Lesson 10: Covariances of causal ARMA Processes

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A causal ARMA(p, q) process

Consider a causal ARMA(p, q) process

$x_t - \phi_1 x_{t-1} - ... - \phi_p x_{t-p} = u_t + \theta_1 u_{t-1} + ... + \theta_q u_{t-q} \ \ \forall t \in \mathbb{Z},$

$u_t \sim WN(0, \sigma_u^2)$

$$\phi(z) = 1 - \phi_1 z - ... - \phi_p z^p$$

and

$$\theta(z) = 1 + \theta_1 z ... + \theta_q z^q$$

A causal ARMA(p, q) process

The causality assumption implies that there exists constants $\psi_0, \psi_1, ...$ such that

$$\sum_{j=0}^{\infty} |\psi_j| < \infty$$

and

$$x_t = \sum_{j=0}^{\infty} \psi_j u_{t-j} \;\; \forall t.$$

The sequence $\{\psi_0, \psi_1, ...\}$ is determined by the identity

$$(1 - \phi_1 z - ... - \phi_p z^p)(\psi_0 + \psi_1 z ...) = 1 + \theta_1 z ... + \theta_q z^q$$

A causal ARMA(p, q) process

Equating coefficients of z^j, j=0,1,..., we obtain

$$\psi_0 = 1$$

$$\psi_1 = \theta_1 + \psi_0\phi_1$$

$$\psi_2 = \theta_2 + \psi_1\phi_1 + \psi_0\phi_2$$

and so on.

A causal ARMA(p, q) process

We have

$$\psi_j = \theta_j + \sum_{k=1}^{p} \phi_k \psi_{j-k}, \quad j = 0, 1...,$$

where $\theta_0 = 1$, $\theta_j = 0$ for $j > q$ and $\psi_j = 0$ for $j < 0$.

The Autocovariance Function of a causal ARMA(p, q) process

Since

$$E(x_t) = \sum_{j=0}^{\infty} \psi_j E(u_{t-j}) = 0 \;\; \forall t.$$

We have

$$\begin{aligned}
\gamma(k) &= E(x_t x_{t-k}) \\
&= E\left(\sum_{j=0}^{\infty} \psi_j u_{t-j} \sum_{i=0}^{\infty} \psi_i u_{t-k-i}\right) \\
&= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \psi_j \psi_i E(u_{t-j} u_{t-k-i})
\end{aligned}$$

The Autocovariance Function of a causal ARMA(p, q) process

Since $u_t \sim WN(0, \sigma^2)$, we have that $E\left(u_{t-j}u_{t-k-i}\right) = \sigma^2$ if $i = j - k$ and 0 otherwise. Therefore

$$\gamma(k) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j-k} \quad k = 0, 1, ...$$

The Autocovariance Function of a causal ARMA(p, q) process

It is possible to show that the sequence $\{\gamma(k)\}$, is absolutely summable, that is

$$\sum_{k=-\infty}^{\infty} |\gamma(k)| < \infty.$$

Further, we observe that the absolute summability of the sequence $\{\gamma(k)\}$ implies that

$$\lim_{k \to \infty} \gamma(k) = 0.$$

The Autocovariance Function of a causal ARMA(p, q) process

We can summarize our findings about the autocovariance function of a causal ARMA(p, q) process as follows:

- The autocovariance function of a causal ARMA(p, q) process is absolutely summable;
- The autocovariance function of a causal ARMA(p, q) process vanishes when the lag tends to infinity.

Ergodicity and ARMA process

Are the causal ARMA processes ergodic?

Ergodic Theorems

Corollary 1 (Sufficient condition for mean-ergodicity). Let x_t be a stationary process with mean μ and autocovariance function $\gamma_x(k)$. If

$$\lim_{k\to\infty}\gamma_x(k) = 0,$$

then x_t is mean-ergodic.

Corollary 2. (Sufficient condition for mean-ergodicity) Let x_t be a stationary process with mean μ and and autocovariance function $\gamma_x(k)$. If

$$\sum_{k=0}^{\infty} |\gamma_x(k)| < \infty,$$

then x_t is mean-ergodic.

Ergodicity under Gaussianity

Let $\{x_t; t \in \mathbb{Z}\}$ be a stationary process with mean μ and autocovariance function $\gamma_x(k)$. If the process is Gaussian, then
1. the condition of absolute summability of covariance function

$$\sum_{k=0}^{\infty} |\gamma_x(k)| < \infty$$

is sufficient to ensure ergodicity for all moments.
2. the condition

$$\lim_{k\to\infty}\gamma_x(k) = 0$$

is necessary and sufficient.

Ergodicity and ARMA process

Thus we have that

- a causal ARMA process is ergodic for the mean.
- a Gaussian causal ARMA process is ergodic for all moments.

