Econometrics Field: ECON7772

Qingsong Yao

Question 1

a) Since $Y_i = a + b\left(X_i + e_i\right)^2$, we get

$$Y_i = \left\{a + b\mathbb{E}\left(e_i^2\right)\right\} + bX_i^2 + \left\{2bX_ie_i + be_i^2 - b\mathbb{E}\left(e_i^2\right)\right\}.$$

If we define $u_i = 2X_ie_ib + be_i^2 - b\mathbb{E}\left(e_i^2\right)$, wet get $\mathbb{E}u_i = 0$ and $\mathbb{E}X_i^2u_i = 0$ due to the fact that e_i and X_i are independent (since e_i is independent of X_j for any i and j). As a result, OLS estimator of a will not be consistent (unless $b = 0$), it will converge to $a + b\mathbb{E}\left(e_i^2\right)$; but $\widehat{b}$ is consistent.

b) Under the condition that $\mathbb{E}X_i^2u_i = 0$, we can calculate its standard error following a regular way. In particular, we first calculate $\widehat{\sigma}_u^2$, which is estimator of the variance of u_i. We have

$$\widehat{\sigma}_u^2 = \frac{1}{n-2}\sum_{i=1}^{n}\left(Y_i - \widehat{a} - \widehat{b}X_i^2\right)^2.$$

Then the standard error of $\widehat{b}$ is given by

$$\sqrt{\frac{n\widehat{\sigma}_u^2}{n\sum_{i=1}^{n}X_i^4 - \left(\sum_{i=1}^{n}X_i^2\right)^2}}.$$

Question ,

a) Given that $Y = a + X + U$ and $X = bY + V$, we have $Y = a + bY + U + V$, so

$$Y = \frac{1}{1-b}\left[a + U + V\right]$$

and

$$X = \frac{b}{1-b}\left[a + U\right] + \frac{1}{1-b}V.$$

b) We can identify a and b under this setup. Note that zero mean of U and V implies that

$$\mathbb{E}Y = a + \mathbb{E}X$$

and

$$\mathbb{E}X = b\mathbb{E}Y$$

So

$$a = \mathbb{E}Y - \mathbb{E}X,\ \ b = \mathbb{E}X/\mathbb{E}Y.$$

So both a and b are identified.

Question 3

a) This is true. We have $\mathbb{E}\overline{Z} = n^{-1} \sum_{i=1}^{n} \mathbb{E} Z_i = \mu$, which validates unbiasedness.

b) This is true. Law of large numbers indicates that as long as $\mathbb{E} |Z_i| < \infty$, we have $n^{-1} \sum_{i=1}^{n} \mathbb{E} Z_i \to_p \mu$ (convergence would hold even in a.s. sense).

c) This is true. Since $\mathbb{E} I (Z_i \le z) = 1 \cdot P (Z_i \le z) + 0 \cdot P (Z_i > z) = P (Z_i \le z) = F (z)$, we have $n^{-1} \sum_{i=1}^{n} \mathbb{E} I (Z_i \le z) = F (z)$.

d) This is true due to the same arguments in (b).

Econometrics Field: ECON8821

Qingsong Yao

Question 1

First of all, since $E\left|X_t\right| < \infty$ and $\sum_j \left|c_j\right| < \infty$, we have $\sum_j \mathbb{E}\left|c_j X_{t-j}\right| < \infty$. This implies that $\sum_j c_j X_{t-j} \rightarrow_{a.s.} Y_t$, and consequently $\sum_j c_j X_{t-j} \rightarrow_p Y_t$. Moreover,

$$\begin{aligned}
\mathbb{E}\left(\sum_{|j|<k} c_j X_{t-j} - Y_t\right)^2 &= \mathbb{E}\left(\sum_{|j|\geq k} c_j X_{t-j}\right)^2 \\
&\leq \sum_{|i|\geq k}\sum_{|j|\geq k} \left|c_i c_j\right| \mathbb{E}\left|X_{t-i}X_{t-j}\right| \\
&\leq \left(\sum_{|i|\geq k}\left|c_i\right|\right)^2 \operatorname{var}\left(X_t\right) \rightarrow 0 \text{ as } k \rightarrow \infty
\end{aligned}$$

This implies that $\sum_j c_j X_{t-j} \rightarrow_{m.s.} Y_t$.

