London theory and Ginzburg-Landau theory of superconductivity

Yanzhi Zhao

Superconductivity is characterized by two fundamental phenomena: zero electrical resistance and the Meissner effect, the latter describing the expulsion of magnetic fields from the bulk of a superconducting material. Before the microscopic BCS theory was developed, phenomenological approaches played a central role in understanding these behaviors. This paper provides a concise review of the London equations and the Ginzburg–Landau theory, two foundational frameworks that successfully describe the Meissner effect and several macroscopic properties of superconductors. Beginning with the London two-fluid model, we discuss how the London equations lead to an exponential decay of magnetic fields within a superconductor. We then introduce the GL free-energy functional and derive the first and second GL equations using variational methods. Applications of the GL theory—including magnetic flux quantization and the definition of the coherence length—are presented to illustrate the physical insights provided by this phenomenological theory.

I. INTRODUCTION

The superconductivity phenomenon was first discovered in 1911 by Dutch physicist Heike Kamerlingh Onnes, when he observed that mercury's electrical resistance vanished as it was cooled to around 4K [1]. Materials that exhibit this phenomenon below a characteristic critical temperature T_c are known as superconductors. Upon cooling through T_c, they undergo a phase transition from the normal state to the superconducting (Meissner) state—characterized by (1) zero electrical resistance and (2) the expulsion of magnetic flux, which is known as the Meissner effect.

Superconductors are classified into Type-I superconductors and Type-II superconductors [2]:

For Type-I superconductors, applying a magnetic field that exceeds the critical field $H_c(T)$ destroys the superconductivity; Above $H_c(T)$, the magnetization decreases suddenly. At the critical field $H_c(T)$, the Meissner state and the normal state are in equilibrium.

Type-II superconductors exhibit perfect diamagnetism up to the lower critical field $H_{c1}(T)$. Above $H_{c1}(T)$, the magnetization decreases continuously rather than suddenly. Magnetic flux begins to penetrate the sample and produces the "mixed state", but does not destroy the superconducting state until the upper critical field $H_{c2}(T)$ is reached.

The occurrence of the Meissner effect indicates that the superconductor cannot be understood simply as a perfect conductor. Before the microscopic BCS theory, the London equations and the Ginzburg–Landau theory provided remarkably successful phenomenological descriptions of superconductivity.

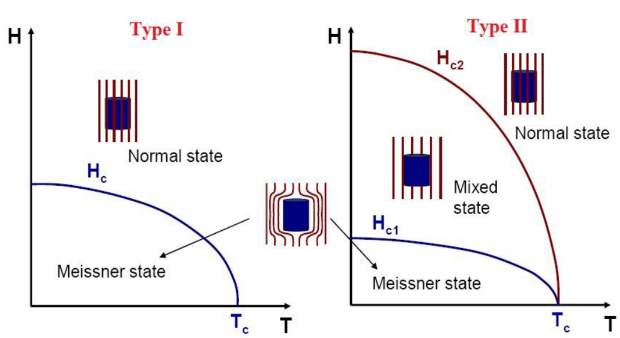

FIG. 1. Phase diagram of Type I and Type II superconductors [3].

II. LONDON THEORY

In 1935, F. and H. London proposed a two-fluid model to predict the Meissner effect [4]. The two-fluid model separates the electron system into a superconducting component with an electron density n_s, and a normal component with an electron density n_n. They assumed the total electron density $n = n_s + n_n$ behaved such that $n_s \to n$ as $T \to 0$, and $n_s = 0$ when $T > T_c$. Hence, in the London brothers' theory, n_s is an order parameter that characterizes the superconducting phase and only depends on the temperature. The superconducting electrons undergo undamped acceleration in the presence of an electric field $\vec{E}$ and form a supercurrent $\vec{J}_s$, and the total current density is $\vec{J} = \vec{J}_n + \vec{J}_s$, $\vec{J}_n$ is the current density contributed by the normal component of the electron system. We can write $\vec{J}_s$ as

$$\vec{J}_s = -en_s\vec{v}_s \tag{1}$$

$\vec{v}_s$ is the velocity of the superconducting electrons.

