3.3.2 Numerical method

In this subsection, we propose a procedure to simulate the dynamics of the whole fluids-membrane system. The procedure mainly contains two FEM based numerical schemes for weak formulations (3.56a)-(3.56f) and (3.57a)-(3.57f), respectively, and the modification of the numerical mesh.

We uniformly partition the time domain $[0,T]$ as $[0,T]=\bigcup_{m=1}^{M}[t_{m-1},t_m]$ with $t_m=m\tau$ and $\tau=T/M$, and the reference domain $D_{\mathbf{r}}=[0,1]$ for the fluid-fluid interface as $D_{\mathbf{r}}=\bigcup_{j=1}^{\mathcal{R}}D_{\mathbf{r},j}$ where $D_{\mathbf{r},j}=[\zeta_{j-1},\zeta_j]$ with $\zeta_j=jh_{\mathbf{r}}$ and $h_{\mathbf{r}}=1/\mathcal{R}$. The reference domain of the membrane at time $t=t_m$ is discretized as $D_{\mathbf{q}}^m=[-1,x_l^m]\cup[x_l^m,x_r^m]\cup[x_r^m,1]$ with $D_{\mathbf{q}}^m=\cup_{j=1}^{\mathcal{Q}}D_{\mathbf{q},j}^m$, where x_l^m, x_r^m denote the left and right contact lines' x value and $D_{\mathbf{q},j}^m=[x_{j-1}^m,x_j^m]$ with $-1=x_0^m<x_1^m<x_2^m<\cdots<x_{j_l}^m=x_l^m<\cdots<x_{j_r}^m=x_r^m<\cdots<x_{\mathcal{Q}}^m=1$. We use the following finite-dimensional spaces to approximate $H^1(D_{\mathbf{q}}^m)$, $H_0^1(D_{\mathbf{q}}^m)$, $V_1([x_l^m,x_r^m])$, $V_2([-1,x_l^m]\cup[x_r^m,1])$, $H^1(D_{\mathbf{r}})$, $H_0^1(D_{\mathbf{r}})$, respectively,

$$V^m:=\left\{f\in C(D_{\mathbf{q}}^m):f|_{D_{\mathbf{q},j}^m}\in\mathcal{P}_2(D_{\mathbf{q},j}^m),\ \forall j=1,2,\cdots,\mathcal{Q}\right\},\tag{3.58a}$$

$$V_0^m:=\{f\in V^m:f(-1)=f(1)=0\},\tag{3.58b}$$

$$V_1^m:=\left\{g\in C([x_l^m,x_r^m]):g|_{D_{\mathbf{q},j}^m}\in\mathcal{P}_2(D_{\mathbf{q},j}^m),\ \forall j=j_l+1,\cdots,j_r\right\},\tag{3.58c}$$

$$V_2^m:=\Big\{g\in C([-1,x_l^m]\cup[x_r^m,1]):g|_{D_{\mathbf{q},j}^m}\in\mathcal{P}_2(D_{\mathbf{q},j}^m),\ \forall j=1,2,\cdots,j_l,j_r+1,\cdots,\mathcal{Q},\ g(-1)=g(1)=0\Big\},\tag{3.58d}$$

$$W^h:=\{\psi\in C(D_{\mathbf{r}}):\psi|_{D_{\mathbf{r},j}}\in\mathcal{P}_2(D_{\mathbf{r},j}),\ \forall j=1,2,\cdots,\mathcal{R}\},\tag{3.58e}$$

$$W_0^h:=\{\psi\in W^h:\psi(0)=\psi(1)=0\},\tag{3.58f}$$

where $\mathcal{P}_2(D)$ denotes the space of polynomials with degrees at most 2 on D.

