Nonlinear Pricing (Second-Degree Price Discrimination)—Lecture Note 2
EC 8854[1]

Two types

Graphical Analysis

See "Nonlinear Pricing - Lecture Note 1 - Graphical Analysis.pdf"

Formal (graduate level) analysis

There are f_L consumers of type L and f_H consumers of type H. The monopolist offers consumers a menu of two quantity/quality-price pairs, either $\{q_L, p_L\}$ or $\{q_H, p_H\}$, and has per-customer costs of $C(q)$. A consumer of type θ who purchases quantity (quality) q has utility

$$u(q,\theta) = V(q,\theta) - P(q),$$

and an outside option of 0. Suppose that $V_\theta > 0$ (high types have high values) and $V_{q\theta} > 0$ (single crossing).

By the revelation principle, it is without loss of generality to consider tariff menus that induce type L to choose option L and type H to choose option H (we can always relabel options so that this is satisfied, allowing for the possibility that we might label $q_L = p_L = 0$ if the low type chose the outside option, or $q_L = q_H$ and $p_L = p_H$ if both chose the same option). Thus the firm's problem is to choose $\{q_L, p_L, q_H, p_H\}$ to maximize expected profits of $\pi = \sum_{\theta\in\{L,H\}} f_\theta (P(q_\theta) - C(q_\theta))$ subject to the constraint that consumers choose the right menu options. That is, the firm solves:

$$\begin{array}{rl} & \max_{\{q_L,p_L,q_H,p_H\}} f_L(p_L - C(q_L)) + f_H(p_H - C(q_H)) \\ & \text{s.t.} \\ \text{Participation / Individual Rationality:} & u(q_L, L) \geq 0 \text{ (IR-L) and } u(q_H, H) \geq 0 \text{ (IR-H)} \\ \text{Incentive Compatibility:} & u(q_L, L) \geq u(q_H, L) \ \text{ (IC-L) and } u(q_H, H) \geq u(q_L, H) \ \text{ (IC-H)} \end{array}$$

Simplifying participation constraint: We can ignore $u(q_H, H) \geq 0$ because $V_\theta > 0$ implies $u(q_L, H) > u(q_L, L)$ and thus $u(q_H, H) \geq u(q_L, H)$ plus $q(q_L, L) \geq 0$ imply $u(q_H, H) \geq 0$. As a result, $u(q_L, L) \geq 0$ must bind with equality, otherwise we could raise $p_L$and p_H both by ε without violating any constraint. Thus the participation constraint simplifies to $u(q_L, L) = 0$, or

$$p_L = V(q_L, L)$$

Simplifying incentive compatibility constraint: Incentive compatibility for type H must hold with equality: $u(q_H, H) = u(q_L, H)$, or

$$p_H = p_L + V(q_H, H) - V(q_L, H) = V(q_L, L) + V(q_H, H) - V(q_L, H).$$

[1] These lecture notes are based on Alessandro Bonatti's which in turn were based on Glenn Ellison's notes and surely draw on Tirole 3.5. A good reference is Fudenberg and Tirole's (1991) Game Theory, Chapter 7. Also see Wilson's (1993) book *Nonlinear Pricing*.

Otherwise we could raise p_H by ε without violating IC-H and this would relax IC-L and not affect IR-L.

Further, $V_{q\theta} > 0$ and $q_H \geq q_L$ imply that $u(q_H, H) - u(q_L, H) \geq u(q_H, L) - u(q_L, L)$. Hence if $u(q_H, H) = u(q_L, H)$, we know that $u(q_H, L) \leq u(q_L, L)$ and incentive compatibility holds for type L. To solve the problem, we first relax the problem, by ignoring the IC-L constraint, and then check at the end that the solution to the relaxed problem satisfies $q_H \geq q_L$, and hence solves the original problem. Following this approach, the relaxed problem is:

$$\max_{\{q_L, q_H\}} \begin{pmatrix} f_L \left(V(q_L, L) - C(q_L)\right) \\ +f_H \left(V(q_L, L) + V(q_H, H) - V(q_L, H) - C(q_H)\right) \end{pmatrix}$$

