APPENDIX A

Proposition: For all natural numbers n, the following position holds: $q_2(n) \le 2n - \lceil 2\sqrt{n} \rceil$.

Proof:

Step 1: Base Case (n=0)

First, we prove that the proposition holds when $n = 0$. That is:

$$q_2(0) = 0 = 2 \times 0 - \lceil 2\sqrt{0} \rceil$$

Therefore, the base case is established.

Step 2: Inductive Hypothesis

We assume that the proposition is true for some positive integer t, which means:

$$q_2(t) \le 2t - \lceil 2\sqrt{t} \rceil$$

$$q_2(t) = \frac{4}{3}(t-1) - \frac{1}{3}q_{20}^{t} + \frac{1}{3}q_{22}^{t}$$

and we also have the number of squares in the corresponding skeleton s^t as $q_2(t) - t + 1$.

Step 3: Inductive Step

We want to prove that if the proposition holds for t, it also holds for $t+1$. Based on the inductive hypothesis, we have:

$$q_2(t+1) = \frac{4}{3}t - \frac{1}{3}q_{20}^{t+1} + \frac{1}{3}q_{22}^{t+1}$$

When changing from t to $t+1$, there will be two cases on the number of squares in the skeleton: increase by 1 or remain unchanged. Therefore, we get:

$$q_2(t+1) = q_2(t) + 1\text{: } q_{20}^{t+1} = q_{20}^{t} + 1, q_{22}^{t+1} = q_{22}^{t}$$

$$q_2(t+1) = q_2(t) + 2\text{: } \begin{cases} q_{20}^{t+1} = q_{20}^{t} - 1, q_{22}^{t+1} = q_{22}^{t} + 1 \\ q_{20}^{t+1} = q_{20}^{t}, q_{22}^{t+1} = q_{22}^{t} + 2 \\ q_{20}^{t+1} = q_{20}^{t} - 2, q_{22}^{t+1} = q_{22}^{t} \end{cases}$$

Let $m = \lceil \sqrt{t} \rceil$, we now characterize $\lceil 2\sqrt{t+1} \rceil - \lceil 2\sqrt{t} \rceil$ for the following cases:

i. $(m-1)^2 < t < m(m-1)$. Since

$$m - 1 = \sqrt{(m-1)^2} < \sqrt{t} < \sqrt{m(m-1)} < m - \frac{1}{2}$$

$$m-1 < \sqrt{(m-1)^2 + 1} < \sqrt{t+1} \le \sqrt{m(m-1)} < m - \frac{1}{2}$$

yielding

$$\lceil 2\sqrt{t+1} \rceil - \lceil 2\sqrt{t} \rceil = 0$$

ii. $t = m(m-1)$. Since

$$\left(m - \frac{1}{2}\right) < \sqrt{\left(m - \frac{1}{2}\right) + \frac{3}{4}} = \sqrt{t+1} = \sqrt{m^2 - m + 1} \le m$$

$$m-1 < \sqrt{t} = \sqrt{m(m-1)} = \sqrt{\left(m - \frac{1}{2}\right) - \frac{1}{4}} < \left(m - \frac{1}{2}\right)$$

yielding

$$\lceil 2\sqrt{t+1} \rceil - \lceil 2\sqrt{t} \rceil = 1$$

iii. $m(m-1) < t < m^2$. Since

$$m - \frac{1}{2} < \sqrt{\left(m - \frac{1}{2}\right)^2 + \frac{3}{4}} = \sqrt{m(m-1)+1} \le \sqrt{t} < m$$

$$m - \frac{1}{2} < \sqrt{m(m-1)+1} < \sqrt{t+1} \le \sqrt{m^2} = m$$

yielding

$$\lceil 2\sqrt{t+1} \rceil - \lceil 2\sqrt{t} \rceil = 0$$

iv. $t = m^2$. Since

$$\sqrt{t} = m$$

$$m < \sqrt{t+1} \le \sqrt{m^2 + m} < \sqrt{\left(m + \frac{1}{2}\right)^2 - \frac{1}{4}} < m + \frac{1}{2}$$

yielding

$$\lceil 2\sqrt{t+1} \rceil - \lceil 2\sqrt{t} \rceil = 1$$

Therefore, the relationship between $\lceil 2\sqrt{t} \rceil$ and $\lceil 2\sqrt{t+1} \rceil$ can be expressed as:

$$\lceil 2\sqrt{t+1} \rceil - \lceil 2\sqrt{t} \rceil = \begin{cases} 0 & (m-1)^2 < t < m(m-1) \\ 1 & t = m(m-1) \\ 0 & m(m-1) < t < m^2 \\ 1 & t = m^2 \end{cases}$$

