AI(Fall 2020)-Assignment 2: CSP and KR

1. Provide formulations for each of the following problems as CSPs: specify the variables, domains and constraints.

(a) Magic square: An order 3 magic square is a 3×3 square grid filled with distinct positive integers in the range $1, 2, \ldots, 9$ such that the sums of the integers in each row, column and diagonal are equal.

- **Variables:** $V_{11}, V_{12}, V_{13}, V_{21}, V_{22}, V_{23}, V_{31}, V_{32}, V_{33}$
- **Domains:**
  - $Dom[V_{ij}] = \{1 - 9\}$
- **Constraints:**
  - Row constraints:
    - $V_{11} + V_{12} + V_{13} = sum$
    - $V_{21} + V_{22} + V_{23} = sum$
    - $V_{31} + V_{32} + V_{33} = sum$
  - Column constraints:
    - $V_{11} + V_{21} + V_{31} = sum$
    - $V_{12} + V_{22} + V_{32} = sum$
    - $V_{13} + V_{23} + V_{33} = sum$
  - Diagonal constraints:
    - $V_{11} + V_{22} + V_{33} = sum$
    - $V_{13} + V_{22} + V_{31} = sum$
  - (Note: sum is a positive integer)

(b) Independent set: Given a graph and a number k, find an independent set of size k, that is, a set of k vertices, no two of which are adjacent.

- **Variables:** $V_1, V_2, \ldots, V_v$ (Note: v is the number of vertices in graph)
- **Domains:** $V_i = \{0, 1\}$ (Note: 0 representing not involved in set, 1 representing involved in set)
- **Constraints:**
  - $V_i = 1, V_j = 1 \Rightarrow \neg \exists e_{ij} \in E(G)$ (Note: $E(G)$ representing the edge set of graph)
  - $\sum\limits_{i=1}^{v} V_i = k$

(c) Crypto-arithmetic puzzle: $INT \times L = AAAI$. We want to replace each letter by a different digit so that the equation is correct.

- **Variables:** I, N, T, L, A
- **Domains:** $I, N, T, L, A = \{0 - 9\}$
- **Constraints:**
  - All-diff(I, N, T, L, A)
  - $(I * 100 + N * 10 + T) * L = A * 1000 + A * 100 + A * 10 + I$

2. Consider the following CSP with binary constraints.

There are 4 variables: A, B, C, D with their respective domains:
$D_A = \{1, 2, 3, 4\}, D_B = \{3, 4, 5, 8, 9\}, D_C = \{2, 3, 5, 6, 7, 9\}, D_D = \{3, 5, 7, 8, 9\}$.

The constraints are: $C_1 : A \geq B$, $C_2 : (B > C) \vee (C - B = 2)$, $C_3 : C \neq D$

(a) Find the first solution by using the Forward Checking algorithm with the MRV heuristic, i.e., always choose the variable with smallest remaining number of elements in the domain to instantiate, breaking ties in alphabetic order. Assign values in the current domain of each variable in increasing order. At each node indicate:

1. The variable being instantiated and the value being assigned to it.
2. The CurDom for each variable.
3. Mark any node with an empty CurDom with DWO.

$D_A = \{1, 2, 3, 4\}$
$D_B = \{3, 4, 5, 8, 9\}$
$D_C = \{2, 3, 5, 6, 7, 9\}$
$D_D = \{3, 5, 7, 8, 9\}$

$D_A = 1, 2$

$D_B = \{\ \}$ ($A \geq B$, DWO)
$D_C = \{2, 3, 5, 6, 7, 9\}$
$D_D = \{3, 5, 7, 8, 9\}$

$D_A = 3$

$D_B = \{3\}$ ($A \geq B$)
$D_C = \{2, 3, 5, 6, 7, 9\}$
$D_D = \{3, 5, 7, 8, 9\}$

