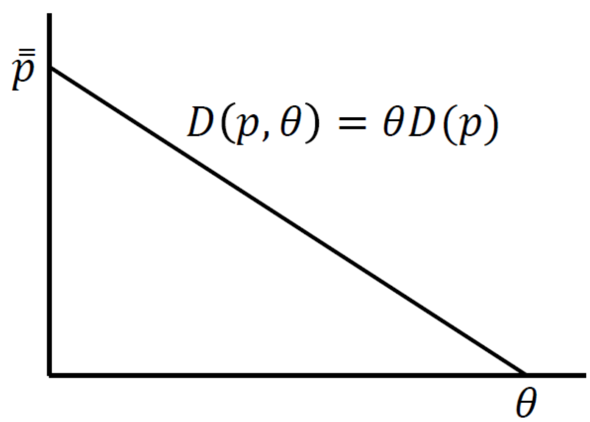
Price Discrimination & Demand Uncertainty
EC 8854—Dana 1999

Dana, James D., Jr. 1999. "Equilibrium Price Dispersion under Demand Uncertainty: The Roles of Costly Capacity and Market Structure." RAND Journal of Economics, 30(4), 632-60.

Model

- Measure $\theta \sim F(\theta)$ consumers with $\theta \in [0, \bar{\theta}]$, each have unit demand with value $v \sim^{iid} G(v)$. Let $D(p) = 1 - G(p)$ be the fraction with value above p. Then demand at a single price p is
$$D(p, \theta) = \theta D(p).$$
Note that $D(\bar{\bar{p}}) = 0$.



- Firm has marginal capacity cost λ and marginal production cost c.
- Before learning θ, firms choose a quantity to sell at each price. This is like choosing a pdf that integrates to total capacity. That is each firm chooses $q^i(p)$ which is positive on $[\underline{p}, \bar{p}]$, and integrates to $Q^i(p) = \int_{\underline{p}}^{p} q^i(x)\,dx$. Total capacity is $K^i = Q^i(\bar{p})$.
- Firm choices aggregate to $q(p) = \sum_i q^i(p)$, and $Q(p) = \sum_i Q^i(p)$.
- Proportional rationing.
- Notice that, were θ a known constant, its value would not affect market pricing. This is due to the *functional form assumption* $D(p, \theta) = \theta D(p)$ and constant marginal costs. Thus variation in θ would not be enough to create price dispersion. Instead, price dispersion will follow costly capacity $\lambda > 0$ and uncertainty about θ at the time prices and capacity are chosen.

Residual Demand

The first result is that residual demand at price p given a price distribution $q(p)$ is

$$RD(p,\theta) = D(p,\theta) - \int_{\underline{p}}^{p} \frac{D(p,\theta)}{D(x,\theta)} q(x)\,dx.$$

This is extremely elegant!

It is intuitive if we imagine that there are sales at one price $\tilde{p}$ below p. Then the formula reduces to

$$RD(p,\theta) = \theta D(p)\left(1 - \frac{q(\tilde{p})}{\theta D(\tilde{p})}\right).$$

$$RD(p,\theta) = D(p,\theta) - \frac{D(p,\theta)}{D(\tilde{p},\theta)} q(\tilde{p}).$$

The idea is that demand would have been $D(p,\theta)$ had no sales taken place at price $\tilde{p}$. However $q(\tilde{p})$ units were purchased at the lower price, and hence there are $q(\tilde{p})$ fewer consumers in the market place. Residual demand is not simply $D(p,\theta) - q(\tilde{p})$, however, because some of those who bought at price $\tilde{p}$ had values $v \in [\tilde{p}, p)$ so were already excluded from $D(p,\theta)$. How many of the sales at price $\tilde{p}$ went to individuals with $v \geq p$? The proportional sales rule means that since individuals with $v \geq p$ made up fraction $\frac{D(p,\theta)}{D(\tilde{p},\theta)}$ of those trying to buy, they also made of fraction $\frac{D(p,\theta)}{D(\tilde{p},\theta)}$ of the sales. Thus we subtract $\frac{D(p,\theta)}{D(\tilde{p},\theta)} q(\tilde{p})$ from demand.

