boxed{\text{If } 0 < b < 1, \text{ then } \lim_{x \to -\infty} b^x = \infty \quad \text{and} \quad \lim_{x \to \infty} b^x = 0.}$$

In both cases, the graph never touches the x-axis since $b^x > 0$ for all x. Thus, for $b \neq 1$, the exponential function $f(x) = b^x$ has domain R and range $(0, \infty)$.

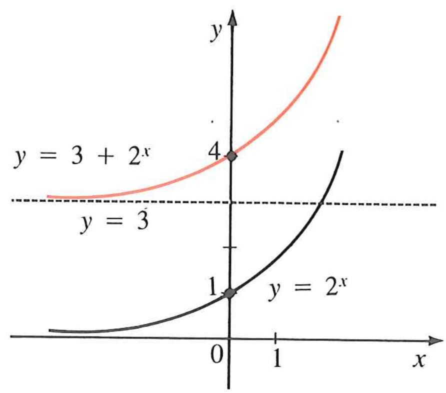
In the next two examples we show how to graph certain functions, not by plotting points but by taking the basic graphs of the exponential functions and applying shifting and reflecting transformations.

Example 2 Use the graph of $y = 2^x$ to sketch the graphs of the following functions.
(a) $y = 3 + 2^x$ (b) $y = -2^x$

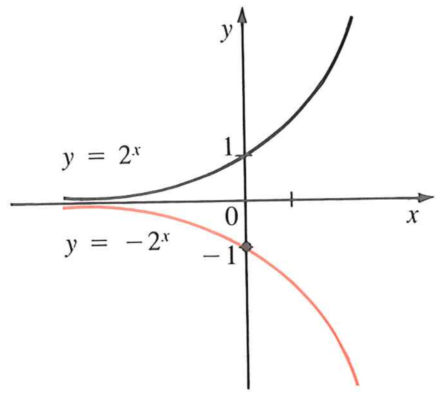
Solution (a) The graph of $y = 3 + 2^x$ is obtained by starting with the graph of $y = 2^x$ and shifting it three units upward.

We see from the graph that the line $y = 3$ is a horizontal asymptote. This can also be seen from the following limit.

$$\lim_{x \to -\infty} (3 + 2^x) = \lim_{x \to -\infty} 3 + \lim_{x \to -\infty} 2^x = 3 + 0 = 3$$

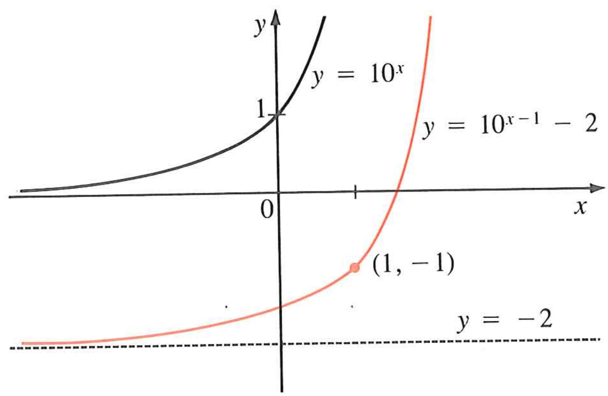


(b) Again we start with the graph of $y = 2^x$, but here we reflect in the x-axis to get the graph of $y = -2^x$. The horizontal asymptote is $y = 0$.



Example 3 (a) Use the graph of $y = 10^x$ to sketch the graph of $y = 10^{x-1} - 2$.
(b) State the asymptote, the domain, and the range of this function.

Solution (a) Recall that we get the graph of $y = f(x - c)$ from the graph of $y = f(x)$ by shifting c units to the right. Thus, we get the graph of $y = 10^{x-1} - 2$ by shifting the graph of $y = 10^x$ one unit to the right and two units downward as shown.



(b) We see from the graph that the horizontal asymptote is $y = -2$. This also follows by computing the following limit.

$$\begin{aligned} \lim_{x \to -\infty} (10^{x-1} - 2) &= \lim_{x \to -\infty} 10^{x-1} - \lim_{x \to -\infty} 2 \\ &= 0 - 2 \qquad (x - 1 \to -\infty \text{ as } x \to -\infty) \\ &= -2 \end{aligned}$$

The domain is R and the range is $\{y \mid y > -2) = (-2, \infty)$

Example 4 Find $\lim_{x \to 3^-} 2^{\frac{1}{x-3}}$

Solution 1 When x is slightly less than 3, $x - 3$ is a small negative number and so $\dfrac{1}{x-3}$ is a large negative number. From ①, or the graph of $y = 2^x$, we see that $2^{\frac{1}{x-3}}$ is small number. Thus we have

x	$2^{\frac{1}{x-3}}$
2.5	0.250 00
2.8	0.031 25
2.9	0.000 98

$$\lim_{x \to 3^-} 2^{\frac{1}{x-3}} = 0$$

Solution 2 A more systematic method is to introduce a new variable t for the exponent and compute its limit first. Let

$$t = \frac{1}{x-3}$$

As $x \to 3^-$, we have $x - 3 \to 0$ and $x - 3$ is negative, so $t \to -\infty$. Thus, by ①, we have

$$\lim_{x \to 3^-} 2^{\frac{1}{x-3}} = \lim_{t \to -\infty} 2^t = 0$$

EXERCISE 8.1

B **1.** Sketch the graph of each function by making a table of values, using a calculator if necessary.

(a) $f(x) = 6^x$ (b) $f(x) = \left(\frac{3}{2}\right)^x$

(c) $g(x) = \left(\frac{1}{4}\right)^x$ (d) $h(x) = (1.1)^x$

2. Use a table of values to graph the functions $y = 4^x$ and $y = 7^x$ using the same axes.

3. Use a table of values to graph the functions $y = \left(\frac{2}{3}\right)^x$ and $y = \left(\frac{4}{3}\right)^x$ using the same axes.

4. Graph the given function, not by plotting points but by starting from the graphs of $y = 2^x$, 10^x, $\left(\frac{1}{2}\right)^x$, and $\left(\frac{1}{10}\right)^x$ given in this section and using transformations. State the domain, range, and asymptote of each function.

(a) $f(x) = -10^x$ (b) $f(x) = 10^{-x}$

(c) $g(x) = 2^x - 5$ (d) $g(x) = 2^{x-5}$

(e) $y = 3 + \left(\frac{1}{2}\right)^x$ (f) $y = 4 - 2^x$

(g) $y = 10^{x+3}$ (h) $y = -\left(\frac{1}{10}\right)^x$

(i) $y = 2^{-2x}$ (j) $y = 1 + 2^{x+1}$

(k) $y = 5 - 2^{x-1}$ (l) $y = 1 + 3(1 - 10^{-x})$

5. Evaluate.

(a) $\lim_{x \to -\infty} 4^x$ (b) $\lim_{x \to \infty} (0.9)^x$

(c) $\lim_{x \to \infty} 10^{2x-1}$ (d) $\lim_{x \to \infty} 3^{-x}$

(e) $\lim_{x \to 0+} 5^{\frac{1}{x}}$ (f) $\lim_{x \to 0^-} 5^{\frac{1}{x}}$

(g) $\lim_{x \to \infty} 10^{-x^2}$ (h) $\lim_{x \to \infty} 4^{\frac{1}{x}}$

(i) $\lim_{x \to -1+} 8^{\frac{x}{x+1}}$ (j) $\lim_{t \to 0^-} 2^{\csc t}$

6. (a) Compare the functions $f(x) = x^2$ and $g(x) = 2^x$ by evaluating each of them for $x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 15$, and 20.

(b) Draw the graphs of f and g for $-4 \leq x \leq 6$ using the same set of axes.

7. If $f(x) = 10^x$, show that

$$\frac{f(x+h) - f(x)}{h} = 10^x\left(\frac{10^h - 1}{h}\right)$$

8. Sketch the graphs of the following functions.

(a) $y = 10^{|x|}$ (b) $y = 10^{-|x|}$

8.2 DERIVATIVES OF EXPONENTIAL FUNCTIONS

Let us try to compute the derivative of the exponential function $f(x) = b^x$ using the definition of a derivative:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} \\ &= \lim_{h \to 0} \frac{b^x b^h - b^x}{h} \\ &= \lim_{h \to 0} \frac{b^x(b^h - 1)}{h} \\ f'(x) &= b^x \lim_{h \to 0} \frac{b^h - 1}{h} \end{aligned} \qquad (1)$$

Notice that we were able to put the factor b^x in front of the limit because it does not depend on h.

The following table helps us to estimate the limit in (1) for $b = 2$ and $b = 3$. (Values are given correct to four decimal places.)

h	$\dfrac{2^h - 1}{h}$	$\dfrac{3^h - 1}{h}$
0.1	0.7177	1.1612
0.01	0.6956	1.1047
0.001	0.6934	1.0992
0.0001	0.6932	1.0987

It appears that

$$\lim_{h \to 0} \frac{2^h - 1}{h} \doteq 0.69 \quad \text{and} \quad \lim_{h \to 0} \frac{3^h - 1}{h} \doteq 1.1$$

and so, from ①,

$$\frac{d}{dx}(2^x) \doteq (0.69)2^x \quad \text{and} \quad \frac{d}{dx}(3^x) \doteq (1.1)3^x$$

This suggests that, of all possible choices for the base b in ①, the simplest differentiation formula occurs when

$$\lim_{h \to 0} \frac{b^h - 1}{h} = 1$$

In view of the estimates of the limit for $b = 2$ and $b = 3$, it seems reasonable that there is a number b for which the limit is 1, and that it lies between 2 and 3. It is traditional to denote this value by the letter e. Thus

e is the number such that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

In Section 8.4 we will see that, correct to five decimal places,

$$e \doteq 2.718\ 28$$

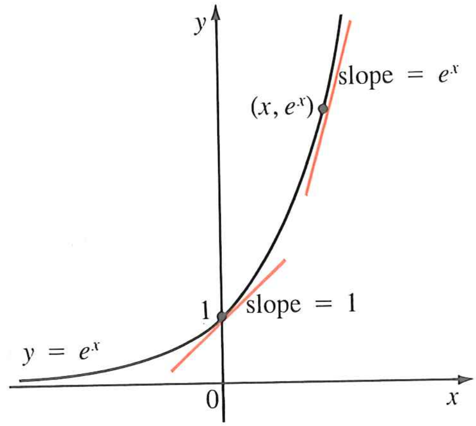
If we put $b = e$ in Formula 1, we have the following simplified formula.

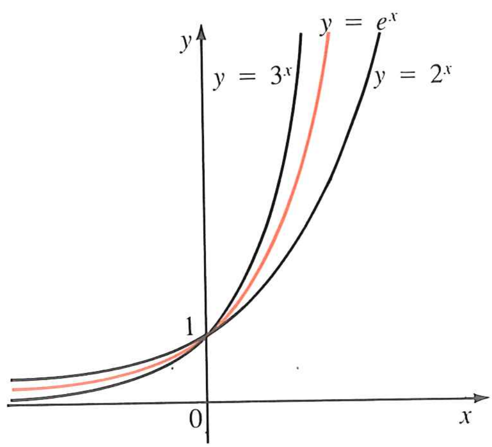
If $f(x) = e^x$, then $f'(x) = e^x$. ②

In Leibniz notation, we have

$$\frac{d}{dx} e^x = e^x$$

Thus the exponential function $f(x) = e^x$ has the property that it is its own derivative. The geometrical significance of this fact is that the slope of a tangent line to the curve $y = e^x$ is equal to the y-coordinate of the point. In particular, if $f(x) = e^x$, then $f'(0) = e^0 = 1$. This means that of all the possible exponential functions $y = b^x$, $y = e^x$ is the one that crosses the y-axis with a slope of 1.





Example 1 Differentiate. (a) $y = x^2e^x$ (b) $y = e^{\sin x}$

Solution (a) Using the Product Rule, we have

$$\begin{aligned}\frac{dy}{dx} &= x^2\frac{d}{dx}e^x + e^x\frac{d}{dx}x^2\\ &= x^2e^x + e^x(2x)\\ &= x(x+2)e^x\end{aligned}$$

(b) To use the Chain Rule, we let $u = \sin x$. Then we have $y = e^u$, so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u \frac{du}{dx} = e^{\sin x} \cos x$$

In general if we combine Formula 2 with the Chain Rule, as in Example 1(b), we get

$$\boxed{\frac{d}{dx} e^u = e^u \frac{du}{dx}} \qquad \text{or} \qquad \boxed{\frac{d}{dx} e^{g(x)} = e^{g(x)} g'(x)}$$

Example 2 Find y' if $y = e^{-3x} \cos 2x$.

Solution

$$\begin{aligned} y' &= e^{-3x} \frac{d}{dx} \cos 2x + \cos 2x \frac{d}{dx} e^{-3x} \\ &= e^{-3x}(-\sin 2x)(2) + (\cos 2x)(e^{-3x})(-3) \\ &= -e^{-3x}(2 \sin 2x + 3 \cos 2x) \end{aligned}$$

Example 3 Find the absolute maximum value of the function $f(x) = xe^{-x}$.

Solution We differentiate to find any critical numbers.

$$f'(x) = xe^{-x}(-1) + e^{-x}(1) = e^{-x}(1 - x)$$

Since exponential functions are always positive, we see that $f'(x) > 0$ when $1 - x > 0$, that is, $x < 1$. Similarly, $f'(x) < 0$ when $x > 1$. By the First Derivative Test for Absolute Extreme Values, f has an absolute maximum value when $x = 1$ and the value is

$$f(1) = (1)e^{-1} = \frac{1}{e} \doteq 0.37$$

Example 4 Sketch the graph of $f(x) = e^{-x^2}$.

Solution We use the headings of Section 5.5.

A. ***Domain.*** The function is defined for all values of x, so the domain is R.

B. ***Intercepts.*** Exponential functions are never 0, so there is no x-intercept. The y-intercept is $f(0) = 1$.

C. ***Symmetry.*** $f(-x) = e^{-x^2} = f(x)$, so f is even and the curve is symmetric about the y-axis.

D. ***Asymptotes.*** As $x \to \infty$ or $x \to -\infty$, we have $-x^2 \to -\infty$, so

$$\lim_{x \to \pm\infty} e^{-x^2} = 0$$

Therefore $y = 0$ is a horizontal asymptote.

E. ***Intervals of Increase or Decrease.*** The derivative of f is

$$f'(x) = -2xe^{-x^2}$$

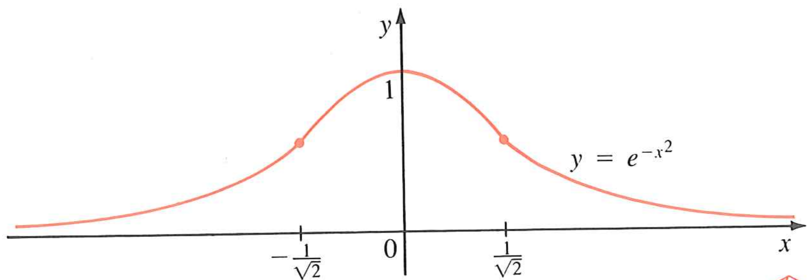
Exponential functions are always positive, so $e^{-x^2} > 0$ for all x. Therefore $f'(x) > 0$ when $x < 0$, so f is increasing on $(-\infty, 0)$. Since $f'(x) < 0$ when $x > 0$, f is decreasing on $(0, \infty)$.

