ECON7772 Econometric Methods

Professor Arthur Lewbel

SAMPLE FINAL EXAM

INSTRUCTIONS:
Write your name on the exam book. Write your answers in the exam book.
You have one hour and fifteen minutes to complete this exam.
You may bring notes on a single sheet of paper to the exam.
Answer all the problems clearly, completely, and concisely. Show all your work.

1. Consider the model $Y_t = a + bX_t + cY_{t-1} + \varepsilon_t$, where $\varepsilon_t = V_t + V_{t-1}$, $E(V_t X_s) = 0$ and $E(V_t) = 0$ for all t,s. Also, $E(V_t V_s) = 0$ for all $t \neq s$, $|c| < 1$, and for all t, V_t and X_t are bounded random variables.

a. Is an ordinary least squares regression of Y_t on a constant, X_t, and Y_{t-1} likely to be consistent? Explain your answer.

b. Is an instrumental variables regression of Y_t on a constant, X_t, and Y_{t-1}, using instruments X_t, X_{t-1}, and a constant likely to be consistent? Explain your answer.

c. Do either of your answers change if the true value of b happens to be zero?

2. Consider a regression model $Y = XB + \varepsilon$ where the k vector B can be efficiently estimated by generalized least squares for any sample size n. Let $\widehat{B}$ be this GLS estimator, and let $\widetilde{B}$ be the ordinary least squares estimator of B. Define $C_n = var(\widetilde{B}) - var(\widehat{B})$.

a. For any n, is C_n symmetric?

b. Derive the simplest expression you can for the matrix $\lim_{n\to\infty} C_n$.

c. Derive the simplest expression you can for the matrix C_k.

3. Consider the following structural model:

$$Y_i = (X_i + Z_i)\, b + e_i$$
$$X_i = (Y_i + Z_i)\, c + u_i$$

where Y_i, X_i, Z_i, e_i, and u_i are random scalars, each having mean zero and finite third moments. Assume $E(e_i u_i) \neq 0$, $E(e_i Z_i) = 0$, $E(u_i Z_i) = 0$, and b and c are finite scalar constants. We have a sample of n independent, identically distributed draws of the triplet Y_i, X_i, Z_i.

a. What is the reduced form equation for X_i?

b. Propose an instrumental variables estimator for b. List any inequality constraints that b and c must satisfy for consistency and identification using your estimator.

SAMPLE FINAL EXAM - ANSWERS

1. a. OLS estimation of $Y_t = a + bX_t + cY_{t-1} + \varepsilon_t$ is *inconsistent*, because

$$E(Y_{t-1}\varepsilon_t) = E\left[(a + bX_{t-1} + cY_{t-2} + V_{t-1} + V_{t-2})(V_t + V_{t-1})\right] = E(V_{t-1}^2) \neq 0$$

The regressor Y_{t-1} is correlated with the error ε_t because both of them depend on V_{t-1}.

b. Instrumental variables estimation of $Y_t = a+bX_t+cY_{t-1}+\varepsilon_t$ using instruments 1, X_t, X_{t-1}, is generally *consistent* because there are enough instruments and they are valid, since $E(V_t X_s) = 0$ makes them uncorrelated with ε_t, and 1 and X_t are instruments for themselves while X_{t-1} is correlated with Y_{t-1} because $Y_{t-1} = a + bX_{t-1} + cY_{t-2} + \varepsilon_{t-1}$.

c. If $b = 0$ then *both OLS and IV are inconsistent (the answer to part a. stays the same and part b. changes)*. OLS is still inconsistent because we still have

$$E(Y_{t-1}\varepsilon_t) = E\left[(a + cY_{t-2} + V_{t-1} + V_{t-2})(V_t + V_{t-1})\right] = E(V_{t-1}^2) \neq 0$$

and now IV also becomes inconsistent because X_{t-1} is no longer a useful instrument for Y_{t-1}, since now $Y_{t-1} = a + cY_{t-2} + \varepsilon_{t-1}$ which does not have X_{t-1} in it (note X_{t-1} is not correlated with Y_{t-2}, since we can substitute out repeatedly to get that Y_{t-1} only depends on past values of V, all of which are uncorrelated with X_{t-1}).

2. a. $C_n = var(\widetilde{B}) - var(\widehat{B})$ must be symmetric because variance matrices are always symmetric, and the difference between two symmetric matrices is symmetric.

b. OLS and GLS both converge in mean square, so the variances $var(\widetilde{B})$ and $var(\widehat{B})$ both go to zero, so $\lim_{n\to\infty} C_n$ is a k by k matrix of zeros.

c. With $n = k$, X and Ω are square, so

$$\begin{aligned}\widehat{B} &= \left(X'\Omega^{-1}X\right)^{-1} X'\Omega^{-1}Y = X^{-1}\Omega X'^{-1}X'\Omega^{-1}Y \\ &= X^{-1}\Omega\Omega^{-1}Y = X^{-1}Y\end{aligned}$$

and similarly

$$\widetilde{B} = (X'X)^{-1} X'Y = X^{-1}X'^{-1}X'Y = X^{-1}Y$$

So with $n = k$ we get $\widetilde{B} = \widehat{B}$, which makes $var(\widetilde{B}) = var(\widehat{B})$, so C_k is again a k by k matrix of zeros.

3. a. Substitute the first equation into the second:

$$
\begin{aligned}
X_i &= \left[(X_i + Z_i)\, b + e_i + Z_i\right] c + u_i \\
X_i &= X_i bc + Z_i(bc + c) + e_i c + u_i
\end{aligned}
$$

and solve for X_i to get the reduced form

$$X_i = Z_i \frac{bc + c}{1 - bc} + \frac{e_i c + u_i}{1 - bc}$$

b. Define $W_i = X_i + Z_i$. Then $Y_i = W_i b + e_i$ and an IV estimator for b is to regress Y on W using Z as the instrument. To see that Z is a valid instrument we were told it is uncorrelated with e, so we only need to show that it is correlated with W. Given the above reduced form for X, we have

$$
\begin{aligned}
W_i = X_i + Z_i &= Z_i \frac{bc + c}{1 - bc} + \frac{e_i c + u_i}{1 - bc} + Z_i \\
&= Z_i \frac{1 + c}{1 - bc} + \frac{e_i c + u_i}{1 - bc}
\end{aligned}
$$

so Z is correlated with W, and so is a valid instrument, as long as $c \neq -1$ and $bc \neq 1$, which makes the coefficient of Z in the equation for W be nonzero.