Sample Questions for Time Series : Solutions

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1. Stationary ARMA Process

EX1: proof

"$\Leftarrow$" $A(z) \neq 0 \ \forall |z| \leq 1 \Rightarrow \exists |z| > 1$ s.t. $A(z) = 0$, i.e., the roots of $A(z)$ are greater than 1 in absolute value. Then, by difference equation theorems, we have

$$-Ab^j \leq C_j \leq Ab^j \quad \exists A > 0, \ b \in (0,1)$$

$$\Rightarrow \sum |C_j| \leq \sum |A| b^j = |A| \sum b^j = \frac{|A|}{1-b} < \infty$$

$\Rightarrow \{X_t\}$ is causal.

"$\Rightarrow$" under $|z| = 1 \Rightarrow C(|z|) = \begin{cases} \sum C_j \\ \sum C_j (-1)^j \end{cases} < \infty$ as $\{X_t\}$ is causal.

$\Rightarrow A(z) \neq 0$

under $|z| < 1 \Rightarrow C(z) = \sum C_j z^j \leq \sup|C_j| \sum z^j$

$$= \sup|C_j| \frac{1}{1-z} < \infty$$

$$\Rightarrow A(z) \neq 0$$

$\Rightarrow \forall |z| \leq 1, \ A(z) \neq 0.$

EX2. Same procedure of proof as EX 1.

EX3. (1) $X_t = \varepsilon_t + \sum_{i=1}^{2} \theta_i \varepsilon_{t-i} = (1 + \theta_1 L + \theta_2 L^2) \varepsilon_t$

~~Denote $\Theta(z) = 1 + \theta_1 z + \theta_2 z^2$. Then we have~~
~~$\Theta(z) = 0 \Rightarrow \theta_2 z^2 + \theta_1 z + 1 = 0$~~
~~① if $\theta_2 > 0$, $\Rightarrow \theta_2 + \theta_1 + 1 >$ $z=1$~~
~~$\theta_2 - \theta_1 + 1$ $z=-1$~~

let $z = \frac{1}{L} \Rightarrow \Theta(L) = 0 \Rightarrow z^2 + \theta_1 z + \theta_2 = 0$

$$\Rightarrow \begin{array}{l} z = 1: \ 1 + \theta_1 + \theta_2 > 0 \Rightarrow \theta_1 + \theta_2 > -1 \\ z = -1: \ 1 - \theta_1 + \theta_2 > 0 \Rightarrow \theta_2 - \theta_1 > -1 \end{array} \Big\} \Rightarrow \theta_2 > -1$$

$$\begin{cases} \theta_1^2 - 4\theta_2 \geq 0 \Rightarrow \theta_2 \leq \frac{\theta_1^2}{4} \\ \frac{|\theta_1|}{2} < 1 \Rightarrow |\theta_1| < 2 \Rightarrow \theta_1^2 < 4 \end{cases} \Big\} \Rightarrow \theta_2 < 1$$

$\Rightarrow |\theta_2| < 1$

(2) $X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \varepsilon_t$

$\Rightarrow (1-\alpha_1 L - \alpha_2 L^2) X_t = \varepsilon_t$

$\Rightarrow A(L) \equiv 1-\alpha_1 L - \alpha_2 L^2$

Denote $z = \frac{1}{L}$

$\Rightarrow A(L)=0 \Rightarrow z^2 - \alpha_1 z - \alpha_2 = 0$

① $z=1 \Rightarrow 1-\alpha_1-\alpha_2 > 0 \Rightarrow \underbrace{\alpha_1+\alpha_2 < 1}$

② $z=-1 \Rightarrow 1+\alpha_1-\alpha_2 > 0 \Rightarrow \underbrace{\alpha_1-\alpha_2 > -1}$

$\Rightarrow \alpha_2 < 1$

③ $\frac{|\alpha_1|}{2} < 1 \Rightarrow |\alpha_1| < 2 \Rightarrow \alpha_1^2 < 4$