F. ***Extreme Values.*** The only critical number is 0. By the First Derivative Test, $f(0) = 1$ is a local (and absolute) maximum.

G. ***Concavity.***
$$\begin{aligned} f''(x) &= -2x(e^{-x^2})(-2x) + e^{-x^2}(-2) \\ &= -2e^{-x^2}(1 - 2x^2) \end{aligned}$$

Thus $f''(x) > 0$ when $1 - 2x^2 < 0$, so $x^2 > \frac{1}{2}$, or $|x| > \dfrac{1}{\sqrt{2}}$. Therefore f is concave upward on $\left(-\infty, -\dfrac{1}{\sqrt{2}}\right)$ and $\left(\dfrac{1}{\sqrt{2}}, \infty\right)$. Also $f''(x) < 0$ when $1 - 2x^2 > 0$ or $|x| < \dfrac{1}{\sqrt{2}}$. Thus f is concave downward on $\left(-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$. The inflection points are $\left(\pm\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{e}}\right)$.

H. ***Sketch of the Curve.***



EXERCISE 8.2

A **1.** Simplify.

(a) $\dfrac{2}{e^{-x}}$ (b) $(e^x)^4$

(c) $e^{1-x}e^{3x}$ (d) $e^x e^{-x}$

(e) $e^{2x}(1 - 5e^{3x})$ (f) $\dfrac{6e^{8x}}{e^{3x}}$

B **2.** (a) Sketch the curve $y = 5^x$.
(b) Use a calculator to evaluate the quantity
$$\frac{5^h - 1}{h}$$
for $h = 0.1, 0.01, 0.001$, and 0.0001. What does the quantity represent?

(c) Estimate the value of the limit

$$\lim_{h\to 0} \frac{5^h - 1}{h}$$

correct to two decimal places.

(d) What does the limit in part (c) represent?

3. Use a calculator to estimate the values of the limits

(a) $\lim_{h\to 0} \frac{2.7^h - 1}{h}$ (b) $\lim_{h\to 0} \frac{2.8^h - 1}{h}$

to two decimal places.

4. Differentiate.

(a) $y = 2e^{-x}$ (b) $y = x^4e^x$

(c) $y = e^{2x} \sin 3x$ (d) $y = e^{\sqrt{x}}$

(e) $y = e^{\tan x}$ (f) $y = \tan(e^x)$

(g) $y = \frac{e^x}{x}$ (h) $y = \frac{e^x}{1 - e^{2x}}$

(i) $y = e^{\sin(x^2)}$ (j) $y = xe^{\cot 4x}$

(k) $y = (1 + 5e^{-10x})^4$ (l) $y = \sqrt{x + e^{1-x^2}}$

5. Find the equation of the tangent line to the curve $y = 1 + xe^{2x}$ at the point where $x = 0$.

6. Find y' if $e^{xy} = 2x + y$.

7. If $f(x) = e^{2x}$, find $f^{(6)}(0)$.

8. Find the intervals of increase and decrease for the function $f(x) = x^2e^{-x}$.

9. Find the absolute minimum value of the function $f(x) = \frac{e^x}{x}$, $x > 0$.

10. For the function $f(x) = xe^x$, find

(a) the absolute minimum value,

(b) the intervals of concavity,

(c) the inflection point.

11. Evaluate.

(a) $\lim_{x\to\infty} e^{-x}$ (b) $\lim_{x\to-\infty} e^{-x}$ (c) $\lim_{t\to\frac{\pi}{2}^+} e^{\tan t}$

12. (a) Draw the graph of the exponential function $y = e^x$.

(b) Use the result of part (a) and transformations to graph the following functions.

(i) $y = e^{-x}$ (ii) $y = 1 - e^x$

13. Discuss each curve under the headings A–H of Section 5.5.

(a) $y = xe^{x^2}$ (b) $y = e^{\frac{1}{x^2}}$

14. (a) Sketch the graphs of the curves $y = e^x$ and $y = -x - 1$ (using the same axes) to show that there is exactly one solution of the equation $e^x = -x - 1$.

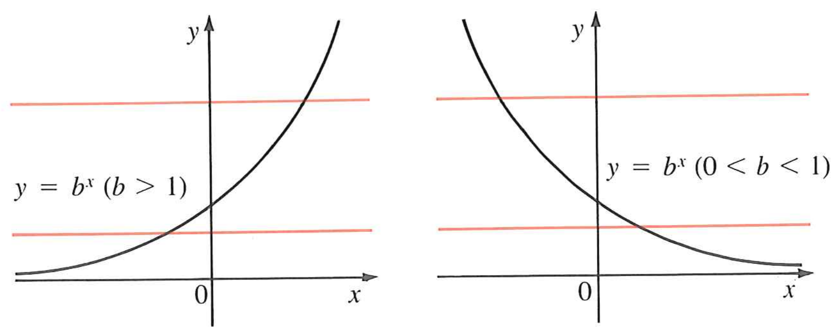
(b) Use Newton's method to find the root of the equation in part (a) correct to six decimal places.

C **15.** Discuss the curve $y = e^{-\frac{1}{x}}$ under the headings of A–H of Section 5.5.

16. Find the millionth derivative of $f(x) = xe^{-x}$.

8.3 LOGARITHMIC FUNCTIONS

If $b > 0$, where $b \neq 1$, then the exponential function $f(x) = b^x$ is a one-to-one function by the Horizontal Line Test (see the diagram) and therefore has an inverse function.



The inverse function f^{-1} of the exponential function $f(x) = b^x$ is called the **logarithmic function with base b** and is denoted by $\log_b$. Recall that the inverse function f^{-1} is defined by

$$f^{-1}(x) = y \quad \text{if and only if} \quad f(y) = x$$

Thus, the definition of the logarithmic function as the inverse of the exponential function means the following.

$$\boxed{\log_b x = y \quad \text{if and only if} \quad b^y = x} \qquad (1)$$

It follows that

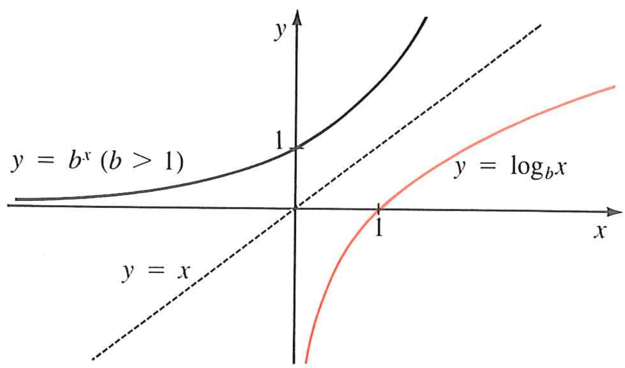
$$\boxed{\begin{aligned} \log_b(b^x) &= x \quad (x \in R) \\ b^{\log_b x} &= x \quad (x > 0) \end{aligned}} \qquad (2)$$

For instance, we have

$$\log_{10}(10^x) = x \quad \text{and} \quad 2^{\log_2 x} = x$$

Recall that if a one-to-one function f has domain A and range B, then its inverse function f^{-1} has domain B and range A. Since the exponential function $f(x) = b^x$, where $b \neq 1$, has domain R and range $(0, \infty)$, we conclude that its inverse function, $f^{-1}(x) = \log_b x$ has domain $(0, \infty)$ and range R.

The graph of $f^{-1}(x) = \log_b x$ is obtained by reflecting the graph of $f(x) = b^x$ in the line $y = x$. The diagram shows the case where $b > 1$. (The most important logarithmic functions have base $b > 1$.) The fact that $y = b^x$ $(b > 1)$ is a very rapidly increasing function for $x > 0$ is reflected in the fact that $y = \log_b x$ is a very slowly increasing function for $x > 1$.



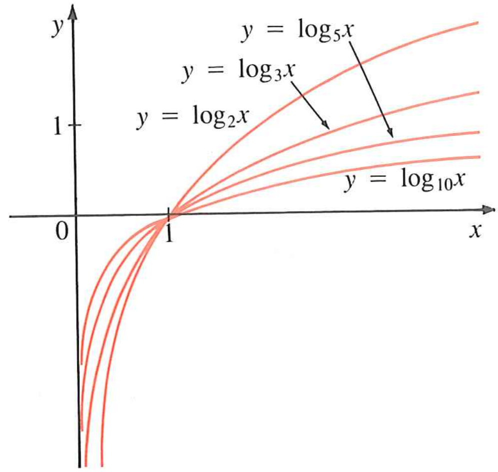
Notice that since $b^0 = 1$, we have

$$\boxed{\log_b 1 = 0}$$

and so the x-intercept of the function $y = \log_b x$ is 1. Notice also that since $y = b^x$ has the x-axis as a horizontal asymptote, the curve $y = \log_b x$ has the y-axis as a vertical asymptote. In fact, we have

$$\boxed{\lim_{x \to 0^+} \log_b x = -\infty \quad \text{and} \quad \lim_{x \to \infty} \log_b x = \infty \text{ for } b > 1} \qquad (3)$$

The diagram shows the relationship among the graphs of the logarithmic functions with bases 2, 3, 5, and 10. These graphs were drawn by reflecting the graphs of $y = 2^x$, $y = 3^x$, $y = 5^x$, and $y = 10^x$ (see Section 8.1) in the line $y = x$.



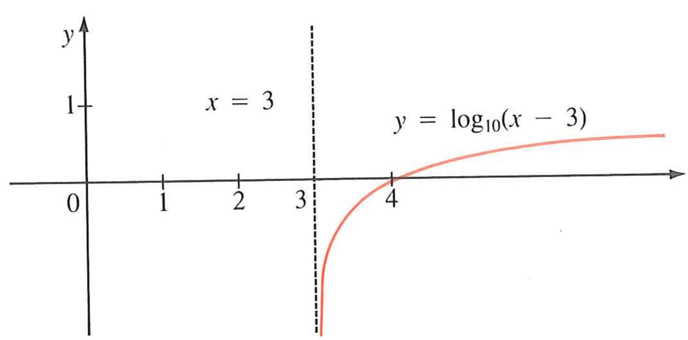
Example 1 Find the domain of the function $f(x) = \log_{10}(x - 3)$ and sketch its graph.

Solution The domain of $y = \log_b x$ is the interval $(0, \infty)$, so $\log_{10} x$ is defined only when $x > 0$. Therefore the domain of $f(x) = \log_{10}(x - 3)$ is

$$\{x \mid x - 3 > 0\} = \{x \mid x > 3\} = (3, \infty)$$

The graph of f is obtained from the graph of $y = \log_{10} x$ by shifting three units to the right. Notice that the line $x = 3$ is a vertical asymptote. This can also be seen from (3) by noting that $x - 3 \to 0^+$ as $x \to 3^+$ and so

$$\lim_{x \to 3^+} \log_{10}(x - 3) = -\infty$$



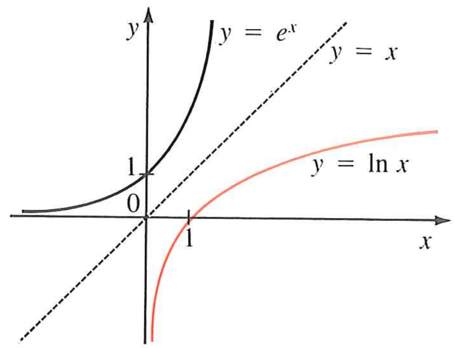
Natural Logarithms

In the next section we will see that, of all possible bases b for logarithms, the most convenient for the purposes of calculus is the number e, which was defined in Section 8.1. The logarithm with base e is called the **natural logarithm** and is given a special notation:

$$\boxed{\log_e x = \ln x}$$

(The abbreviation ln is short for *logarithmus naturalis*.)

Thus, the natural logarithmic function $y = \ln x$ is the inverse function of the exponential function $y = e^x$; they are both graphed in the following diagram.



If we put $b = e$ and write ln for $\log_e$ in ① and ②, then the defining properties of the natural logarithm become

$$\boxed{\ln x = y \quad \Leftrightarrow \quad e^y = x} \tag{4}$$

and

$$\boxed{\begin{aligned} \ln(e^x) &= x \quad (x \in R) \\ e^{\ln x} &= x \quad (x > 0) \end{aligned}} \tag{5}$$

In particular, it is worth noting that

$$\boxed{\ln e = 1} \qquad \text{and} \qquad \boxed{\ln 1 = 0}$$

Example 2 Solve for x: $\ln x = 8$.

Solution 1 From ④ we see that

$$\ln x = 8 \quad \text{means} \quad e^8 = x$$

Therefore $x = e^8$.

Solution 2 Start with the equation

$$\ln x = 8$$

and apply the natural exponential function to both sides of the equation:

$$e^{\ln x} = e^8$$

The second equation in ⑤ says that $e^{\ln x} = x$. Therefore $x = e^8$.

Example 3 Solve the equation $e^{3-2x} = 4$.

Solution We take natural logarithms of both sides of the equation and use ⑤:

$$\begin{aligned} \ln(e^{3-2x}) &= \ln 4 \\ 3 - 2x &= \ln 4 \\ 2x &= 3 - \ln 4 \\ x &= \tfrac{1}{2}(3 - \ln 4) \end{aligned}$$

Since the natural logarithm is found on scientific calculators, we can give an approximation to the solution:

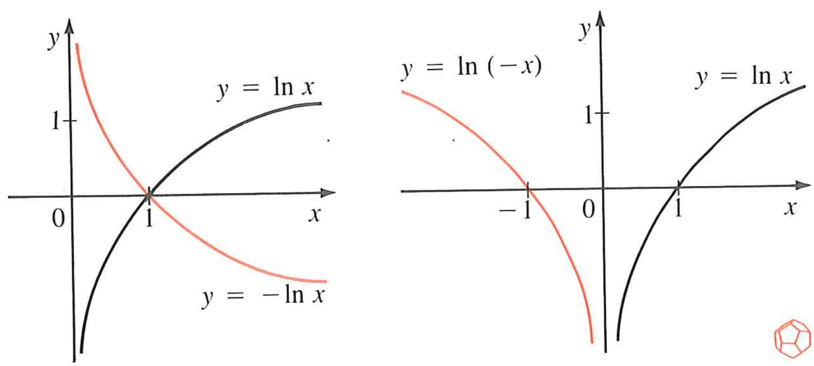
$$x \doteq 0.807$$

Example 4 Sketch the graphs of the following functions.