We next show that $\{Y_t\}$ is covariance stationary, and derive the variance. We have

$$\begin{aligned}
\gamma_Y(h) &= \operatorname{cov}\left(Y_t, Y_{t+h}\right) \\
&= \operatorname{cov}\left(\sum_j c_j X_{t-j}, \sum_j c_j X_{t+h-j}\right) \\
&= \sum_{j,k} c_j c_k \operatorname{cov}\left(X_{t-k}, X_{t+h-j}\right) \\
&= \sum_{j,k} c_j c_k \gamma\left(h-j+k\right),
\end{aligned}$$

where the last equation comes from the definition of the covariance function of $\{X_t\}$. When X_t is IID, we have $\gamma(l) = \sigma^2$ when $l = 0$ and $\gamma(l) = 0$ when $l \neq 0$. So (without loss of generality, assume $h \geq 0$)

$$\begin{aligned}
\gamma_Y(h) &= \sum_{j,k} c_j c_k \gamma\left(h-j+k\right) \\
&= \sum_{h+k=j} c_j c_k \sigma^2 = \sum_k c_k c_{k+h} \sigma^2.
\end{aligned}$$

Question 2

(1) Since

$$
\begin{aligned}
& f\left(Y_n, Y_{n-1}, \cdots, Y_{p+1} | Y_p, Y_{p-1}, \cdots, Y_1\right) \\
&= \prod_{t=p+1}^{n} f\left(Y_t, | Y_{t-1}, Y_{t-2}, \cdots, Y_{t-p}\right) \\
&= \prod_{t=p+1}^{n} (2\pi)^{-\frac{m}{2}} |\Omega|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\left(Y_t - C - \sum_{i=1}^{p} A_i Y_{t-i}\right)' \Omega^{-1} \left(Y_t - C - \sum_{i=1}^{p} A_i Y_{t-i}\right)\right)
\end{aligned}
$$

we have

$$
L = \sum_{t=p+1}^{n} \left\{-\frac{m}{2}\log(2\pi) - \frac{1}{2}\log\Omega - \frac{1}{2}\left(Y_t - C - \sum_{i=1}^{p} A_i Y_{t-i}\right)' \Omega^{-1} \left(Y_t - C - \sum_{i=1}^{p} A_i Y_{t-i}\right)\right\}.
$$

(2)We first show that $\widehat{B}$ do not change with Ω. Note that the log-likelihood is given by

$$
L(B, \Omega) = \sum_{t=p+1}^{n} \left\{-\frac{m}{2}\log(2\pi) - \frac{1}{2}\log\Omega - \frac{1}{2}\left(Y_t - B'X_t\right)' \Omega^{-1} \left(Y_t - B'X_t\right)\right\}.
$$

Take derivative with respect to B, we have

$$
\frac{\partial L(B, \Omega)}{\partial B} = -\sum_{t=1}^{n} \left(X_t X_t' B - X_t Y_t'\right) \Omega^{-1} = 0.
$$

So

$$
\widehat{B} = \left(\sum_{t=1}^{n} X_t X_t'\right)^{-1} \sum_{t=1}^{n} X_t Y_t'.
$$

This indicates that we can show the results under the condition that $\Omega = I$. Under this condition, we have