$\alpha_1^2 + 4\alpha_2 \geq 0 \Rightarrow \alpha_2 \geq -\frac{\alpha_1^2}{4}$

$\Rightarrow \alpha_2 \geq -1$

$\Rightarrow \underbrace{|\alpha_2| < 1}$

EX4. $X_t + 0.2 X_{t-1} - 0.48 X_{t-2} = \varepsilon_t$

$\Rightarrow (1+0.2L - 0.48L^2) X_t = \varepsilon_t$

$\Rightarrow z^2 + 0.2z - 0.48 = 0$

$= z_1 = 0.6 \quad z_2 = -0.8 \Rightarrow |z|$ are both $<1 \Rightarrow$ causal.

EX5. $X_t + 0.6 X_{t-2} = \varepsilon_t + 1.2 \varepsilon_{t-1}$

$\Rightarrow \underbrace{(1+0.6L^2)}_{A(L)} X_t = \underbrace{(1+1.2L)}_{\Theta(L)} \varepsilon_t$

Let $z^2 + 0.6 = 0 \Rightarrow z = \pm\sqrt{0.6}\, i \Rightarrow |z| = 0.6 < 1$ ✓ Causal

$z^2 + 1.2z = 0 \Rightarrow z(z+1.2) = 0 \Rightarrow$ $z_1 = 0 \Rightarrow |z_1| < 1$ ✓ not invertable.

$z_2 = -1.2 \Rightarrow |z_2| > 1$ ✗

EX6. & EX7

$\{X_t\}$ is covariance stationary

$\Rightarrow \mathbb{E}[X_t] = \mu_X < \infty$

$\mathbb{E}[X_t X_{t+h}] = \mu_X^2 + \gamma_X(h) < \infty \qquad (\gamma_X(h) = \sigma_X^2)$

Likewise, $\{Y_t\}$ is covariance stationary

$\mathbb{E}[Y_t] = \mu_Y < \infty$

$\mathbb{E}[Y_t Y_{t+h}] = \mu_Y^2 + \gamma_Y(h) < \infty \qquad (\gamma_Y(h) = \sigma_Y^2)$

Denote $\{Z_t\} = \{aX_t + bY_t\}$

$\Rightarrow \mathbb{E}[Z_t] = a\mathbb{E}[X_t] + b\mathbb{E}[Y_t] = a\mu_X + b\mu_Y < \infty$

$$\begin{aligned}
\mathbb{E}[Z_t Z_{t+h}] &= \mathbb{E}[(aX_t + bY_t)(aX_{t+h} + bY_{t+h})] \\
&= a^2\mathbb{E}[X_t X_{t+h}] + b^2\mathbb{E}[Y_t Y_{t+h}] \\
&\quad + ab\,\mathbb{E}[X_t Y_{t+h}] + ab\,\mathbb{E}[X_{t+h} Y_t] \\
&= a^2(\gamma_X(h) + \mu_X^2) + b^2(\gamma_Y(h) + \mu_Y^2) \\
&\quad + ab\,(\underbrace{\mathbb{E}[X_t Y_{t+h}] + \mathbb{E}[X_{t+h} Y_t]})
\end{aligned}$$

Thus:

EX7. if $\{X_t\}$ and $\{Y_t\}$ are jointly covariance stationary

$\mathbb{E}[X_t Y_{t+h}] + \mathbb{E}[X_{t+h} Y_t] = \gamma_{XY}(h)$ ✓

EX6 if not, - - — — — - - = can be function of t.