What if there are sales at two lower prices $\tilde{p}_1 < \tilde{p}_2 < p$? Well, residual demand at price p given sales at all prices at $\tilde{p}_1$ have taken place is as above:

$$RD(p,\theta;\tilde{p}_1) = D(p,\theta) - \frac{D(p,\theta)}{D(\tilde{p}_1,\theta)} q(\tilde{p}_1).$$

What is residual demand at price p given sales at all prices $\tilde{p}_2$ and lower have taken place? We want to subtract off an additional $q(\tilde{p}_2)$ units discounted by the fraction of those sales that went to consumers with values $v \geq p$. By the same logic as above, that fraction will be

$$\frac{RD(p,\theta;\tilde{p}_1)}{RD(\tilde{p}_2,\theta;\tilde{p}_1)}$$

which happily simplifies as

$$\frac{RD(p,\theta;\tilde{p}_1)}{RD(\tilde{p}_2,\theta;\tilde{p}_1)} = \frac{D(p,\theta)\left(1 - \frac{1}{D(\tilde{p}_1,\theta)} q(\tilde{p}_1)\right)}{D(\tilde{p}_2,\theta)\left(1 - \frac{1}{D(\tilde{p}_1,\theta)} q(\tilde{p}_1)\right)} = \frac{D(p,\theta)}{D(\tilde{p}_2,\theta)}.$$

(Thereby saving us from a recursive formula!)

Thus

$$RD(p_i,\theta) = D(p_i,\theta) - \sum_{j<i} \frac{D(p_i,\theta)}{D(p_j,\theta)} q(p_j).$$

In the continuous case, this corresponds to

$$RD(p,\theta) = D(p,\theta) - \int_{\underline{p}}^{p} \frac{D(p,\theta)}{D(x,\theta)} q(x).$$

Definitions:

- $\rho(\theta)$ is the maximum sales price given θ. $(RD(\rho(\theta), \theta) = 0)$
- $\theta(p)$ is the demand state that leads to p being the maximum sales price. $(RD(p, \theta(p)) = 0)$
- Re writing

$$RD(p, \theta) = \theta D(p) \left(1 - \int_{\underline{p}}^{p} \frac{q(x)}{\theta D(x)} dx\right), \tag{1}$$

we have

$$\theta(p) = \int_{\underline{p}}^{p} \frac{q(x)}{D(x)} dx.$$

Perfect Competition

Definition: $q^*(p)$ and $y^*(p)$ are equilibrium market price distribution and probability of sale function. ($y^*(p)$ is the equilibrium probability that items with price p are sold)

$$y(p) = 1 - F(\theta(p)) = 1 - F\left(\int_{\underline{p}}^{p} \frac{q(x)}{D(x)} dx\right)$$

In a competitive market profit on each unit of capacity must be zero

$$(p - c)\, y(p) - \lambda = 0$$

So

$$p = c + \frac{\lambda}{1 - F(\theta(p))} = c + \frac{\lambda}{1 - F\left(\int_{\underline{p}}^{p} \frac{q(x)}{D(x)} dx\right)}$$

Thus each unit of capacity should have a price related to the probability of sale. Re-arranging terms we have

$$\begin{aligned}
p - c &= \frac{\lambda}{1 - F\left(\int_{\underline{p}}^{p} \frac{q(x)}{D(x)} dx\right)} \\
1 - F\left(\int_{\underline{p}}^{p} \frac{q(x)}{D(x)} dx\right) &= \frac{\lambda}{p - c} \\
F\left(\int_{\underline{p}}^{p} \frac{q(x)}{D(x)} dx\right) &= 1 - \frac{\lambda}{p - c} \\
\int_{\underline{p}}^{p} \frac{q(x)}{D(x)} dx &= F^{-1}\left(1 - \frac{\lambda}{p - c}\right)
\end{aligned}$$

Differentiating this equation wrt p on both sides gives

$$\frac{q(p)}{D(p)} = \frac{1}{f\left(F^{-1}\left(1 - \frac{\lambda}{p-c}\right)\right)} \frac{d}{dp}\left(1 - \frac{\lambda}{p - c}\right) = \frac{\lambda (p - c)^{-2}}{f\left(F^{-1}\left(1 - \frac{\lambda}{p-c}\right)\right)}$$

and therefore

$$q(p) = D(p) \frac{\lambda (p-c)^{-2}}{f\left(F^{-1}\left(1-\frac{\lambda}{p-c}\right)\right)}$$

What is support? Answer $\left[c+\lambda, \overline{\overline{p}}\right]$. Why?

