Questions

1. For each of pair of demand curves described in (a), (b), and (c), using the setup of Dana's (1999) game, calculate equilibrium prices and quantities under both monopoly and perfect competition. For each of (a), (b), and (c), comment on how price dispersion varies with the level of competition. Explain (including intuition) why this coincides or differs from Dana's (1999) result. In each case assume that the low and high demand states are equally likely, so that $\Pr(\theta = L) = \Pr(\theta = H) = 1/2$.

   (a) $P = 10 - Q/\theta$; $Q = \theta(10 - P)$; $L = 100$; $H = 200$; $\lambda = 1$; $c = 0$.

   (b) $P = \sqrt{\theta/Q}$; $Q = \theta/P^2$; $L = 1600$; $H = 3200$; $\lambda = 1$; $c = 0$.

   (c) $P = \theta - Q$; $Q = (\theta - P)$; $L = 10$; $H = 20$; $\lambda = 0$; $c = 1$.

2. Tirole (1988) states on page 138 of Chapter 3 that third-degree "price discrimination reduces welfare if it does not increase total output."

   (a) Explain why.

   (b) The result assumes quasi-linear utility, as does much welfare analysis in IO. What is the implicit assumption about the relative weights on utility functions of low- and high-income individuals in the social welfare function?

   (c) Leslie (2004) does not assume quasi-linear utility. Do you think Leslie's (2004) social welfare function appropriately weights utility of rich versus poor individuals?

Solutions to Problem Set 2

Question 1

Q1 part a

Part (a) satisfies $D(\theta, P) = \theta D(P)$, and thus the solution follows that in Dana (1999). In particular, P_L coincides with the optimal price assuming the low demand state occurs with certainty and residual demand above P_L in the high state is $(H-L)\,D(p)$. Thus, starting with the competitive case,

$$\begin{aligned}
P_L^{a,PC} &= \lambda = 1 \\
P_H^{a,PC} &= \frac{\lambda}{1/2} = 2\lambda = 2 \\
Q_L^{a,PC} &= L\left(10 - P_L^{a,PC}\right) = 100\,(10-1) = 900
\end{aligned}$$

Residual demand in the high state at prices above P_L is $(H-L)(10-P)$.[1] Thus

$$Q_H^{a,PC} = (H-L)\left(10 - P_H^{a,PC}\right) = 100\,(10-2) = 800.$$

Turning to the monopoly case, we compute

$$\begin{aligned}
R &= PQ = Q\,(10 - Q/\theta) \\
MR &= 10 - 2Q/\theta
\end{aligned}$$

and

$$\begin{aligned}
MR_L &= 10 - 2Q/L = \lambda \\
&\rightarrow\ Q_L^{a,M} = (10-\lambda)(L/2) = (10-1)(100/2) = 450 \\
&\rightarrow\ P_L^{a,M} = 10 - Q_L^{a,M}/L = 10 - 450/100 = 5.5
\end{aligned}$$

Next, inverse residual demand is $P = 10 - Q/(H-L)$ so

$$\begin{aligned}
MR_{RD} &= 10 - 2Q/(H-L) = 2\lambda \\
&\rightarrow\ Q_H^{a,M} = (10-2\lambda)(H-L)\frac{1}{2} = (10-2)\,100\left(\frac{1}{2}\right) = 400 \\
&\rightarrow\ P_H^{a,M} = 10 - Q_H^{a,M}/(H-L) = 10 - 400/100 = 6
\end{aligned}$$

To summarize

Part (a)	Perfect Competition	Monopoly
Low Price Tickets (always sell)	900 at \$1.00	450 at \$5.50
High Price Tickets (sell half the time)	800 at \$2.00	400 at \$6.00

In this case price dispersion is higher with competition than with monopoly, which coincides with Dana's (1999) results. This is because demand is linear, so the monopoly PTR is 1/2. Thus while capacity cost adjusted for sale probability varies from 1 to 2, only a half of this variation is passed through into price variation. Moving to competition increases PTR to 1, and leads to full pass-through of the $\lambda/\Pr(sale)$ variation into price variation.

[1] To compute residual demand in the high demand state at prices above P_L, note that demand at P_L is $H\left(10 - P_L^{a,PC}\right)$ of which $L\left(10 - P_L^{a,PC}\right)$ is served, meaning fraction $\frac{H-L}{H}$ of demand remains unserved. Thus residual demand above P_L is $RD(P > P_L) = \frac{H-L}{H}H(10-P) = (H-L)(10-P)$.