The optimal quantities are $q_H^* = q_H^{FB}$: $V_q(q_H^*, H) = C_q(q_H^*)$ and $q_L^* < q_L^{FB}$:

$$\frac{V_q(q_L, L) - C_q(q_L)}{V_q(q_L, H) - V_q(q_L, L)} = \frac{f_H}{f_L}.$$

Finally, to verify that this solution to the relaxed problem also solves the original problem requires verifying that $q_H \geq q_L$.

Continuous types

Consider a model with a population of heterogeneous types with a density $f(\theta)$ on $[\underline{\theta}, \bar{\theta}]$. The monopolist offers a tariff $P(q)$. Consumer of type θ who purchases quantity (quality) q has utility

$$u(q, \theta) = V(q, \theta) - P(q).$$

Type θ has outside option $\underline{u}(\theta) = 0$. Thus the tariff $P(q)$ induces consumer type θ to buy $q(\theta) = \arg\max_q u(q, \theta)$ and pay $P(q(\theta))$ if $u(q(\theta), \theta) \geq \underline{u}(\theta)$.

We will assume (1) $V_\theta > 0$ (a normalization), (2) $V_{q\theta} > 0$ (single crossing), (3) $\frac{d}{d\theta}\left[f(\theta)/(1 - F(\theta))\right] \geq 0$ (non-decreasing hazard rate), (4) $V_{qq} \leq 0$ (decreasing marginal value of quantity), (5) $C_{qq} \geq 0$ (convex costs), (6) $V_{qq\theta} \geq 0$ and $V_{q\theta\theta} \leq 0$ (technical/uninterpretable).

Firm costs are $C(q)$ and expected profits are $E\left[P(q(\theta)) - C(q(\theta))\right]$. While the firm actually chooses a tariff $P(q)$ that maps quantities into prices, it is helpful in solving the problem to follow the mechanism design approach. We imagine the firm designing a *direct revelation mechanism* in which a consumer reports her type θ to the firm and the firm assigns her a quantity and payment as a function of the reported type. Thus the firm chooses a pair of functions $\{q(\theta), T(\theta)\}$ that map types into quantity and payment pairs. These functions must satisfy the *incentive compatibility* (IC) constraint that it be optimal for the consumer to truthfully reveal her type θ.[2] A consumer of true type θ who reports her type to be $\hat{\theta}$ will receive

$$u\left(\theta, \hat{\theta}\right) = V\left(q\left(\hat{\theta}\right), \theta\right) - T\left(\hat{\theta}\right).$$

[2] The *relevation principle* says that we need only consider truthful mechanisms. Any mechanism in which agents misreported their types as $\tau(\theta)$ and assigned $\{q(\tau), T(\tau)\}$ could be replaced by one in which agents were truthful are were assigned $\left\{\hat{q}(\theta), \hat{T}(\theta)\right\} = \{q(\tau(\theta)), T(\tau(\theta))\}$.

Thus the IC constraint is that $u(\theta,\theta) \geq u\left(\theta,\hat{\theta}\right)$ for all θ and $\hat{\theta}$. Note that, letting $\theta(q)$ be the inverse of $q(\theta)$, the tariff that implements this mechanism will be $P(q) = T(\theta(q))$.

