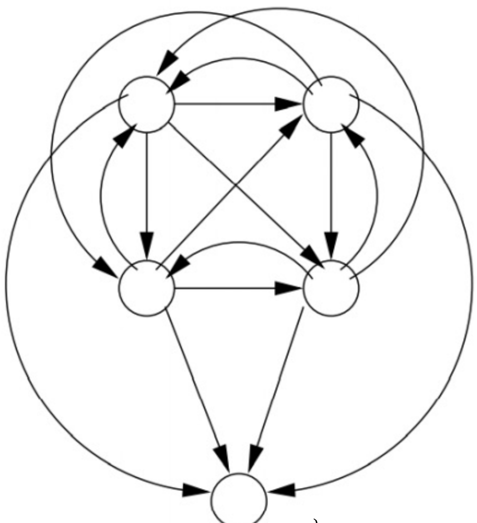
Question 1:

The graph is symmetric, so all the nodes in the clique will have the same pagerank.




So:
$$\begin{cases} Pr(i) = Pr(j) \quad \text{for } i, j \le n & ① \\ n \cdot Pr(i) + Pr(n+1) = 1 & ② \\ Pr(i) = Pr(i)\left(\frac{\beta}{n} + \frac{1-\beta}{n+1}\right)\cdot(n-1) + Pr(i)\cdot\frac{1-\beta}{n+1} + Pr(n+1)\frac{1}{n+1} & ③ \\ Pr(n+1) = \left(\frac{\beta}{n} + \frac{1-\beta}{n-1}\right) Pr(i)\, n + \frac{1}{n+1} Pr(n+1) & ④ \end{cases}$$

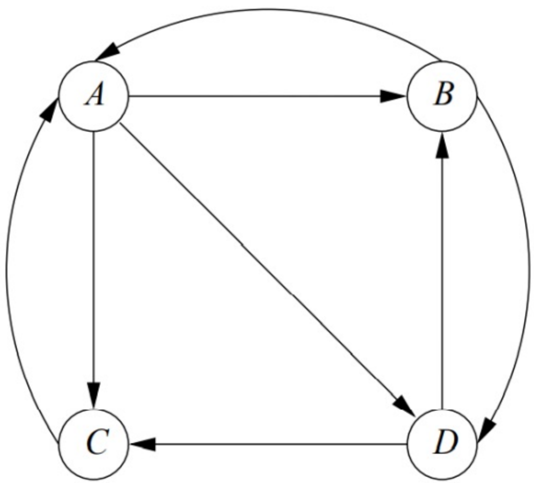
③ ④ is based on the one-step transition

Solve all the formulations:
$$\begin{cases} Pr(i) = \frac{n}{n^2+n+\beta} \\ Pr(n+1) = \frac{n+\beta}{n^2+n+\beta} \end{cases}$$

Question 2:

(a):
$$A=\begin{bmatrix} 0 & \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix} \times 0.8 + 0.2 \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{3}{5} & 1 & \frac{1}{5} \\ \frac{4}{15} & 0 & 0 & \frac{2}{5} \\ \frac{4}{15} & 0 & 0 & \frac{2}{5} \\ \frac{4}{15} & \frac{2}{5} & 0 & 0 \end{bmatrix}$$

故:
$$\begin{cases} \frac{1}{5}x_1 + \frac{3}{5}x_2 + x_3 + \frac{1}{5}x_4 = x_1 \\ \frac{4}{15}x_1 + \frac{2}{5}x_4 = x_2 \\ \frac{4}{15}x_1 + \frac{2}{5}x_4 = x_3 \\ \frac{4}{15}x_1 + \frac{2}{5}x_2 = x_4 \end{cases} \Longrightarrow \begin{cases} x_1 = \frac{3}{7} \\ x_2 = \frac{4}{21} \\ x_3 = \frac{4}{21} \\ x_4 = \frac{4}{21} \end{cases}$$