Combining eqn. (1) with the Newton's law $m_e\frac{d\vec{v}_s}{dt} = -e\vec{E}$ and the Maxwell equation (in cgs unit) $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$, we obtain

$$\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{J} + \frac{n_s e^2}{m_e c}\vec{B}) = 0 \tag{2}$$

Eqn. (2) is satisfied if imposing the following equation, which is now known as the **London equation**

$$\vec{\nabla}\times\vec{J} + \frac{n_s e^2}{m_e c}\vec{B} = 0 \tag{3}$$

Substitute $\vec{B} = \vec{\nabla}\times\vec{A}$ into eqn. (3), we obtain another form of the London equation

$$\vec{J} = -\frac{n_s e^2}{m_e c}(\vec{A} + \vec{\nabla}\chi) \tag{4}$$

Where χ is a scalar field.

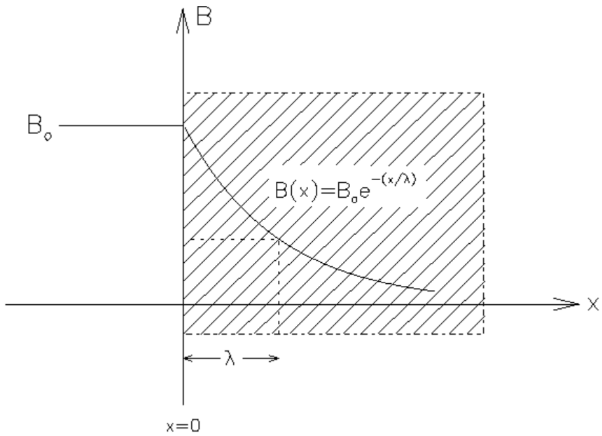

FIG. 2. The London equation predicts the exponential decay of the magnetic field inside the superconductor [4].

It will soon become clear that imposing the London equation forces the magnetic field to decay exponentially as it penetrates the bulk of the superconductor.

To simplify the discussion, assume the static limit where the electromagnetic field distribution outside the superconductor remains unchanged with time. In the static limit, the electric field inside the superconductor has to be zero, otherwise $\vec{v}_s$ and $\vec{j}_s$ will keep ramping to infinity large. So that $\vec{J}_n = 0$, $\vec{J}_s = \vec{J}$. Combining eqn.(3) with the Maxwell equation $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}_s$ and $\vec{\nabla}\cdot\vec{B} = 0$, and , we obtain

$$\nabla^2\vec{B} = \frac{1}{\lambda^2}\vec{B} \tag{5}$$

Where λ is known as the **London penetration depth**, and its formula is given by

$$\lambda = \sqrt{\frac{m_e c^2}{4\pi n_s e^2}} \tag{6}$$

In the static limit, $\vec{\nabla}\cdot\vec{J} = -\partial_t\rho = 0$, so that the scalar field χ can be determined by $\vec{\nabla}^2\chi = -\vec{\nabla}\cdot\vec{A}$. If we apply the **London gauge** $\vec{\nabla}\cdot\vec{A} = 0$, in which case we can choose $\chi = 0$, then the London equation can be simplified as

$$\vec{J}_s = -\frac{n_s e^2}{m_e c}\vec{A} \tag{7}$$

A. Exponential decay of magnetic fields

Considering a superconductor that fills the region $x > 0$ and apply a uniform magnetic field $B_0\hat{x}$ outside, the solution of eqn. (5) indicates the magnetic field inside the superconductor to be $\vec{B}(x) = B_0 e^{-x/\lambda}\hat{x}$. The magnetic field decays exponentially as it penetrates inside the superconductor, and this is basically consistent with the Meissner effect. In the London theory, the supercurrent $\vec{J}_s$ is in charge of expelling the magnetic field from the superconductor.

III. GINZBURG-LANDAU (GL) THEORY OF SUPERCONDUCTIVITY

Although the London theory gives a reasonably well-phenomenological description of the Meissner effect, its order parameter n_s only depends on the temperature. In reality, the superconductivity is also affected by the magnetic field and the spatial variations; therefore, n_s should also depend on the magnetic field $\vec{B}$ and the spatial coordinate $\vec{r}$.