We will assume that it is optimal for the firm to serve all consumers, so we will also impose a *participation* or *individual rationality* (IR) constraint that $u(\theta,\theta) \geq \underline{u}(\theta)$ for all θ. The firm's profit maximization problem is then:

$$\begin{aligned}
&\max_{q(\theta),T(\theta)} \int_{\underline{\theta}}^{\bar{\theta}} \left[T(\theta) - C(q(\theta))\right] f(\theta)\, d\theta \\
&\text{s.t.} \\
\text{IR:} \quad & u(\theta,\theta) \geq \underline{u}(\theta) \quad \forall\theta \\
\text{IC:} \quad & u(\theta,\theta) \geq u\left(\theta,\hat{\theta}\right) \quad \forall\theta,\hat{\theta}
\end{aligned}$$

Simplifying the problem:

(1) We can separate the IC constraint into a local and a global condition. The local IC constraint is that the consumer should not wish to misreport her type by ε. In other words, reporting truthfully must satisfy the local first-order condition for optimality,

$$\frac{\partial}{\partial\hat{\theta}} u(\theta,\theta) = 0,$$

which in turn implies that

$$\frac{d}{d\theta} u(\theta,\theta) = \frac{\partial}{\partial\theta} u(\theta,\theta) = V_\theta(q(\theta),\theta).$$

(This is an application of the envelope theorem.) Let $u(\theta) = u(\theta,\theta)$. As $u(\theta) = u(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \frac{d}{d\theta} u(x)\, dx$, we can write

$$u(\theta) = u(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} V_\theta(q(x),x)\, dx. \tag{1}$$

This is an important equation! It illustrates the trade-off between efficiency and rent extraction. I'd like to sell the efficient quantity $q^*(\theta)$ to maximize surplus if I didn't have to give any of the surplus to consumers. But this equation shows that the higher $q(\theta)$ the higher is $u(\theta')\ \forall\theta' > \theta$. I have no reason to distort quantity of the highest type, but as we get lower in the type space there is more and more of a motivation to extract rents.

As, $u(\theta) = V(q(\theta),\theta) - T(\theta)$, we can solve for $T(\theta)$ as

$$T(\theta) = V(q(\theta),\theta) - u(\theta) = -u(\underline{\theta}) + V(q(\theta),\theta) - \int_{\underline{\theta}}^{\theta} V_\theta(q(x),x)\, dx.$$

Thus, given a proposed allocation rule $q(\theta)$, the payment rule $T(\theta)$ is pinned down up to a constant $u(\underline{\theta})$. That is marginal prices are entirely determined by the allocation $q(\theta)$.

(2) We begin by assuming a zero outside option ($\underline{u}(\theta) = 0$). In this case, notice that if participation is satisfied at the bottom ($u(\underline{\theta}) \geq 0$) then local IC implies participation is satisfied for all higher types $(u(\theta) \geq 0\ \forall\theta)$. This follows from equation (1) as

$\int_{\underline{\theta}}^{\theta} V_\theta(q(x), x)\, dx \geq 0$. (Recall we assumed $V_\theta \geq 0$). Thus the participation constraint reduces to

$$u(\underline{\theta}) = 0.$$

(If the constraint were not binding the firm could raise all prices by ε without violating IR or IC.)

(3) Next, notice that a sufficient condition for global incentive compatibility is the following second order condition

$$\frac{\partial^2}{\partial\theta\partial\hat{\theta}} u\left(\theta, \hat{\theta}\right) \geq 0.$$

As $\frac{\partial}{\partial\hat{\theta}} u(\theta, \theta) = 0$, the condition $\frac{\partial^2}{\partial\theta\partial\hat{\theta}} u\left(\theta, \hat{\theta}\right) \geq 0$ implies that (1) $\frac{\partial}{\partial\hat{\theta}} u\left(\theta, \hat{\theta}\right) \geq 0$ for $\theta > \hat{\theta}$ (meaning it is optimal to increase the reported type $\hat{\theta}$ if it is below the true type) and (2) $\frac{\partial}{\partial\hat{\theta}} u\left(\theta, \hat{\theta}\right) \leq 0$ for $\theta < \hat{\theta}$ (meaning it is optimal to decrease the reported type $\hat{\theta}$ if it is above the true type) Thus $\frac{\partial^2}{\partial\theta\partial\hat{\theta}} u\left(\theta, \hat{\theta}\right) \geq 0$ is sufficient for global IC given local IC.

