Questions

1. In class, we derived optimal nonlinear pricing by a monopolist. Consider a similar setting in which the buyer has monopsony power and the seller has private information. Let the buyer's valuation be $V(q)$, the seller's cost be $C(q,\theta)$, and the seller's type θ have density $f(\theta)$ on $[\underline{\theta},\overline{\theta}]$. Assume V and C are continuously differentiable several times and that the seller's outside option is 0. Further, assume that (1) $C_\theta > 0$, (2) $C_{q\theta} > 0$, (3) $\frac{d}{d\theta}[f(\theta)/F(\theta)] \leq 0$ (non-increasing reverse hazard rate), (4) $V_{qq} \leq 0$, and (5) $C_{qq} \geq 0$. Making an appropriate assumption on the signs of $C_{qq\theta}$ and $C_{q\theta\theta}$, characterize the optimal monopsony purchase quantity $q(\theta)$ and corresponding marginal price $P'(q)$ offered by the monopsonist (your derivation should constitute a proof of optimality). How is quantity distorted from first best?

2. Suppose a car rental company has costs of renting a car for q miles of $C(q) = (25 + 0.05q)$. There are two customer segments: 2/3 are business travelers who have value $V^B(q)$ and 1/3 are tourists who have value $V^T(q)$, where

$$V^B(q) = \begin{cases} 30q - 1.5q^2 & 0 \leq q \leq 10 \\ 150 & 10 \leq q \end{cases},$$
$$V^T(q) = \begin{cases} \frac{1}{2}q - \frac{1}{2}\frac{1}{1000}q^2 & 0 \leq q \leq 500 \\ 125 & 500 \leq q \end{cases}$$

   (a) Assuming second-degree price discrimination, what is the optimal car rental contract to offer, i.e. the optimal price as a function of mileage $P(q)$? How is consumption distorted away from first-best?

   (b) Does this coincide with the analysis in class? Why or why not? [**Hint:** if you get stuck on (a), please still attempt (b).]

   (c) How would your answer to (a) differ if $V^B(q) = 30q - 1.5q^2$ and $V^T(q) = \frac{1}{2}q - \frac{1}{2}\frac{1}{1000}q^2$?

3. Crawford and Shum (2007) and Attanasio and Pastorino (2020) both assume consumer preferences satisfy the single crossing property, as in the standard theory. Is this a reasonable assumption in the cable TV market? Is this a reasonable assumption in rural markets for food staples? Is it a reasonable assumption for any market?

   In the Attanasio and Pastorino (2020) setting you may wish to think about how family size (number of children) might affect preferences recalling that Attanasio and Pastorino (2020) state that "poorer households tend to have more children".

Bibliography

[1] Attanasio, Orazio, and Elena Pastorino. 2020. "Nonlinear Pricing in Village Economies." *Econometrica* 88 (1):207-263. doi: 10.3982/ECTA13918.

[2] Crawford, Gregory S., and Matthew Shum. 2007. "Monopoly Quality Degradation and Regulation in Cable Television." *The Journal of Law and Economics* 50 (1):181-219. doi: 10.1086/508310.

Solutions to Problem Set 1

Question 1

Monopsonist expected profits are $E\left[V\left(q\left(\theta\right)\right)-P\left(q\left(\theta\right)\right)\right]$. While the firm actually chooses a tariff $P\left(q\right)$ that maps quantities into prices, it is helpful in solving the problem to follow the mechanism design approach. We imagine the firm designing a *direct revelation mechanism* in which the seller reports her type θ to the buyer and the buyer assigns her a quantity and payment as a function of the reported type. Thus the buyer chooses a pair of functions $\{q\left(\theta\right),T\left(\theta\right)\}$ that map types into quantity and payment pairs. These functions must satisfy the *incentive compatibility* (IC) constraint that it be optimal for the consumer to truthfully reveal her type θ.[1] A seller of true type θ who reports her type to be $\hat{\theta}$ will receive

$$\pi_s\left(\theta,\hat{\theta}\right)=T\left(\hat{\theta}\right)-C\left(q\left(\hat{\theta}\right),\theta\right).$$

Thus the IC constraint is that $\pi_s\left(\theta,\theta\right)\geq\pi_s\left(\theta,\hat{\theta}\right)$ for all θ and $\hat{\theta}$. Note that, letting $\theta\left(q\right)$ be the inverse of $q\left(\theta\right)$, the tariff that implements this mechanism will be $P\left(q\right)=T\left(\theta\left(q\right)\right)$.