(a) $y = -\ln x$ (b) $y = \ln(-x)$

Solution (a) We start with the graph of $y = \ln x$ and reflect in the x-axis to get the graph of $y = -\ln x$.

(b) To obtain the graph of $y = \ln(-x)$ we reflect the graph of $y = \ln x$ in the y-axis.



Example 5 Find the domain of the function $f(x) = \ln(4 - x^2)$.

Solution As with any logarithmic function, $\ln x$ is defined when $x > 0$. Thus the domain of f is

$$\{x \mid 4 - x^2 > 0\} = \{x \mid x^2 < 4\} = \{x \mid |x| < 2\} = (-2, 2)$$

As a special case of (3) we have the following limits.

$$\boxed{\lim_{x \to 0^+} \ln x = -\infty \qquad \lim_{x \to \infty} \ln x = \infty} \qquad (6)$$

Example 6 Find $\lim_{x \to 2^-} \ln(4 - x^2)$.

Solution If we let $t = 4 - x^2$, then $t \to 0^+$ as $x \to 2^-$. So by (6) we have

$$\lim_{x \to 2^-} \ln(4 - x^2) = \lim_{t \to 0^+} \ln t = -\infty$$

Since ln is just a special logarithm, the laws of logarithms (see the Review and Preview to this chapter) also hold for the natural logarithm:

Suppose that $x > 0$, $y > 0$, and r is any real number. Then

1. $\ln(xy) = \ln x + \ln y$
2. $\ln\left(\dfrac{x}{y}\right) = \ln x - \ln y$
3. $\ln(x^r) = r \ln x$

Example 7 Express $\frac{1}{2} \ln x - 4 \ln y + \ln(x^2 + 1)$ as a single logarithm.

Solution

$$\begin{aligned}\tfrac{1}{2} \ln x - 4 \ln y + \ln(x^2 + 1) &= \ln x^{\frac{1}{2}} - \ln y^4 + \ln(x^2 + 1)\\ &= \ln \frac{\sqrt{x}(x^2 + 1)}{y^4}\end{aligned}$$

Change of Base

For some purposes it is useful to be able to change from logarithms in one base to logarithms in another base. Suppose that we are given $\log_b x$ and want to find $\log_a x$. Let

$$y = \log_a x$$

We write this in exponential form and take logarithms, with base b, of both sides.

$$\begin{aligned} a^y &= x \\ \log_b(a^y) &= \log_b x \\ y \log_b a &= \log_b x \\ y &= \frac{\log_b x}{\log_b a} \end{aligned}$$

Thus we have proved the following formula.

Change of Base Formula

$$\log_a x = \frac{\log_b x}{\log_b a}$$

In particular, if we put $x = b$, then $\log_b b = 1$ and the change of base formula becomes

$$\boxed{\log_a b = \frac{1}{\log_b a}}$$

Example 8 Evaluate $\log_8 5$ correct to six decimal places.

Solution There is no $\log_8$ key on a calculator, but we use the change of base formula with $a = 8$ and $b = e$ to convert to natural logarithms:

$$\log_8 5 = \frac{\ln 5}{\ln 8} \doteq 0.773\,976$$

Logarithms with base 10 are called **common logarithms** and are often denoted by omitting the base:

$$\boxed{\log x = \log_{10} x}$$

Common logarithms are also found on scientific calculators, so an alternative solution to Example 8 is as follows:
Take $a = 8$ and $b = 10$ in the change of base formula. Then

$$\log_8 5 = \frac{\log_{10} 5}{\log_{10} 8} = \frac{\log 5}{\log 8} \doteq 0.773\,976$$

EXERCISE 8.3

B **1.** Most scientific calculators have keys for both LN and LOG ($= \log_{10}$). Use such a calculator to draw the graphs of $y = \ln x$ and $y = \log_{10} x$, $0.1 \leqslant x \leqslant 10$, on the same axes.

2. Graph each function, not by plotting points, but by starting from the graphs of $y = \log_2 x$, $\log_{10} x$, and $\ln x$ given in this section and using transformations. State the domain, range, and asymptote of each function.

(a) $f(x) = \log_2(x - 4)$ (b) $f(x) = -\log_{10} x$
(c) $g(x) = \log(-x)$ (d) $g(x) = \ln(x + 2)$
(e) $y = 2 + \log_{10} x$ (f) $y = \log_2(x - 1) - 2$
(g) $y = 1 - \ln x$ (h) $y = 1 + \ln(-x)$
(i) $y = |\ln x|$ (j) $y = \ln|x|$

3. Evaluate without using a calculator.

(a) $e^{\ln 5}$ (b) $\ln e^2$
(c) $2 \ln e$ (d) $e^{5 \ln 2}$
(e) $\ln \sqrt{e}$ (f) $\ln 2 + 2 \ln 3 - \ln 18$

4. Solve for x.

(a) $e^x = 4$ (b) $\ln x = 6$
(c) $\ln(2x - 1) = 1$ (d) $e^{3x+5} = 10$
(e) $\ln(e^{3-x}) = 8$ (f) $\ln x = \ln 4 + \ln 7$
(g) $\ln(\ln x) = 2$ (h) $e^{e^x} = 5$

5. Find the solution of each equation correct to six decimal places.

(a) $\ln(x + 1) = 3$ (b) $e^{-x} = \frac{1}{2}$
(c) $e^{5x+3} = 10$ (d) $2^{x-5} = 3$

6. Express as a single logarithm.

(a) $\frac{1}{3} \ln x + 2 \ln(3x - 5)$

(b) $2 \ln x - \frac{1}{2} \ln(x^2 - 1) + 3 \ln(x^2 + 1)$

7. Find the domain of each function.

(a) $f(x) = \log_{10}(2 + 5x)$ (b) $f(x) = \log_2(10 - 3x)$
(c) $g(x) = \log_3(x^2 - 1)$ (d) $g(x) = \ln(x - x^2)$
(e) $h(x) = \ln x + \ln(2 - x)$
(f) $h(x) = \sqrt{x - 2} - \ln(10 - x)$

8. Compare the domains of the functions $f(x) = \ln x^2$ and $g(x) = 2 \ln x$.

9. Find each limit.

(a) $\lim_{x \to -4^+} \ln(x + 4)$ (b) $\lim_{x \to \infty} \ln(x + 4)$
(c) $\lim_{x \to 1^+} \log_{10}(x^2 - x)$ (d) $\lim_{t \to \pi^-} \ln(\sin t)$

10. Use the change of base formula and a calculator to evaluate the logarithm correct to six decimal places.
(a) $\log_2 7$ (b) $\log_5 2$
(c) $\log_3 11$ (d) $\log_6 92$

11. Use the change of base formula to show that $\log e = \dfrac{1}{\ln 10}$.

12. Simplify $(\log_2 5)(\log_5 7)$.

C **13.** A **learning curve** is a graph of a function $P(t)$ that measures the performance of someone learning a skill as a function of the training time t. At first, the rate of learning is rapid. Then, as performance increases and approaches a maximal value M, the rate of learning decreases. It has been found that the function

$$P(t) = M - Ce^{-kt}$$

where k and C are positive constants and $C < M$, is a reasonable model for learning.
(a) Sketch the graph of P.
(b) Express the learning time t as a function of the performance level P.

14. Which is larger, $\log_4 17$ or $\log_5 24$?

15. (a) Find the domain of the function $f(x) = \log_2(\log_{10} x)$.
(b) Find the inverse function of f.

16. (a) Find the domain of the function $f(x) = \ln(\ln(\ln x))$.
(b) Find the inverse function of f.

17. Solve the equation $4^x - 2^{x+1} = 3$. (Hint: First write the equation as a quadratic equation in 2^x.)

18. Solve the equation $\log_2 x + \log_4 x + \log_8 x = 11$.

8.4 DERIVATIVES OF LOGARITHMIC FUNCTIONS

In this section we find the derivatives of the logarithmic functions $y = \log_b x$. First we differentiate the natural logarithmic function $y = \ln x$.

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}} \qquad (1)$$

Proof

Let $\quad y = \ln x$

Then $\quad e^y = x$

Differentiating this equation implicitly with respect to x, we get

$$e^y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

Example 1 Differentiate. (a) $y = x \ln x$ (b) $y = \ln(x^2 + 2x - 5)$

Solution (a) Using the Product Rule, we have

$$\begin{aligned}\frac{dy}{dx} &= x\frac{d}{dx}(\ln x) + \ln x\frac{d}{dx}(x)\\ &= (x)\left(\frac{1}{x}\right) + (\ln x)(1)\\ &= 1 + \ln x\end{aligned}$$

(b) To use the Chain Rule we let $u = x^2 + 2x - 5$. Then $y = \ln u$, so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u}\frac{du}{dx} = \frac{1}{x^2 + 2x - 5}(2x + 2) = \frac{2(x + 1)}{x^2 + 2x - 5}$$

In general, if we combine Formula 1 with the Chain Rule, we get

$$\boxed{\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}} \quad \text{or} \quad \boxed{\frac{d}{dx}\ln[g(x)] = \frac{g'(x)}{g(x)}}$$

Example 2 Find the derivative of $f(x) = \ln(\cos x)$.

Solution

$$\begin{aligned}f'(x) &= \frac{d}{dx}\ln(\cos x)\\ &= \frac{1}{\cos x}\frac{d}{dx}\cos x\\ &= \frac{1}{\cos x}(-\sin x)\\ &= -\tan x\end{aligned}$$

Example 3 Differentiate $y = (\ln x)^4$.

Solution This time the logarithm is the inner function, so the Chain Rule gives

$$y' = 4(\ln x)^3 \frac{d}{dx}(\ln x) = \frac{4(\ln x)^3}{x}$$

Example 4 Find $\dfrac{d}{dx} \ln \dfrac{x}{\sqrt{x+1}}$.

Solution 1 We use the Chain Rule and then the Quotient Rule:

$$\begin{aligned}
\frac{d}{dx} \ln \frac{x}{\sqrt{x+1}} &= \frac{1}{\dfrac{x}{\sqrt{x+1}}} \frac{d}{dx} \frac{x}{\sqrt{x+1}} \\
&= \frac{\sqrt{x+1}}{x} \left[\frac{\sqrt{x+1}(1) - (x)\dfrac{1}{2\sqrt{x+1}}}{x+1} \right] \\
&= \frac{x + 1 - \frac{1}{2}x}{x(x+1)} \\
&= \frac{x+2}{2x(x+1)}
\end{aligned}$$

Solution 2 If we first use the Laws of Logarithms to rewrite the function, then the differentiation becomes easier:

$$\begin{aligned}
\frac{d}{dx} \ln \frac{x}{\sqrt{x+1}} &= \frac{d}{dx} [\ln x - \tfrac{1}{2} \ln(x+1)] \\
&= \frac{1}{x} - \frac{1}{2(x+1)}
\end{aligned}$$

(This answer can be left as it is, but if we were to use a common denominator we would see that it gives the same answer as in Solution 1.)

Example 5 If $f(x) = \ln |x|$, find $f'(x)$.

Solution If x is positive, then $|x| = x$, so

$$f'(x) = \frac{d}{dx} (\ln x) = \frac{1}{x}$$

If x is negative, then $|x| = -x$, so

$$f'(x) = \frac{d}{dx} \ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}$$

Therefore $f'(x) = \dfrac{1}{x}$ for all $x \neq 0$.

The result of Example 5 is worth remembering:

$$\boxed{\frac{d}{dx} \ln|x| = \frac{1}{x}} \qquad (2)$$

In Section 8.2 we showed that if $f(x) = b^x$ then

$$f'(x) = b^x \lim_{h \to 0} \frac{b^h - 1}{h}$$

Now we can show that the value of the limit is $\ln b$.

$$\boxed{\frac{d}{dx} b^x = b^x \ln b} \qquad (3)$$

Proof

We use the fact that $e^{\ln b} = b$.

$$\begin{aligned} \frac{d}{dx} b^x &= \frac{d}{dx} (e^{\ln b})^x \\ &= \frac{d}{dx} e^{(\ln b)x} \\ &= e^{(\ln b)x} \frac{d}{dx} (\ln b)x \\ &= (e^{\ln b})^x (\ln b) \\ &= b^x \ln b \end{aligned}$$

Example 6 Find y' if $y = 2^{x^2}$.

Solution Combining Formula 3 with the Chain Rule, we get

$$y' = 2^{x^2} (\ln 2) \frac{d}{dx} x^2 = (2 \ln 2)\, x 2^{x^2}$$

Next we differentiate the general logarithmic function $y = \log_b x$.

$$\boxed{\frac{d}{dx} \log_b x = \frac{1}{x \ln b}} \qquad (4)$$

Proof

Let $\quad y = \log_b x$

Then $\quad b^y = x$

Using Formula 3 we differentiate this equation implicitly with respect to x:

$$b^y (\ln b)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{b^y \ln b} = \frac{1}{x \ln b}$$

(Another proof of Formula 4 could be given using the Change of Base Formula. See Question 11 in Exercise 8.4.)

Example 7 Find $f'(x)$ if $f(x) = \log_{10}(3x + 1)^4$.

Solution First we rewrite f as

$$f(x) = 4 \log_{10}(3x + 1)$$

Then we combine Formula 4 with the Chain Rule.

$$f'(x) = (4)\frac{1}{(3x + 1)\ln 10}\frac{d}{dx}(3x + 1) = \frac{12}{(3x + 1)\ln 10}$$

From Formula 4 we see one of the main reasons that natural logarithms (logarithms with base e) are used in calculus: The differentiation formula is simplest when $b = e$ because $\ln e = 1$.

Example 8 Find the absolute maximum value of the function $f(x) = \dfrac{\ln x}{x}$.

Solution The Quotient Rule gives

$$f'(x) = \frac{x\left(\dfrac{1}{x}\right) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2}$$

The critical numbers occur when $1 - \ln x = 0$, so $\ln x = 1$, or $x = e$.

Since $x^2 \geqslant 0$, we see that $f'(x) > 0$ when $1 - \ln x > 0$, so $\ln x < 1$, or $x < e$. Similarly $f'(x) < 0$ when $x > e$. Thus, by the First

Derivative Test for Absolute Extreme Values, the absolute maximum value occurs when $x = e$ and is

$$f(e) = \frac{\ln e}{e} = \frac{1}{e}$$

Example 9 Discuss the curve $y = f(x) = \ln(x^2 - 1)$ under the headings A–H of Section 5.5.

Solution A. ***Domain.*** The function is defined when $x^2 - 1 > 0$, so the domain is

$$\begin{aligned}\{x \mid x^2 > 1\} &= \{x \mid |x| > 1\} \\ &= \{x \mid x > 1 \text{ or } x < -1\} \\ &= (-\infty, -1) \cup (1, \infty)\end{aligned}$$

B. ***Intercepts.*** The x-intercepts occur when $\ln(x^2 - 1) = 0$, so $x^2 - 1 = 1$, $x^2 = 2$, $x = \pm\sqrt{2}$. There is no y-intercept since $f(0)$ is undefined.