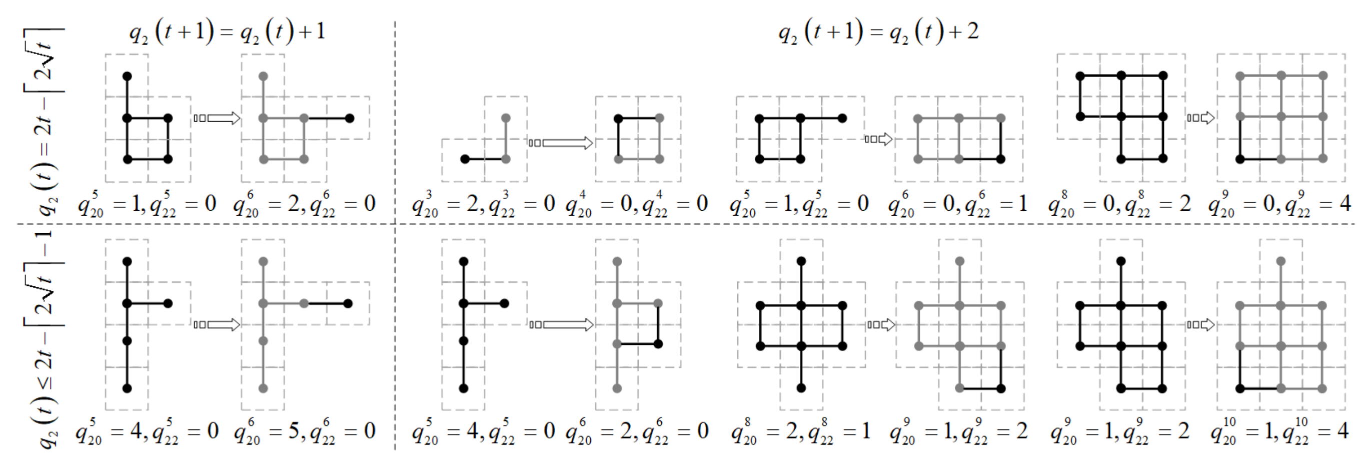

Fig. 1. Changes of $q_2(t)$ as t changes.

As for $q_2\left(t+1\right) = q_2\left(t\right)+1$, based on the inductive hypothesis, we have:

$$q_2\left(t+1\right) \le 2\left(t+1\right) - \lceil 2\sqrt{t} \rceil - 1 \le 2\left(t+1\right) - \lceil 2\sqrt{t+1} \rceil$$

As for $q_2\left(t+1\right) = q_2\left(t\right)+2$, two cases are considered:

a. $q_2\left(t\right)$ cannot be minimized, i.e., $q_2\left(t\right) \le 2t - \lceil 2\sqrt{t} \rceil - 1$, we get:

$$q_2\left(t+1\right) \le 2t - \lceil 2\sqrt{t} \rceil - 1 + 2 \le 2\left(t+1\right) - \lceil 2\sqrt{t+1} \rceil$$

b. $q_2\left(t\right)$ is minimized, i.e., $q_2\left(t\right) = 2t - \lceil 2\sqrt{t} \rceil$. Since $q_2\left(t+1\right) = q_2\left(t\right)+1$ when t is equal to m^2 or $m\left(m-1\right)$, we always have $\lceil 2\sqrt{t+1} \rceil - \lceil 2\sqrt{t} \rceil = 0$ under $minq_2\left(t\right)$ and $q_2\left(t+1\right) = q_2\left(t\right)+2$. Thus,

$$q_2\left(t+1\right) = 2t - \lceil 2\sqrt{t} \rceil + 2 = 2\left(t+1\right) - \lceil 2\sqrt{t+1} \rceil$$

Combining above cases, we have $q_2\left(t+1\right) \le 2\left(t+1\right) - \lceil 2\sqrt{t+1} \rceil$, i.e., the proposition is true for t and $t+1$. Therefore, we can conclude that he proposition holds for all natural numbers n.