Ginzburg and Landau developed a more advanced phenomenological theory to describe the Meissner effect in the superconductors, which is now known as the **Ginzburg-Landau theory**. They proposed an order parameter $\phi(\vec{r})$ that depends on the temperature, the magnetic field, and the spatial coordinate. $\phi(\vec{r})$ is a complex number that

$$\phi(\vec{r}) = |\phi(\vec{r})|e^{i\theta(\vec{r})},\ |\phi(\vec{r})|^2 = n_s(\vec{r}) \tag{8}$$

Where $|\phi(\vec{r})|$ is zero in the normal state, and is non-zero in the superconducting state.

The GL free energy functional is

$$F[\vec{A},\phi] = \int d^3\vec{r} f[\vec{A},\phi] \tag{9}$$

with free energy density [2]

$$\begin{aligned}f[\vec{A},\phi] &= \frac{1}{2m^*}|(\frac{\hbar}{i}\vec{\nabla} - \frac{q^*}{c}\vec{A})\phi|^2 + \alpha|\phi|^2 + \frac{\beta}{2}|\phi|^4 + \frac{B^2}{8\pi}\\ &= \frac{\hbar^2}{2m^*}(\vec{\nabla}\phi)\cdot(\vec{\nabla}\phi^*) - \frac{1}{2m^*}\frac{\hbar q^*}{ic}\vec{A}\cdot(\vec{\nabla}\phi)\phi^* + \frac{1}{2m^*}\frac{\hbar q^*}{ic}\vec{A}\cdot(\vec{\nabla}\phi^*)\phi\\ &+ (\frac{q^*}{c})^2 A^2|\phi|^2 + \alpha|\phi|^2 + \frac{\beta}{2}|\phi|^4 + \frac{B^2}{8\pi}\end{aligned} \tag{10}$$

q^* and m^* are the charge and the mass of the superconducting carriers. [1] The parameter α is positive above T_c, and negative below T_c; The parameter β is always positive. The parameter α varies with the temperature through

$$\alpha = \alpha_0(t-1),\ t \equiv \frac{T}{T_c},\ \alpha_0 > 0 \tag{11}$$

Minimizing the free energy requires

$$\delta F = 0 \tag{12}$$

Solving Eqn. (12) will lead to the first and the second Ginzburg-Landau equations.

To visualize the profile of the free energy density $f[\vec{A},\phi]$, consider a simple situation where $\vec{A} = 0$, we can then choose the gauge where the order parameter ϕ is real. The free energy density $f[\phi] = \alpha_0(t-1)\phi^2 + \frac{\beta}{2}\phi^4$ takes the minimum value at $\phi = 0$ when $T > T_c$, and at $\phi = \pm\sqrt{\alpha_0(1-t)/\beta}$ when $T < T_c$, as shown in Fig. 3.

[1] The microscopic BCS theory can be expanded into a GL form, and proves $m^* = 2m_e$, $q^* = -2e$.

A. First Ginzburg-Landau equation

Keeping $\vec{A}$ constant and varying ϕ,

$$\begin{aligned}\delta F = \int d^3\vec{r}[\frac{\partial f}{\partial \phi}\delta\phi + \frac{\partial f}{\partial \phi^*}\delta\phi^* \\ + \frac{\partial f}{\partial(\vec{\nabla}\phi)}\delta(\vec{\nabla}\phi) + \frac{\partial f}{\partial(\vec{\nabla}\phi^*)}\delta(\vec{\nabla}\phi^*)]\end{aligned} \quad (13)$$

According to eqn. (10),

$$\begin{aligned}&\frac{\partial f}{\partial \phi}\delta\phi = \\ &\frac{1}{2m^*}[\frac{\hbar q^*}{ic}\vec{A}\cdot(\vec{\nabla}\phi^*) + (\frac{q^*}{c})^2A^2\phi^* + \alpha\phi^* + \beta\phi^*|\phi|^2]\delta\phi\end{aligned} \quad (14)$$

$$\frac{\partial f}{\partial(\vec{\nabla}\phi)}\delta(\vec{\nabla}\phi) = \frac{1}{2m^*}(\hbar^2\vec{\nabla}\phi^* - \frac{\hbar q^*}{ic}\phi^*\vec{A})\delta(\vec{\nabla}\phi) \quad (15)$$