The counterexample: $Z \sim unif[-\pi, \pi]$

$\{X_t\} = \{\cos(Z+t)\} \qquad \{Y_t\} = \{\cos(2Z+t)\}$

$\mathbb{E}[X_t] = \mathbb{E}[Y_t] = 0$

$\mathbb{E}[X_t X_{t+h}] = \frac{1}{2}\cos h, \quad \mathbb{E}[Y_t Y_{t+h}] = \frac{1}{2}\cos h$

$\mathbb{E}[X_t Y_{t+h}] =$

The counter-example

$\{X_t\}$ is covariance stationary s.t

$\mathbb{E}[X_t]=0$

$\mathbb{E}[X_t^2]=\sigma^2$

$\mathbb{E}[X_tX_{t+h}]=\gamma(h)$

Define $Y_t=(-1)^tX_t$

$\Rightarrow \mathbb{E}[Y_t]=(-1)^t\mathbb{E}[X_t]=0$

$\mathbb{E}[Y_t^2]=(-1)^{2t}\mathbb{E}[X_t^2]=\mathbb{E}[X_t^2]=\sigma^2$

$\mathbb{E}[Y_tY_{t+h}]=(-1)^{2t}\mathbb{E}[X_tX_{t+h}]=\gamma(h)$

But $\mathbb{E}[X_tY_{t+h}]=(-1)^t\mathbb{E}[X_tX_{t+h}]=(-1)^t\gamma(h)$ (has t)

EX8. $\sum|C_j|<\infty \Rightarrow \sum C_j<\infty$

$\Rightarrow \mathbb{E}[Y_t]=\mathbb{E}[\sum C_jX_{t-j}]=\sum C_j\mathbb{E}[X_{t-j}]=\mu_x\sum C_j<\infty$

And

$\mathbb{E}[Y_tY_{t+h}]=\mathbb{E}[(\sum C_jX_{t-j})(\sum C_iX_{t-i+h})]$

$=\sum\sum C_iC_j\mathbb{E}[X_{t-j}X_{t-i+h})]$

$=\sum\sum C_iC_j(\mu_x^2+\gamma(j+h-i))]$

$\leq(\mu_x^2+\sigma_x^2)\sum\sum C_iC_j$

$=(\mu_x^2+\sigma_x^2)(\sum C_i)^2$

$<\infty$

EX9 (1) $\gamma_X(0)=\mathbb{E}[X_t^2]=\mathbb{E}[(\varepsilon_t+\theta\varepsilon_{t-1})^2]=\mathbb{E}[\varepsilon_t^2]+\theta^2\mathbb{E}[\varepsilon_{t-1}^2]=(1+\theta^2)\sigma^2$

$\begin{cases}\gamma_X(1)=\mathbb{E}[X_tX_{t+1}]=\mathbb{E}[(\varepsilon_t+\theta\varepsilon_{t-1})(\varepsilon_{t+1}+\theta\varepsilon_t)]=\theta\mathbb{E}[\varepsilon_t^2]=\theta\sigma^2\\ \gamma_X(-1)=\gamma_X(1)\ \text{symmetry}\end{cases}$

$\gamma_X(h)=\mathbb{E}[X_tX_{t+h}]=\mathbb{E}[(\varepsilon_t+\theta\varepsilon_{t-1})(\varepsilon_{t+h}+\theta\varepsilon_{t+h-1})]=0 \quad |h|>1$

(2) In general

$\gamma_X(-h)=\gamma_X(h)=\mathbb{E}[X_tX_{t+h}]=\mathbb{E}[(\sum\theta_i\varepsilon_{t-i})(\sum\theta_i\varepsilon_{t+h-i})]$

$= \sum_i \sum_j \theta_i \theta_j \mathbb{E}[\varepsilon_{t-i} \varepsilon_{t-j+h}]$