Let $\Xi^m:=\mathbf{q}^m=(x^m,y^m)\in D_{\mathbf{q}}^m\times V^m$ be the numerical approximation to the whole membrane Ξ at the time step t_m. Then it is straightforward to define Σ_1^m and Σ_2^m with $\Xi^m=\Sigma_1^m\cup\Sigma_2^m$ as separated by x_l^m and x_r^m. For piecewise continuous functions u and v defined on $D_{\mathbf{q}}^m$, the inner products on Σ_1^m, Σ_2^m and Ξ^m can be approximated as

$$(u,\ v)_{\Sigma_1^m} := \sum_{j=j_l+1}^{j_r} |\partial_x \mathbf{q}^m|_j \int_{x_{j-1}^m}^{x_j^m} u(x)v(x)\mathrm{d}x, \tag{3.59a}$$

$$(u,\ v)_{\Sigma_2^m} := \left(\sum_{j=1}^{j_l} + \sum_{j=j_r+1}^{\mathcal{Q}}\right) |\partial_x \mathbf{q}^m|_j \int_{x_{j-1}^m}^{x_j^m} u(x)v(x)\mathrm{d}x, \tag{3.59b}$$

$$(u,\ v)_{\Xi^m} := (u,\ v)_{\Sigma_1^m} + (u,\ v)_{\Sigma_2^m}, \tag{3.59c}$$

where $|\partial_x \mathbf{q}^m|_j = \sqrt{1 + \left(\frac{y^m(x_j^m) - y^m(x_{j-1}^m)}{x_j^m - x_{j-1}^m}\right)^2}$, $j = 1, 2, \cdots, \mathcal{Q}$. We could further compute them using the Boole's rule as

$$\begin{aligned}\int_{x_{j-1}^m}^{x_j^m} u(x)v(x)\mathrm{d}x =& \frac{x_j^m - x_{j-1}^m}{90}\Big[7(u\cdot v)(x_{j-1}^{m,+}) + 32(u\cdot v)(x_{j-3/4}^m) \\ &+ 12(u\cdot v)(x_{j-\frac{1}{2}}^m) + 32(u\cdot v)(x_{j-1/4}^m) + 7(u\cdot v)(x_j^{m,-})\Big],\end{aligned} \tag{3.60}$$

where $u(x_j^{m,\pm})$ are the one-sided limits of u approaching x_j^m from the right and left sides, $x_{j-3/4}^m = \frac{1}{4}(3x_{j-1}^m + x_j^m)$, $x_{j-\frac{1}{2}}^m = \frac{1}{2}(x_{j-1}^m + x_j^m)$ and $x_{j-1/4}^m = \frac{1}{4}(x_{j-1}^m + 3x_j^m)$. Let $\boldsymbol{\tau}^m$, $\mathbf{n}^m$, κ^m, ν^m be the numerical approximations to the membrane's unit tangential and normal vectors, the membrane curvature, and the membrane inner tension, respectively, at the time step t_m.

Let $\Sigma_3^m := \mathbf{r}^m = (x_{\mathbf{r}}^m(\zeta), y_{\mathbf{r}}^m(\zeta)) \in W^h \times W^h$ be the numerical approximation to the fluid-fluid interface Σ_3 at the time step t_m. For piecewise continuous functions u and v defined on $D_{\mathbf{r}}$, the inner products on Σ_3^m can be approximated as

$$(u,\ v)_{\Sigma_3^m} := \sum_{j=1}^{\mathcal{R}} |\mathbf{r}^m(\zeta_j) - \mathbf{r}^m(\zeta_{j-1})| \int_{\zeta_{j-1}}^{\zeta_j} u(\zeta)v(\zeta)\mathrm{d}\zeta, \tag{3.61}$$

where $|\mathbf{r}^m(\zeta_j) - \mathbf{r}^m(\zeta_{j-1})| = \sqrt{(x_{\mathbf{r}}^m(\zeta_j) - x_{\mathbf{r}}^m(\zeta_{j-1}))^2 + (y_{\mathbf{r}}^m(\zeta_j) - y_{\mathbf{r}}^m(\zeta_{j-1}))^2}$, $j = 1, 2, \cdots, \mathcal{R}$. Similarly, we could further compute it using the Boole's rule.