Integrating by parts, we obtain

$$\int d^3\vec{r}(\hbar^2\vec{\nabla}\phi^* - \frac{\hbar q^*}{ic}\phi^*\vec{A})\delta(\vec{\nabla}\phi) = -\int d^3\vec{r}\delta\phi\vec{\nabla}\cdot(\hbar^2\vec{\nabla}\phi + \frac{\hbar q^*}{ic}\phi\vec{A})$$

Thus eqn.(15) can be written as

$$\int d^3\vec{r}\frac{\partial f}{\partial(\vec{\nabla}\phi)}\delta(\vec{\nabla}\phi) = -\int d^3\vec{r}[\frac{1}{2m^*}\vec{\nabla}\cdot(\hbar^2\vec{\nabla}\phi + \frac{\hbar q^*}{ic}\phi\vec{A})]\delta\phi \quad (16)$$

Notice that $(\partial f/\partial\phi^*)\delta\phi^*$ and $(\partial f/\partial(\vec{\nabla}\phi^*))\delta(\vec{\nabla}\phi^*)$ are the complex conjugates of $(\partial f/\partial\phi)\delta\phi$ and $(\partial f/\partial(\vec{\nabla}\phi))\delta(\vec{\nabla}\phi)$, respectively. So we obtain

$$\begin{aligned}\delta F &= \int d^3\vec{r}\{\frac{1}{2m^*}[\frac{\hbar q^*}{ic}\vec{A}\cdot(\vec{\nabla}\phi^*) + (\frac{q^*}{c})^2A^2\phi^* - \hbar^2\vec{\nabla}\cdot(\vec{\nabla}\phi^*) \\ &+ \frac{1}{2m^*}\frac{\hbar q^*}{ic}\vec{\nabla}\cdot(\phi^*\vec{A})] + \alpha\phi^* + \beta\phi^*|\phi|^2\}\delta\phi + c.c. \\ &= \int d^3\vec{r}[\frac{1}{2m^*}(\frac{\hbar}{i}\vec{\nabla} - \frac{q^*}{c}\vec{A})^2\phi^* + \alpha\phi^* + \beta\phi^*|\phi|^2]\delta\phi + c.c.\end{aligned} \quad (17)$$

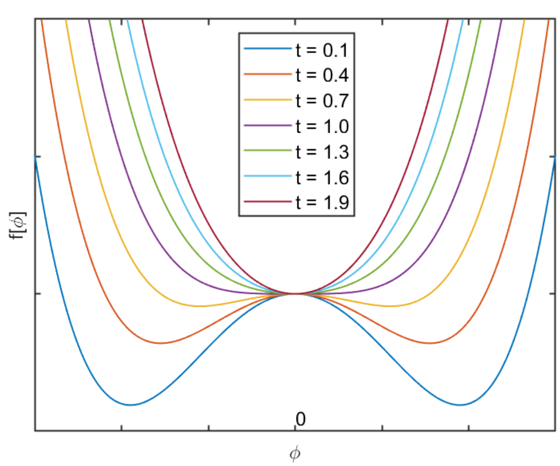

FIG. 3. Free energy density $f[\phi] = \alpha_0(t-1)\phi^2 + \frac{\beta}{2}\phi^4$

For δF to always be zero, we impose

$$\frac{1}{2m^*}(\frac{\hbar}{i}\vec{\nabla} - \frac{q^*}{c}\vec{A})^2\phi + \alpha\phi + \beta\phi|\phi|^2 = 0 \quad (18)$$