Q1 part b

Part (b) also satisfies $D(\theta, p) = \theta D(p)$, and thus the solution follows that in Dana (1999). In particular, P_L coincides with the optimal price assuming the low demand state occurs with certainty and residual demand above P_L in the high state is $(H - L) D(p)$. Thus, starting with the competitive case,

$$\begin{aligned} P_L^{a,PC} &= \lambda = 1 \\ P_H^{a,PC} &= \frac{\lambda}{1/2} = 2\lambda = 2 \\ Q_L^{a,PC} &= L/\left(P_L^{a,PC}\right)^2 = \frac{1600}{1^2} = 1600 \end{aligned}$$

Residual demand in the high demand state at prices above P_L is $(H - L)/P^2$. Thus

$$Q_H^{a,PC} = (H - L)/\left(P_H^{a,PC}\right)^2 = \frac{1600}{2^2} = 400.$$

Turning to the monopoly case, we compute

$$\begin{aligned} R &= PQ = Q\sqrt{\theta/Q} = \sqrt{\theta Q} \\ MR &= \frac{1}{2}\sqrt{\theta/Q} = \frac{1}{2}P \end{aligned}$$

and

$$\begin{aligned} MR_L &= \frac{1}{2}P_L = \lambda \rightarrow P_L^{b,M} = 2\lambda = 2 \\ &\rightarrow Q_L^{b,M} = L/\left(P_L^{b,M}\right)^2 = \frac{1600}{2^2} = 400 \end{aligned}$$

Next,

$$\begin{aligned} MR_{RD} &= \frac{1}{2}P = 2\lambda \rightarrow P_H^{b,M} = 4\lambda = 4 \\ &\rightarrow Q_H^{b,M} = (H - L)/\left(P_H^{b,M}\right)^2 = \frac{1600}{4^2} = 100 \end{aligned}$$

To summarize

Part (b)	Perfect Competition	Monopoly
Low Price Tickets (always sell)	1600 at \$1.00	400 at \$2.00
High Price Tickets (sell half the time)	400 at \$2.00	100 at \$4.00

In this case price dispersion is lower with competition than with monopoly, which is opposite to Dana's (1999) results. This is because demand is shaped such that the monopoly PTR is 2. Thus while capacity cost adjusted for sale probability varies from 1 to 2, double this variation is passed through into price variation. Moving to competition *decreases* PTR to 1, and leads to *only* full pass-through of the $\lambda/\Pr(sale)$ variation into price variation. This appears to violate Proposition 5. However Proposition 5 only holds for the continuous case, and is likely misleading by focusing on price support as a measure of dispersion.

Q1 part c

$P = \theta - Q$; $Q = \theta - P$; $L = 10$; $H = 20$; $\lambda = 0$; $c = 1$.

Note that the perfectly competitive case is easy in part (c) because $\lambda = 0$. As a result, under perfect competition, there is only one price, $P = c = 1$. In the low demand state, sales are $Q_L = L - P = 10 - 1 = 9$. With probability $1/2$ and additional $Q_H = H - P - Q_L = 20 - 1 - 9 = 10$ units sell, but at the same price.

The monopoly case requires more work, in particular because $D(\theta, p) \neq \theta D(p)$ so it need not be the case that P_L maximizes profits in the low state, and residual demand will differ. What is residual demand? Well at P_L demand in the high state is $(H - P_L)$, but all that is served is fraction $\frac{(L-P_L)}{(H-P_L)}$, leaving residual fraction