$D_B = 3$

$D_C = \{2, 5\}$ ($B > C \vee C - B = 2$)
$D_D = \{3, 5, 7, 8, 9\}$

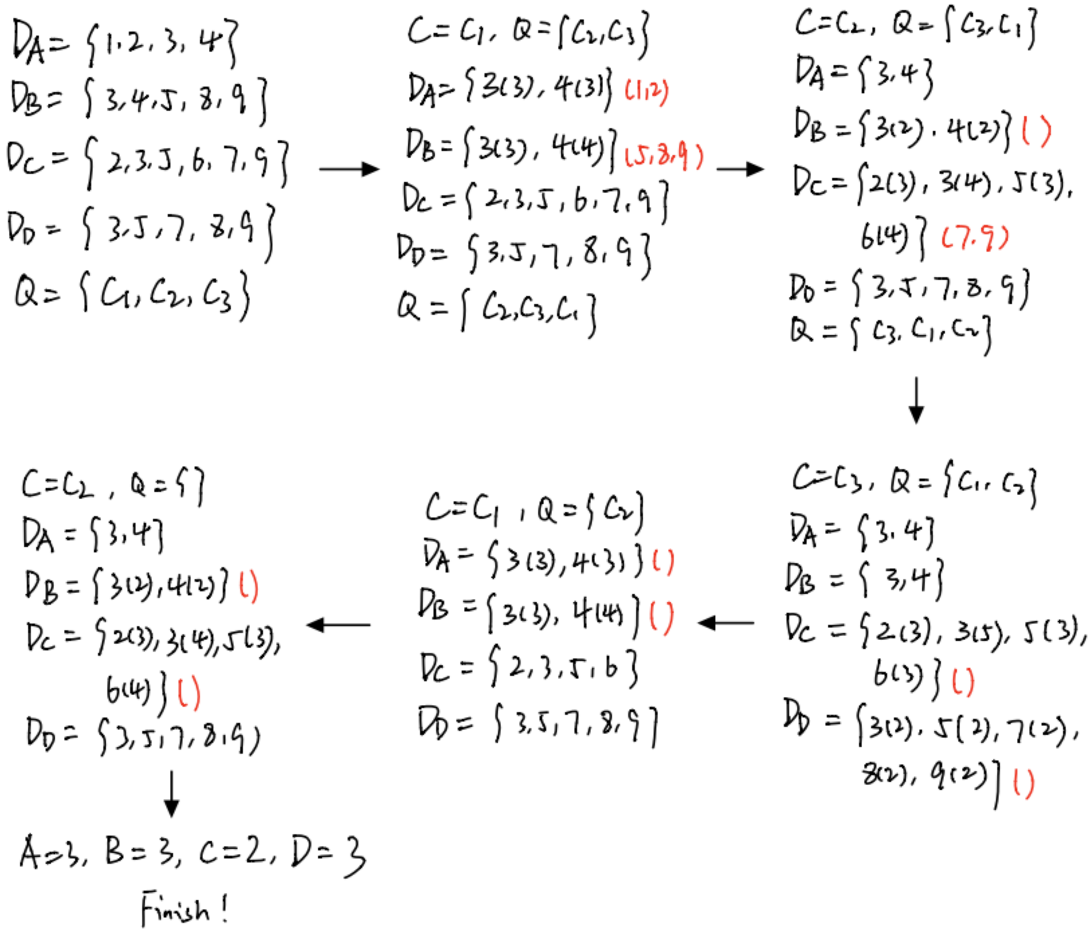
$D_C = 2$

$D_D = \{3, 5, 7, 8, 9\}$ ($C \neq D$)

$D_D = 3$

Finished!

(b) Enforce GAC on he constraints and give the resultant variable domains. You should show which values of a domain are removed at each step, and which arc is responsible for removing the value. After this first step, use the GAC algorithm to find the first solution.



3. Consider the following facts about the Elm Street Bridge Club:

- Joe, Sally, Bill, and Ellen are the only members of the club.
- Joe is married to Sally. Bill is Ellen's brother.
- The spouse of every married person in the club is also in the club.

From these facts, most people would be able to determine that Ellen is not married.

(a) Represent these facts as sentences in FOL, and show semantically that by themselves they do not entail that Ellen is not married.

Vocabulary:

- Member(x): true when person x is a member of the Elm Street Bridge Club
- Brother(x, y): true when person x is brother of person y
- Married(x, y): true when person x is married to person y
- Equal(x, y): true when person x is equal to person y

Sentences:

$$Member(Joe)$$
$$Member(Sally)$$
$$Member(Bill)$$
$$Member(Ellen)$$
$$\forall x(Member(x)) \rightarrow (Equal(x, Joe) \vee Equal(x, Sally) \vee Equal(x, Bill) \vee Equal(x, Ellen))$$
$$Married(Joe, Sally)$$
$$Brother(Ellen, Bill)$$
$$\forall x, y(Member(x) \wedge Married(x, y)) \rightarrow (Member(y))$$

Because there is no sentence describing the relationship between spouse and brother, it's impossible for us to entail that Ellen is not married. In fact, if someone is brother of another, they won't be spouse either.

(b) Write in FOL some additional facts that most people would be expected to know, and show that the augmented set of sentences now entails that Ellen is not married.