This is known as the **First Ginzburg-Landau equation**

B. Second Ginzburg-Landau equation

Keeping ϕ constant and varying $\vec{A}$,

$$\begin{aligned}\delta F &= \int d^3\vec{r}\frac{\partial f}{\partial\vec{A}}\delta\vec{A} \\ &= \int d^3\vec{r}\frac{1}{2m^*}[\frac{\hbar q^*}{ic}\phi(\vec{\nabla}\phi^*) - \frac{\hbar q^*}{ic}\phi^*(\vec{\nabla}\phi) + 2(\frac{q^*}{c})^2|\phi|^2\vec{A}]\cdot\delta\vec{A} \\ &+ \int d^3\vec{r}\frac{1}{4\pi}(\vec{\nabla}\times\delta\vec{A})\cdot(\vec{\nabla}\times\vec{A})\end{aligned} \quad (19)$$

Applying the London gauge $\vec{\nabla}\cdot\vec{A} = 0$, we have

$$\begin{aligned}&\int d^3\vec{r}(\vec{\nabla}\times\delta\vec{A})\cdot(\vec{\nabla}\times\vec{A}) \\ &\equiv \int d^3\vec{r}[\vec{\nabla}\cdot(\delta\vec{A}\times(\vec{\nabla}\times\vec{A})) + \delta\vec{A}\cdot(\vec{\nabla}\times(\vec{\nabla}\times\vec{A}))] \\ &= \int d^3\vec{r}\delta\vec{A}\cdot(\vec{\nabla}\times(\vec{\nabla}\times\vec{A})) = -\int d^3\vec{r}(\nabla^2\vec{A})\cdot\delta\vec{A}\end{aligned}$$

Thus

$$\begin{aligned}&\delta F = \\ &\int d^3\vec{r}\{\frac{1}{2m^*}[\frac{\hbar q^*}{ic}\phi\vec{\nabla}\phi^* - \frac{\hbar q^*}{ic}\phi^*\vec{\nabla}\phi + 2(\frac{q^*}{c})^2|\phi|^2\vec{A}] - \frac{\nabla^2\vec{A}}{4\pi}\}\cdot\delta\vec{A}\end{aligned} \quad (20)$$

For δF to always be zero, we impose

$$\frac{1}{2m^*}[\frac{\hbar q^*}{ic}\phi\vec{\nabla}\phi^* - \frac{\hbar q^*}{ic}\phi^*\vec{\nabla}\phi + 2(\frac{q^*}{c})^2|\phi|^2\vec{A}] - \frac{\nabla^2\vec{A}}{4\pi} = 0 \quad (21)$$

Assuming the static limit and according to the Maxwell equation $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}_s$, the supercurrent

$$\vec{J}_s = \frac{c}{4\pi}\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = -\frac{c}{4\pi}\nabla^2\vec{A} \quad (22)$$

Substitute eqn. (22) into eqn. (21)

$$\vec{J}_s = \frac{\hbar q^*}{2m^*i}(\phi^*\vec{\nabla}\phi - \phi\vec{\nabla}\phi^*) - \frac{q^{*2}}{m^*c}|\phi|^2\vec{A} \quad (23)$$

This is known as the **Second Ginzburg-Landau equation**. Substitute eqn. (8) into eqn. (23), the second GL equation can be written as

$$\vec{J}_s = \frac{q^*\hbar}{m^*}|\phi|^2(\vec{\nabla}\theta - \frac{q^*}{c\hbar}\vec{A}) \quad (24)$$

If ignoring the spatial variation of the order parameter ϕ, and substitute $|\phi|^2$ by n_s, then the second GL equation (eqn. (24)) reduces to the London equation (eqn. (7)) that predicts the exponential decay and expulsion of a magnetic field inside a superconductor.

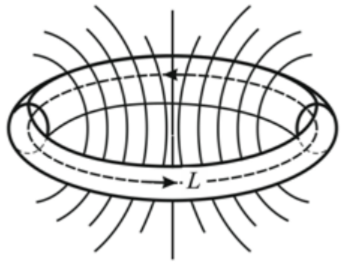

FIG. 4. Magnetic flux through a superconducting ring [5].