Based on the discretization of $D_{\mathbf{r}}$ and $D_{\mathbf{q}}^m$, we denote $\mathbf{r}_j^m = \mathbf{r}^m(\zeta_j)$, $j = 0, 1, \cdots, \mathcal{R}$

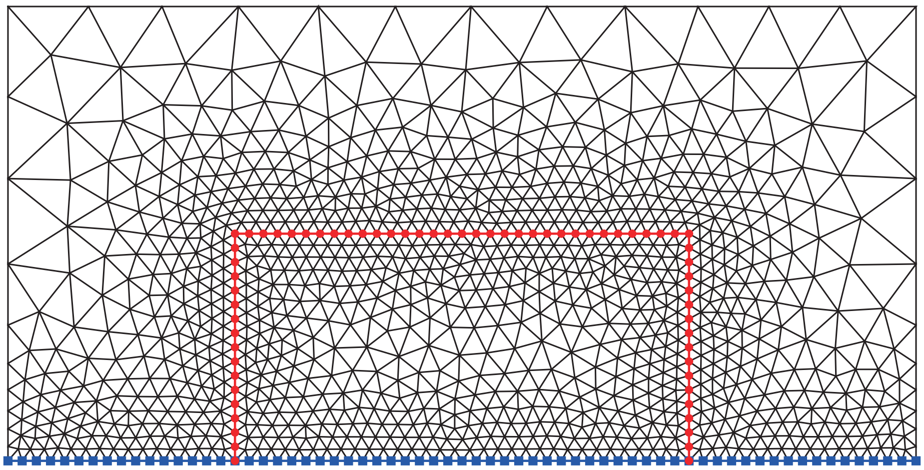

Figure 3.3: The initial mesh used for wetting and nonwetting examples with $\mathcal{R} = 64,\ \mathcal{Q} = 64,\ J^0 = 1120,\ N^0 = 2142$. The red line with circle markers represents the fluid-fluid interface. The blue line with square markers represents the membrane.

and $\mathbf{q}_j^m = (x_j^m, y^m(x_j^m)),\ j = 0, 1, \cdots, \mathcal{Q}$ as markers to represent the fluid-fluid interface Σ_3^m and the membrane Ξ^m, respectively. Especially, we have $\mathbf{r}_0^m = \mathbf{q}_{jl}^m = (x_l^m, y^m(x_l^m))$ marking the left contact line Λ_l^m and $\mathbf{r}_{\mathcal{R}}^m = \mathbf{q}_{jr}^m = (x_r^m, y^m(x_r^m))$ marking the right contact line Λ_r^m.

Let $\mathcal{T}^m := \bigcup_{j=1}^{N^m} \bar{o}_j^m$ be a triangulation of Ω^m at the time step $t = t_m$. The mesh contains J^m vertices denoted by $\{\mathbf{p}_k^m\}_{k=1}^{J^m}$. We use a fitted mesh (see Fig. 3.3) such that line segments of Ξ^m and Σ_3^m are edges from $\mathcal{T}^m$, i.e., $\Xi^m \cup \Sigma_3^m \subset \bigcup_{j=1}^{N^m} \partial o_j^m$. Besides, markers $\mathbf{r}_j^m,\ j = 0, 1, \cdots, \mathcal{R}$ and markers $\mathbf{q}_j^m,\ j = 0, 1, \cdots, \mathcal{Q}$ are vertices of the mesh. We define the following finite element spaces over $\mathcal{T}^m$,

$$S_k^m := \{\varphi_h \in C(\bar{\Omega}^m) : \varphi_h|_{o_j^m} \in \mathcal{P}_k(o_j^m),\ j = 1, \cdots, N^m\}, \tag{3.62a}$$

$$S_0^m := \{\varphi_h \in L^2(\Omega^m) : \varphi_h|_{o_j^m} \in \mathcal{P}_0(o_j^m),\ j = 1, \cdots, N^m\}, \tag{3.62b}$$

where $\mathcal{P}_k(D)$ denotes the space of polynomials with degrees at most k on D.