Now, as $\frac{\partial}{\partial\theta} u\left(\theta, \hat{\theta}\right) = V_\theta\left(q\left(\hat{\theta}\right), \theta\right)$

$$\frac{\partial^2}{\partial\theta\partial\hat{\theta}} u(\theta, \theta) = V_{q\theta}\left(q\left(\hat{\theta}\right), \theta\right) \frac{d}{d\hat{\theta}} q\left(\hat{\theta}\right)$$

Given our single crossing assumption ($V_{q\theta} > 0$) our sufficient second-order condition is then satisfied as long as $q(\theta)$ is non-decreasing (*monotonicity*). Our approach will be to impose the local incentive constraint, solve the relaxed problem that ignores the global constraint, and check afterwards to see if the solution $q(\theta)$ is non-decreasing (and hence solves the full problem).

Given the three simplifying steps above, we can re-write the (relaxed) problem as an unconstrained maximization over the allocation $q(\theta)$:

$$\max_{q(\theta)} \int_{\underline{\theta}}^{\bar{\theta}} \left[V(q(\theta), \theta) - \int_{\underline{\theta}}^{\theta} V_\theta(q(x), x)\, dx - C(q(\theta))\right] f(\theta)\, d\theta.$$

Notice that payments $T(\theta)$ have totally dropped out. They are determined entirely by $q(\theta)$, IR, and IC. Thus profits depend only on the chosen allocation rule $q(\theta)$. [This got Myerson the Nobel...]

To solve this problem, the next step is to eliminate the nested integral using integration by parts. Recall that $\int_a^b u dv = uv|_a^b - \int_a^b v du$. Thus

$$\begin{aligned}
\int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\theta} V_\theta(q(x), x)\, dx f(\theta)\, d\theta &= \left.\int_{\underline{\theta}}^{\theta} V_\theta(q(x), x)\, dx F(\theta)\right|_{\underline{\theta}}^{\bar{\theta}} - \int_{\underline{\theta}}^{\bar{\theta}} V_\theta(q(\theta), \theta) F(\theta)\, d\theta \\
&= \int_{\underline{\theta}}^{\bar{\theta}} V_\theta(q(\theta), \theta)\, d\theta - \int_{\underline{\theta}}^{\bar{\theta}} V_\theta(q(\theta), \theta) F(\theta)\, d\theta \\
&= \int_{\underline{\theta}}^{\bar{\theta}} (1 - F(\theta)) V_\theta(q(\theta), \theta)\, d\theta.
\end{aligned}$$

Substituting this back in to our objective function, the monopolists problem finally is

$$\max_{q(\theta)} \int_{\underline{\theta}}^{\bar{\theta}} \left[V\left(q\left(\theta\right),\theta\right) - C\left(q\left(\theta\right)\right) - \frac{1-F\left(\theta\right)}{f\left(\theta\right)} V_{\theta}\left(q\left(\theta\right),\theta\right) \right] f\left(\theta\right) d\theta.$$

The term $V\left(q,\theta\right) - C\left(q\right)$ is surplus. The adjusted term inside the integrand is called virtual surplus

$$\psi\left(q,\theta\right) = V\left(q,\theta\right) - C\left(q\right) - \frac{1-F\left(\theta\right)}{f\left(\theta\right)} V_{\theta}\left(q,\theta\right).$$

Thus we find that the firm, rather than maximizing expected surplus should maximize expected virtual surplus $E\left[\psi\left(q\left(\theta\right),\theta\right)\right]$. The beautiful thing about this expression as it can be maximized point-wise. So we want to solve the FOC $\frac{d}{dq}\psi\left(q,\theta\right) = 0$ to find the optimal $q\left(\theta\right)$ and then check SOC $\frac{d^2}{dq^2}\psi\left(q,\theta\right) \leq 0$. Finally, we will then need to check whether the solution $q\left(\theta\right)$ is non-decreasing. To do so, we use a result from monotone comparative statics (MCS). The result says that if $\frac{d^2}{dqd\theta}\psi\left(q,\theta\right) \geq 0$ then $q\left(\theta\right) = \arg\max_q \psi\left(q,\theta\right)$ is non-decreasing in θ. So we need to solve $\frac{d}{dq}\psi\left(q,\theta\right) = 0$ and check $\frac{d^2}{dq^2}\psi\left(q,\theta\right) \leq 0$ and $\frac{d^2}{dqd\theta}\psi\left(q,\theta\right) \geq 0$.