C. Magnetic Flux Quantization

Consider a superconducting ring shown in Fig. 4, from eqn. (24),

$$\vec{\nabla}\theta = \frac{m^*}{q^*\hbar|\phi|^2}\vec{J}_s + \frac{q^*}{c\hbar}\vec{A}$$

Integrate $\vec{\nabla}\theta$ along a closed loop L inside the superconducting ring, as shown in Fig. 4. Along the loop, $\vec{J}_s = 0$, so that

$$\oint_L (\vec{\nabla}\theta)\cdot d\vec{l} = \frac{q^*}{c\hbar}\oint_L \vec{A}\cdot d\vec{l} = \frac{q^*}{c\hbar}\Phi_0 \tag{25}$$

Where $\Phi_0 \equiv \iint \vec{B}\cdot d\vec{a}$ is the magnetic flux through the closed loop L. Since the phase θ of the order parameter $\phi(\vec{r})$ must be unique, or differ by a multiple of 2π at each point, we obtain

$$2\pi n = \oint_L (\vec{\nabla}\theta)\cdot d\vec{l} \tag{26}$$

$$\Phi_0 = \frac{2\pi c\hbar}{q^*}n,\ n = 0,1,2,... \tag{27}$$

The magnetic flux Φ_0 is quantized.

D. Coherence Length

First, considering the simplest case where the external magnetic field of the superconductor is zero, and the order parameter ϕ does not have spatial variations, we choose the gauge such that ϕ is real, then the first GL equation reduces to

$$\alpha\phi + \beta\phi^3 = 0 \tag{28}$$

The solution of eqn. (28) is $\phi = 0$ and $\phi^2 = -\alpha/\beta$. Denote $\phi_0^2 \equiv -\alpha/\beta$. From Fig. 3, when $\phi^2 = \phi_0^2$, the superconductor has the minimum free energy and is in the ground state.

If the external magnetic field of the superconductor is zero, but the order parameter ϕ has a small spatial variation, the first GL equation reduces to

$$-\frac{\hbar^2}{2m^*}\nabla^2\phi + \alpha\phi + \beta\phi^3 = 0 \tag{29}$$

The order parameter can be expressed as $\phi = \phi_0 + f$, where f is a small perturbation to ϕ_0, which satisfies eqn. (28). Substitute $\beta = -\alpha/\phi_0^2$ into eqn. (29), we obtain

$$\nabla^2 f - \frac{4|\alpha|m^*}{\hbar^2}f = 0 \tag{30}$$

Define the coherence length ξ as

$$\xi = \sqrt{\frac{\hbar^2}{4m^*|\alpha|}} \tag{31}$$

Then eqn. (30) can be written as

$$\nabla^2 f - \frac{1}{\xi^2}f = 0 \tag{32}$$

Eqn. (32) indicates that $f \propto e^{-\xi r}$, so the coherence length ξ characterizes the scale over which the order parameter ϕ and the superconducting carrier density n_s varies.

IV. CONCLUSION

The London theory and the GL theory together form the backbone of our phenomenological understanding of superconductivity. The London equations successfully predict the exponential decay of magnetic fields inside superconductors, thereby explaining the Meissner effect and introducing the penetration depth as an essential physical scale. However, because the London order parameter is treated as spatially uniform, the theory cannot account for variations in the superconducting state or for phenomena arising from spatial inhomogeneity.

The GL theory addresses these limitations by introducing a complex order parameter $\phi(r)$ whose magnitude and phase encode both the density and coherence of the superconducting condensate. Minimization of the GL free-energy functional yields the first and second GL equations, from which important physical consequences naturally follow—including magnetic flux quantization and the identification of the coherence length as a second fundamental length scale. Although phenomenological, the GL theory remains widely used due to its flexibility and accuracy near T_c.

[1] D. V. Delft and P. Kes, The discovery of superconductivity, Europhysics News **42(1)**, 21 (2011).

[2] S. Girvin and K. Yang, *Modern condensed matter physics* (Cam-

bridge University Press, 2019).
[3] P. C. Llosa, M.Sc. thesis, Autonomous University of Barcelona (2012).
[4] J. E. Sonier, M.Sc. thesis, University of British Columbia (1994).
[5] N. Podoprygora1 and A. Tkachenko, How the cycle of scientific knowledge is reflected in the course of solid state physics: the effect of magnetic flux quantization, American Journal of Educational Research **2 (No. 12B)**, 61 (2014).