The fluid-fluid interface Σ_3^m divides the domain Ω^m into Ω_1^m and Ω_2^m. Correspondingly, the mesh $\mathcal{T}^m$ is divided into $\mathcal{T}_1^m$ and $\mathcal{T}_2^m$. Based on the spatial discretization, we define the viscosity η^m and the surface tension γ^m as

$$\eta^m = \eta_1 \chi_{\Omega_1^m} + \eta_2 \chi_{\Omega_2^m}, \quad \gamma^m = \gamma_1 \chi_{\Sigma_1^m} + \gamma_2 \chi_{\Sigma_2^m}. \tag{3.63}$$

We denote the corresponding finite element spaces for $\mathbb{U}_i$ $(i = 1, 2, 3)$ and $\mathbb{P}$ as $\mathbb{U}_i^m$ $(i = 1, 2, 3)$ and $\mathbb{P}^m$, respectively. These finite element spaces are chosen to use the standard P2-(P1+P0) elements as

$$\mathbb{U}_1^m = [S_2^m]^2 \cap \mathbb{U}_1, \quad \mathbb{U}_2^m = [S_2^m]^2 \cap \mathbb{U}_2, \quad \mathbb{P}^m = (S_1^m + S_0^m) \cap \mathbb{P}, \tag{3.64a}$$

$$\mathbb{U}_3^m = \left\{ \boldsymbol{\omega} \in \mathbb{U}_1^m : \boldsymbol{\omega} \cdot \mathbf{n}^m = -\frac{1}{|\partial_x \mathbf{q}^m|} \frac{y^m - y^{m-1}}{\tau} \text{ on } \Xi^m \right\}, \tag{3.64b}$$

where $|\partial_x \mathbf{q}^m| = \sqrt{1 + (\frac{\partial y^m}{\partial x})^2}$. These choices satisfy the inf-sup stability condition,

$$\inf_{\varphi \in \mathbb{P}^m} \sup_{0 \neq \boldsymbol{\omega} \in \mathbb{U}_i^m} \frac{(\varphi, \, \nabla \cdot \boldsymbol{\omega})_{\Omega^m}}{\|\varphi\|_0 \|\boldsymbol{\omega}\|_1} \geq C_0 > 0, \quad i = 1, 2, \tag{3.65}$$

where C_0 is a constant, and $\| \cdot \|_0$ and $\| \cdot \|_1$ denote the L^2 and H^1-norm on Ω^m, respectively.

The overall procedure of the numerical method is summarized as follows. Given the initial configuration of the system including $D_{\mathbf{q}}^0$, y^0 and κ^0 for the membrane Ξ^0, $\mathbf{r}^0$ for the fluid-fluid interface Σ_3^0, and $\mathcal{T}^0$ for the triangulation of Ω^0. Set $m = 0$ and then go through the following steps.

(1) Fix the current configuration of fluid-fluid interface $\mathbf{r}^m$ and the positions of the contact lines x_l^m and x_r^m, based on the mesh $\mathcal{T}^m$, find the fluid velocity $\mathbf{u}^{m+\frac{1}{2}} \in \mathbb{U}_1^m$, the fluid pressure $p^{m+\frac{1}{2}} \in \mathbb{P}^m$, the membrane configuration $y^{m+\frac{1}{2}} \in V^m$, the curvature of the membrane $\kappa^{m+\frac{1}{2}} \in V_0^m$, and the inner tension of the membrane $\nu^{m+\frac{1}{2}} \in V_1^m \cup V_2^m$ such that

