Lecture Note: Two fundamental Welfare Theorems

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1 Second Welfare Theorem

1.1 Target:

1.2 Step one (Monotonicity, Separating Hyperplane Theorem)

Claim 1

$p^* \geqq 0$*(so that p^* doesn't contain any strictly negative prices for any of the goods)*

Proof

Let $u_j = j^{th}$ unit basis vector in R^l, $u_1 = (1, 0, ..., 0), u_2 = (0, 1, 0, ..., 0), ..., u_l = (0, 0, ..., 0, 1)$ By monotonicity of preferences, add ϵ_{u_j}(little bid of commodity j) leads to strictly preferred allocation for any good j and any person i.

So

$$\bar{x}_i + \frac{1}{n} u_j \succ_i \bar{x}_i, \forall j, \forall i$$

$$\sum_{i=1}^{n} \bar{x}_i + \sum_{i=1}^{n} \frac{1}{n} u_j \in U(\bar{x}), (since\ each\ \bar{x}_i + \frac{1}{n} u_j \in U(\bar{x}))$$

$$(E + u_j \in U(\bar{x}))$$

By easy separating hyperplane theorem:

$$p^*.(E+u_j) \geqq p^*.E$$

$$P^*.u_j \geqq 0 \Longrightarrow P_j^* \geqq 0 \; for \; any \; commodity \; j$$

$$therefore \; p^* \geqq 0$$

$$Done \; Claim 1$$

1.3 Step two (Continuity, Monotonicity, Separating Hyperplane Theorem)

$$Claim \; 2$$

For any individual trader $\bar{i}$, $x_{\bar{i}} \succ_{\bar{i}} \bar{x}_{\bar{i}}$, *where* $\bar{x}_{\bar{i}}$ *is the PO allocation. Then* $p^* x_{\bar{i}} \geqq p^* \bar{x}_{\bar{i}}$ *(something strictly better than* $\bar{x}_{\bar{i}}$ *cannot cost strictly less under price* p^**)*

$$Proof$$

We have:

By continuity:

$$x_{\bar{i}} - (\epsilon, ..., \epsilon) =^{def} \bar{\bar{x}}_{\bar{i}} \succ_{\bar{i}} \bar{x}_{\bar{i}}$$

By monotonicity:

$$\bar{x}_i + \frac{1}{n-1}.(\epsilon, ..., \epsilon) =^{def} \bar{\bar{x}}_i \succ_i \bar{x}_i, (i \neq \bar{i})$$

Hence by separating hyperplane, there is a vector p^* *that:*

$$p^*.\sum_{i=1}^{n} \bar{\bar{x}}_i \geqq p^*.\sum_{i=1}^{n} \bar{x}_i = p^*.E$$

By 1 and 2, we have:

$$p^*.\sum_{i=1}^{n}\bar{\bar{x}}_i \geqq p^*.\sum_{i=1}^{n}\bar{x}_i$$

$$p^*.(\sum_{i=\bar{i}}\bar{\bar{x}}_{\bar{i}}+\sum_{i\neq\bar{i}}\bar{\bar{x}}_i)\geqq p^*.\sum_{i=1}^{n}\bar{x}_i$$

$$p^*.(x_{\bar{i}}-(\epsilon,...,\epsilon))+p^*\sum_{i\neq\bar{i}}(\bar{x}_i+\frac{1}{n-1}.(\epsilon,...,\epsilon))\geqq p^*.\sum_{i=1}^{n}\bar{x}_i$$

$$p^*.x_{\bar{i}}+p^*\sum_{i\neq\bar{i}}\bar{x}_i\geqq p^*.\sum_{i=1}^{n}\bar{x}_i$$

$$p^*.x_{\bar{i}}\geqq p^*.\bar{x}_{\bar{i}}$$

$$Done\ Claim 2$$

1.4 Step three (Continuity, Proof by contradiction)

$$Claim\ 3$$

Stengthen above from $\geqq$ to $>$. If $x_i \succ_i \bar{x}_i$, then $p^*.x_i > p^*.\bar{x}_i$

$$Proof$$

Strengthen above from $\geq$ to $>$ Claim that if $x_i \succ_i \bar{x}_i$, then $p^*.x_i > p^*.\bar{x}_i$

By continuity of preference, if $\lambda\in (1,0)$ (scalar) is sufficiently close to 1, then $\lambda.x_i \succ_i \bar{x}_i$ (shrink the vector a little bit). By the above, for all $i=1,...,n$:

$$p^*.(\lambda.x_i) \geq p^*.\bar{x}_i$$

(Proof by contraction) Suppose that weak inequility holds with equality for some particular traders, call person $\bar{i}$

I.e.,

suppose $p^*.x_{\bar{i}} = p^*.\bar{x}_{\bar{i}}$ for some person $\bar{i}$ where $x_{\bar{i}} \succ_{\bar{i}} \bar{x}_{\bar{i}}$ Then

$$\lambda p^*.x_{\bar{i}} = p^*.(\lambda x_{\bar{i}}) < p^*.\bar{x}_{\bar{i}}$$

Because $\lambda < 1$, which contradicts $p^*.(\lambda x_i) \geq p^*.\bar{x}_i$ for all i as above.

$$Down\ Claim\ 3$$

We've shown that for any i, $p^*.x_i > p^*.\bar{x}_i$ whenever $x_i \succ_i \bar{x}_i$. So i's demand at price vector p^* includes allocation $\bar{x}_i$ when i has initial endowment $\bar{e}_i = \bar{x}_i$ (note that $\bar{x}_i$ is affordable at prices p^* or indeed for any proice vector) Therefore $(p^*, \bar{x})$ is a competitive equilibrium if endowments are $\bar{e}_i = \bar{x}_i$ for all i (no trade equilibrium at supporting prices p^*)

$$Down\ the\ theorem!$$