The seller always has an outside option not to sell and to earn 0, thus the solution must satisfy the *participation* or *individual rationality* (IR) constraint that $\pi_s\left(\theta,\theta\right)\geq 0$ for all θ. The buyer's profit maximization problem is then:

$$\begin{array}{ll} & \displaystyle\max_{q(\theta),T(\theta)}\int_{\underline{\theta}}^{\bar{\theta}}\left[V\left(q\left(\theta\right)\right)-T\left(\theta\right)\right]f\left(\theta\right)d\theta \\ & \text{s.t.} \\ \text{IR:} & \pi_s\left(\theta,\theta\right)\geq 0 \ \ \forall\theta \\ \text{IC:} & \pi_s\left(\theta,\theta\right)\geq\pi_s\left(\theta,\hat{\theta}\right) \ \ \forall\theta,\hat{\theta} \end{array}$$

Simplifying the problem:

(1) We can separate the IC constraint into a local and a global condition. The local IC constraint is that the consumer should not wish to misreport her type by ε. In other words, reporting truthfully must satisfy the local first-order condition for optimality,

$$\frac{\partial}{\partial\hat{\theta}}\pi_s\left(\theta,\theta\right)=0,$$

which in turn implies that

$$\frac{d}{d\theta}\pi_s\left(\theta,\theta\right)=\frac{\partial}{\partial\theta}\pi_s\left(\theta,\theta\right)=-C_\theta\left(q\left(\theta\right),\theta\right).$$

(This is an application of the envelope theorem.) Let $\pi_s\left(\theta\right)=\pi_s\left(\theta,\theta\right)$. As $\pi_s\left(\theta\right)=\pi_s\left(\bar{\theta}\right)-\int_\theta^{\bar{\theta}}\frac{d}{d\theta}\pi_s\left(x\right)dx$, we can write

$$\pi_s\left(\theta\right)=\pi_s\left(\bar{\theta}\right)+\int_\theta^{\bar{\theta}}C_\theta\left(q\left(x\right),x\right)dx. \tag{1}$$

[1] The *relevation principle* says that we need only consider truthful mechanisms. Any mechanism in which agents misreported their types as $\tau\left(\theta\right)$ and assigned $\{q\left(\tau\right),T\left(\tau\right)\}$ could be replaced by one in which agents were truthful are were assigned $\left\{\hat{q}\left(\theta\right),\hat{T}\left(\theta\right)\right\}=\{q\left(\tau\left(\theta\right)\right),T\left(\tau\left(\theta\right)\right)\}$.

As, $\pi_s(\theta) = T(\theta) - C(q(\theta), \theta)$, we can solve for $T(\theta)$ as

$$T(\theta) = C(q(\theta), \theta) + \pi_s(\theta) = C(q(\theta), \theta) + \pi_s(\bar{\theta}) + \int_{\theta}^{\bar{\theta}} C_{\theta}(q(x), x)\, dx.$$

Thus, given a proposed allocation rule $q(\theta)$, the payment rule $T(\theta)$ is pinned down up to a constant $\pi_s(\bar{\theta})$. That is marginal prices are entirely determined by the allocation $q(\theta)$.

(2) Notice that if the participation constraint is satisfied at the top $(\pi_s(\bar{\theta}) \geq 0)$ then local IC implies participation is satisfied for all lower types $(\pi_s(\theta) \geq 0\ \forall\theta)$. This follows from equation (1) as $\int_{\theta}^{\bar{\theta}} C_{\theta}(q(x), x)\, dx \geq 0$. (Recall we assumed $C_{\theta} \geq 0$). Thus the participation constraint reduces to $\pi_s(\bar{\theta}) \geq 0$. This will be satisfied with equality, so that

$$\pi_s(\bar{\theta}) = 0,$$

as otherwise the buyer could lower all prices by ε without violating IR or IC. Thus $T(\theta) = C(q(\theta), \theta) + \int_{\theta}^{\bar{\theta}} C_{\theta}(q(x), x)\, dx$.