B'21 B'22 B11 B12 B13 B14

$$- \left(p^{m+\frac{1}{2}}, \, \nabla \cdot \boldsymbol{\omega}^h \right)_{\Omega^m} + \left(\eta^m (\nabla \mathbf{u}^{m+\frac{1}{2}} + (\nabla \mathbf{u}^{m+\frac{1}{2}})^T), \, \nabla \boldsymbol{\omega}^h \right)_{\Omega^m}$$
$$- \frac{1}{Ca} \left(\nu^{m+\frac{1}{2}}, \, \nabla_s^m \cdot \boldsymbol{\omega}^h \right)_{\Xi^m} - \frac{1}{Ca} \left(\gamma^m \partial_s^m y^{m+\frac{1}{2}}, \, \partial_s^m \left(|\partial_x \mathbf{q}^m| \boldsymbol{\omega}^h \cdot \mathbf{n}^m \right) \right)_{\Xi^m}$$

M21' M22' M51 M52

M31 M32

$$+\frac{c_b}{Ca}\left(\partial_s^m\Big(\frac{\kappa^{m+\frac{1}{2}}}{|\partial_x\mathbf{q}^m|}\Big),\ \partial_s^m(|\partial_x\mathbf{q}^m|\boldsymbol{\omega}^h\cdot\mathbf{n}^m)\right)_{\Xi^m} \tag{3.66a}$$

M11 M12 I1

$$+\frac{3c_b}{2Ca}\left((\kappa^m)^2\partial_s^m y^m,\ \partial_s^m(|\partial_x\mathbf{q}^m|\boldsymbol{\omega}^h\cdot\mathbf{n}^m)\right)_{\Xi^m}+\frac{1}{Ca}\left(\partial_s\mathbf{r}^m,\ \partial_s\boldsymbol{\omega}^h\right)_{\Sigma_3^m}$$

cl1

$$+\frac{\gamma_1-\gamma_2}{Ca}\left(\left(|\partial_x\mathbf{q}^m|\omega_1^h\right)\Big|_{\Lambda_r^m}-\left(|\partial_x\mathbf{q}^m|\omega_1^h\right)\Big|_{\Lambda_l^m}\right)=0,\quad \forall\boldsymbol{\omega}^h=(\omega_1^h,\omega_2^h)\in\mathbb{U}_1^m,$$

B21 B22

$$\left(\nabla\cdot\mathbf{u}^{m+\frac{1}{2}},\ \varphi^h\right)_{\Omega^m}=0,\quad \forall\varphi^h\in\mathbb{P}^m, \tag{3.66b}$$

M41 / τ M42 M43

$$\left(\frac{y^{m+\frac{1}{2}}-y^m}{\tau},\ f^h\right)_{\Xi^m}+\left(|\partial_x\mathbf{q}^m|\mathbf{u}^{m+\frac{1}{2}}\cdot\mathbf{n}^m,\ f^h\right)_{\Xi^m}=0,\quad \forall f^h\in V^m, \tag{3.66c}$$

M44 M45

$$\left(\frac{\kappa^{m+\frac{1}{2}}}{|\partial_x\mathbf{q}^m|},\ \beta^h\right)_{\Xi^m}+\left(\partial_s^m y^{m+\frac{1}{2}},\ \partial_s^m\beta^h\right)_{\Xi^m}=0,\quad \forall\beta^h\in V_0^m, \tag{3.66d}$$

M6 M23 M24

cl4

$$\frac{l_s}{\mu_1}\left(\partial_s^m\nu^{m+\frac{1}{2}},\ \partial_s^m g_1^h\right)_{\Sigma_1^m}+Ca\left(\nabla_s^m\cdot\mathbf{u}^{m+\frac{1}{2}},\ g_1^h\right)_{\Sigma_1^m}+\frac{1}{\mu_\Lambda}\left([\![\nu^{m+\frac{1}{2}}]\!]_2^1 g_1^h\right)\Big|_{\Lambda^m}=0,\quad \forall g_1^h\in V_1^m, \tag{3.66e}$$