$= \sum_{i=j-h}^{q} \theta_i \theta_j \sigma^2$

$= \sigma^2 \sum_0^{q-|h|} \theta_i \theta_{i+|h|}$

And under $|h| > q$, $q-|h| < 0$ $\Rightarrow$ $\gamma(h) = 0$

EX 10

~~$X_t = \alpha X_t +$~~ $X_t = \alpha X_{t-1} + \varepsilon_t$

$\Rightarrow (1-\alpha L) X_t = \varepsilon_t$

$\Rightarrow X_t = \frac{1}{1-\alpha L} \varepsilon_t$

$= (1 + \alpha L + \alpha^2 L^2 + \cdots) \varepsilon_t$

$= \sum_0^{\infty} \alpha^j \varepsilon_{t-j}$

$\Rightarrow \mathbb{E}[X_t X_{t+h}] = \mathbb{E}[(\sum \alpha^j \varepsilon_{t-j})(\sum \alpha^j \varepsilon_{t+h-j})]$

$= \sigma^2 \sum_0^{\infty} \alpha^j \alpha^{j+|h|}$

$= \alpha^{|h|} \sigma^2 \sum_0^{\infty} \alpha^{2j}$

$= \alpha^{|h|} \sigma^2 \frac{1}{1-\alpha^2}$

$= \frac{\sigma^2 \alpha^{|h|}}{1-\alpha^2} = \gamma(h)$

$\gamma(0) = \frac{\sigma^2}{1-\alpha^2}$

$\Rightarrow \rho_X(h) = \frac{\gamma(h)}{\gamma(0)} = \alpha^{|h|}$

EX 11

$Y_t = \sum c_j X_{t-j}$

$\gamma_Y(h) = \mathbb{E}[Y_t Y_{t+h}] = \mathbb{E}[(\sum c_j X_{t-j})(\sum c_k X_{t-k+h})]$

$= \sum_j \sum_k c_j c_k \mathbb{E}[X_{t-j} X_{t-k+h}]$

$= \sum_j \sum_k c_j c_k \gamma(h-k+j) = \sum_j \sum_k c_j c_k \gamma(|h|-k+j)$

And under $\{X_t\}$ IID

$$\gamma(h-k+j) = \begin{cases} \sigma^2 & k=|h|+j \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \gamma_\xi(h) = \sigma^2 \sum_j c_j c_{j+|h|}$$

EX12. For ARMA(p, q)

$$X_t = \sum_1^p a_i X_{t-i} + \varepsilon_t + \sum_1^q \theta_i \varepsilon_{t-i}$$

$$\Rightarrow A(L) X_t = \Theta(L) \varepsilon_t$$

where $A(L) = 1 - \alpha_1 L - \alpha_2 L^2 - \cdots \alpha_p L^p$

$\Theta(L) = 1 + \theta_1 L + \theta_2 L + \cdots \theta_q L^q$

$$\Rightarrow X_t = A^{-1}(L)\Theta(L)\varepsilon_t$$
$$= \psi(L)\varepsilon_t$$

where $\psi(L) \equiv A^{-1}(L)\Theta(L) = (\psi_0 + \psi_1 L + \cdots)$

$$\Rightarrow X_t = \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \cdots = \sum_0^\infty \psi_j \varepsilon_{t-j}$$

$$\Rightarrow \gamma(h) = E[X_t X_{t+h}] = \sigma^2 \sum_0^\infty \psi_j \psi_{j+|h|}$$

$$\gamma(0) = \sigma^2 \sum \psi_j^2$$

$$\Rightarrow \rho(h) = \frac{\gamma(h)}{\gamma(0)} = \frac{\sum \psi_j \psi_{j+|h|}}{\sum \psi_j^2}$$

EX13

$$\psi_j - \psi_{j-1} + 0.25\psi_{j-2} = 0$$

initial condition $\psi_0 = 1,\ \psi_1 = 2$

$$\Rightarrow \psi_j = \left(\tfrac{1}{2}\right)^j (3j+1)$$

$$\Rightarrow \rho(h) = \left(\tfrac{1}{2}\right)^h + 3|h|\left(\tfrac{1}{2}\right)^h \left(\frac{\sum \left(\frac{1}{2}\right)^{2j}(3j+1)}{\sum \left(\frac{1}{2}\right)^{2j}(3j+1)^2}\right)$$