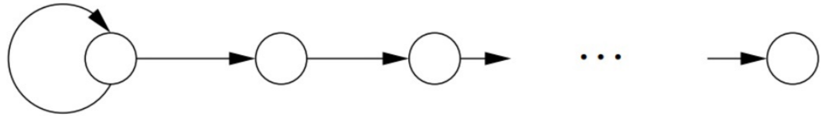

(2) similarly

$$A=\begin{bmatrix} 0+\frac{1-\beta}{2} & \frac{1}{2}\beta+\frac{1-\beta}{2} & \beta+\frac{1-\beta}{2} & 0+\frac{1-\beta}{2} \\ \frac{1}{2}\beta & 0 & 0 & \frac{1}{2}\beta \\ \frac{1}{3}\beta+\frac{1-\beta}{2} & 0+\frac{1-\beta}{2} & 0+\frac{1-\beta}{2} & \frac{1}{2}\beta+\frac{1-\beta}{2} \\ \frac{1}{3}\beta & \frac{1}{2}\beta & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} & \frac{1}{2} & \frac{9}{10} & \frac{1}{10} \\ \frac{4}{15} & 0 & 0 & \frac{2}{5} \\ \frac{11}{30} & \frac{1}{10} & \frac{1}{10} & \frac{1}{2} \\ \frac{14}{15} & \frac{2}{5} & 0 & 0 \end{bmatrix}$$

Using the same method:

$Pr(A) = \frac{27}{70}$ $\quad Pr(B) = \frac{6}{35}$

$Pr(C) = \frac{19}{70}$ $\quad Pr(D) = \frac{6}{35}$



Question 3:

$$A = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 1 \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 \cdots & 1 \\ 0 & 0 & 0 & 0 \cdots & 0 \end{bmatrix}_{N\times N} \quad A^T = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & 1 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}_{N\times N}$$

$$A^TA = \begin{bmatrix} \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} & 0 \\ 0 & I_{(N-2)\times(N-2)} \end{bmatrix} \quad AA^T = \begin{bmatrix} 2 & 0 \\ 0 & I_{(N-1)\times(N-1)} \end{bmatrix}$$

$a$ is the eigenvector of $A^TA$

solve $\lambda a = A^TAa \rightarrow |A^TA - \lambda I| = 0 \rightarrow (1-\lambda)^n - (1-\lambda)^{n-2} = 0 \Rightarrow \lambda = 1/2 \Rightarrow \lambda = 2$ is the principle eigenvalue

$\rightarrow$ solve $a = \left[\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0, \cdots 0\right]^T$

$h$ is the eigenvector of $AA^T$

$\rightarrow$ solve $h = [1, 0, 0, \cdots 0]^T$

Question 4: Assume $M$ is a matrix of $n \times n$

$\lambda_1 \cdots \lambda_n$ are its eigenvalues, $V_1 V_2 \cdots V_n$ are the corresponding eigenvector

So $r^{(0)} = \sum_{i=1}^{n} c_i V_i$

$$M r^{(0)} = \sum_{i=1}^{n} c_i \lambda_i V_i$$

$$M^2 r^{(0)} = \sum_{i=1}^{n} c_i \lambda_i^2 V_i$$

$$\vdots$$

$$M^k r^{(0)} = \lambda_1^k \left[ c_1 V_1 + c_2 \left( \lambda_2 / \lambda_1 \right)^k V_2 + \cdots + c_n \left( \lambda_n / \lambda_1 \right)^k V_n \right]$$

because $\lambda_1$ is the biggest eigenvalue

so: $\lim_{k \to \infty} \left( \frac{\lambda_i}{\lambda_1} \right)^k = 0$ (for $\forall m > 1$)

so $\lim_{k \to \infty} M^k r^{(0)} = c_1 \lambda_1^k V_1$

So $M^k r^{(0)}$ approachs the principal eigenvector of $M$