$$
\begin{aligned}
\sum_{t=1}^{n} \left(Y_t - B'X_t\right)' \left(Y_t - B'X_t\right) &= \sum_{t=1}^{n} \left(\begin{bmatrix} Y_{1t} \\ \vdots \\ Y_{mt} \end{bmatrix} - \begin{bmatrix} b_1' \\ \vdots \\ b_m' \end{bmatrix} X_t \right)' \left(\begin{bmatrix} Y_{1t} \\ \vdots \\ Y_{mt} \end{bmatrix} - \begin{bmatrix} b_1' \\ \vdots \\ b_m' \end{bmatrix} X_t \right) \\
&= \sum_{t=1}^{n} \begin{bmatrix} Y_{1t} - b_1'X_t \\ \vdots \\ Y_{mt} - b_m'X_t \end{bmatrix}' \begin{bmatrix} Y_{1t} - b_1'X_t \\ \vdots \\ Y_{mt} - b_m'X_t \end{bmatrix} \\
&= \sum_{t=1}^{n} \sum_{j=1}^{m} \left(Y_{jt} - b_j'X_t\right)' \left(Y_{jt} - b_j'X_t\right) \\
&= \sum_{j=1}^{m} \sum_{t=1}^{n} \left(Y_{jt} - b_j'X_t\right)' \left(Y_{jt} - b_j'X_t\right).
\end{aligned}
$$

This implies that

$$
\widehat{b}_j = \arg\min \sum_{t=1}^{n} \left(Y_{jt} - b_j'X_t\right)' \left(Y_{jt} - b_j'X_t\right) = \widehat{b}_{OLS}.
$$

Question 3

We have

$$
\widehat{\alpha} = 1 + \frac{\sum u_t y_{t-1}}{\sum y_{t-1}^2}.
$$

Note that

$$
T^{-2} \sum y_{t-1}^2 = \sum \left(\frac{y_{t-1}}{\sqrt{T}} \right)^2 \frac{1}{T} \Rightarrow \omega^2 \int_0^1 W(r)\, dr,
$$

where ω is longrun variance of y_t. And

$$
\begin{aligned}
T^{-1} \sum_{t=2}^{T} u_t y_{t-1} &= \frac{1}{2} T^{-1} \sum_{t=2}^{T} \left[(y_{t-1} + u_t)^2 - y_{t-1}^2 - u_t^2 \right] \\
&= \frac{1}{2} \left[T^{-1} y_T^2 - T^{-1} y_1^2 - T^{-1} \sum_{t=2}^{T} u_t^2 \right] \\
&\Rightarrow \frac{1}{2} \left[\omega^2 W(1) - \sigma^2 \right].
\end{aligned}
$$

Then we have

$$
T(\widehat{\alpha} - 1) \Rightarrow \frac{\frac{1}{2} \left[\omega^2 W(1) - \sigma^2 \right]}{\omega^2 \int_0^1 W(r)\, dr}.
$$

Question 4

We have

$$
\begin{aligned}
T\left(\widehat{\beta}-\beta\right) &= T\frac{\sum_{t=1}^{n} x_t u_t}{\sum_{t=1}^{n} x_t^2} \\
&= \frac{\sum_{t=1}^{n} \frac{x_t}{\sqrt{T}} \frac{u_t}{\sqrt{T}}}{\sum_{t=1}^{n} \left(\frac{x_t}{\sqrt{T}}\right)^2 \cdot \frac{1}{T}} \Rightarrow \frac{\int_0^1 B_x\left(r\right) dB_u + \Delta_{xu}}{\int_0^1 B_x^2\left(r\right) dr}
\end{aligned}
$$

where $\Delta_{xu} = \sum_{t=0}^{\infty} \mathbb{E}\left(\Delta x_t u_0\right)$.

Econometrics Field: ECON8825

Qingsong Yao

Question 1

a) First of all, since $(\eta_i, \varepsilon_i) \sim N(0, \Sigma)$, we have that

$$\varepsilon_i | \, \eta_i \sim N\left(\frac{\gamma_0}{\sigma_1^2}\eta_i, \sigma_2^2 - \frac{\gamma_0^2}{\sigma_1^2}\right).$$