EX15 Denote $\gamma(h)$ be the autocovariance function of Z_t

(1) $\Rightarrow \gamma_X(h) = \mathbb{E}[X_t X_{t+h}]$

$$= \mathbb{E}[(Z_t - 0.4 Z_{t-1})(Z_{t+h} - 0.4 Z_{t+h-1})]$$

$$= \mathbb{E}[Z_t Z_{t+h}] - 0.4\,\mathbb{E}[Z_{t-1} Z_{t+h}] - 0.4\,\mathbb{E}[Z_t Z_{t+h-1}] + 0.16\,\mathbb{E}[Z_{t-1} Z_{t+h-1}]$$

$$= 1.16\gamma(h) - 0.4\gamma(h+1) - 0.4\gamma(h-1)$$

Likewise

$$\gamma_Y(h) = 7.25\gamma(h) - 2.5\gamma(h+1) - 2.5\gamma(h-1)$$

$$= 6.25(1.16\gamma(h) - 0.4\gamma(h+1) - 0.4\gamma(h-1))$$

(2) $\Rightarrow \gamma_Y(h) = 6.25\gamma_X(h)$

$\Rightarrow \gamma_Y(0) = 6.25\gamma_X(0)$

$$\Rightarrow \rho_Y(h) = \frac{\gamma_Y(h)}{\gamma_Y(0)} = \frac{6.25\gamma_X(h)}{6.25\gamma_X(0)} = \rho_X(h)$$

EX17 $X_t - \sum_{i=1}^{p} \alpha_i X_{t-i} = \varepsilon_t$

$$\Rightarrow X_t X_{t-h} - \sum_{i=1}^{p} \alpha_i X_{t-i} X_{t-h} = \varepsilon_t X_{t-h}$$

$$\Rightarrow \mathbb{E}[X_t X_{t-h}] - \sum_{i=1}^{p} \alpha_i \mathbb{E}[X_{t-i} X_{t-h}] = \mathbb{E}[\varepsilon_t X_{t-h}]$$

$$\Rightarrow \gamma_X(h) - \sum_{i=1}^{p} \alpha_i \gamma_X(h-i) = \mathbb{E}[\mathbb{E}[\varepsilon_t | X_{t-h}] X_{t-h}] = 0$$

$$\Rightarrow \gamma_X(h) - \sum_{i=1}^{p} \alpha_i \gamma_X(h-i) = 0$$

$$\Rightarrow \frac{\gamma_X(h)}{\gamma_X(0)} - \sum \alpha_i \frac{\gamma_X(h-i)}{\gamma_X(0)} = 0$$

$$\Rightarrow \rho_X(h) = \sum_{i=1}^{p} \alpha_i \rho_X(h-i)$$

EX19 (i)
$$\begin{pmatrix} \rho(0) & \rho(1) \\ \rho(1) & \rho(0) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \rho(1) \\ \rho(2) \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \rho(0) & \rho(1) \\ \rho(1) & \rho(0) \end{pmatrix}^{-1} \begin{pmatrix} \rho(1) \\ \rho(2) \end{pmatrix}$$

$$= \begin{pmatrix} \rho(0) & -\rho(1) \\ -\rho(1) & \rho(0) \end{pmatrix} \begin{pmatrix} \rho(1) \\ \rho(2) \end{pmatrix} \frac{1}{\rho(0)^2 - \rho(1)^2}$$

$$= \begin{pmatrix} \dfrac{\rho(1)\rho(0) - \rho(1)\rho(2)}{\rho(0)^2 - \rho(1)^2} \\ \dfrac{\rho(2)\rho(0) - \rho(1)\rho(1)}{\rho^2(0) - \rho^2(1)} \end{pmatrix}$$