Augmented set:

$$Member(Joe)$$
$$Member(Sally)$$
$$Member(Bill)$$
$$Member(Ellen)$$
$$\forall x(Member(x)) \rightarrow (Equal(x, Joe) \vee Equal(x, Sally) \vee Equal(x, Bill) \vee Equal(x, Ellen))$$
$$Married(Joe, Sally)$$
$$Brother(Ellen, Bill)$$
$$\forall x, y(Member(x) \wedge Married(x, y)) \rightarrow (Member(y))$$
$$\forall x, y(Brother(x, y)) \rightarrow (\neg Married(x, y))$$
$$\forall x, y, u, v(Married(x, y) \wedge Married(u, v)) \rightarrow (\neg Equal(y, v))$$
$$\forall x, y(Married(x, y)) \rightarrow (Married(y, x))$$
$$\forall x, y, z(Married(x, y) \wedge Equal(y, z)) \rightarrow (Married(x, z))$$

Proof:

$$1. Member(Joe)$$
$$2. Member(Sally)$$
$$3. Member(Bill)$$
$$4. Member(Ellen)$$
$$5. (\neg Member(x), Equal(x, Joe), Equal(x, Sally), Equal(x, Bill), Equal(x, Ellen))$$
$$6. Married(Joe, Sally)$$
$$7. Brother(Ellen, Bill)$$
$$8. (\neg Member(y), \neg Married(y, z), Member(z))$$
$$9. (\neg Brother(u, v), \neg Married(u, v))$$
$$10. (\neg Married(a, b), \neg Married(c, d), \neg Equal(b, d))$$
$$11. (\neg Married(e, f), Married(f, e))$$
$$12. Married(Ellen, w)$$
$$13. R[12, 8b](y = Ellen, z = w)\ (\neg Member(Ellen), Member(w))$$
$$14. R[13, 4]\ Member(w)$$
$$15. R[7, 9a](u = Ellen, v = Bill)\ \neg Married(Ellen, Bill)$$
$$16. R[12, 10a](a = Eleen, b = w)\ (\neg Married(c, d), \neg Equal(w, d))$$
$$17. R[6, 16a](c = Joe, d = Sally)\ (\neg Married(Joe, Sally), \neg Equal(w, Sally))$$
$$18. R[6, 17a]\ \neg Equal(w, Sally)$$
$$19. R[6, 11a](e = Joe, f = Sally)\ Married(Sally, Joe)$$
$$20. R[19, 16a](c = Sally, d = Joe)\ \neg Equal(w, Joe)$$
$$21. R[12, 11a](e = Ellen, f = w)\ Married(w, Ellen)$$
$$22. R[21, 16a](c = w, d = Ellen)\ \neg Equal(w, Ellen)$$
$$23. R[14, 5a](x = w)\ (Equal(w, Joe), Equal(w, Sally), Equal(w, Bill), Equal(w, Ellen))$$
$$24. R[20, 23a]\ (Equal(w, Sally), Equal(w, Bill), Equal(w, Ellen))$$
$$25. R[18, 24a]\ (Equal(w, Bill), Equal(w, Ellen))$$

$$26. R[22, 25b]\ Equal(w, Bill)$$
$$27. (\neg Married(g, h), \neg Equal(h, i), Married(g, i))$$
$$28. R[12, 27a](g = Ellen, h = w)\ (\neg Equal(w, i), Married(Ellen, i))$$
$$29. R[26, 28a](i = Bill)\ Married(Ellen, Bill)$$
$$30. R[29, 15]\ ()$$

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4. Consider the following formulae asserting that a binary relation is symmetric, transitive, and serial:

- $S_1 : \forall x \forall y (P(x, y) \to P(y, x))$
- $S_2 : \forall x \forall y \forall z (P(x, y) \land P(y, z) \to P(x, z))$
- $S_3 : \forall x \exists y P(x, y)$

Prove by resolution that

$$S_1 \land S_2 \land S_3 \models \forall x P(x, x)$$

In other words, if a binary relation is symmetric, transitive and serial, then it is reflexive.

$$1. (\neg P(x, y), P(y, x))$$
$$2. (\neg P(u, v), \neg P(v, w), P(u, w))$$
$$3. P(z, a)$$
$$4. \neg P(b, b)$$
$$5. [4, 2c](u = b, w = b)\ (\neg P(b, v), \neg P(v, b))$$
$$6. [3, 5a](z = b, v = a)\ \neg P(a, b)$$
$$7. [6, 1b](x = b, y = a)\ \neg P(b, a)$$
$$8. [3, 7](z = b)\ ()$$