$$\begin{aligned}
\frac{d}{dq}\psi\left(q,\theta\right) &= V_q\left(q,\theta\right) - C_q\left(q\right) - \frac{1-F\left(\theta\right)}{f\left(\theta\right)} V_{q\theta}\left(q,\theta\right), \\
\frac{d^2}{dq^2}\psi\left(q,\theta\right) &= V_{qq}\left(q,\theta\right) - C_{qq}\left(q\right) - \frac{1-F\left(\theta\right)}{f\left(\theta\right)} V_{qq\theta}\left(q,\theta\right), \\
\frac{d^2}{dqd\theta}\psi\left(q,\theta\right) &= V_{q\theta}\left(q,\theta\right)\left(1 - \frac{d}{d\theta}\left(\frac{1-F\left(\theta\right)}{f\left(\theta\right)}\right)\right) - \frac{1-F\left(\theta\right)}{f\left(\theta\right)} V_{q\theta\theta}\left(q,\theta\right).
\end{aligned}$$

Lets get the technical details out of the way first, and then turn to the solution. First $\frac{d^2}{dq^2}\psi\left(q,\theta\right) \leq 0$ holds given $V_{qq}\left(q,\theta\right) \leq 0$ (diminishing marginal value of quantity), $C_{qq}\left(q\right) \geq 0$ (convex costs), and $V_{qq\theta}\left(q,\theta\right) \geq 0$ (uninterpretable), which is why we have made these assumptions. Second, $\frac{d^2}{dqd\theta}\psi\left(q,\theta\right) \geq 0$ holds given $V_{q\theta} > 0$ (single crossing), $\frac{d}{d\theta}\left[f\left(\theta\right)/\left(1-F\left(\theta\right)\right)\right] \geq 0$ (non-decreasing hazard rate), and $V_{q\theta\theta} \leq 0$ (uninterpretable). This is why we have assumed $V_{q\theta\theta} \leq 0$ and non-decreasing hazard.

Now, on to the solution! The FOC with respect to $q\left(\theta\right)$ is

$$V_q\left(q,\theta\right) = C_q\left(q\right) + \frac{1-F\left(\theta\right)}{f\left(\theta\right)} V_{q\theta}\left(q,\theta\right). \tag{2}$$

Notice that the left-hand side is the marginal price $P'\left(q\right)$. (This follows from the consumers maximization problem $q = \arg\max_q \left(V\left(q,\theta\right) - P\left(q\right)\right)$ which sets $P'\left(q\right) = V_q\left(q,\theta\right)$.) Thus equation (2) says that marginal price is equal to marginal cost plus an upwards distortion $\frac{1-F(\theta)}{f(\theta)} V_{q\theta}\left(q,\theta\right)$, which necessarily distorts the quantity down below first best. Some intuition for the distortion:

1. We can think of $f\left(\theta\right) V_q\left(q,\theta\right) - \left(1-F\left(\theta\right)\right) V_{q\theta}\left(q,\theta\right)$ as marginal revenue and $f\left(\theta\right) C_q\left(q\right)$ the marginal cost of selling an extra unit to type θ. When we sell an extra unit to type θ, we earn the marginal price $V_q\left(q,\theta\right)$ and pay marginal cost $C_q\left(q,\theta\right)$, with probability $f\left(\theta\right)$. However, we also have to given an additional information rent $V_{q\theta}\left(q,\theta\right)$ to all higher types, which arise with probability $\left(1-F\left(\theta\right)\right)$.