C. ***Symmetry.*** $f(-x) = f(x)$, so f is even and the curve is symmetric about the y-axis.

D. ***Asymptotes.*** As $x \to 1^+$ or $x \to -1^-$, we have $x^2 - 1 \to 0^+$, so

$$\lim_{x \to 1^+} \ln(x^2 - 1) = -\infty \quad \text{and} \quad \lim_{x \to -1^-} \ln(x^2 - 1) = -\infty$$

Thus the lines $x = 1$ and $x = -1$ are vertical asymptotes. There is no horizontal asymptote since

$$\lim_{x \to \pm\infty} \ln(x^2 - 1) = \infty$$

E. ***Intervals of Increase and Decrease.***

$$f'(x) = \frac{2x}{x^2 - 1}$$

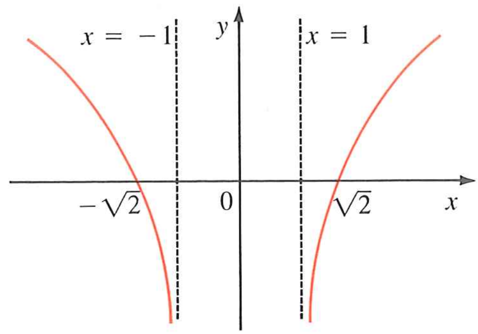
Recall that $f(x)$ is defined only when $x > 1$ or $x < -1$. We have $f'(x) > 0$ for $x > 1$ and $f'(x) < 0$ for $x < -1$. Thus f is increasing on $(1, \infty)$ and decreasing on $(-\infty, -1)$.

F. ***Extreme Values.*** There is no maximum or minimum value.

G. ***Concavity.***
$$f''(x) = \frac{(x^2 - 1)(2) - (2x)(2x)}{(x^2 - 1)^2} = \frac{-2x^2 - 2}{(x^2 - 1)^2}$$

Thus $f''(x) < 0$, so f is concave downward on $(-\infty, -1)$ and $(1, \infty)$.

H. ***Sketch of the Curve.***



The Number e as a Limit

We have shown that if $f(x) = \ln x$, then $f'(x) = \dfrac{1}{x}$. Thus $f'(1) = 1$. We now use this fact to express the number e as a limit.

From the definition of a derivative as a limit, we have

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{x\to 0} \frac{f(1+x) - f(1)}{x} \\ &= \lim_{x\to 0} \frac{\ln(1+x) - \ln 1}{x} \\ &= \lim_{x\to 0} \frac{1}{x}\ln(1+x) && \text{(since } \ln 1 = 0\text{)} \\ &= \lim_{x\to 0} \ln(1+x)^{\frac{1}{x}} && \text{(by Law 3 of logarithms)} \\ &= \ln\left[\lim_{x\to 0} (1+x)^{\frac{1}{x}}\right] && \text{(since ln is continuous)} \end{aligned}$$

Since $f'(1) = 1$, we have

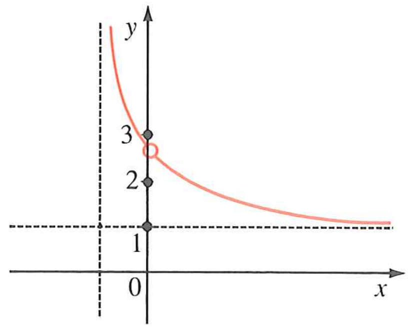
$$\ln\left[\lim_{x\to 0} (1+x)^{\frac{1}{x}}\right] = 1$$

Therefore

$$\boxed{\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e} \qquad (5)$$

Formula 5 is illustrated by the graph of the function $y = (1+x)^{\frac{1}{x}}$ and a table of values for small values of x.

x	$(1 + x)^{\frac{1}{x}}$
0.1	2.593 742 46
0.01	2.704 813 83
0.001	2.716 923 93
0.000 1	2.718 145 93
0.000 01	2.718 268 24
0.000 001	2.718 280 47
0.000 000 1	2.718 281 69
0.000 000 01	2.718 281 82



It appears that

$$e \doteq 2.718\ 282$$

In fact it can be shown that, correct to 15 decimal places,

$$e \doteq 2.718\ 281\ 828\ 459\ 045$$

The decimal expansion of e is nonrepeating because e is an irrational number. The notation e for this number was chosen by the Swiss mathematician Leonhard Euler in 1727 probably because it is the first letter of the word *exponential*.

See the biography of Euler at the end of this chapter.

If we put $n = \frac{1}{x}$ in Formula 5, then $n \to \infty$ as $x \to 0$ and an alternative expression for e is

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

EXERCISE 8.4

B **1.** Differentiate.

(a) $f(x) = x^2 \ln x$ (b) $f(x) = \sqrt{\ln x}$

(c) $g(x) = \ln(x^3 + 1)$ (d) $g(x) = \ln(5x)$

(e) $y = \sin(\ln x)$ (f) $y = \ln(\sin x)$

(g) $y = \dfrac{\ln x}{x^3}$ (h) $y = (x + \ln x)^3$

(i) $y = \ln|2x + 1|$ (j) $y = \ln\left(\frac{x+1}{x-1}\right)$

(k) $y = \ln\sqrt{\frac{x}{2x+3}}$ (l) $y = \ln\frac{x}{\sqrt{x^2+1}}$

(m) $y = \ln(\sec x + \tan x)$ (n) $y = \tan[\ln(1 - 3x)]$

2. (a) If $f(x) = \ln(\ln x)$, find $f'(x)$.
(b) Find the domains of f and f'.

3. Find the derivative of each function.
(a) $f(x) = \log_2(x^2 + 1)$ (b) $g(x) = x\log_{10} x$
(c) $F(x) = \log_5(3x - 8)$ (d) $G(x) = \frac{1 + \log_3 x}{x}$

4. Differentiate.
(a) $y = x^3 + 3^x$ (b) $y = 2^{x^4 - x}$
(c) $y = x5^{\sqrt{x}}$ (d) $y = 10^{\tan \pi x}$

5. Find the equation of the tangent line to each curve at the given point.
(a) $y = \ln(x - 1)$, $(2, 0)$ (b) $y = x^2 \ln x$, $(1, 0)$
(c) $y = 10^x$, $(1, 10)$ (d) $y = \log_{10} x$, $(100, 2)$

6. Find y' if $\ln(x + y) = y - 1$.

7. (a) Find the absolute minimum value of the function $f(x) = x \ln x$.
(b) On what interval is it concave upward?

8. (a) Find the local maximum and minimum values of the function $f(x) = x(\ln x)^2$.
(b) Find the inflection point of this function.

9. Discuss each curve under the headings A–H of Section 5.5.
(a) $y = \ln(9 - x^2)$ (b) $y = x + \ln x$
(c) $y = (\ln x)^2$ (d) $y = \ln(\cos x)$

10. (a) Sketch the graphs of the curves $y = \ln x$ and $y = 2 - x$ (using the same axes) to show that there is exactly one solution of the equation $\ln x = 2 - x$.
(b) Use Newton's method to find the root of the equation in part (a) correct to six decimal places.

11. Use Formula 1, together with the Change of Base Formula from Section 8.3, to prove Formula 4.

C **12.** Evaluate $\lim_{x \to 0} (1 + 3x)^{\frac{1}{x}}$. (Hint: Make the change of variable $t = 3x$ and use Formula 5.)

PROBLEMS PLUS

Prove that the number $\log_2 5$ is an irrational number.

8.5 EXPONENTIAL GROWTH AND DECAY

In many of the sciences, certain quantities grow or decay at a rate that is proportional to their size. In this section we see that such quantities occur in biology, chemistry, physics, and economics and are modelled by exponential functions.

If $y = f(t)$ is the value of a quantity y at time t and if the rate of change of y with respect to t is proportional to its size $f(t)$ at any time, then

$$\boxed{f'(t) = kf(t)} \quad \text{or} \quad \boxed{\frac{dy}{dt} = ky} \qquad (1)$$

where k is a constant. This equation is called the **Law of Natural Growth** (if $k > 0$) or the **Law of Natural Decay** (if $k < 0$).

To solve (1), all we have to do is think of a function whose derivative is a constant multiple of itself. Any exponential function has this property. In fact if $f(t) = Ce^{kt}$, where C is a constant, then

$$f'(t) = Ce^{kt}(k) = k(Ce^{kt}) = kf(t)$$

It can also be shown that *any* solution of (1) must be of the form $f(t) = Ce^{kt}$.

Notice that

$$f(0) = Ce^{k(0)} = C$$

and so the value of the constant C is the size of the quantity at time $t = 0$. We call it the **initial value** of the function and we denote it by y_0.

> The solution of the equation
>
> $$\frac{dy}{dt} = ky$$
>
> is given by
>
> $$y = y_0e^{kt} \qquad (2)$$
>
> where y_0 is the initial value of the function.

Exponential Growth

If $y = f(t)$ is the number of individuals in a population of animals or bacteria cells at time t, then it seems reasonable to expect that the rate of growth $f'(t)$ is proportional to the population $f(t)$:

$$f'(t) = kf(t)$$

(In a bacteria culture, for instance, the growth is a result of division of the bacteria and the more bacteria present at a given moment, the greater the opportunity for division.) This reasoning is valid under ideal conditions of adequate nutrition, freedom from disease, and unlimited environment. We make these assumptions throughout this chapter. Thus, from ②, the size of the population at time t is

$$y = y_0 e^{kt}$$

where y_0 is the initial size of the population.

Example 1 A bacteria culture starts with 2000 bacteria and after 3 h the estimated count is 10 000 bacteria.

(a) Find the number of bacteria after t hours.
(b) Find the number of bacteria after 2 h.
(c) Find the rate of growth after 2 h.
(d) When will the bacteria population reach 18 000?

Solution (a) Let $y = f(t)$ be the number of bacteria after t hours. We are given that the initial population size is $y_0 = f(0) = 2000$ and also $f(3) = 10\ 000$. From ② we have

$$f(t) = y_0 e^{kt} = 2000\ e^{kt}$$

and putting $t = 3$ we get

$$\begin{aligned} 2000\ e^{3k} &= 10\ 000 \\ e^{3k} &= 5 \\ 3k &= \ln 5 \\ k &= \tfrac{1}{3} \ln 5 \end{aligned}$$

Putting this value of k back into the expression for $f(t)$, we get

$$y = f(t) = 2000\ e^{\left(\frac{1}{3} \ln 5\right)t}$$

Since $e^{\ln 5} = 5$, another way to write the answer is

$$y = (2000)(e^{\ln 5})^{\frac{t}{3}} = (2000)5^{\frac{t}{3}}$$

(b) The number of bacteria after 2 h is

$$y = f(2) = (2000)5^{\frac{2}{3}} \doteq 5848$$

(c) The rate of growth is

$$f'(t) = 2000\ e^{\left(\frac{1}{3} \ln 5\right)t} \left(\tfrac{1}{3} \ln 5\right)$$

After 2 h it is

$$f'(2) = 2000\ e^{\left(\frac{1}{3} \ln 5\right)(2)} \left(\tfrac{1}{3} \ln 5\right) \doteq 3137$$

The rate of growth after 2 h is about 3137 bacteria/h.

(d) We want to find the value of t such that $f(t) = 18\ 000$; that is,

$$2000\ e^{\left(\frac{1}{3}\ln 5\right)t} = 18\ 000$$
$$e^{\left(\frac{1}{3}\ln 5\right)t} = 9$$

We solve this equation for t by taking the natural logarithm of both sides.

$$\left(\tfrac{1}{3}\ln 5\right)t = \ln 9$$
$$t = \frac{3\ln 9}{\ln 5} \doteq 4.0$$

The population will reach 18 000 after about 4 h.

Radioactive Decay

Radioactive substances decompose by spontaneously emitting radiation. The rate of decay is proportional to the amount of the substance that has not yet disintegrated. Therefore if $y = f(t)$ is the mass of the substance that remains at time t, we have

$$\frac{dy}{dt} = ky$$

(Here k is a negative constant because f is a decreasing function.) From (2) we then have

$$y = y_0 e^{kt}$$

where y_0 is the mass at time $t = 0$.

The **half-life** of a radioactive substance is the period of time during which any given amount decays until half of it remains.

Example 2 The half-life of Polonium-210 is 140 d and a sample of this element has a mass of 300 mg.

(a) Find the mass that remains after t days.
(b) Find the mass that remains after 50 d.
(c) Find the rate of decrease of the mass after 50 d.
(d) How long will the sample take to decay to a mass of 200 mg?

Solution (a) Let $y = f(t)$ be the mass of Polonium-210, in milligrams, that remains after t days. We are given that the initial mass is $y_0 = 300$ mg, so

$$f(t) = 300\ e^{kt}$$

To determine the value of k we use the fact that the half-life is 140 d and so $f(140) = 150$. Thus

$$300\, e^{140k} = 150$$
$$e^{140k} = \tfrac{1}{2}$$
$$140k = \ln\left(\tfrac{1}{2}\right) = -\ln 2$$
$$k = -\frac{\ln 2}{140}$$

Therefore the mass remaining after t days is

$$f(t) = 300e^{-\frac{\ln 2}{140}t}$$

Since $e^{\ln 2} = 2$, an alternative form of the answer is

$$f(t) = (300)2^{-\frac{t}{140}}$$

(b) The mass after 50 d is

$$f(50) = (300)2^{-\frac{50}{140}} \doteq 234 \text{ mg}$$

(c) The rate of change of the mass is

$$f'(t) = 300e^{-\frac{\ln 2}{140}t}\left(-\frac{\ln 2}{140}\right)$$

After 50 d it is

$$f'(50) = (300)2^{-\frac{50}{140}}\left(-\frac{\ln 2}{140}\right) \doteq -1.2$$

Thus the rate of decrease is about 1.2 mg/d.

(d) The mass will be 200 mg when

$$300\, e^{-\frac{\ln 2}{140}t} = 200$$
$$e^{-\frac{\ln 2}{140}t} = \tfrac{2}{3}$$
$$-\frac{\ln 2}{140}t = \ln \tfrac{2}{3}$$
$$t = -140\frac{\ln \frac{2}{3}}{\ln 2} \doteq 82$$

The time required is about 82 d.

Compound Interest

If an amount of money P, called the **principal**, is invested at an interest rate r, then the interest after one time period is Pr and the amount of money is

$$A = P + Pr = P(1 + r)$$

For instance, if $P = \$1000$ and the interest rate is 12% per annum, then $r = 0.12$ and the amount after one year is $\$1000(1.12) = \$1120$.

If the interest is reinvested, then the new principal is $P(1 + r)$ and the amount after another time period is

$$A = P(1 + r)(1 + r) = P(1 + r)^2$$

Similarly, after a third time period the amount is $P(1 + r)^3$, and in general after k periods it is

$$A = P(1 + r)^k$$

Notice that this is an exponential function with base $1 + r$.