$$1 - \frac{L - P_L}{H - P_L} = \frac{H - L}{H - P_L},$$

for residual demand of

$$RD = Q_H = \frac{H - L}{H - P_L}(H - P_H).$$

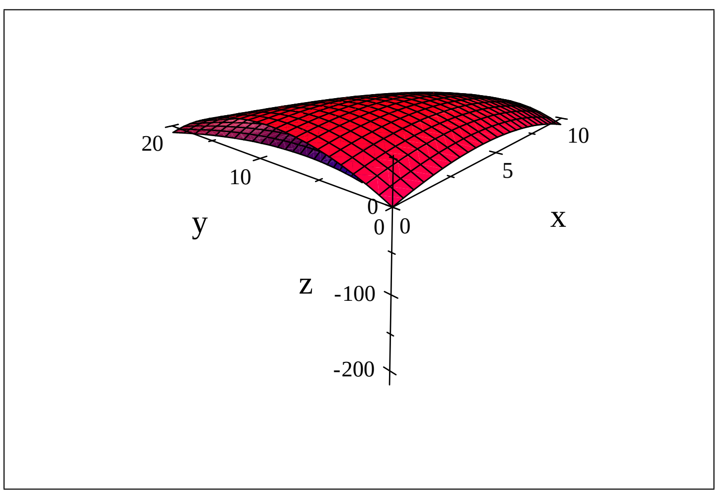
or the inverse residual demand

$$P_H = H - Q_H \frac{H - P_L}{H - L} = H - Q_H \frac{H - (L - Q_L)}{H - L} = H - Q_H \frac{H - L + Q_L}{H - L}$$

Profits are then

$$\begin{aligned} \Pi &= (P_L - c) Q_L + \frac{1}{2}(P_H - c) Q_H \\ &= (L - Q_L - c) Q_L + \frac{1}{2}\left(H - Q_H \frac{H - L + Q_L}{H - L} - c\right) Q_H \\ &= (10 - Q_L - 1) Q_L + \frac{1}{2}\left(20 - Q_H \frac{10 + Q_L}{10} - 1\right) Q_H. \end{aligned}$$

Check SOC graphically:



First-order conditions are

$$\frac{d\Pi}{dQ_L} = (L - 2Q_L - c) - \frac{1}{2} \cdot \frac{1}{H - L} Q_H^2 = 0$$

$$\frac{d\Pi}{dQ_H}=\frac{1}{2}\left(H-2Q_H\frac{H-L+Q_L}{H-L}-c\right)=0$$

Solving $\frac{d\Pi}{dQ_L}=0$ for Q_L gives

$$Q_L=\frac{1}{2}(L-c)-\frac{1}{4}\frac{1}{H-L}Q_H^2$$

Substituting into $\frac{d\Pi}{dQ_H}=0$ gives

$$H-2Q_H\frac{H-L+\frac{1}{2}(L-c)-\frac{1}{4}\frac{1}{H-L}Q_H^2}{H-L}-c=0$$

$$20-2Q_H\frac{10+0.5(10-1)-\frac{1}{4}\frac{1}{10}Q_H^2}{10}-1=0$$

There are three solutions:

$$19.667, 7.1935, -26.86$$

The positive roots correspond to

$$\begin{aligned}
Q_L(19.667) &= \frac{1}{2}(10-1)-\frac{1}{4}\frac{1}{10}(19.667)^2=-5.1698 \\
Q_L(7.1935) &= \frac{1}{2}(10-1)-\frac{1}{4}\frac{1}{10}(7.1935)^2=3.2063
\end{aligned}$$

So only one root is positive for both quantities, this is

$$\begin{aligned}
Q_L &\approx 3.2063 \\
Q_H &\approx 7.1935
\end{aligned}$$

With corresponding prices

$$\begin{aligned}
P_L &= L-Q_L=10-3.2063=6.7937 \\
P_H &= H-Q_H\frac{H-L+Q_L}{H-L}=20-7.1935\frac{10+3.2063}{10}=10.5
\end{aligned}$$

To summarize,

Part (c)	Perfect Competition	Monopoly
Low Price Tickets (always sell)	9 at \$1.00	3.2 at \$6.79
High Price Tickets (sell half the time)	10 at \$1.00	7.2 at \$10.50

Thus in this example, market power increases price dispersion. This example differs from Dana (1999) because the optimal monopoly price with known demand varies with θ and $\lambda=0$. Setting $\lambda=0$ eliminates all the effects that Dana (1999) considers, and leaves no price dispersion under monopoly if $D(\theta,p)=\theta D(p)$. However, eliminating the $D(\theta,p)=\theta D(p)$ assumption adds a new reason to vary price under market power–the fact that optimal monopoly price with known demand varies with θ. This translates into multiple prices when demand is uncertain.