(3) Next, notice that a sufficient condition for global incentive compatibility is the following second order condition

$$\frac{\partial^2}{\partial\theta\partial\hat{\theta}}\pi_s(\theta, \hat{\theta}) \geq 0.$$

As $\frac{\partial}{\partial\hat{\theta}}\pi_s(\theta, \theta) = 0$, the condition $\frac{\partial^2}{\partial\theta\partial\hat{\theta}}\pi_s(\theta, \hat{\theta}) \geq 0$ implies that (1) $\frac{\partial}{\partial\hat{\theta}}\pi_s(\theta, \hat{\theta}) \geq 0$ for $\theta > \hat{\theta}$ (meaning it is optimal to increase the reported type $\hat{\theta}$ if it is below the true type) and (2) $\frac{\partial}{\partial\hat{\theta}}\pi_s(\theta, \hat{\theta}) \leq 0$ for $\theta < \hat{\theta}$ (meaning it is optimal to decrease the reported type $\hat{\theta}$ if it is above the true type) Thus $\frac{\partial^2}{\partial\theta\partial\hat{\theta}}\pi_s(\theta, \hat{\theta}) \geq 0$ is sufficient for global IC given local IC.

Now, as $\frac{\partial}{\partial\theta}\pi_s(\theta, \hat{\theta}) = -C_{\theta}(q(\hat{\theta}), \theta)$,

$$\frac{\partial^2}{\partial\theta\partial\hat{\theta}}\pi_s(\theta, \theta) = -C_{q\theta}(q(\hat{\theta}), \theta)\frac{d}{d\hat{\theta}}q(\hat{\theta})$$

Given our single crossing assumption $(C_{q\theta} > 0)$ our sufficient second-order condition is then satisfied as long as $q(\theta)$ is non-increasing (*monotonicity*). Our approach will be to impose the local incentive constraint, solve the relaxed problem that ignores the global constraint, and check afterwards to see if the solution $q(\theta)$ is non-increasing (and hence solves the full problem).

Given the three simplifying steps above, we can re-write the (relaxed) problem as an unconstrained maximization over the allocation $q(\theta)$:

$$\max_{q(\theta)} \int_{\underline{\theta}}^{\bar{\theta}} \left[V(q(\theta)) - C(q(\theta), \theta) - \int_{\theta}^{\bar{\theta}} C_{\theta}(q(x), x)\, dx\right] f(\theta)\, d\theta.$$

Notice that payments $T(\theta)$ have totally dropped out. They are determined entirely by $q(\theta)$, IR, and IC. Thus profits depend only on the chosen allocation rule $q(\theta)$.

To solve this problem, the next step is to eliminate the nested integral using integration by parts. Recall that $\int_a^b u dv = uv|_a^b - \int_a^b v du$. Thus

$$\begin{aligned}
\int_{\underline{\theta}}^{\bar{\theta}} \int_{\theta}^{\bar{\theta}} C_\theta (q(x), x) dx f(\theta) d\theta &= \left. \int_{\theta}^{\bar{\theta}} C_\theta (q(x), x) dx F(\theta) \right|_{\underline{\theta}}^{\bar{\theta}} + \int_{\underline{\theta}}^{\bar{\theta}} C_\theta (q(\theta), \theta) F(\theta) d\theta \\
&= \int_{\underline{\theta}}^{\bar{\theta}} F(\theta) C_\theta (q(\theta), \theta) d\theta.
\end{aligned}$$

Substituting this back in to our objective function, the monopolists problem finally is

$$\max_{q(\theta)} \int_{\underline{\theta}}^{\bar{\theta}} \left[V(q(\theta)) - C(q(\theta), \theta) - \frac{F(\theta)}{f(\theta)} C_\theta (q(\theta), \theta) \right] f(\theta) d\theta.$$

The term $V(q) - C(q, \theta)$ is surplus. The adjusted term inside the integrand is called virtual surplus

$$\psi(q, \theta) = V(q) - C(q, \theta) - \frac{F(\theta)}{f(\theta)} C_\theta (q, \theta).$$