If the interest rate is stated as 12% per annum compounded semiannually, then the time period is six months and the interest rate per time period is

$$i = \frac{0.12}{2} = 0.06$$

If interest is compounded n times per annum, then in each time period the interest rate is

$$i = \frac{r}{n}$$

There are nt time periods in t years, so the amount after t years is

$$\boxed{A = P\left(1 + \frac{r}{n}\right)^{nt}} \qquad (3)$$

Example 3 A sum of \$1000 is invested at an interest rate of 12% per annum. Find the amount in the account after three years if interest is compounded
(a) annually (b) semiannually (c) quarterly
(d) monthly (e) daily

Solution We use Formula 3 with $P = \$1000$, $r = 0.12$, and $t = 3$.

(a) With annual compounding, $n = 1$:

$$A = 1000(1.12)^3 = \$1404.93$$

(b) With semiannual compounding, $n = 2$:

$$A = 1000\left(1 + \frac{0.12}{2}\right)^{2(3)} = 1000(1.06)^6 = \$1418.52$$

(c) With quarterly compounding, $n = 4$:

$$A = 1000\left(1 + \frac{0.12}{4}\right)^{4(3)} = 1000(1.03)^{12} = \$1425.76$$

(d) With monthly compounding, $n = 12$:

$$A = 1000\left(1 + \frac{0.12}{12}\right)^{12(3)} = 1000(1.01)^{36} = \$1430.77$$

(e) With daily compounding, $n = 365$:

$$A = 1000\left(1 + \frac{0.12}{365}\right)^{365(3)} = \$1433.24$$

We see from Example 3 that the interest paid increases as the number of compounding periods (n) increases. In general, let us see what happens as n increases indefinitely. If we let $m = \frac{n}{r}$, then

$$A = P\left(1 + \frac{r}{n}\right)^{nt} = P\left[\left(1 + \frac{r}{n}\right)^{\frac{n}{r}}\right]^{rt} = P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt}$$

Recall that as m becomes large, the quantity $\left(1 + \frac{1}{m}\right)^{m}$ approaches the number e. Thus the amount approaches

$$\boxed{A = Pe^{rt}} \qquad (4)$$

When interest is paid according to Formula 4, we say that interest is **compounded continuously**.

Example 4 Find the amount after three years if \$1000 is invested at an interest rate of 12% per annum compounded continuously.

Solution Using Formula 4 with $P = \$1000$, $r = 0.12$, and $t = 3$, we have

$$A = 1000\, e^{(0.12)3} = 1000\, e^{0.36} = \$1433.33$$

EXERCISE 8.5

B **1.** A bacteria culture starts with 1000 bacteria. After 2 h the estimated count is 10 000 bacteria.
(a) Find the number of bacteria after t hours.
(b) Find the number of bacteria after 5 h.
(c) Find the rate of growth after 5 h.
(d) When will the bacteria population reach 15 000?

2. The initial size of a bacteria culture is 400. After an hour there are 1200 bacteria.
(a) Find the number of bacteria after t hours.
(b) In what period of time does the population double?

3. A cell of the bacterium Escherichia coli in a nutrient broth medium divides into two cells every 20 min. Suppose that there are initially 500 cells. Find
(a) the number of cells after t hours
(b) the number of cells after 8 h
(c) the time required for the size to reach 6000 cells

4. The count in a bacteria culture was 5000 after 15 min and 40 000 after 1 h.
(a) What was the initial size of the culture?
(b) Find the population after t hours.
(c) Find the rate of growth after 15 min.
(d) When will the size of the population be 150 000?

5. The population of a certain city grows at a rate of 4% per year. The population in 1980 was 275 000.
(a) What was the population in 1985?
(b) Predict the population in the year 2000, assuming the growth rate remains constant.

6. The population of the world is doubling about every 35 a. In 1987 the total population reached 5 billion.
(a) Find the projected world population
(i) for the year 2001 (ii) for the year 2100
(b) When will the world population reach 50 billion?

7. Uranium-238 has a half-life of 4.5×10^9 a.
(a) Find the mass that remains from a 100 mg sample after t years.
(b) Find the mass that remains from this sample after 10 000 a.
(c) Find the rate of decrease of the mass after 10 000 a.

8. An isotope of sodium, ^{24}Na, has a half-life of 15 h. A sample of this isotope has a mass of 2 g.
(a) Find the mass that remains after t hours.
(b) Find the mass that remains after 5 h.
(c) Find the rate of decrease of the mass after 5 h.
(d) How long will the sample take to decay to a mass of 0.4 g?

9. Uranium-234 has a half-life of 2.5×10^5 a.
(a) Find the amount remaining from a 10 mg sample after a thousand years.
(b) How long would it take this sample to decompose until its mass is 7 mg?

10. A sample of Bismuth-210 decayed to 33% of its original mass after eight days.
(a) Find the half-life of this element.
(b) Find the mass remaining after twelve days.

11. \$1000 is borrowed at a rate of 16% interest per annum. Find the amount due at the end of two years if the interest is compounded

(a) annually (b) quarterly (c) monthly
(d) weekly (e) daily (f) continuously

12. \$10 000 is invested at an interest rate of 10% per annum. Find the amount of the investment at the end of four years if the interest is compounded

(a) annually (b) semiannually (c) monthly
(d) daily (e) hourly (f) continuously

13. Which of the following would be the better investment?

(a) An account paying $9\frac{1}{4}$% per annum compounded semiannually, or

(b) an account paying 9% per annum compounded continuously.

14. How long would it take for an investment to double in value if the interest rate is 8.5% per annum compounded continuously?

15. If the chemical reaction

$$2N_2O_5 \rightarrow 4\ NO_2 + O_2$$

takes place at 45°C, the rate of reaction of dinitrogen pentoxide is proportional to its concentration:

$$-\frac{d[N_2O_5]}{dt} = 0.0005[N_2O_5]$$

(See Section 3.3.)

(a) If the initial concentration is C, find the concentration $[N_2O_5]$ after t seconds.

(b) After what period of time will the concentration be reduced to half its original value?

16. Scientists can determine the age of ancient objects by a method called **radiocarbon dating**. The bombardment of the upper atmosphere by cosmic rays converts nitrogen to a radioactive isotope of carbon, ^{14}C, with a half-life of about 5570 a. Vegetation absorbs carbon dioxide through the atmosphere and animal life assimilates ^{14}C through food chains. When a plant or animal dies it stops replacing its carbon and the amount of ^{14}C begins to decrease through radioactive decay. Therefore the level of radioactivity must also decay exponentially.

A parchment fragment was discovered that had about 77% as much ^{14}C radioactivity as does plant material on earth today. Estimate the age of the parchment.

PROBLEMS PLUS

Show that if $x > 0$ and $x \neq 1$, then

$$\frac{1}{\log_2 x} + \frac{1}{\log_3 x} + \frac{1}{\log_5 x} + \frac{1}{\log_7 x} = \frac{1}{\log_{210} x}$$

8.6 LOGARITHMIC DIFFERENTIATION

The calculation of derivatives of complicated functions involving products, quotients, and powers can often be simplified by taking logarithms. The method used in the following example is called **logarithmic differentiation**.

Example 1 Differentiate $y = \dfrac{e^x\sqrt{x^2+1}}{(x^2+2)^3}$.

Solution It is possible to differentiate this function using the Quotient and Product Rules, but it is easier to take natural logarithms of both sides.

$$\begin{aligned}
\ln y &= \ln\left[\frac{e^x(x^2+1)^{\frac{1}{2}}}{(x^2+2)^3}\right] \\
&= \ln(e^x) + \ln(x^2+1)^{\frac{1}{2}} - \ln(x^2+2)^3 \\
&= x + \tfrac{1}{2}\ln(x^2+1) - 3\ln(x^2+2)
\end{aligned}$$

If we now differentiate implicitly with respect to x, we get

$$\frac{1}{y}\frac{dy}{dx} = 1 + \left(\frac{1}{2}\right)\frac{2x}{x^2+1} - (3)\frac{2x}{x^2+2}$$

Solving for the derivative, we have

$$\begin{aligned}
\frac{dy}{dx} &= y\left[1 + \frac{x}{x^2+1} - \frac{6x}{x^2+2}\right] \\
&= \frac{e^x\sqrt{x^2+1}}{(x^2+2)^3}\left[1 + \frac{x}{x^2+1} - \frac{6x}{x^2+2}\right]
\end{aligned}$$

Steps in Logarithmic Differentiation

1. Take logarithms of both sides of an equation $y = f(x)$.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y'.

Notice that, in Example 1, y was positive and so there was no problem using logarithms. If it happens that $f(x) < 0$ for some values of x, then $\ln y$ is not defined. However we can always write $|y| = |f(x)|$ and use Formula 2 from Section 8.4:

$$\frac{d}{dx} \ln |x| = \frac{1}{x}$$

This procedure is illustrated in the following example.

Example 2 Find y' if $y = \sqrt[3]{\dfrac{x \cos x}{x^2 - 1}}$.

Solution Since y can be negative, we write

$$|y| = \left|\sqrt[3]{\frac{x \cos x}{x^2 - 1}}\right| = \left[\frac{|x|\,|\cos x|}{|x^2 - 1|}\right]^{\frac{1}{3}}$$

Then $\quad \ln |y| = \frac{1}{3}[\ln |x| + \ln |\cos x| - \ln |x^2 - 1|]$

Differentiating with respect to x, we have

$$\frac{1}{y} y' = \frac{1}{3}\left[\frac{1}{x} + \frac{-\sin x}{\cos x} - \frac{2x}{x^2 - 1}\right]$$

$$y' = \frac{1}{3}\sqrt[3]{\frac{x \cos x}{x^2 - 1}}\left[\frac{1}{x} - \tan x - \frac{2x}{x^2 - 1}\right]$$

The technique of logarithmic differentiation can be used to prove the General Power Rule as stated in Section 2.2:

> If n is any real number, then $\dfrac{d}{dx}(x^n) = nx^{n-1}$.

Proof

Let $\quad y = x^n$

Then
$$\begin{aligned}
|y| &= |x^n| \\
&= |x|^n \\
\ln|y| &= \ln(|x|^n) \\
&= n \ln |x| \\
\frac{y'}{y} &= n\left(\frac{1}{x}\right) \\
y' &= n\left(\frac{y}{x}\right) = n\frac{x^n}{x} = nx^{n-1}
\end{aligned}$$

We should note carefully the difference between the Power Rule

$$\frac{d}{dx}x^n = nx^{n-1} \quad \text{(variable base, constant exponent)}$$

and the rule for differentiating exponential functions

$$\frac{d}{dx}b^x = b^x \ln b \quad \text{(constant base, variable exponent)}$$

But what do we do when faced with a function such as

$$y = x^{\sin x}$$

where both the base and exponent are variable? There is no differentiation rule for such a function, but we can use logarithmic differentiation.

Example 3 Differentiate $y = x^{\sin x}$, $x > 0$.

Solution Using logarithmic differentiation, we have

$$\begin{aligned}
\ln y &= \ln(x^{\sin x}) \\
&= \sin x \ln x \\
\frac{y'}{y} &= (\sin x)\frac{1}{x} + \cos x \ln x \\
y' &= y\left[\frac{\sin x}{x} + \cos x \ln x\right] \\
y' &= x^{\sin x}\left[\frac{\sin x}{x} + \cos x \ln x\right]
\end{aligned}$$

EXERCISE 8.6

B **1.** Use logarithmic differentiation to find the derivative of each function.

(a) $y = (x^2 + 1)^2(x^2 + x + 1)^3$

(b) $y = (x - 1)^4(2x + 3)^5(x^2 - 2x + 3)^3$

(c) $y = e^{x^2}x^3(x^2 + 8)^4$

(d) $y = \dfrac{(x + 1)^3}{(x + 2)^5(x + 3)^7}$

(e) $y = \dfrac{x\sqrt{x + 1}}{(x + 2)(x^3 + 1)}$

(f) $y = \sqrt{\dfrac{x^2 + 1}{x^2 + 4}}$

2. Differentiate.

(a) $y = x^{x^2}$

(b) $y = x^{\sqrt{x}}$

(c) $y = x^{\cos x}$

(d) $y = (\cos x)^x$

(e) $y = (\ln x)^x$

(f) $y = (\cos x)^{\sin x}$

3. Find the equation of the tangent line to the curve $y = x^x$ at the point $(2, 4)$.

C 4. Let $f(x) = x^{-\ln x}$.
   (a) Compute $\lim_{x \to 0^+} x^{-\ln x}$ and $\lim_{x \to \infty} x^{-\ln x}$ by writing
   $$x^{-\ln x} = (e^{\ln x})^{-\ln x} = e^{-(\ln x)^2}$$
   (b) Use logarithmic differentiation to find $f'(x)$.
   (c) Find the interval on which f is increasing or decreasing.
   (d) Find the absolute maximum value of f.
   (e) Find the intervals of concavity and inflection points.
   (f) Sketch the graph of f.

PROBLEMS PLUS

Solve the inequality
$$\log_2\left(1 + \frac{1}{x}\right) + \log_{\frac{1}{2}}(1 + x) \geq 1$$

8.7 REVIEW EXERCISE

1. Graph each function starting from the graphs of the basic exponential and logarithmic functions.
   (a) $y = 1 + 2^x$ (b) $y = \log_{10}(x - 1)$
   (c) $y = \ln(-x)$ (d) $y = e^{-x}$
   (e) $y = 1 - 10^x$ (f) $y = -\log_2 x$

2. Evaluate.
   (a) $\lim_{x \to -\infty} (1 + 2^x)$ (b) $\lim_{x \to 1^+} \log_{10}(x - 1)$
   (c) $\lim_{x \to \infty} \ln(x^2 + x + 1)$ (d) $\lim_{x \to -1^-} e^{\frac{2}{x+1}}$
   (e) $\lim_{x \to \frac{\pi}{2}^-} e^{\tan x}$ (f) $\lim_{x \to 10^-} \ln(10 - x)$

3. Find the domain, range, and asymptote of the following functions.
   (a) $y = 1 + \ln(x + 2)$ (b) $y = 1 + 3e^{2x}$
   (c) $y = 10 - e^{-x}$ (d) $y = \ln(1 - 2x)$

4. Evaluate.
   (a) $\ln 1$ (b) $e^{\ln 10}$
   (b) $e^{3 \ln 2}$ (d) $\ln\left(\frac{1}{e}\right)$

5. Solve each equation. State your answer exactly and also correct to six decimal places.
   (a) $\ln x = \frac{1}{2}$ (b) $e^x = 7$
   (c) $e^{5-3x} = 2$ (d) $\ln(4x + 7) = 4$

6. Express as a single logarithm.
   (a) $2 \ln x + 3 \ln(1 + x) - 4 \ln(2 + x)$
   (b) $\frac{1}{2} \ln x - 2 \ln(x^2 + x + 1)$

7. Differentiate.
   (a) $f(x) = \ln(x^2 + 1)$
   (b) $f(x) = e^{x^3}$
   (c) $f(x) = \sqrt{x}\, e^x$
   (d) $f(x) = \dfrac{\ln x}{x^2}$
   (e) $y = x^4 - 4^x$
   (f) $y = \ln \sqrt{\dfrac{2x + 3}{4x - 5}}$
   (g) $y = \sin(e^{2x})$
   (h) $y = e^{2 \sin x}$
   (i) $y = \log_{10}(1 - x + x^3)$
   (j) $y = e^x \ln x$
   (k) $y = \dfrac{e^{x^2}}{x^2}$
   (l) $y = \sqrt{1 + (\ln x)^4}$

8. Find the equation of the tangent line to each curve at the given point.
   (a) $y = 2^x$, $(0, 1)$
   (b) $y = \dfrac{\ln x}{x}$, $(1, 0)$

9. If $f(x) = e^{-x} \cos 2x$, find $f''(0)$.

10. Use logarithmic differentiation to find the derivative of each function.
    (a) $y = x^5 e^x \sqrt{x^2 - x + 1}$
    (b) $y = \sqrt{x}^{\,x}$

11. On what intervals is the function $f(x) = 2x^2 - \ln x$ increasing or decreasing?

12. Find the absolute minimum value of the function $g(x) = e^x - x$.

13. Discuss each curve under the headings A–H of Section 5.5.
    (a) $y = e^x + e^{-2x}$
    (b) $y = \ln(1 + x^2)$

14. (a) Use the change of base formula and a calculator to evaluate $\log_2 93.5$ correct to six decimal places.
    (b) Use the change of base formula to show that

$$\ln b = \frac{1}{\log_b e}$$

    Deduce that the formula for differentiating a logarithmic function can be written as

$$\frac{d}{dx} \log_b x = \frac{1}{x} \log_b e$$

15. The initial size of a bacteria culture is 800. After 4 h there are 7200 bacteria.
    (a) Find the number of bacteria after t hours.
    (b) Find the number of bacteria after 6 h.
    (c) Find the rate of growth after 6 h.
    (d) When will the bacteria population reach 20 000?