Q1 summary table

		(a)	(b)	(c)
Perfect	$Q_L^{PC}; P_L^{PC}$	900; \$1.00	1600; \$1.00	9; \$1.00
Competition	$Q_H^{PC}; P_H^{PC}$	800; \$2.00	400; \$2.00	10; \$1.00
	spread	\$1.00	\$1.00	\$0.00
Monopoly	$Q_L^M; P_L^M$	450; \$5.50	400; \$2.00	3.2; \$6.79
	$Q_H^M; P_H^M$	400; \$6.00	100; \$4.00	7.2; \$10.50
	spread	\$0.50	\$2.00	\$3.71
Matches paper's prediction		yes	no	no
Rationale		$1 = PTR_{PC} >$ $PTR_M = 1/2$	$1 = PTR_{PC} <$ $PTR_M = 2$	ε_D varies with θ

Question 2

Q2 part a

Price discrimination raises prices to high value consumers and lowers prices to low value consumers. This tends to reallocate goods from high to low value consumers, which lowers welfare. To increase welfare, third-degree price discrimination must offset this loss by increasing total sales, thereby ameliorating the typical downward distortion in consumption due to market power.

Q2 part b

For many purposes, quasi-linear utility is a good assumption. This is because everyone's (standard) utility function is locally approximately linear. However, the assumption can be problematic for welfare calculations. When we calculate consumer surplus as the area under a demand curve, we are weighting individual i's utility such that marginal *social* value of a dollar to individual i is equal to 1 independent of i. Thus, if our social welfare function is

$$SW = \sum_i \alpha_i U(y_i),$$

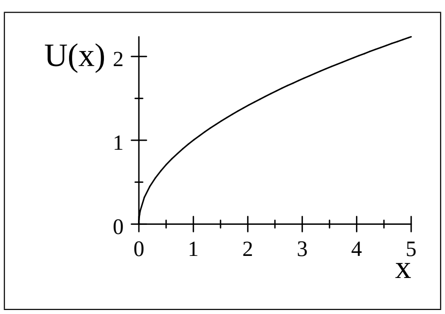
we are assuming that

$$\frac{d}{dy_i} SW = \alpha_i U'(y_i) = 1,$$

or that the weight on i's utility in social welfare is

$$\alpha_i = \frac{1}{U'(y_i)} = \frac{1}{\text{marginal value of income}}.$$

Given that we believe the marginal value of income is decreasing, this means that whenever one integrates over a demand curve to compute consumer surplus, one is implicitly weighting utility of the rich above utility of the poor. This will bias us against price discrimination in our welfare analysis whenever high income market segments are less elastic, because in that case price discrimination tends to raise prices to the rich and lower them to the poor. Do not overlook this common bias in IO welfare analysis.



More formally, let us assume that $U(q, x, \theta) = q\theta + u(x)$ is the utility of type θ who consumes q units of the good being studied and x units of the numeraire. Here x is income left over from buying the good: $x = y - qx$. Preference heterogeneity is entirely captured by θ. I assume θ is additively separable from the numeraire x so that θ can be interpreted as the value of the good in utils, and this value is independent of the level of the numeraire. Moreover, I assume that θ is independent of income y in the population. These are overly strong assumptions made for illustrative purposes.

Since $U(q, x, \theta)$ is approximately linear locally, we will approximate it as

$$U(q, x, \theta) \approx \beta_0 + q(\theta - u'(y)p)$$

This would lead to an approximate social welfare function

$$W \approx \sum_i \beta_{0,i} + q_i(\theta_i - u'(y_i)p)$$

I can add any individual specific constant to i's utility, and multiply all utilities by the same constant without affecting individual consumer choices, or those of the social planner. Anticipating what comes next, let me subtract off $\beta_{0,i}$ from each utility, and multiply all utilities by the constant $\frac{\pi}{\sigma_\theta\sqrt{6}}$. After this normalization, the variance of $\hat{\theta} = \frac{\pi}{\sigma_\theta\sqrt{6}}\theta$ is $\pi^2/6$ and $\hat{u}'(y_i) = \frac{\pi}{\sigma_\theta\sqrt{6}}u'(y_i)$. Moreover, adopting common discrete choice notation, I can write utility with $q_i = 0$ as $U_{i0} = 0$ and utility with $q_i = 1$ as

$$U_{i1} = \delta_1 - \alpha_i p + \varepsilon_i$$

for $\delta_1 = E\left[\hat{\theta}_i\right]$, $\alpha_i = \hat{u}'(y_i)$, and $\varepsilon_i = \hat{\theta}_i - E\left[\hat{\theta}_i\right]$.