2. The first-order condition balances a trade-off between creating surplus and extracting rents. It balances additional surplus $(V_q - C_q)$ for type θ with probability $f(\theta)$ against additional information rents $V_{q\theta}$ for higher types with probability $(1 - F(\theta))$.

Some results that follow from equation (2):

1. No distortion at the top: At $\theta = \bar{\theta}$, $\frac{1-F(\theta)}{f(\theta)} = 0$ so there is no distortion at the top.

2. Quantity discounts: Suppose that $V = q\theta - q^2/2$ so that $V_q = \theta - q$ and $V_{q\theta} = 1$, and strictly increasing hazard $\left(\frac{d}{d\theta}\frac{1-F(\theta)}{f(\theta)} < 0\right)$. Then the distortion is decreasing in θ. This is because distortions are imposed to minimize information rents to higher types, and there are fewer higher types above θ as θ increases. In particular, the absolute markup
$$p - c = V_q(q, \theta) - C_q(q) = \frac{1 - F(\theta)}{f(\theta)}$$
is decreasing in θ and q. Suppose also that $C(q) = cq$ so that marginal cost is constant. Then the decreasing markup implies marginal price is decreasing in θ (and hence q). Thus $P''(q) < 0$. Concavity of the tariff implies that the average price per unit $P(q)/q$ is decreasing. A quantity discount in a stronger sense.

3. Quantity discounts: Suppose that $V = \theta q^{\gamma}$, for $\gamma \in (0, 1)$, so that $V_q = \theta\gamma/q^{1-\gamma}$ and $V_{q\theta} = \gamma/q^{1-\gamma}$. Assume constant marginal cost c. Then $p = c + \gamma\frac{1-F(\theta)}{f(\theta)q^{1-\gamma}(\theta)}$. Notice that (by $q(\theta)$ non-decreasing). The same quantity discount results apply: $p - c$ is decreasing, p is decreasing, and $P(q)/q$ is decreasing. (This example is inspired by McManus, Brian. 2007. "Nonlinear Pricing in an Oligopoly Market: The Case of Specialty Coffee." The RAND Journal of Economics, 38(2), 512-32.)

Examples

Now lets look at some special cases.

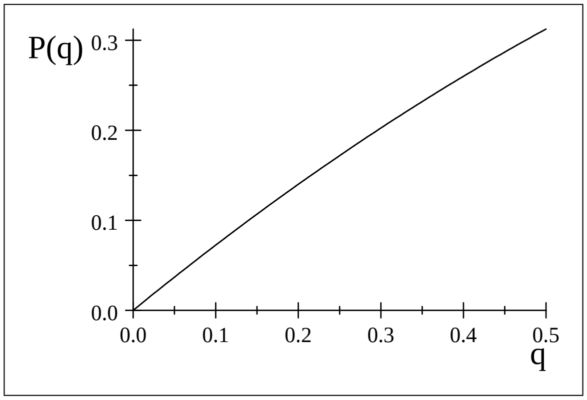
(1) Suppose that $V = q\theta - q^2/2$ and $C = cq$. Then we have $V_q = \theta - q$ and $V_{q\theta} = 1$. In this case
$$\theta - q = c + \frac{1 - F(\theta)}{f(\theta)}.$$

(1a) Suppose θ is uniform on $[0, 1]$. Then $\theta - q = c + (1 - \theta)$, or
$$q(\theta) = 2\theta - 1 - c,$$
with inverse
$$\theta(q) = \frac{1 + c + q}{2}$$
We see that the minimum type served is
$$\theta^* = (1 + c)/2.$$
Marginal price is $V_q(q, \theta(q)) = \theta - q$ or
$$P'(q) = \frac{1 + c + q}{2} - q = \frac{1 + c - q}{2}.$$