Thus we find that the firm, rather than maximizing expected surplus should maximize expected virtual surplus $E[\psi(q(\theta), \theta)]$. The beautiful thing about this expression as it can be maximized point-wise. So we want to solve the FOC $\frac{d}{dq}\psi(q, \theta) = 0$ to find the optimal $q(\theta)$ and then check SOC $\frac{d^2}{dq^2}\psi(q, \theta) \leq 0$. Finally, we will then need to check whether the solution $q(\theta)$ is non-increasing. To do so, we use a result from monotone comparative statics (MCS). The result says that if $\frac{d^2}{dqd\theta}\psi(q, \theta) \leq 0$ then $q(\theta) = \arg\max_q \psi(q, \theta)$ is non-increasing in θ. So we need to solve $\frac{d}{dq}\psi(q, \theta) = 0$ and check $\frac{d^2}{dq^2}\psi(q, \theta) \leq 0$ and $\frac{d^2}{dqd\theta}\psi(q, \theta) \leq 0$.

$$\begin{aligned}
\frac{d}{dq}\psi(q, \theta) &= V_q(q) - C_q(q, \theta) - \frac{F(\theta)}{f(\theta)} C_{q\theta}(q, \theta), \\
\frac{d^2}{dq^2}\psi(q, \theta) &= V_{qq}(q) - C_{qq}(q, \theta) - \frac{F(\theta)}{f(\theta)} C_{qq\theta}(q, \theta), \\
\frac{d^2}{dqd\theta}\psi(q, \theta) &= -C_{q\theta}(q, \theta) \left(1 + \frac{d}{d\theta} \left(\frac{F(\theta)}{f(\theta)} \right) \right) - \frac{F(\theta)}{f(\theta)} C_{q\theta\theta}(q, \theta).
\end{aligned}$$

First $\frac{d^2}{dq^2}\psi(q, \theta) \leq 0$ holds given $V_{qq}(q) \leq 0$, $C_{qq}(q) \geq 0$, and $C_{qq\theta}(q, \theta) \geq 0$. Second, $\frac{d^2}{dqd\theta}\psi(q, \theta) \leq 0$ holds given $C_{q\theta} > 0$, $\frac{d}{d\theta}[f(\theta)/F(\theta)] \leq 0$ (non-increasing reverse hazard rate), and $C_{q\theta\theta} \geq 0$. Thus buyer's SOC and seller's global IC are satisfied.

Now, on to the solution! The FOC with respect to $q(\theta)$ is

$$C_q(q, \theta) = V_q(q) - \frac{F(\theta)}{f(\theta)} C_{q\theta}(q, \theta). \tag{2}$$

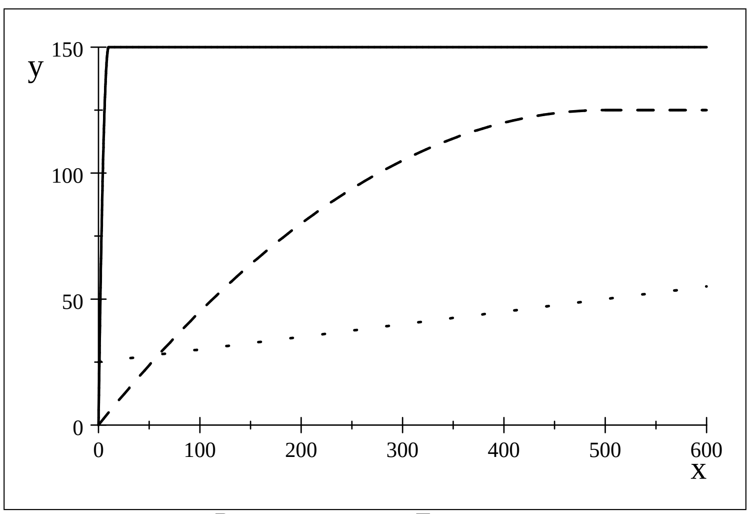
Equation (2) characterizes the optimal allocation $q(\theta)$. Moreover, it also characterizes the marginal price because the seller's maximization problem $(q = \arg\max_q (P(q) - C(q, \theta)))$ implies that $P'(q) = C_q(q, \theta)$. Thus equation (2) says that marginal price is equal to buyer's marginal value less a downwards distortion $\frac{F(\theta)}{f(\theta)} C_{q\theta}(q, \theta)$, which necessarily distorts the quantity down below first best for all but the bottom type $\underline{\theta}$ for whom $F(\underline{\theta}) = 0$.