What is the minimum price? That for which $y\left(\underline{p}\right) = 1$. Why? Well obviously $y\left(\underline{p}\right) \leq 1$. Next, if $y\left(\underline{p}\right) < 1$, that must mean $\underline{p} > \overline{\overline{p}}$ ($\overline{\overline{p}}$ being the price at which $D\left(\overline{\overline{p}}\right) = 0$) and there is never any sale at any price. [Alternatively, equation $\theta(p) = \int_{\underline{p}}^{p} \frac{q(x)}{D(x)} dx$ implies $\theta\left(\underline{p}\right) = 0$, which implies $y\left(\underline{p}\right) = 1 - F\left(\theta\left(\underline{p}\right)\right) = 1 - F(0)$ and since $\underline{\theta} = 0$, that means $y(\underline{p}) = 1$.] Given that $(p-c)\, y(p) - \lambda = 0$, $y(\underline{p}) = 1$ gives

$$\left(\underline{p} - c\right) - \lambda = 0 \rightarrow \underline{p} = c + \lambda.$$

What is the maximum price? Well it must be the maximum price $\overline{\overline{p}}$. If the maximum price were $p^{\max} < \overline{\overline{p}}$, we could charge $p^{\max} + \varepsilon$ and make money. Why? We would get $RD\left(p^{\max}+\varepsilon, \theta\right) = \theta D\left(p^{\max}+\varepsilon\right)\left(1 - \int_{\underline{p}}^{p^{\max}} \frac{q(x)}{D(x,\theta)} dx\right) = \frac{D(p^{\max}+\varepsilon)}{D(p^{\max})} RD\left(p^{\max}, \theta\right)$ just a little lower but still strictly positive in all the same states of θ, so essentially the same probability of sale at a higher price. Only when $p^{\max} = \overline{\overline{p}}$, does this argument not apply. Note, this relies on $y\left(p^{\max}\right) > 0$, which must be true for $p^{\max}$ to be finite and equal $c + \lambda / y\left(p^{\max}\right)$. In fact, we see that $y\left(\overline{\overline{p}}\right) = \frac{\lambda}{\overline{\overline{p}} - c} > 0$. This is clearly positive.[1]

Monopoly

If a monopolist chooses $q(p)$, then

$$\pi(q) = \int_{\underline{\theta}}^{\bar{\theta}} \left(\int_{\underline{p}}^{\rho(\theta)} (p-c)\, q(p)\, dp - \lambda \int_{\underline{p}}^{\bar{p}} q(p)\, dp \right) f(\theta)\, d\theta$$

$$\pi(q) = \int_{\underline{p}}^{\bar{p}} \left((1 - F(\theta(p)))(p-c) - \lambda \right) q(p)\, dp$$

FOC: Standard Monopoly problem

$$\begin{aligned} MR &= MC \\ P + P'Q &= MC \\ P &= MC - P'Q \end{aligned}$$

$P'Q$ is the revenue lost on inframarginal units when reduce price enough to sell one more unit. Equivalently, as $P' = 1/D'$ we have

$$P = MC - D/D'$$

FOC here

[1] It also has to be less than 1, this is true when $\lambda < \overline{\overline{p}} - c$ or $\overline{\overline{p}} > c + \lambda$, which is required for the market to operate.