The Autocovariance Function of a causal ARMA(1, 1) process

Consider a causal ARMA(1, 1) process:

$$x_t - \phi x_{t-1} = u_t + \theta u_{t-1}, \quad u_t \sim WN(0, \sigma^2)$$

The Autocovariance Function of a causal ARMA$(1, 1)$ process

We find that

$$\psi_0 = 1 \text{ and } \psi_j = (\phi + \theta)\, \phi^{j-1} \text{ for } j \geq 1.$$

The Autocovariance Function of a causal ARMA$(1,1)$ process

Thus we have

$$\begin{aligned}\gamma(0) &= \sigma^2 \textstyle\sum_{j=0}^{\infty} \psi_j^2 \\ &= \sigma^2 \left[1 + (\phi+\theta)^2 \textstyle\sum_{j=0}^{\infty} \phi^{2j}\right] \\ &= \sigma^2 \left[1 + \frac{(\phi+\theta)^2}{1-\phi^2}\right]\end{aligned}$$

$$\begin{aligned}\gamma(1) &= \sigma^2 \textstyle\sum_{j=0}^{\infty} \psi_j \psi_{j-1} \\ &= \sigma^2 \left[\phi + \theta + (\phi+\theta)^2 \phi \textstyle\sum_{j=0}^{\infty} \phi^{2j}\right] \\ &= \sigma^2 \left[\phi + \theta + \frac{(\phi+\theta)^2 \phi}{1-\phi^2}\right]\end{aligned}$$

and

$$\gamma(k) = \phi^{k-1}\gamma(1), \quad k \geq 2.$$

The Autocovariance Function of a causal ARMA(p, q) process

The autocovariance function, $\gamma(k)$, of a causal ARMA process tends to zero, as $k \to \infty$, at an exponential rate, that is, there exist constants, D and δ such that, as $k \to \infty$, $\gamma(k) \approx D\delta^k$, with $-1 < \delta < 1$

The Autocovariance Function of a causal AR(1) process

Consider a causal AR(1) process:

$$x_t - \phi x_{t-1} = u_t, \;\; u_t \sim WN(0, \sigma^2)$$

The Autocovariance Function of a causal AR(1) process

We have

$$\gamma(0) \quad = \sigma^2 \left(\frac{1}{1-\phi^2} \right)$$

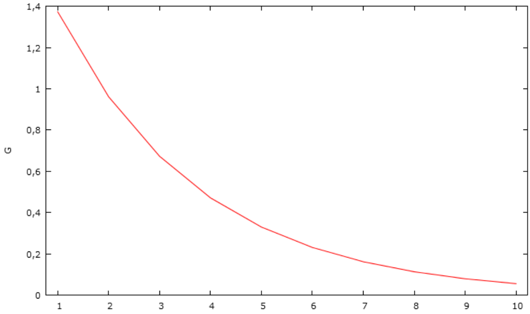
and

$$\gamma(k) = \phi^k \gamma(0), \quad k \geq 1.$$

The Autocovariance Function of a causal AR(1) process

Consider a causal AR(1) process:

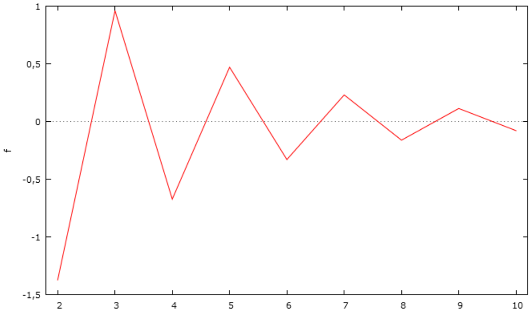
$$x_t - 0.7x_{t-1} = u_t, \quad u_t \sim WN(0,1)$$



The Autocovariance Function of a causal AR(1) process

Consider a causal AR(1) process:

$$x_t + 0.7x_{t-1} = u_t, \quad u_t \sim WN(0,1)$$



The Autocovariance Function of a causal AR(1) process

Thus the autocovariances decay exponentially in one of two forms.

1. If $0 < \phi < 1$, the autocovariances are positive;
2. If $-1 < \phi < 0$ they oscillate between positive and negative values.