16. Yeast in a sugar solution is growing at such a rate that 1 g becomes 1.2 g after 10 h.
(a) Find the mass of the yeast after t hours.
(b) Find the mass of the yeast after 24 h.
(c) In what period of time does the mass double?

17. An isotope of lead, ^{214}Pb, has a half-life of 26.8 min. A sample of this isotope has a mass of 15 g.
(a) Find the mass that remains after t minutes.
(b) Find the mass that remains after an hour.
(c) Find the rate of decrease after an hour.
(d) How long would it take this sample to decay until its mass is 1 g?

18. (a) A sum of \$5000 is invested at an interest rate of 8% per year. Find the amount due at the end of three years if the interest is compounded
(i) annually (ii) semiannually
(iii) daily (iv) continuously
(b) How long would it take for the value of the investment to reach \$8000 if the interest is compounded continuously?

PROBLEMS PLUS

Let $f(x) = \log_2(\log_3(\log_4 x))$.
(a) Find the domain of f. (b) Find $f'(x)$.

8.8 CHAPTER 8 TEST

1. Draw the graphs of the functions $f(x) = e^x$ and $g(x) = \ln x$ using the same axes.
2. By applying transformations to your graph in Question 1, draw the graph of the function $y = 6 - e^{-x}$. Find the domain, range, and asymptote of this function.
3. Evaluate the following limits.

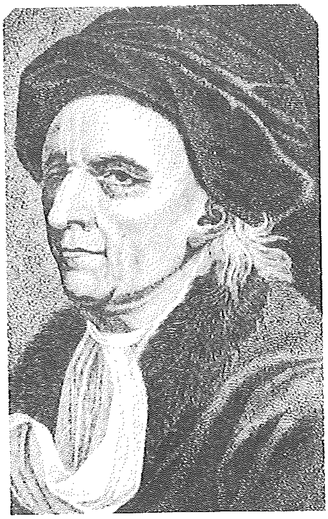
   (a) $\lim_{x\to 8^+} \ln(x - 8)$ (b) $\lim_{x\to\infty} e^{-x^2}$
4. Evaluate $e^{-2\ln 3}$.
5. Find the exact solution of the equation $e^{1-2x} = 5$.
6. Differentiate.

   (a) $y = x^2 \ln(x^3 + 2x - 1)$ (b) $y = \dfrac{e^{4x}}{x^2 + 1}$

   (c) $y = e^{\tan\sqrt{x}}$ (d) $y = 2^{-\frac{1}{x}}$

   (e) $y = x^{x^3}$
7. A bacteria culture starts with 1000 bacteria and grows to a population of 7000 after an hour.
   (a) Find the size of the population after t hours.
   (b) Find the size of the population after 3 h.
   (c) Find the rate of growth of the population after 3 h.
   (d) When will the population reach 10 000?
8. How long does it take for an investment to double in value if the interest rate is 9% compounded continuously?
9. Discuss the curve $y = \ln(9 - x^2)$ under the following headings.
   (a) Domain
   (b) Intercepts
   (c) Symmetry
   (d) Asymptotes
   (e) Intervals of increase or decrease
   (f) Local maximum and minimum values
   (g) Concavity
   (h) Sketch of the curve

FOUNDERS OF CALCULUS



Leonhard Euler (1707–1783) was born in Basel, Switzerland, and studied there under Johann Bernoulli. He left Basel for the last time when he was twenty to assume a research position in St. Petersburg in Russia. Euler contributed new ideas in calculus, algebra, geometry, number theory, and probability. He also worked in many areas of applied mathematics, including acoustics, optics, astronomy, ship design, navigation, mechanics, statistics, and finance. He is the most prolific scientist ever; his *Collected Works* fill three or four rows of shelves in a university library. Much of his work was done after he became blind in one eye in 1735, and totally blind in 1766.

Mathematical notation owes much to Euler. For example, e, i, π and Σ, the summation symbol, were either invented or popularized by him through his many textbooks. These texts were written in Latin, German, or French but translated into all European languages. The three numbers just named are related by his famous formula:

$$e^{i\pi} + 1 = 0$$

which is a special case of

$$e^{ix} = \cos x + i \sin x$$

Euler competed with the Bernoullis in the solving of many problems. A problem that he solved, which they could not, was to evaluate the sum of the reciprocals of the squares of the natural numbers. By a very ingenious method, he proved

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

We say that two natural numbers are *relatively prime* if they have no common divisor greater than one. Euler invented the function $\phi(n)$, called to this day *Euler's phi function*, which counts the number of natural numbers less than n and relatively prime to it. He proved

$$\lim_{n\to\infty} \frac{\phi(n)}{n} = \frac{6}{\pi^2}$$

We can restate this in probabilistic terms: the probability that two numbers chosen at random are relatively prime is

$$\frac{6}{\pi^2}$$

Euler was the first to show the deep interconnection between calculus (analysis) and number theory. His ideas still have an impact on these areas of mathematics.

ANSWERS

CHAPTER 8 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

REVIEW AND PREVIEW TO CHAPTER 8

EXERCISE 1

1. -243 (b) $\frac{1}{64}$ (c) $\frac{625}{8}$ (d) $-\frac{8}{9}$ (e) 6
(f) -4 (g) 25 (h) $\frac{1}{2187}$

2. (a) 2^7 (b) 2^{18} (c) 2^{36} (d) 2^{-2} (e) $2^{-\frac{3}{2}}$
(f) $2^{\frac{1}{2}}$ (g) $2^{\frac{5}{2}}$ (h) 2^0

3. (a) $6x^7y^5$ (b) $8s^9t^3$ (c) $16x^{10}$ (d) $\dfrac{a^2}{b}$ (e) $64r^7s$

(f) $\dfrac{8v^7}{9}$ (g) $\dfrac{x^3}{y}$ (h) $\dfrac{d^7}{c^6}$ (i) $\dfrac{(a + b)^2}{ab}$ (j) $\dfrac{y^{\frac{8}{3}}}{z^2}$

(k) $\dfrac{3t^{\frac{25}{6}}}{s^{\frac{1}{2}}}$ (l) $\dfrac{a^{\frac{9}{2}}x}{b^{10}y^{\frac{19}{3}}}$

EXERCISE 2

1. (a) $2^6 = 64$ (b) $5^0 = 1$ (c) $10^{-2} = 0.01$
(d) $8^{\frac{2}{3}} = 4$ (e) $8^3 = 512$ (f) $2^{-4} = \frac{1}{16}$
(g) $a^c = b$ (h) $r^w = v$

2. (a) $\log_2 8 = 3$ (b) $\log_{10} 100\,000 = 5$
(c) $\log_{10} 0.0001 = -4$ (d) $\log_{81} 9 = \frac{1}{2}$
(e) $\log_4 0.125 = -\frac{3}{2}$ (f) $\log_6 \frac{1}{6} = -1$
(g) $\log_r t = s$ (h) $\log_{10} n = m$

3. (a) 4 (b) 5 (c) 3 (d) 17 (e) 1 (f) 0
(g) -3 (h) $\frac{3}{2}$ (i) $-\frac{2}{3}$ (j) $\frac{1}{4}$

4. (a) 1024 (b) 625 (c) $\frac{95}{3}$ (d) -25
(e) $1 - \log_2 3$ (f) $\frac{1}{2}(1 + \log_3 5)$
(g) 43 046 721 (h) $\log_5(\log_{10} 3)$

EXERCISE 3

1. (a) $\log_2 x + \log_2(x - 1)$ (b) $\log_5 x - \log_5 2$
(c) $\log_2 A + 2 \log_2 B$ (d) $\frac{1}{4} \log_6 17$
(e) $\log_3 x + \frac{1}{2} \log_3 y$ (f) $10 \log_2 x + 10 \log_2 y$

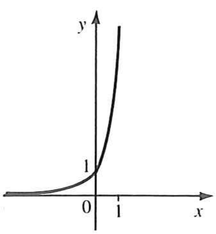
(g) $\frac{1}{3}\log_5(x^2 + 1)$ (h) $2\log_b x - \log_b y - 3\log_b z$ (i) $3\log_{10}x + 4\log_{10}y - 6\log_{10}z$
(j) $2\log_{10}a - 4\log_{10}b - \frac{1}{2}\log_{10}c$

2. (a) $\frac{3}{2}$ (b) 4 (c) 1 (d) $-\frac{1}{2}$ (e) 3 (f) 2

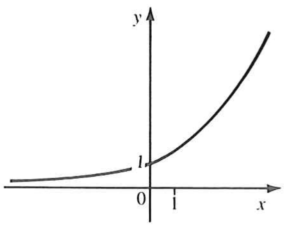
3. (a) $\log_{10}(6\sqrt{7})$ (b) $\log_2\left(\frac{AB}{C^2}\right)$ (c) $\log_5(x + 1)$
(d) $\log_2\left[\frac{x^4(x-1)}{\sqrt[3]{x^2+1}}\right]$ (e) $\log_5\left(y\sqrt{\frac{x}{z^3}}\right)$
(f) $\log_a\left(\frac{bd^c}{s^r}\right)$

EXERCISE 8.1

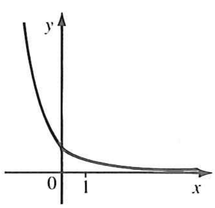
1. (a)



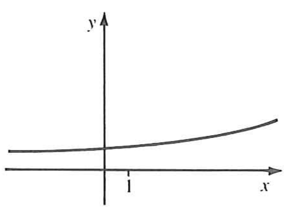
(b)



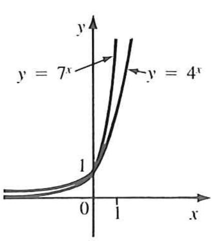
(c)



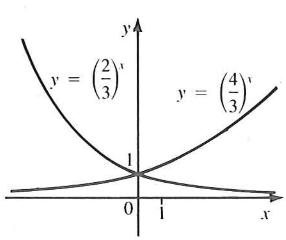
(d)



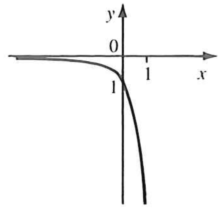
2.



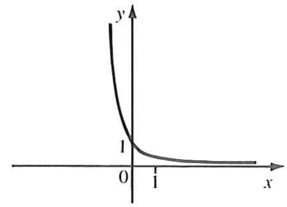
3.



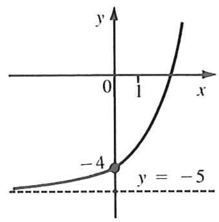
4. (a) domain R, range $(-\infty, 0)$ HA: $y = 0$



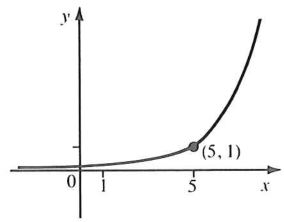
(b) domain R, range $(0, \infty)$ HA: $y = 0$



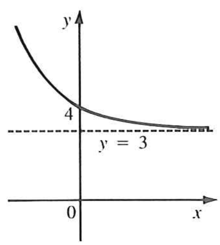
(c) domain R, range $(-5, \infty)$ HA: $y = -5$



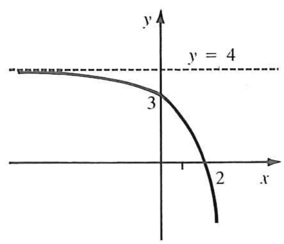
(d) domain R, range $(0, \infty)$ HA: $y = 0$



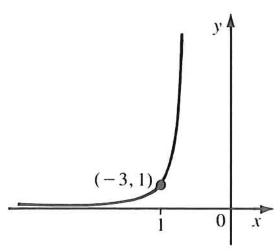
(e) domain R, range $(3, \infty)$ HA: $y = 3$



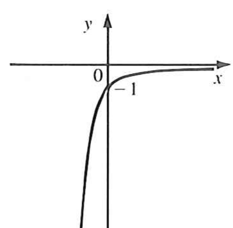
(f) domain R, range $(-\infty, 4)$ HA: $y = 4$



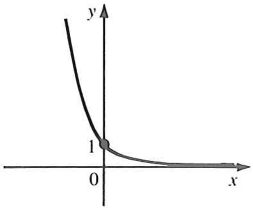
(g) domain R, range $(0, \infty)$ HA: $y = 0$



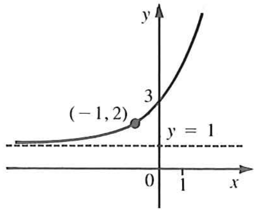
(h) domain R, range $(-\infty, 0)$ HA: $y = 0$



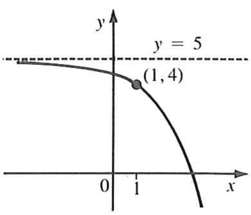
(i) domain R,
range $(0, \infty)$
HA: $y = 0$