$$\frac{l_s}{\mu_2}\left(\partial_s^m\nu^{m+\frac{1}{2}},\ \partial_s^m g_2^h\right)_{\Sigma_2^m}+Ca\left(\nabla_s^m\cdot\mathbf{u}^{m+\frac{1}{2}},\ g_2^h\right)_{\Sigma_2^m}-\frac{1}{\mu_\Lambda}\left([\![\nu^{m+\frac{1}{2}}]\!]_2^1 g_2^h\right)\Big|_{\Lambda^m}=0,\quad \forall g_2^h\in V_2^m, \tag{3.66f}$$

where $\partial_s^m=\frac{1}{|\partial_x\mathbf{q}^m|}\frac{\partial}{\partial x}$, $\nabla_s^m=\boldsymbol{\tau}^m\partial_s^m$, and the superscript $m+\frac{1}{2}$ is used to denote the temporary state from the time step t_m to t_{m+1}.

(2) Due to the condition that the fluid-fluid interface always attaches the membrane, update the y value of $\mathbf{r}^m(0)$ and $\mathbf{r}^m(1)$ as

$$\mathbf{r}^{m+\frac{1}{2}}(\zeta)=\begin{cases}(x_l^m,y^{m+\frac{1}{2}}(x_l^m)), & \zeta=0,\\ (x_r^m,y^{m+\frac{1}{2}}(x_r^m)), & \zeta=1,\\ \mathbf{r}^m(\zeta), & 0<\zeta<1.\end{cases} \tag{3.67}$$

Use the computed membrane configuration $y^{m+\frac{1}{2}}$ to update the numerical mesh from $\mathcal{T}^m$ to $\mathcal{T}^{m+\frac{1}{2}}$. Since $x_j^m, j=0,1,\cdots,\mathcal{Q}$ are not modified, set $D_{\mathbf{q}}^{m+\frac{1}{2}}$ as $D_{\mathbf{q},j}^{m+\frac{1}{2}}=D_{\mathbf{q},j}^m$ for $j=1,2,\cdots,\mathcal{Q}$.

(3) Using the computed $y^{m+\frac{1}{2}}$, based on the mesh $\mathcal{T}^{m+\frac{1}{2}}$, find the fluid velocity

$\mathbf{u}^{m+1} \in \mathbb{U}_3^{m+\frac{1}{2}}$, the fluid pressure $p^{m+1} \in \mathbb{P}^{m+\frac{1}{2}}$, the inner tension of the membrane $\nu^{m+1} \in V_1^{m+\frac{1}{2}} \cup V_2^{m+\frac{1}{2}}$, and the fluid-fluid interface $\mathbf{r}^{m+1} \in W^h \times W^h$ such that