Question 2

Q2 part b

I have written question 2 so that consumer preferences violate the single crossing property that is normally assumed. The standard nonlinear pricing model assumes $V_{q\theta} > 0$ and $V_\theta > 0$. In this problem, preferences satisfy $V_\theta > 0$ if we label the business travelers as the high type and the tourists as the low type, as $V^B > V^T$ for all q. However, the same cannot be said for V_q, as $V_q^B(0) > V_q^T(0)$ but $V_q^B(10) > V_q^T(10)$. In other words, while business travelers always have the highest total value, and have the highest marginal value for the first mile, they do not have the highest marginal value for all later miles. This is best illustrated graphically. First, plotting costs and values shows Business travelers always have higher values for consumption:

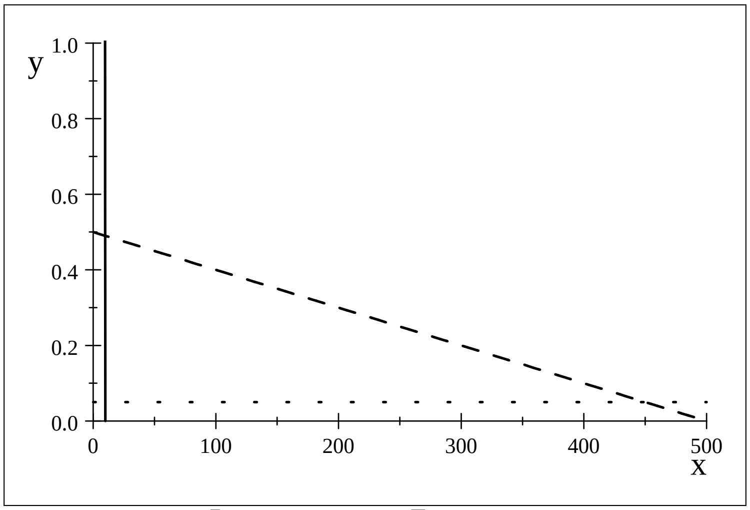
$$
\begin{aligned}
C(q) &= (25 + 0.05q) \\
V^B(q) &= \left\{ \begin{array}{cc} 30q - 1.5q^2 & 0 \le q \le 10 \\ 150 & 10 \le q \end{array} \right. , \\
V^T(q) &= \left\{ \begin{array}{cc} \left(\frac{1}{2}q - \frac{1}{2}\frac{1}{1000}q^2\right) & 0 \le q \le 500 \\ 125 & 500 \le q \end{array} \right. ,
\end{aligned}
$$



Solid: $V^B(q)$. Dash: $V^T(q)$. Dot: $C(q)$.

Next, plotting marginal values and marginal costs shows that Business travelers only have the higher marginal valuation for the first ~10 miles or so:

$$
\begin{aligned}
C_q(q) &= 0.05 \\
V_q^B(q) &= \left\{ \begin{array}{cc} 30 - 3q & 0 \le q \le 10 \\ 0 & 10 \le q \end{array} \right. , \\
V_q^T(q) &= \left\{ \begin{array}{cc} \left(\frac{1}{2} - \frac{1}{1000}q\right) & 0 \le q \le 500 \\ 0 & 500 \le q \end{array} \right. ,
\end{aligned}
$$



Solid: $V_q^B(q)$. Dash: $V_q^T(q)$. Dot: $C_q(q)$.

Much in the solution to the problem remains similar, including the "no distortion at the top" result. However, relaxing the single crossing property has led to some differences. In particular, the low type's consumption is distorted upwards rather than downwards. In particular $q_T^* \approx 483 > q_T^{FB} = 450$. Keeping in mind that the assumptions of the standard model are not satisfied, we will have to solve parts (a) and (c) from scratch, rather than applying the results given in class.