The Autocovariance Function of a causal AR(2) process

Consider a causal AR(2) process:

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + u_t, \quad u_t \sim WN(0, \sigma_u^2)$$

The autocovariance function of this process is given by the following recursive formula:

$$\gamma_x(k) = \phi_1 \gamma_x(k-1) + \phi_2 \gamma_x(k-2) \quad k = 2, 3, ...$$

with the initial conditions

$$\gamma_x(0) = \frac{(1-\phi_2)\,\sigma_u^2}{(1+\phi_2)\left[(1-\phi_2)^2 - \phi_1^2\right]}$$

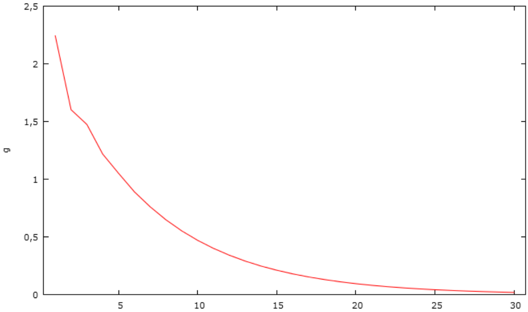
and

$$\gamma_x(1) = \frac{\phi_1 \gamma_x(0)}{(1-\phi_2)}.$$

The Autocovariance Function of a causal AR(2) process

Consider a causal AR(2) process:

$$x_t = 0.5x_{t-1} + 0.3x_{t-2} + u_t, \quad u_t \sim WN(0,1)$$



The Autocovariance Function of a causal AR(2) process

In general, the autocovariance function of AR(p) processes decays exponentially.

The Autocovariance Function of an MA(1) process

Consider an MA(1) process:

$$x_t = u_t + \theta u_{t-1}, \quad u_t \sim WN(0, \sigma^2)$$

We have

$$\gamma(0) \quad = \sigma^2 \left(1 + \theta^2\right)$$

$$\gamma(1) \quad = \sigma^2 \theta$$

and

$$\gamma(k) = 0, \quad k \geq 2.$$

The Autocovariance Function of a causal MA(q) process

In general, the autocovariance function of MA(q) models is zero for all lags greater than q.

The Autocovariance Function of a causal ARMA(p, q) process

Causal ARMA processes are *short memory* processes. This means that their autocovariance function dies out quickly, and hence

$$\sum_{k=-\infty}^{\infty} |\gamma(k)| < \infty$$

Long memory processes

A stationary process x_t with an autocovariance function $\gamma(k)$ is called a long memory process, if

$$\sum_{k=-\infty}^{\infty} |\gamma(k)| = \infty$$

Conclusion

In this lecture, we have considered the calculation of the autocovariance function $\gamma(\cdot)$ of a causal ARMA process.
We remember that the autocorrelation function is readily found from the autocovariance function on dividing by $\gamma(0)$.
The partial autocorrelation function is also found from the function $\gamma(\cdot)$.

Conclusion

We close this lecture by introducing the lag operator L.

Definition. $Lx_t = x_{t-1}$

Properties.

1. $L^j = x_t = x_{t-j}$
2. $L(\phi x_t) = \phi L x_t = \phi x_{t-1}$
3. $L^i L^j x_t = L^i x_{t-j} = x_{t-j-i} = L^{i+j} x_t$
4. $L(x_t + y_t) = Lx_t + Ly_t = x_{t-1} + y_{t-1}$

Conclusion

Finally note that by convention we define the zeroth power of L to be the identity operator, i. e.

$$L^0 x_t = x_t$$

Thus we can consider polynomials of degree $n \geq 0$ in the lag operator L, defined as

$$a(L) = a_0 L^0 + a_1 L + ... + a_n L^n$$

where $a_0, a_1, ..., a_n$ are real or complex coefficients. We have

$$a(L)x_t = a_0 x_t + a_1 x_{t-1} + ... + a_n x_{t-n}$$

Conclusion

It may shown that the algebra of the polynomials of degree $n \geq 0$ in the lag operator L, over the field of the real or complex numbers (i.e. with real or complex coefficients) is isomorphic to the algebra of polynomials in the real or complex indeterminate, say z.

We can manipulate the polynomials $a(L)$, $b(L)$,... like the polynomials $a(z)$, $b(z)$,...

Conclusion

ARMA(1,1) in terms of the lag operator L.

Consider an ARMA(1,1) process defined by the equation

$$x_t - \phi x_{t-1} = u_t + \theta u_{t-1} \quad u_t \sim WN(0, \sigma_u^2)$$

By using the lag operator this equation can be rewritten as

$$\phi(L) x_t = \theta(L) u_t$$

where $\phi(L) = 1 - \phi L$ and $\theta(L) = 1 + \theta L$

Conclusion

ARMA(p,q) in terms of the lag operator L.

We can write

$$\phi(L)x_t = \theta(L)u_t$$

where

$$\phi(L) = 1 - \phi_1 L - \phi_2 L^2 - ... - \phi_1 L^p$$

and

$$\phi(L) = 1 + \theta_1 L + \theta_2 L^2 - ... + \theta_q L^q$$

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