So

$$P(\varepsilon_i < x|\eta_i) = P\left(\frac{\varepsilon_i - \frac{\gamma_0}{\sigma_1^2}\eta_i}{\sqrt{\sigma_2^2 - \frac{\gamma_0^2}{\sigma_1^2}}} < \frac{x - \frac{\gamma_0}{\sigma_1^2}\eta_i}{\sqrt{\sigma_2^2 - \frac{\gamma_0^2}{\sigma_1^2}}}|\eta_i\right) = \Phi\left(\frac{x - \frac{\gamma_0}{\sigma_1^2}\eta_i}{\sqrt{\sigma_2^2 - \frac{\gamma_0^2}{\sigma_1^2}}}\right)$$

where $\Phi(\cdot)$ is the cdf of normal distribution. Then

$$P\left(\varepsilon_i < x|\eta_i \geq -w_i'\delta_0\right) = \int_{-w_i'\delta_0}^{+\infty} \Phi\left(\frac{x - \frac{\gamma_0}{\sigma_1^2}\eta_i}{\sqrt{\sigma_2^2 - \frac{\gamma_0^2}{\sigma_1^2}}}\right)\frac{1}{\sigma_1}\phi\left(\frac{\eta_i}{\sigma_1}\right)d\eta_i.$$

Denote the density of ε_i conditional on $\eta_i \geq -w_i'\delta_0$ as $f(x|\eta_i \geq -w_i'\delta_0)$. We have

$$\begin{aligned} f\left(x|\eta_i \geq -w_i'\delta_0\right) &= \frac{\partial P(\varepsilon_i < x|\eta_i \geq -w_i'\delta_0)}{\partial x} \\ &= \int_{-w_i'\delta_0}^{+\infty} \frac{1}{\sqrt{\sigma_2^2\sigma_1^2 - \gamma_0^2}}\phi\left(\frac{x - \frac{\gamma_0}{\sigma_1^2}\eta_i}{\sqrt{\sigma_2^2 - \frac{\gamma_0^2}{\sigma_1^2}}}\right)\phi\left(\frac{\eta_i}{\sigma_1}\right)d\eta_i. \end{aligned}$$

Given $f(x|\eta_i \geq -w_i'\delta_0)$, we can now give the likelihood function. We have

$$\begin{aligned} L &= \prod_{i=1}^{n} f\left(y_i - x_i'\beta_0|\eta_i \geq -w_i'\delta_0\right) \\ &= \prod_{i=1}^{n}\int_{-w_i'\delta_0}^{+\infty} \frac{1}{\sqrt{\sigma_2^2\sigma_1^2 - \gamma_0^2}}\phi\left(\frac{y_i - x_i'\beta_0 - \frac{\gamma_0}{\sigma_1^2}\eta_i}{\sqrt{\sigma_2^2 - \frac{\gamma_0^2}{\sigma_1^2}}}\right)\phi\left(\frac{\eta_i}{\sigma_1}\right)d\eta_i. \end{aligned}$$

(b) Since we know the distribution of ε_i given $\eta_i \geq -w_i'\delta_0$, we can simply calculate the conditional mean of y_i given x_i and $\eta_i \geq -w_i'\delta_0$, which is given by

$$\begin{aligned}\mathbb{E}\left[y_i|\eta_i \geq -w_i'\delta_0, x_i\right] &= x_i'\beta_0 + \mathbb{E}\left[\varepsilon_i|\eta_i \geq -w_i'\delta_0, x_i\right] \\ &= x_i'\beta_0 + \mathbb{E}\left[\varepsilon_i|\eta_i \geq -w_i'\delta_0\right],\end{aligned}$$

where we can recover the expression of $\mathbb{E}\left[\varepsilon_i|\eta_i \geq -w_i'\delta_0\right]$ based on the conditional distribution $f\left(\varepsilon_i|\eta_i \geq -w_i'\delta_0\right)$. So the NLS estimator is given by

$$\widehat{\theta}_{NLS} = \arg\min \sum_{i=1}^{n} \left(y_i - x_i'\beta - \mathbb{E}\left[\varepsilon_i|\eta_i \geq -w_i'\delta\right]\right)^2.$$

Note that $\mathbb{E}\left[\varepsilon_i|\eta_i \geq -w_i'\delta\right]$ will have the form of $\mathbb{E}\left[\varepsilon_i|\eta_i \geq -w_i'\delta\right] = \sigma_2 g\left(\frac{\delta}{\sigma_1}\right)$, where g is a known function.