Q2 part a

We can write the firm's problem as

$$\begin{array}{ll} \max\limits_{q_B, P_B, q_T, P_T} & \frac{2}{3}\left(P_B - C\left(q_B\right)\right) + \frac{1}{3}\left(P_T - C\left(q_T\right)\right) \\ \text{such that} & \begin{array}{ll} \text{IC-H} & V_B\left(q_B\right) - P_B \geq V_B\left(q_T\right) - P_T \\ \text{IC-L} & V_T\left(q_T\right) - P_T \geq V_T\left(q_B\right) - P_B \\ \text{IR-L} & V_T\left(q_T\right) - P_T \geq 0 \\ \text{IR-H} & V_B\left(q_B\right) - P_B \geq 0 \end{array} \end{array}$$

One can verify that $V_B(q_T) \geq V_T(q_T)$ for any q_T. Therefore IC-H and IR-L imply IR-H:

$$V_B\left(q_B\right) - P_B \geq_{\text{IC-H}} V_B\left(q_T\right) - P_T \geq V_T\left(q_T\right) - P_T \geq_{\text{IC-L}} 0.$$

Thus we can drop IR-H from the problem. As a result IR-L must bind with equality. Otherwise I could raise P_B and P_T both by ε and raise profits without violating IR-L or affecting IC constraints. Hence

$$P_T = V_T\left(q_T\right).$$

Moreover, dropping IR-H from the problem, we see that IC-H must bind with equality. If it did not, raising P_B by epsilon would raise profits while not violating IC-H, relaxing IC-L, and not affecting IR-L. Hence $V_B(q_B) - P_B = V_B(q_T) - P_T$, which when substituting $P_T = V_T(q_T)$ implies that

$$P_B = V_B\left(q_B\right) - \left(V_B\left(q_T\right) - V_T\left(q_T\right)\right),$$

where the second term is type B's information rent. Substituting prices into into IC-L yields

$$\text{IC-L} \quad V_T(q_T) - V_T(q_B) \geq V_B(q_T) - V_B(q_B).$$

To solve the problem we guess that IC-L will not bind, solve the "relaxed" problem without the constraint, and then check at the end that it is satisfied. The relaxed problem is

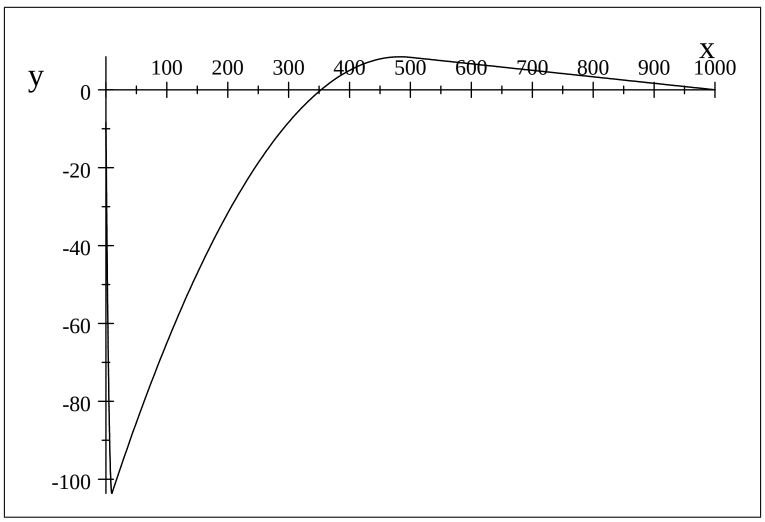
$$\max_{q_B, q_T} \frac{2}{3}\left(V_B(q_B) - V_B(q_T) + V_T(q_T) - C(q_B)\right) + \frac{1}{3}\left(V_T(q_T) - C(q_T)\right).$$

Grouping q_B and q_T terms together, this is

$$\max_{q_B, q_T} \frac{2}{3}\left(V_B(q_B) - C(q_B)\right) + \frac{1}{3}\left(V_T(q_T) - C(q_T) + 2\left(V_T(q_T) - V_B(q_T)\right)\right).$$