Now the coefficients in U_{i1} are only identified up to multiplication by a constant, so estimation requires a normalization. In discrete choice demand estimation, a common normalization is that $var(\varepsilon_i) = \pi^2/6$. Given my prior normalization this is exactly right. So estimation correctly recovers δ_1 and α_i. Now an estimate of consumer welfare can be obtained by computing:

$$\hat{W} \approx \sum_i q_i(\delta_1 - \alpha_i p + \varepsilon_i)$$

Naturally, this will require integrating out over ε_i, perhaps via simulation, as they are unobserved. (Calculating the social welfare of firm profits in comparable units is a separate issue.) Alternatively, to compute social surplus, we compute

$$S \approx \sum_i \frac{1}{\alpha_i} q_i(\delta_1 - \alpha_i p + \varepsilon_i)$$

which normalizes utility to be in dollars for each i. As $\alpha_i = \hat{u}'(y_i)$, relative to social welfare, social surplus overweights the preferences of high income individuals for whom $1/\hat{u}'(y_i)$ is large.

This example makes it look easy to get social welfare right—just don't divide by α_i. This is misleading, however. It only works nicely because I have assumed that the value of the good, $U(1, x, \theta) - U(0, x, \theta) = \theta$ is independent of the level of the numeraire x for any given individual, and is distributed independently of income in the population.[2] Relaxing these assumptions realistically leads to difficulty. For instance, suppose there are only two income

[2] For a given individual, we might expect that $U(1, x, \theta) - U(0, x, \theta)$ is increasing in x for Hawaiian time shares (that you can only take advantage of with high x) or decreasing in x for bus tickets (that you only need if you don't have a car).

groups, $y_i \in \{L, H\}$ and that the variance of θ_i in the population varies across groups, being either σ_L^2 or σ_H^2. Following the same estimation approach as above, whereby $var\left(\varepsilon_i|y_i\right)$ is normalized to $\pi^2/6$, will mean that instead of applying equal weights to low and high income, high income get weighted by σ_L^2/σ_H^2 relative to low income (if we don't divide by α_i). If valuation for the good is lower variance (in utils) for low income individuals, this means we now have the opposite problem of under-weighting high income preferences.

Q2 part c

Recall from Leslie (2004) the utility specification is

$$\begin{aligned} U &= q\left(B\left(y\right) - p\right)^{\eta} \\ B\left(y\right) &= \delta_1 y^{\delta_2} \end{aligned}$$

For $\delta_1 > 0$ and $\delta_2 \in (0, 1]$. Note that this model is a little funny because, for someone who buys a ticket ($q = 1$), money enters in two places. When $\delta_2 < 1$ but $\eta > 0$ we could have U concave in y but convex in $(-p)$. However, as prices are endogenous but income is exogenous in this model, the key issue is how changes in price affect utility at different income levels (rather than how changes in income affect utility at different income levels). In other words, the question of interest is, while holding allocations fixed, does social welfare increase, decrease, or remain constant when we cut price paid by \$1 for a poor person but raise it by \$1 for a rich person? In the quasi-linear model this would leave social welfare constant. Given declining marginal utility of money, we would think that this change should actually increase social welfare. To answer this question we want to sign $d^2U/dpdy$. If this is negative, it means that at larger income (larger y) dU/dp is more negative, so a price cut for the rich is more socially valuable than a price cut for the poor, which overweights the rich even more than the quasi-linear case. If this is positive it means the reverse, that the interests of rich and poor are being weighted more equally.

$$\frac{dU}{dp} = -\eta q\left(B\left(y\right) - p\right)^{\eta - 1} < 0$$

$$\frac{d^2U}{dydp} = -\eta\left(\eta - 1\right)\delta_1\delta_2 y^{\delta_2 - 1} q\left(B\left(y\right) - p\right)^{\eta - 2}$$

$$\frac{d^2U}{dydp}\begin{cases} > 0, & \eta < 1 \quad \text{price cuts to rich weighted less} \\ = 0, & \eta = 1 \quad \text{price cuts to rich weighted equally (quasi linear case)} \\ < 0, & \eta > 1 \quad \text{price cuts to rich weighted more} \end{cases}$$

As Leslie (2004) estimates $\hat{\eta} = 1.03 > 1$, his counter-factual welfare calculations would put positive value on transferring money from poor to rich through ticket prices. This exacerbates the problem with quasi-linear utility, that rich are weighted more in social welfare. I do not think rich and poor are weighted appropriately in the utility function.