Thus total price

$$P(q) = \frac{1+c}{2}q - \frac{1}{4}q^2$$

For $c = 1/2$, this is $P(q) = \frac{1}{4}(3q - q^2)$



(1b) Suppose that θ is exponential. Then

$$\frac{1-F(\theta)}{f(\theta)} = \frac{\exp(-\lambda\theta)}{\lambda\exp(-\lambda\theta)} = \lambda$$

and

$$\theta - q = c + \lambda$$

or $q(\theta) = \theta - c - \lambda$ (minimum type served is $\theta^* = c + \lambda$) and $\theta(q) = q + c + \lambda$, implying linear pricing:

$$P'(q) = \theta - q = c + \lambda$$

$$P(q) = (c+\lambda)q.$$

(2) Another example would be $V = q\theta$ and $C = cq$. This one leads to no price discrimination. In particular, virtual surplus ψ is linear (rather than concave) in q. Thus the FOC is bang-bang.

$$\begin{aligned}\frac{d}{dq}\psi(q,\theta) &= V_q(q,\theta) - C_q(q) - \frac{1-F(\theta)}{f(\theta)}V_{q\theta}(q,\theta) \\ &= \theta - c - \frac{1-F(\theta)}{f(\theta)}\end{aligned}$$

is independent of q. It is positive for $\theta > \theta^* = c + \frac{1-F(\theta)}{f(\theta)}$ and negative for $\theta < \theta^*$. Thus types $\theta \geq \theta^*$ should all be served the maximum quantity (quality) at the same price with no price discrimination.[3] To have meaningful price discrimination we need to have a strictly concave value function V if costs are linear.

[3]Recall from Leslie (2004) that the model assumed $U_{ij} = q_{ij}[B(y_i) - p_j]^\eta$. Had he set $\eta = 1$, giving $U_{ij} = q_{ij}B(y_i) - p_j$, he would have found no discrimination optimal (given he assumed no costs, $C(q) = 0$). (Note he would still have found price dispersion within the theater because seat quality was exogenously disperse. However, he would not have found the discount booth optimal. So η drives the discount booth results.) It is not clear, however, why Leslie couldn't have used an arguably simpler specifications such as $U_{ij} = q_{ij}B(y_i) - \eta q_{ij}^2 - p_j$ or perhaps $U_{ij} = (q_{ij} - \eta q_{ij}^2)B(y_i) - p_j$ (with $\eta \leq 1$ and $\bar{q} = 1$).

Applications

- Nonlinear pricing
- Regulated firm (Ramsey pricing)
- Optimal taxation

When is price discrimination profitable?

Anderson, Eric T. and James D. Dana. 2009. "When Is Price Discrimination Profitable?" Management Science, 55(6), 980-89.

- Anderson & Dana (2009) show that whether price discrimination is profitable or not depends on whether surplus is log supermodular (profitable) or log submodular (unprofitable). An important question for empirical work: Does the model assume that surplus is log supermodular or log submodular or nest both as special cases depending on an estimated parameter? If the former, the empirical work has imposed the profitability of price discrimination by assumption.

- Note that if $S(q,\theta)$ is continuously differentiable then it is log supermodular if and only if $\frac{d^2}{dqd\theta}\ln S > 0$ (equivalently $S_{q\theta}S - S_q S_\theta > 0$) and log submodular if and only if $\frac{d^2}{dqd\theta}\ln S < 0$ (equivalently $S_{q\theta}S - S_q S_\theta < 0$).

- The linear example above, $V = q\theta$ and $C = cq$, falls into the unprofitable submodular case: $S = q(\theta - c)$, $S_q = (\theta - c)$, $S_\theta = q$, $S_{q\theta} = 0$ so $S_{q\theta}S - S_q S_\theta = -q(\theta - c) = -S < 0$.

- This result does not directly apply to Leslie (2004) because Anderson & Dana (2009) assume quasi-linear utility but Leslie (2004) does not. (For what it's worth, for fixed p, Leslie (2004) does assume log supermodular demand.)