Paper states that:

$$(1-F(\theta(p)))(p-c)-\lambda-\int_{\theta(p)}^{\theta(\bar{p})}(\rho(\theta)-c)\frac{D(\rho(\theta))}{D(p)}f(\theta)\,d\theta=0$$

This is equivalent to

$$p=c+\frac{\lambda}{1-F(\theta(p))}+\int_{\theta(p)}^{\theta(\bar{p})}(\rho(\theta)-c)\frac{D(\rho(\theta))}{D(p)}\frac{f(\theta)}{1-F(\theta(p))}d\theta$$

or

$$p=c+\frac{\lambda}{1-F(\theta(p))}+E\left[(\rho(\theta)-c)\frac{D(\rho(\theta))}{D(p)}\middle|\,\theta\geq\theta(p)\right] \tag{2}$$

This looks like the standard formula, where marginal cost is $c+\lambda/(1-F(\theta(p)))$, which inflates cost of capacity by $1/\Pr(sale)$ rather than deflating margin $(p-c)$ by $\Pr(sale)$. Also, in state $\theta>\theta(p)$, the maximum sales price is $\rho(\theta)>p$. The fact that I sold one more unit at price p, means that $D(\rho(\theta))/D(p)$ units were sold to folks with value $\rho(\theta)$. So I sell that many fewer units at price $\rho(\theta)$. The last terms is the expected such loss given $\theta\geq\theta(p)$.[2] So $MR=p-E\left[(\rho(\theta)-c)\frac{D(\rho(\theta))}{D(p)}\middle|\,\theta\geq\theta(p)\right]$, and the markup over MC is the difference between p and MR, just as in the standard monopoly problem.

To derive (8) and (9), recall $y(p)=1-F(\theta(p))$, multiple both sides by $D(p)$ and write FOC as

$$y(p)(p-c)D(p)=\lambda D(p)+\int_{\theta(p)}^{\theta(\bar{p})}(\rho(\theta)-c)D(\rho(\theta))f(\theta)\,d\theta$$

Also, $y'(p)=-f(\theta(p))\theta'(p)$, so we can do a change of variables and write FOC as

$$y(p)(p-c)D(p)=\lambda D(p)+\int_{p}^{\bar{p}}(x-c)D(x)f(\theta(x))\theta'(x)\,dx$$

$$y(p)(p-c)D(p)=\lambda D(p)-\int_{p}^{\bar{p}}(x-c)D(x)y'(x)\,dx$$

Now, differentiate both sides wrt p

$$y'(p)D(p)(p-c)+y(p)(D'(p)(p-c)+D(p))=\lambda D'(p)+(p-c)D(p)y'(p)$$

y' terms cancel

$$y(p)(D'(p)(p-c)+D(p))=\lambda D'(p)$$

Solve for $y(p)$ to get eq(8):

$$y(p)=\frac{\lambda D'(p)}{D'(p)(p-c)+D(p)}$$

Now, recall that $\theta(p)=\int_{\underline{p}}^{p}\frac{q(x)}{D(x)}dx$, so $\theta'(p)=\frac{q(p)}{D(p)}$ and $q(p)=D(p)\theta'(p)$. Moreover, from $y'(p)=-f(\theta(p))\theta'(p)$ we have $\theta'(p)=-y'(p)/f(\theta(p))$. Therefore:

$$q(p)=D(p)\theta'(p)=-D(p)\frac{y'(p)}{f(\theta(p))}$$

[2] We could also write this as $\lambda=(1-F(\theta(p)))\left(p-c-E\left[(\rho(\theta)-c)\frac{D(\rho(\theta))}{D(p)}\middle|\,\theta\geq\theta(p)\right]\right)$.

Substituting in $\theta(p) = F^{-1}(1 - y(p))$ we get eq(9).

Now we can re-write eq (8) a few ways. First,

$$D'(p)(p-c) + D(p) = \frac{\lambda D'(p)}{y(p)}$$

$$p = c + \frac{\lambda}{y(p)} - \frac{D(p)}{D'(p)} = c + \frac{\lambda}{y(p)} - \frac{p}{\varepsilon_d} \tag{3}$$

Notice that this is the same as in the standard monopoly problem, except that mc cost is adjusted for cost of capacity, inflated by $1/\Pr(sale)$. For $\lambda = 0$ there is a unique solution at the standard monopoly price, and this is the unique price charged. For $\lambda > 0$, $y(p)$ is determined so that this holds at all p in $[\underline{p}, \bar{p}]$.