$$
\begin{aligned}
&-\left(\overset{B'_{21}}{p^{m+1}},\ \overset{B'_{22}}{\nabla\cdot\boldsymbol{\omega}^h}\right)_{\Omega^{m+\frac{1}{2}}} + \left(\eta^{m+\frac{1}{2}}(\overset{B_{11}\ B_{12}}{\nabla\mathbf{u}^{m+1}} + \overset{B_{13}}{(\nabla\mathbf{u}^{m+1})^T}),\ \overset{B_{14}}{\nabla\boldsymbol{\omega}^h}\right)_{\Omega^{m+\frac{1}{2}}} && (3.68a)\\
&+\frac{1}{Ca}\left(\overset{I_1}{\partial_s\mathbf{r}^{m+1}},\ \partial_s\boldsymbol{\omega}^h\right)_{\Sigma_3^{m+\frac{1}{2}}} - \frac{1}{Ca}\left(\overset{M'_{21}}{\nu^{m+1}},\ \overset{M'_{22}}{\nabla_s^{m+\frac{1}{2}}\cdot\boldsymbol{\omega}^h}\right)_{\Xi^{m+\frac{1}{2}}}\\
&+\frac{\gamma_1-\gamma_2}{Ca}\left(\overset{c_{15}}{\left(\boldsymbol{\omega}^h\cdot\boldsymbol{\tau}^{m+\frac{1}{2}}\right)}\overset{c_{16}}{\Big|_{\Lambda_r^{m+\frac{1}{2}}}} - \left(\boldsymbol{\omega}^h\cdot\boldsymbol{\tau}^{m+\frac{1}{2}}\right)\Big|_{\Lambda_l^{m+\frac{1}{2}}}\right) = 0, \quad \forall\boldsymbol{\omega}^h \in \mathbb{U}_2^{m+\frac{1}{2}},\\
&\left(\overset{B_{21}}{\nabla\cdot\mathbf{u}^{m+1}},\ \overset{B_{22}}{\varphi^h}\right)_{\Omega^{m+\frac{1}{2}}} = 0, \quad \forall\varphi^h \in \mathbb{P}^{m+\frac{1}{2}}, && (3.68b)\\
&\frac{l_s}{\mu_1}\left(\partial_s^{m+\frac{1}{2}}\overset{M_{23}}{\nu^{m+1}},\ \overset{M_{24}}{\partial_s^{m+\frac{1}{2}}g_1^h}\right)_{\Sigma_1^{m+\frac{1}{2}}} + Ca\left(\overset{M_6}{\nabla_s^{m+\frac{1}{2}}\cdot\mathbf{u}^{m+1}},\ g_1^h\right)_{\Sigma_1^{m+\frac{1}{2}}}\\
&\qquad\qquad + \frac{1}{\mu_\Lambda}\left(\overset{c_{14}}{[\![\nu^{m+1}]\!]_2^1}g_1^h\right)\Big|_{\Lambda^{m+\frac{1}{2}}} = 0, \quad \forall g_1^h \in V_1^{m+\frac{1}{2}}, && (3.68c)\\
&\frac{l_s}{\mu_2}\left(\partial_s^{m+\frac{1}{2}}\nu^{m+1},\ \partial_s^{m+\frac{1}{2}}g_2^h\right)_{\Sigma_2^{m+\frac{1}{2}}} + Ca\left(\nabla_s^{m+\frac{1}{2}}\cdot\mathbf{u}^{m+1},\ g_2^h\right)_{\Sigma_2^{m+\frac{1}{2}}}\\
&\qquad\qquad - \frac{1}{\mu_\Lambda}\left([\![\nu^{m+1}]\!]_2^1 g_2^h\right)\Big|_{\Lambda^{m+\frac{1}{2}}} = 0, \quad \forall g_2^h \in V_2^{m+\frac{1}{2}}, && (3.68d)\\
&\left(\overset{I_2}{\frac{\mathbf{r}^{m+1}-\mathbf{r}^{m+\frac{1}{2}}}{\tau}},\ \boldsymbol{\psi}^h\right)_{\Sigma_3^{m+\frac{1}{2}}} - \left(\overset{I'_3}{\mathbf{u}^{m+1}},\ \boldsymbol{\psi}^h\right)_{\Sigma_3^{m+\frac{1}{2}}} = 0, \quad \forall\boldsymbol{\psi}^h \in W^h\times W_0^h,\\
& && (3.68e)\\
&\frac{y_{\mathbf{r}}^{m+1}(0)-y_{\mathbf{r}}^{m+\frac{1}{2}}(0)}{\tau} = \frac{\partial y^{m+\frac{1}{2}}}{\partial x}\Big|_{x=x_l^m}\frac{x_{\mathbf{r}}^{m+1}(0)-x_{\mathbf{r}}^{m+\frac{1}{2}}(0)}{\tau}, && (3.68f)\\
&\frac{y_{\mathbf{r}}^{m+1}(1)-y_{\mathbf{r}}^{m+\frac{1}{2}}(1)}{\tau} = \frac{\partial y^{m+\frac{1}{2}}}{\partial x}\Big|_{x=x_r^m}\frac{x_{\mathbf{r}}^{m+1}(1)-x_{\mathbf{r}}^{m+\frac{1}{2}}(1)}{\tau}, && (3.68g)
\end{aligned}
$$

where Eq. (3.68f) and Eq. (3.68g)) are numerical approximations to the time derivatives of $y_{\mathbf{r}}(0,t) = y^{m+\frac{1}{2}}(x_{\mathbf{r}}(0,t))$ and $y_{\mathbf{r}}(1,t) = y^{m+\frac{1}{2}}(x_{\mathbf{r}}(1,t))$, respectively. These two equations are approximating the motion of contact lines along the membrane.