It is apparent by inspection that $q_B^* = q_B^{FB}$, which means that $30 - 3q_B^* = 5/100$, or $q_B^* = \frac{599}{60} = 9.9833$. Plotting the q_T term, $V_T(q_T) - \frac{1}{3}C(q_T) - \frac{2}{3}V_B(q_T)$,

$$V_T(q) - \frac{1}{3}C(q) - \frac{2}{3}V_B(q) = \begin{cases} \left(\frac{1}{2}q - \frac{1}{2}\frac{1}{1000}q^2\right) - \frac{1}{3}(25 + 0.05q) - \frac{2}{3}(30q - 1.5q^2) & 0 \leq q \leq 10 \\ \left(\frac{1}{2}q - \frac{1}{2}\frac{1}{1000}q^2\right) - \frac{1}{3}(25 + 0.05q) - \frac{2}{3}150 & 10 \leq q \leq 500 \\ 125 - \frac{1}{3}(25 + 0.05q) - \frac{2}{3}150 & 500 \leq q \end{cases}$$



we see that it is maximized at $q_T^* \in (10, 500)$. Thus the first-order condition for optimality is

$$\frac{d}{dq}\left(\left(\frac{1}{2}q - \frac{1}{2}\frac{1}{1000}q^2\right) - \frac{1}{3}(25 + 0.05q) - \frac{2}{3}150\right) = 0$$

$$\left(\frac{1}{2} - \frac{1}{1000}q - \frac{1}{3}0.05\right) = 0$$

$$q_T^* = 1000\left(\frac{1}{2} - \frac{1}{3}\frac{5}{100}\right) = 1450/3 \approx 483.33$$

Plugging these values back into earlier expressions yields prices of

$$P_T = V_T(q_T) = \left(\frac{1}{2}\left(\frac{1450}{3}\right) - \frac{1}{2}\frac{1}{1000}\left(\frac{1450}{3}\right)^2\right) = \frac{4495}{36} \approx 124.86$$

$$
\begin{aligned}
P_B &= V_B\left(q_B\right) - V_B\left(q_T\right) + V_T\left(q_T\right) \\
&= \left(30\left(\frac{599}{60}\right) - \frac{3}{2}\left(\frac{599}{60}\right)^2\right) - 150 + \frac{4495}{36} \\
&= \frac{4495}{36} - \frac{1}{2400} = \frac{898\,997}{7200} \approx 124.86.
\end{aligned}
$$

Note that $P_T - P_B = \frac{1}{2400}$. It remains to check if IC-L is satisfied. Would type L be willing to forgo 473 miles to save $\frac{1}{2400}$? Clearly not, so yes, IC-L is satisfied.

Important take-away: Notice that this optimal contract can be closely approximated by "Unlimited mileage for \$124.99 per day!", which is similar to what we often observe in practice. Thus price discrimination and violation of single crossing may explain why rental car firms offer unlimited mileage despite positive marginal costs of several cents per mile.

Q2 part c

In this case neither IC-L nor IC-H bind. As a result, both types are given FB and pay until IR binds. Thus $q_B^* = q_B^{FB} = \frac{599}{60} \approx 10$ and $P_B^* = \left(30\left(\frac{599}{60}\right) - 1.5\left(\frac{599}{60}\right)^2\right) \approx 150$. Also $q_T^* = q_T^{FB} = 450$ and $P_T^* = \left(\frac{1}{2}450 - \frac{1}{2}\frac{1}{1000}\left(450\right)^2\right) = 123.75$. The difference with respect to part (b) is that IC-H does not bind. Notice that if a business traveler chose the tourist option and were forced to drive 450 miles, she would have payoff

$$
V_B\left(q_T\right) - P_T = 30\left(450\right) - 1.5\left(450\right)^2 - 123.75 \approx -2.9037 \times 10^5
$$

Notice the interpretation of the contract is that consuming mileage other than 450 has an infinite price, so is not feasible. If the firm is restricted to offering a contract with less than or equal to q miles (rather than equal to q miles) then we are back in a situation like part (a) where the preferences essentially assumed free disposal of mileage.

Question 3

Single crossing may be a reasonable assumption in Cable TV markets or staple foods markets, but the rental car market studied in question 2 is a good example of a market where the assumption likely fails. Careful thought should be applied to such "standard" assumptions in any particular application.

Notice that in the staple foods markets, if poor households tend to have more children, we might think that single crossing is violated as in question 2. Consider a single meal. Poor households might have a lower willingness to pay for the first pound of corn than rich households, due to their high marginal value of money, but a higher willingness to pay for the n^{th} pound of corn because the n^{th} child needs to eat.