What is the support? minimum price is again at $y(\underline{p}) = 1$ for the same reason. So $\underline{p} = c + \lambda - \frac{D(\underline{p})}{D'(\underline{p})}$, which is akin to the standard monopoly price. What is the maximum price? For similar reasons it is $\bar{\bar{p}}$. So we have narrower support of prices, because top of support is the same, but bottom is shifted up due to monopoly markup.

Oligopoly

- Paper extends results to oligopoly case, conditional on λ sufficiently large
- Monopoly and competition results are limiting cases
- Timing: Oligopoly model assumes firms choose prices and capacity at the same time given a capacity cost λ. The results are the same as if there were a **binding** capacity constraint that led to shadow cost of capacity λ. This is not the same as a sequential game where capacity is chosen at cost λ before price. That game may be more natural, and would correspond more closely to Cournot competition. Model in paper closer to Bertrand.

Price dispersion and Market Structure

- Paper: Price dispersion increases with competition
  - Prop 5: support of prices widens with competition
  - Prop 6: variance of price higher with perfect competition than monopoly given linear demand.
- Intuition for Proposition 6: Proposition 6 assumes linear demand for which monopoly PTR=1/2. PTR is 1 with perfect competition given constant marginal cost (supply perfectly elastic). This implies that competition increases the PTR. An increase in the PTR means that changes in $\lambda/y(p)$ get passed through more, leading to more price dispersion.

- The more general result is that price dispersion increases with PTR. This may or many not coincide with increasing competition.
- Glen Weyl has pointed out that in IO we often inadvertently assume our conclusions by specifying a demand system with PTR<1. See homework for an example with $PTR > 1$.

- Intuition for Proposition 5:

  - Intuition 1: Look at equation (2) in these notes. The monopoly markup is the lost revenue from reduced sales of higher priced units when selling one more unit at the price p. At the highest price $p^{\max}$, there are not higher priced units, so the monopoly markup is zero. Thus the highest monopoly price will be the same as the competitive price as long as $y\left(p^{\max}\right)$ is the same in both cases. Moreover, looking at the expression for the monopoly markup in equation (3) in these notes, we see that for the monopoly markup to be zero at the maximum price we need $p^{\max} = \bar{\bar{p}}$ so that $D\left(p^{\max}\right) = 0$. This is the same maximum price derived in the competitive case.
  - Intuition 2: The minimum price is higher under monopoly than competition due to the monopoly markup. The maximum prices are $\bar{\bar{p}}$ under either market structure because the firm may offer a continuum of prices. Thus for any p, there is always a high enough demand state such that $c + \lambda / \left(1 - F\left(\theta\right)\right) > p$. That is, if there were no finite demand intercept, the firm would always set some capacity at price ∞. As a result the upper bound of prices does not change, but the lower bound increases with market power. However, this result is misleading. First, the support of prices is not a good measure of dispersion. A narrow support can be consistent with a higher variance. Second, if we impose that the firm must offer a finite number k prices, then the maximum prices can vary with market structure. As a result, examining the discrete demand states model may be a better guide. See homework.
  - Intuition 3: We need $y\left(p^{\max}\right) > 0$ to cover the cost of capacity. That requires $\theta\left(p^{\max}\right) < \bar{\theta}$. That implies, however that we have a positive probability (actually just $y\left(p^{\max}\right)$) of selling out. So if $p^{\max} < \bar{p}$, there would be demand to buy at $p^{\max}+\varepsilon$ with probability close to $y\left(p^{\max}\right)$, which would be profitable for sufficiently small units $q\left(p^{\max}\varepsilon\right)$.

- Bottom line: Price dispersion likely increases with PTR, which often (but not always) increases with competition.

Uncertain Demand but Costless Capacity

Suppose that capacity costs are sunk at the time prices are chosen, and capacity is all used with probability 0–the firm never sells out. Then $\lambda = 0$. Dana predicts no price dispersion under either monopoly or perfect competition. However, in the case of market power, this result relies on the assumption $D\left(\theta, p\right) = \theta D\left(p\right)$. Relaxing this assumption can lead to price dispersion under monopoly but not perfect competition. See homework.