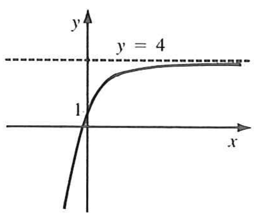
(j) domain R,
range $(1, \infty)$
HA: $y = 1$



(k) domain R,
range $(-\infty, 5)$
HA: $y = 5$



(l) domain R,
range $(-\infty, 4)$
HA: $y = 4$

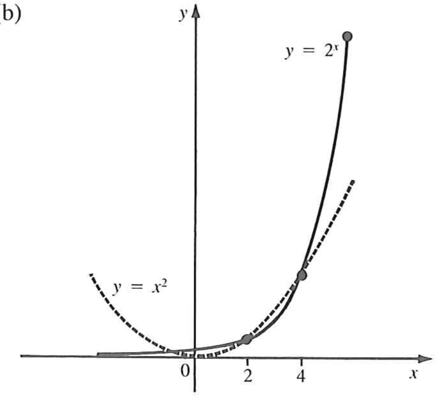


5. (a) 0 (b) 0 (c) ∞ (d) 0 (e) ∞ (f) 0 (g) 0
(h) 1 (i) 0 (j) 0

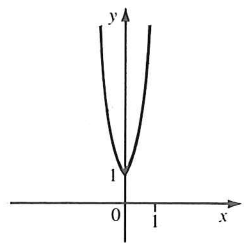
6. (a)

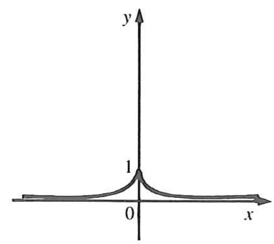
x	x^2	2^x
0	0	1
1	1	2
2	4	4
3	9	8
4	16	16
5	25	32
6	36	64
7	49	128
8	64	256
9	81	512
10	100	1 024
15	225	32 768
20	400	1 048 576

(b)



8. (a) $y = 10^{|x|}$ (b) $y = 10^{-|x|}$

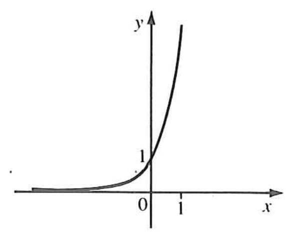




EXERCISE 8.2

1. (a) $2e^x$ (b) e^{4x} (c) e^{2x+1} (d) 1
(e) $e^{2x} - 5e^{5x}$ (f) $6e^{5x}$

2. (a) $y = 5^x$



(b)

h	$\dfrac{5^h - 1}{h}$
0.1	1.746 189
0.01	1.622 459
0.001	1.610 734
0.0001	1.609 567

Slope of secant line

(c) 1.61 (d) The slope of the tangent line to $y = 5^x$ at $(0, 1)$.

3. (a) 0.99 (b) 1.03

4. (a) $y' = -2e^{-x}$ (b) $y' = x^3e^x(4 + x)$
(c) $y' = e^{2x}(2 \sin 3x + 3 \cos 3x)$
(d) $y' = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}$ (e) $y' = (\sec^2 x)e^{\tan x}$
(f) $y' = (e^x) \sec^2(e^x)$ (g) $y' = \dfrac{e^x}{x^2}(x - 1)$
(h) $y' = \dfrac{e^x(1 + e^{2x})}{(1 - e^{2x})^2}$
(i) $y' = 2x \cos(x^2)e^{\sin(x^2)}$
(j) $y' = e^{\cot 4x}(1 - 4x \csc^2 4x)$
(k) $y' = -200e^{-10x}(1 + 5e^{-10x})^3$
(l) $y' = \dfrac{1 - 2xe^{1-x^2}}{2\sqrt{x + e^{1-x^2}}}$

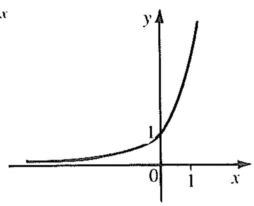
5. $x - y + 1 = 0$ **6.** $\dfrac{2 - ye^{xy}}{xe^{xy} - 1}$ **7.** 64

8. increasing on $(0,2)$, decreasing on $(-\infty, 0)$, $(2,\infty)$ **9.** e

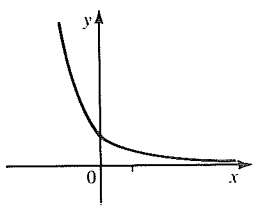
10. (a) $-\dfrac{1}{e}$ (b) CU on $(-2, \infty)$, CD on $(-\infty, -2)$
(c) $(-2, -2e^{-2})$

11. (a) 0 (b) ∞ (c) 0

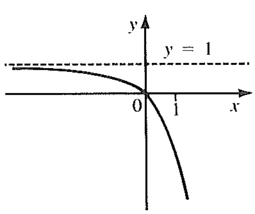
12. (a) $y = e^x$



(b) (i) $y = e^{-x}$

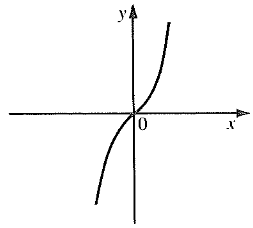


(ii) $y = 1 - e^x$



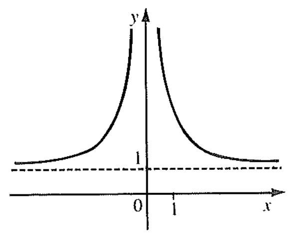
13. (a) A. R B. y-intercept 0 C. about the origin D. none E. increasing on $(-\infty, \infty)$ F. none G. CU on $(0, \infty)$, CD on $(-\infty, 0)$, IP $(0, 0)$

H.

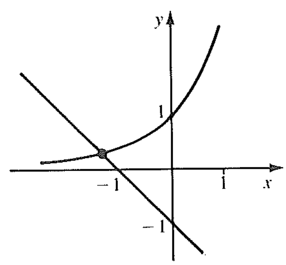


(b) A. $(-\infty, 0) \cup (0, \infty)$ B. none C. about the y-axis D. HA: $y = 1$, VA: $x = 0$ E. increases on $(-\infty, 0)$, decreases on $(0, \infty)$ F. none G. CU on $(-\infty, 0)$, $(0, \infty)$, no IP

H.



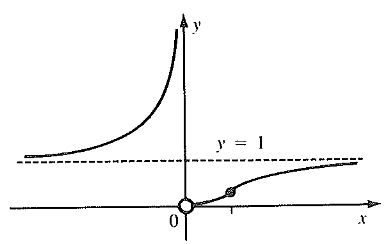
14. (a)



(b) $-1.278\,465$

15. A. $(-\infty, 0) \cup (0, \infty)$ B. none C. none D. HA: $y = 1$, VA: $x = 0$ E. increasing on $(-\infty, 0)$, $(0, \infty)$ F. none G. CU on $(-\infty, 0)$, $\left(0, \frac{1}{2}\right)$ CD on $\left(\frac{1}{2}, \infty\right)$, IP $\left(\frac{1}{2}, e^{-2}\right)$

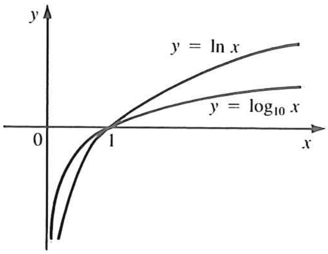
H.



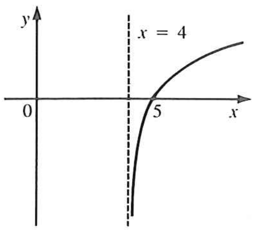
16. $f^{(1\,000\,000)}(x) = e^{-x}(x - 1\,000\,000)$

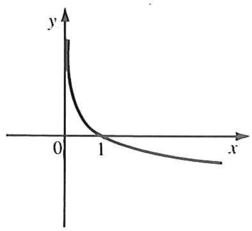
EXERCISE 8.3

1.

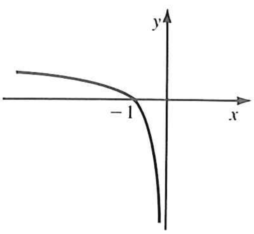


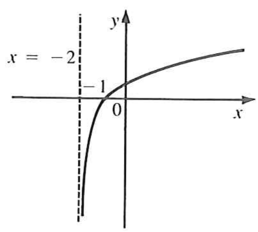
2. (a) domain $(4, \infty)$, range R, asymptote $x = 4$ (b) domain $(0, \infty)$, range R, asymptote $x = 0$



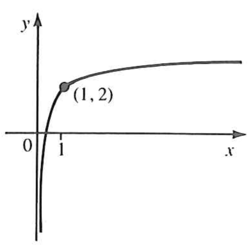


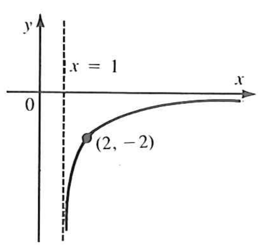
(c) domain $(-\infty, 0)$, range R, asymptote $x = 0$ (d) domain $(-2, \infty)$, range R, asymptote $x = -2$



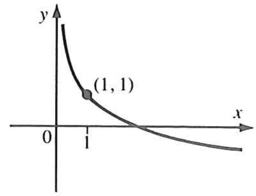


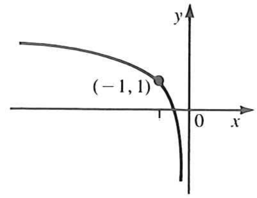
(e) domain $(0, \infty)$, range R, asymptote $x = 0$ (f) domain $(1, \infty)$, range R, asymptote $x = 1$



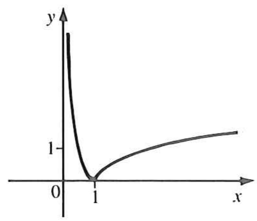


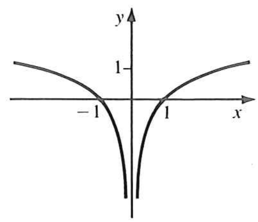
(g) domain $(0, \infty)$, range R, asymptote $x = 0$ (h) domain $(-\infty, 0)$, range R, asymptote $x = 0$





(i) domain $(0, \infty)$, range $[0, \infty)$, asymptote $x = 0$ (j) domain $(-\infty, 0) \cup (0, \infty)$, range R, asymptote $x = 0$





3. (a) 5 (b) 2 (c) 2 (d) 32 (e) $\frac{1}{2}$ (f) 0

4. (a) ln 4 (b) e^6 (c) $\frac{1}{2}(e + 1)$
(d) $\frac{1}{3}(\ln 10 - 5)$ (e) -5 (f) 28
(g) e^{e^2} (h) $\ln(\ln 5)$

5. (a) 19.085 537 (b) 0.693 147
(c) $-0.139\,483$ (d) 6.584 963

6. (a) $\ln[\sqrt[3]{x}(3x - 5)^2]$ (b) $\ln\left(\dfrac{x^2(x^2 + 1)^3}{\sqrt{x^2 - 1}}\right)$

7. (a) $\left(-\frac{2}{5}, \infty\right)$ (b) $\left(-\infty, \frac{10}{3}\right)$
(c) $(-\infty, -1) \cup (1, \infty)$ (d) $(0, 1)$ (e) $(0, 2)$
(f) $[2, 10)$

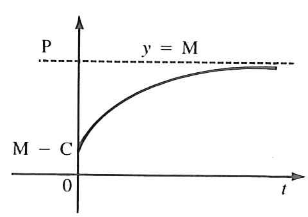
8. $\text{dom}(f) = (-\infty, 0) \cup (0, \infty)$, $\text{dom}(g) = (0, \infty)$

9. (a) $-\infty$ (b) ∞ (c) $-\infty$ (d) $-\infty$

10. (a) 2.807 355 (b) 0.430 677 (c) 2.182 658
(d) 2.523 658

12. $\log_2 7$

13. (a) $P(t) = M - Ce^{-kt}$



(b) $t = -\dfrac{1}{k}\ln\left(\dfrac{M-P}{C}\right)$

14. $\log_4 17$ **15.** (a) $(1, \infty)$ (b) $f^{-1}(x) = 10^{2^x}$

16. (a) (e, ∞) (b) $f^{-1}(x) = e^{e^{e^x}}$ **17.** $\log_2 3$

18. 64

EXERCISE 8.4

1. (a) $f'(x) = x(2\ln x + 1)$

(b) $f'(x) = \dfrac{1}{2x\sqrt{\ln x}}$ (c) $g'(x) = \dfrac{3x^2}{x^3+1}$

(d) $g'(x) = \dfrac{1}{x}$ (e) $y' = \dfrac{1}{x}\cos(\ln x)$

(f) $y' = \cot x$ (g) $y' = \dfrac{1 - 3\ln x}{x^4}$

(h) $y' = 3(x + \ln x)^2\left(1 + \dfrac{1}{x}\right)$

(i) $y' = \dfrac{2}{2x+1}$

(j) $y' = -\dfrac{2}{x^2-1}$ (k) $y' = \dfrac{3}{2x(2x+3)}$

(l) $y' = \dfrac{1}{x(x^2+1)}$ (m) $y' = \sec x$

(n) $y' = \dfrac{-3\sec^2[\ln(1-3x)]}{1-3x}$

2. (a) $f'(x) = \dfrac{1}{x\ln x}$ (b) $(1, \infty)$, $(1, \infty)$

3. (a) $f'(x) = \dfrac{2x}{(x^2+1)\ln 2}$

(b) $g'(x) = \log_{10} x + \dfrac{1}{\ln 10}$

(c) $F'(x) = \dfrac{3}{(3x-8)\ln 5}$

(d) $G'(x) = \dfrac{1 - \ln x - \ln 3}{(\ln 3)\,x^2}$

4. (a) $y' = 3x^2 + 3^x \ln 3$

(b) $y' = (4x^3 - 1)(\ln 2)\,2^{x^4 - x}$

(c) $y' = \frac{1}{2}5^{\sqrt{x}}(\sqrt{x}\ln 5 + 2)$

(d) $y' = \pi(\sec^2 \pi x)\ln 10[10^{\tan \pi x}]$

5. (a) $x - y - 2 = 0$ (b) $x - y - 1 = 0$

(c) $10(\ln 10)\,x - y - 10(\ln 10 - 1) = 0$

(d) $x - 100(\ln 10)\,y + 100(2\ln 10 - 1) = 0$

6. $\dfrac{1}{x+y-1}$ **7.** (a) $-\dfrac{1}{e}$ (b) $(0, \infty)$

8. (a) local maximum $f(e^{-2}) = 4e^{-2}$, local minimum $f(1) = 0$ (b) $\left(\dfrac{1}{e}, \dfrac{1}{e}\right)$

9. (a) A. $(-3, 3)$ B. y-intercept $\ln 9$, x-intercepts $\pm 2\sqrt{2}$ C. about the y-axis

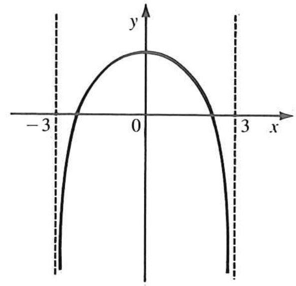
D. VA: $x = 3$, $x = -3$

E. increasing on $(-3, 0)$, decreasing on $(0, 3)$

F. local maximum $f(0) = \ln 9$

G. CD on $(-3, 3)$

H.