(c) As we can see in (b), $\mathbb{E}\left[\varepsilon_i|\eta_i \geq -w_i'\delta_0\right]$ is a complicated (but known) function. Then we can use Heckman two-step. In particular, we use the following steps:

Step1: Estimate $\widehat{\delta/\sigma_1}$ based on the binary variable of whether working hours is censored. Since $P\left(\eta_i > -w_i'\delta_0|w_i, x_i\right) = P\left(-\eta_i < w_i'\delta_0|w_i, x_i\right) = \Phi\left(\frac{w_i'\delta_0}{\sigma_1}\right)$, we maximize the following log-likelihood function

$$\widehat{\delta/\sigma_1} = \arg\max\left[d_i \cdot \log\left(\Phi\left(w_i'\frac{\delta}{\sigma_1}\right)\right) + (1-d_i)\cdot\log\left(1-\Phi\left(w_i'\frac{\delta}{\sigma_1}\right)\right)\right].$$

Step 2: Plug such $\widehat{\delta/\sigma_1}$ into the sum of squared in part (b), that is,

$$\widehat{\theta}_{NLS} = \arg\min \sum_{i=1}^{n} \left(y_i - x_i'\beta - \sigma_2 g\left(\widehat{\delta/\sigma_1}\right)\right)^2$$

Since $g\left(\widehat{\delta/\sigma_1}\right)$ is known, it can be regarded as a covariate. So the second step is simple as OLS, which guarantees closed form solution for β and σ_2.

Question 2

(a) Note that

$$\begin{aligned}\mathbb{E}\left[y|x\right] = \mathbb{E}\left[y|x, \varepsilon > -\frac{x'\beta}{\sigma}\right] \cdot P\left(\varepsilon > -\frac{x'\beta}{\sigma}|x\right) \\ + \mathbb{E}\left[y|x, \varepsilon \leq -\frac{x'\beta}{\sigma}\right] \cdot P\left(\varepsilon \leq -\frac{x'\beta}{\sigma}|x\right) \\ = \left\{x'\beta + \sigma\mathbb{E}\left[\varepsilon|\varepsilon > -\frac{x'\beta}{\sigma}\right]\right\} \cdot \left(1 - F\left(-\frac{x'\beta}{\sigma}\right)\right).\end{aligned}$$

Since

$$\mathbb{E}\left[\varepsilon|\varepsilon > -\frac{x'\beta}{\sigma}\right] = \frac{\int_{-\frac{x'\beta}{\sigma}}^{+\infty} \varepsilon f\left(\varepsilon\right) d\varepsilon}{1 - F\left(-\frac{x'\beta}{\sigma}\right)},$$

we have

$$\mathbb{E}\left[y|x\right] = x'\beta \cdot \left(1 - F\left(-\frac{x'\beta}{\sigma}\right)\right) + \sigma \int_{-\frac{x'\beta}{\sigma}}^{+\infty} \varepsilon f\left(\varepsilon\right) d\varepsilon.$$

So

$$\begin{aligned}\frac{\partial \mathbb{E}\left[y|x\right]}{\partial x} &= \left(1 - F\left(-\frac{x'\beta}{\sigma}\right)\right)\beta + x'\beta f\left(-\frac{x'\beta}{\sigma}\right)\frac{\beta}{\sigma} \\ &+ \sigma \cdot (-1) \cdot \left(-\frac{x'\beta}{\sigma}\right) f\left(-\frac{x'\beta}{\sigma}\right) \cdot \left(-\frac{\beta}{\sigma}\right) \\ &= \left(1 - F\left(-\frac{x'\beta}{\sigma}\right)\right)\beta + x'\beta f\left(-\frac{x'\beta}{\sigma}\right)\frac{\beta}{\sigma} - x'\beta f\left(-\frac{x'\beta}{\sigma}\right)\frac{\beta}{\sigma} \\ &= \left(1 - F\left(-\frac{x'\beta}{\sigma}\right)\right)\beta\end{aligned}$$

(b) We have

$$\overline{\delta} = \mathbb{E}_x \delta_x = \mathbb{E}_x\left[1 - F\left(-\frac{x'\beta}{\sigma}\right)\right]\beta$$