(4) Update x_l^{m+1}, x_r^{m+1} with

$$x_l^{m+1} = x_{\mathbf{r}}^{m+1}(0), \quad x_r^{m+1} = x_{\mathbf{r}}^{m+1}(1), \tag{3.69}$$

Then we could update markers for the membrane $\mathbf{q}_j^{m+1}$, $j = 0, 1, \cdots, \mathcal{Q}$ as

$$\mathbf{q}_j^{m+1} = \begin{cases} (x_l^{m+1}, y_{\mathbf{r}}^{m+1}(0)), & j = j_l, \\ (x_r^{m+1}, y_{\mathbf{r}}^{m+1}(1)), & j = j_r, \\ (x_j^m, y^{m+\frac{1}{2}}(x_j^m)), & j = 0, 1, \cdots, \mathcal{Q} \text{ but } j \neq j_l, j_r. \end{cases} \tag{3.70}$$

(5) Fix the markers $\mathbf{r}_0^{m+1} = \mathbf{q}_{j_l}^{m+1}$, $\mathbf{r}_{\mathcal{R}}^{m+1} = \mathbf{q}_{j_r}^{m+1}$ for the contact lines. According to the equal arclength, redistribute markers $\mathbf{r}_j^{m+1}$, $j = 1, 2, \cdots, \mathcal{R} - 1$ for the fluid-fluid interface Σ_3^{m+1}, markers $\mathbf{q}_j^{m+1}$, $j = j_l + 1, \cdots, j_r - 1$ for the membrane Σ_1^{m+1} under the droplet, and markers $\mathbf{q}_j^{m+1}$, $j = 0, 1, \cdots, j_l - 1, j_r + 1, \cdots, \mathcal{Q}$ for the membrane Σ_2^{m+1} outside the droplet. A ratio ρ between the average arclengths of Σ_1^{m+1} and Σ_2^{m+1} is computed. If the ratio $\rho \in [1/1.3, 1.3]$, no further redistribution is conducted. If the ratio $\rho > 1.3$ or $\rho < 1/1.3$, to guarantee a good approximation accuracy of markers, our operation is to redetermine the numbers of markers for Σ_1^{m+1} and Σ_2^{m+1} to make the ratio ρ as closed to one as possible, while the total number of markers for the membrane remains unchanged. Then, new markers can be determined by interpolation. After these redistribution, we obtain $D_{\mathbf{q}}^{m+1}$ and $y^{m+1} \in V^{m+1}$, and then we could compute the membrane curvature $\kappa^{m+1} \in V_0^{m+1}$.

(6) If no redetermination is conducted in the previous step, use the moving mesh method to generate the new fitted mesh $\mathcal{T}^{m+1}$ from $\mathcal{T}^m$, which is introduced in Appendix B.3. Otherwise, use the same method for the initial mesh to generate the new mesh $\mathcal{T}^{m+1}$. Then, go to step (1) with $m = m + 1$.

3.3.3 Well-posedness of the FEM schemes

We show that the FEM schemes (3.66a)-(3.66f) and (3.68a)-(3.68g) both yield a unique solution.

Theorem 1. *(Well-posedness). Let* $(\mathbb{U}_1^m, \ \mathbb{P}^m)$ *satisfy the inf-sup condition* (3.65). *Then the FEM scheme* (3.66a)-(3.66f) *admits a unique solution.*