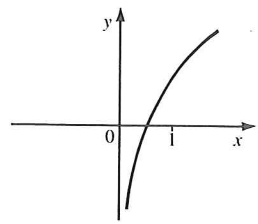


(b) A. $(0, \infty)$ C. none D. VA: $x = 0$

E. increasing on $(0, \infty)$ F. none

G. CD on $(0, \infty)$

H.

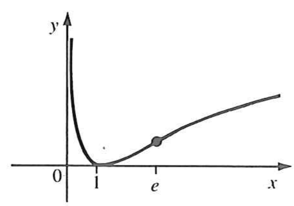


(c) A. $(0, \infty)$ B. x-intercept 1 C. none

D. VA:$x = 0$ E. increasing on $(1, \infty)$, decreasing on $(0, 1)$ F. local minimum $f(1) = 0$

G. CU on $(0, e)$, CD on (e, ∞), IP $(e, 1)$

H.



(d) A. $\{x \mid (4n-1)\frac{\pi}{2} < x < (4n+1)\dfrac{\pi}{2}, n \in I\}$

B. x-intercepts $2n\pi$, $n \in I$, y-intercept 0

C. about the y-axis; period 2π

D. VA: $x = (2n+1)\dfrac{\pi}{2}$, $n \in I$

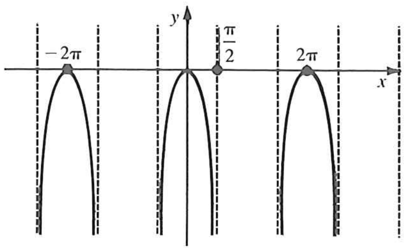
E. increasing on $\left((4n-1)\dfrac{\pi}{2}, 2n\pi\right)$,

decreasing on $\left(2n\pi, (4n+1)\dfrac{\pi}{2}\right)$

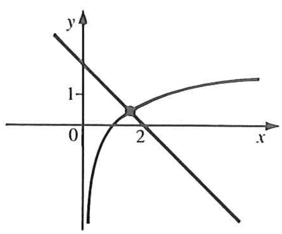
F. local maxima $f(2n\pi) = 0$

G. CD on $\left((4n - 1)\frac{\pi}{2}, (4n + 1)\frac{\pi}{2}\right)$

H.



10. (a)



(b) 1.557 146

12. e^3

EXERCISE 8.5

1. (a) $(1000)\,10^{\frac{t}{2}}$ (b) 316 228 (c) 364 071 bacteria/h (d) after 2 h 21 min

2. (a) $(400)3^t$ (b) $\frac{\ln 2}{\ln 3}$ h $\doteq$ 37.5 min

3. (a) $(500)2^{3t}$ (b) 8.389×10^9
(c) $\frac{\ln 12}{3 \ln 2}$ h $\doteq$ 1 h 11 m

4. (a) 2500 (b) $(2500)\,16^t$ (c) 13 863 bacteria/h
(d) $\frac{\ln 60}{4 \ln 2}$ h $\doteq$ 1 h 28 min

5. (a) 334 580 (b) 602 559

6. (a) (i) 6.6 billion (ii) 47 billion (b) 2103

7. (a) $(100)\,2^{-\frac{t}{4.5 \times 10^9}}$ (b) 99.999 846 mg
(c) 1.540×10^{-8}mg/a

8. (a) $(2)\,2^{-\frac{t}{15}}$ (b) $2^{\frac{2}{3}} \doteq 1.587$ g (c) 0.073 g/h
(d) $\frac{15 \ln 5}{\ln 2} \doteq 34.8$ h

9. (a) 9.972 mg (b) 128 643 a

10. (a) 5 d (b) 19% of original mass

11. (a) \$1345.60 (b) \$1368.57 (c) \$1374.22 (d) \$1376.45 (e) \$1377.03 (f) \$1377.13

12. (a) \$14 641.00 (b) \$14 774.55
(c) \$14 893.54 (d) \$14 917.43
(e) \$14 918.21 (f) \$14 918.25

13. (a) is better **14.** $\frac{\ln 2}{0.085} \doteq 8$ a, 56 d

15. (a) $Ce^{-0.0005t}$ (b) $2000 \ln 2 \doteq 1386$ s

16. 2100 a

EXERCISE 8.6

1. (a) $y' = (x^2 + 1)^2(x^2 + x + 1)^3 \times \left[\frac{4x}{x^2 + 1} + \frac{6x + 3}{x^2 + x + 1}\right]$

(b) $y' = (x - 1)^4(2x + 3)^5(x^2 - 2x + 3)^3 \times \left[\frac{4}{x - 1} + \frac{10}{2x + 3} + \frac{6x - 6}{x^2 - 2x + 3}\right]$

(c) $y' = e^{x^2}x^3(x^2 + 8)^4\left[2x + \frac{3}{x} + \frac{8x}{x^2 + 8}\right]$

(d) $y' = \frac{(x + 1)^3}{(x + 2)^5(x + 3)^7} \times \left[\frac{3}{x + 1} - \frac{5}{x + 2} - \frac{7}{x + 3}\right]$

(e) $y' = \frac{x\sqrt{x + 1}}{(x + 2)(x^3 + 1)} \times \left[\frac{1}{x} + \frac{1}{2(x + 1)} - \frac{1}{x + 2} - \frac{3x^2}{x^3 + 1}\right]$

(f) $y' = \sqrt{\frac{x^2 + 1}{x^2 + 4}}\left[\frac{x}{x^2 + 1} - \frac{x}{x^2 + 4}\right]$

2. (a) $y' = x^{x^2}(2x \ln x + x)$

(b) $y' = x^{\sqrt{x} - \frac{1}{2}}\left(\frac{1}{2} \ln x + 1\right)$

(c) $y' = x^{\cos x}\left[-(\sin x) \ln x + \frac{\cos x}{x}\right]$

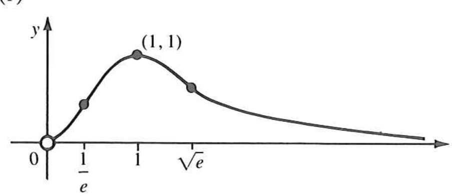
(d) $y' = (\cos x)^x[\ln(\cos x) - x \tan x]$

(e) $y' = (\ln x)^x\left[\ln(\ln x) + \frac{1}{\ln x}\right]$

(f) $y' = (\cos x)^{\sin x}[(\cos x) \ln(\cos x) - (\sin x) \tan x]$

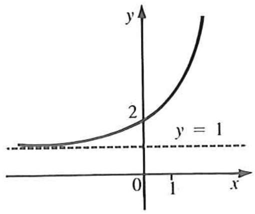
3. $(4 \ln 2 + 4)x - y - 4(2 \ln 2 + 1) = 0$

4. (a) 0, 0 (b) $-2x^{-1 - \ln x}(\ln x)$
(c) increasing on $(0, 1)$, decreasing on $(1, \infty)$
(d) $f(1) = 1$
(e) CU on $\left(0, \frac{1}{e}\right)$, $(\sqrt{e}, \infty)$, CD on $\left(\sqrt{e}, \frac{1}{\sqrt[4]{e}}\right)$
(f)

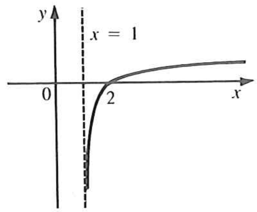


8.7 REVIEW EXERCISE

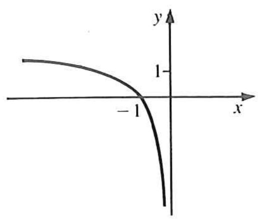
1. (a)



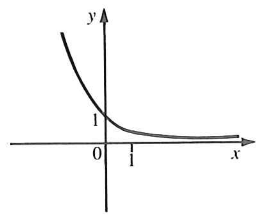
(b)



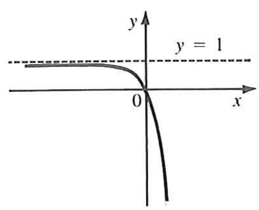
(c)



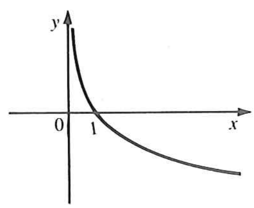
(d)



(e)



(f)



2. (a) 1 (b) $-\infty$ (c) ∞ (d) 0 (e) ∞ (f) $-\infty$

3. (a) $(-2, \infty)$, R, $x = -2$
(b) R, $(1, \infty)$, $y = 1$
(c) R, $(-\infty, 10)$, $y = 10$
(d) $\left(-\infty, \frac{1}{2}\right)$, R, $x = \frac{1}{2}$

4. (a) 0 (b) 10 (c) 8 (d) -1

5. (a) $\sqrt{e} \doteq 1.648\ 721$
(b) $\ln 7 \doteq 1.945\ 910$
(c) $\frac{1}{3}(5 - \ln 2) \doteq 1.435\ 618$
(d) $\frac{1}{4}(e^4 - 7) \doteq 11.899\ 538$

6. (a) $\ln\left[\dfrac{x^2(1 + x)^3}{(2 + x)^4}\right]$ (b) $\ln\left(\dfrac{\sqrt{x}}{(x^2 + x + 1)^2}\right)$

7. (a) $f'(x) = \dfrac{2x}{x^2 + 1}$
(b) $f'(x) = 3x^2e^{x^3}$
(c) $f'(x) = \dfrac{e^x}{2\sqrt{x}}(1 + 2x)$
(d) $f'(x) = \dfrac{1 - 2\ln x}{x^3}$
(e) $y' = 4x^3 - 4^x \ln 4$
(f) $y' = \dfrac{-11}{(2x + 3)(4x - 5)}$
(g) $y' = 2e^{2x}\cos(e^{2x})$
(h) $y' = (2\cos x)e^{2\sin x}$
(i) $y' = \dfrac{3x^2 - 1}{(\ln 10)(1 - x + x^3)}$
(j) $y' = e^x\left(\ln x + \dfrac{1}{x}\right)$
(k) $y' = \dfrac{2e^{x^2}}{x^3}(x^2 - 1)$
(l) $y' = \dfrac{2(\ln x)^3}{x\sqrt{1 + (\ln x)^4}}$

8. (a) $(\ln 2)x - y + 1 = 0$ (b) $x - y - 1 = 0$

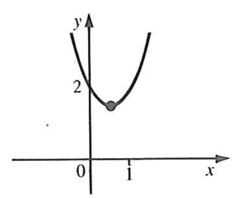
9. -3

10. (a) $y' = x^5e^x\sqrt{x^2 - x + 1} \times \left[\dfrac{5}{x} + 1 + \dfrac{2x - 1}{2(x^2 - x + 1)}\right]$
(b) $y' = \dfrac{\sqrt{x^x}}{2}(\ln x + 1)$

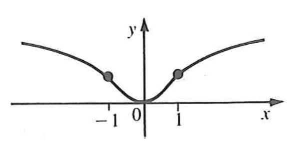
11. increases on $\left(\frac{1}{2}, \infty\right)$, decreases on $\left(0, \frac{1}{2}\right)$

12. $g(0) = 1$

13. (a) A. R B. y-intercept 2 C. none
D. none E. increasing on $\left(\frac{1}{3}\ln 2, \infty\right)$,
decreasing on $\left(-\infty, \frac{1}{3}\ln 2\right)$
F. local minimum $f\left(\frac{1}{3}\ln 2\right) = \frac{3}{2}\sqrt[3]{2}$
G. CU on $(-\infty, \infty)$
H.



(b) A. R B. intercepts 0 C. about y-axis
D. none E. increasing on $(0, \infty)$, decreasing on $(-\infty, 0)$ F. local minimum $f(0) = 0$
G. CU on $(-1, 1)$, CD on $(-\infty, -1)$, $(1, \infty)$, IP $(\pm 1, \ln 2)$
H.



14. (a) 6.546 984

15. (a) $800(3)^{\frac{t}{2}}$ (b) 21 600 (c) 11 865 bacteria/h (d) $\dfrac{2 \ln 25}{\ln 3} \doteq 5$ h, 51 min

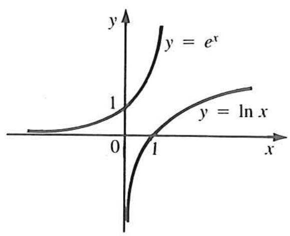
16. (a) $(1.2)^{\frac{t}{10}}$ (b) $(1.2)^{2.4} \doteq 1.549$ g (c) 38 h

17. (a) $15(2)^{\frac{-t}{26.8}}$ (b) 3.178 g (c) 0.082 193 g/min (d) $\dfrac{26.8(\ln 15)}{\ln 2} \doteq 105$ min

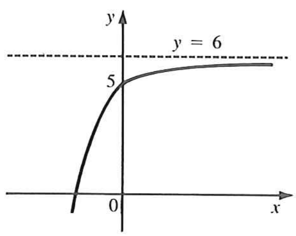
18. (a) (i) \$6298.56 (ii) \$6326.60 (iii) \$6356.08 (iv) \$6356.25 (b) $\dfrac{\ln 1.6}{0.08} \doteq 5.9$ a

8.8 CHAPTER 8 TEST

1.



2. R, $(-\infty, 6)$, $y = 6$



3. (a) $-\infty$ (b) 0 **4.** $\frac{1}{9}$ **5** $\frac{1}{2}(1 - \ln 5)$

6. (a) $y' = \dfrac{x^2(3x^2 + 2)}{x^3 + 2x - 1} + 2x \ln(x^3 + 2x - 1)$

(b) $y' = \dfrac{2e^{4x}}{(x^2 + 1)^2}(2x^2 - x + 2)$

(c) $y' = \dfrac{1}{2\sqrt{x}}(\sec^2 \sqrt{x})e^{\tan\sqrt{x}}$ (d) $y' = \dfrac{\ln 2}{x^2} 2^{-\frac{1}{x}}$

(e) $y' = x^{x^3+2}(3 \ln x + 1)$

7. (a) $(1000)7^t$ (b) 343 000 (c) 667 447 bacteria/h (d) $\dfrac{\ln 10}{\ln 7}$ h $\doteq 1$ h, 11 min

8. $\dfrac{\ln 2}{0.09} \doteq 7.7$ a

9. A. $(-3, 3)$ B. y-intercept $2 \ln 3$, x-intercepts $\pm 2\sqrt{2}$ C. about the y-axis D. VA: $x = \pm 3$ E. decreasing on $(0, 3)$, increasing on $(-3, 0)$ F. maximum $f(0) = 2 \ln 3$ G. CD on $(-3, 3)$

H.