(c) Given an estimator of β, we only need to obtain an estimator of $\alpha = \beta/\sigma$. Denote $d_i = 0$ if censored and $d_i = 1$ if not censored. We have the log-likelihood function of y is given by

$$\begin{aligned}L\left(\alpha\right) &= \sum_{i=1}^{n} d_i \log\left(P\left(\varepsilon_i > -x_i'\alpha\right)\right) + (1 - d_i)\log\left(1 - P\left(\varepsilon_i > -x_i'\alpha\right)\right) \\ &= \sum_{i=1}^{n}\left\{d_i \log\left(1 - F\left(-x_i'\alpha\right)\right) + (1 - d_i)\log\left(F\left(-x_i'\alpha\right)\right)\right\}\end{aligned}$$

We have $\widehat{\alpha} = \arg\max L(\alpha)$. Then we have $\widehat{\alpha} \to_p \alpha_0$.

With $\widehat{\beta}$ and $\widehat{\alpha}$ in hand, the estimator of $\overline{\delta}$ is given by

$$\widehat{\overline{\delta}} = \frac{1}{n}\sum_{i=1}^{n}\left[1 - F\left(-x_i'\widehat{\alpha}\right)\right]\widehat{\beta}.$$

The asymptotic property of $\widehat{\overline{\delta}}$ is given as follows. Note that

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\left[1 - F\left(-x_i'\widehat{\alpha}\right)\right]\widehat{\beta} &= \frac{1}{n}\sum_{i=1}^{n}\left[1 - F\left(-x_i'\widehat{\alpha}\right)\right]\left(\widehat{\beta} - \beta\right) \\ &\quad + \frac{1}{n}\sum_{i=1}^{n}\left[F\left(-x_i'\alpha\right) - F\left(-x_i'\widehat{\alpha}\right)\right]\beta \\ &\quad + \frac{1}{n}\sum_{i=1}^{n}\left[1 - F\left(-x_i'\alpha\right)\right]\beta.\end{aligned}$$

For the third part, we have

$$\frac{1}{n}\sum_{i=1}^{n}\left[F\left(-x_i'\alpha\right)\right]\beta - \overline{\delta} \to_p 0$$

For the first part, we have

$$\left|\frac{1}{n}\sum_{i=1}^{n}\left[1 - F\left(-x_i'\widehat{\alpha}\right)\right]\left(\widehat{\beta} - \beta\right)\right| \leq \left|\widehat{\beta} - \beta\right| \to_p 0$$

For the second part, we have

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}\left[F\left(-x_i'\alpha\right) - F\left(-x_i'\widehat{\alpha}\right)\right]\beta &= -\frac{1}{n}\sum_{i=1}^{n}\left[f\left(-x_i'\alpha\right)\right]\beta\left(\alpha - \widehat{\alpha}\right) + o_p(1) \\ &\to_p 0.\end{aligned}$$

So $\widehat{\overline{\delta}} \to_p \overline{\delta}$ holds. But note that $\sqrt{n}\left(\widehat{\overline{\delta}} - \overline{\delta}\right) \Rightarrow N(0, V)$does not hold in general. In particular, the asymptotic mean of $\sqrt{n}\left(\widehat{\overline{\delta}} - \overline{\delta}\right)$ depends on the distribution of $\sqrt{n}\left(\widehat{\beta} - \beta\right)$ and $\sqrt{n}\left(\widehat{\alpha} - \alpha\right)$.

(d) When F and f is unknown, we can directly use nonparametric method to estimate $\mathbb{E}[y|x]$ and $\partial\mathbb{E}[y|x]/\partial x$. Suppose the estimator is given by $\partial\mathbb{E}\,\widehat{[y|x]}/\partial x$, then $\overline{\delta}$ is given by

$$\frac{1}{n}\sum_{i=1}^{n}\partial\mathbb{E}\,\widehat{[y|x]}/\partial x$$

As a matter of fact, when using nonparametric method, we can expect that

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\partial\mathbb{E}\widehat{[y|x]}/\partial x-\overline{\delta}\right)\Rightarrow N\left(0,V\right)$$

for some V.