(ii) By solving difference equation:

$$\rho(h) = \frac{\alpha_1}{1-\alpha_2} \left(\sum_{i=0}^{h-1} p^i q^{h-1-i} \right) - \left(\sum_{i=0}^{h-2} p^{i+1} q^{h-1-i} \right)$$

where $q, p = \dfrac{\alpha_1 \pm \sqrt{\alpha_1^2 + 4\alpha_2}}{2}$ $\quad (\alpha_1^2 + 4\alpha_2 \geq 0)$

EX20
$$X_t - \sum_{i=1}^{p} \alpha_i X_{t-i} = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots \theta_q \varepsilon_{t-q}$$

$$\Rightarrow X_t X_{t-h} - \sum_{i=1}^{p} \alpha_i X_{t-i} X_{t-h} = \varepsilon_t X_{t-h} + \cdots \theta_q \varepsilon_{t-q} X_{t-h}$$

$$\Rightarrow \gamma_X(h) - \sum_{i=1}^{p} \alpha_i \gamma_X(h-i) = \underbrace{E[\varepsilon_t X_{t-h}] + \cdots \theta_q E[\varepsilon_{t-q} X_{t-h}]}_{=0}$$

if $h > q$ $(h \geq q+1)$

EX21

$$L(\alpha,\sigma^2) = f_{X_1}(x_1|\alpha,\sigma^2)\prod_{t=2}^{n} f_{X_t|X_{t-1}}(x_t|x_{t-1},\alpha,\sigma^2)$$

where $f_{X_t|X_{t-1}}(x_t|x_{t-1},\alpha,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(x_t-\alpha x_{t-1})^2}{2\sigma^2}\right\}$

EX22

$$L(\alpha_1,\dots\alpha_p,\sigma^2) = f_{X_1\dots X_p}(x_1,\dots x_p|\alpha_1,\dots\alpha_p,\sigma^2)\prod_{t=p+1}^{n} f_{X_t|X_{t-1},\dots X_{t-p}}(x_t|x_{t-1},\dots x_{t-p};\alpha_1,\dots\alpha_p,\sigma^2)$$

where $f_{X_t|X_{t-1}\dots X_{t-p}}(x_t|x_{t-1}\dots x_{t-p},\alpha_1,\dots\alpha_p,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(x_t-(-\alpha_1 x_{t-1}\dots-\alpha_p x_{t-p}))^2}{2\sigma^2}\right\}$

$f_{X_1,\dots X_p}(x_1,\dots x_p|\alpha_1,\dots\alpha_p,\sigma^2)$

$= (2\pi)^{-\frac{p}{2}}|V_p|^{-\frac{1}{2}}\exp\left\{-\frac{1}{2}(X_p^T V_p^{-1} X_p)\right\}$

where V_p = variance-covariance matrix $(p\times p)$ of $X_p=(x_1,\dots x_p)^T$

EX24

$$L = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(y_1-\alpha x_1)^2}{2\sigma^2}\right\}\prod_{t=2}^{n}\frac{1}{\sqrt{2\pi\sigma^2(1-\rho^2)}}\exp\left\{-\frac{(u_t-\rho u_{t-1})^2}{2(1-\rho^2)\sigma^2}\right\}$$

$$\begin{cases}\frac{\rho(y_1-\alpha x_1)^2}{n-1} + 2(1-\rho^2)\rho\sigma^2 + \frac{1}{n-1}\sum(y_t-\alpha x_t-\rho y_{t-1}+\alpha\rho x_{t-1})(y_{t-1}-\alpha x_{t-1}) = 0 \\ (1-\rho^2)(y_1-\alpha x_1)x_1 + \sum(y_t-\alpha x_t-\rho y_{t-1}+\alpha\rho x_{t-1})(x_t-\rho x_{t-1}) = 0\end{cases}$$

EX25

$$\mathbb{E}[(X-g(Z))^2]$$

$$= \mathbb{E}[(X-\mathbb{E}[X|Z]+\mathbb{E}[X|Z]-g(Z))^2]$$

$$\cancel{= \mathbb{E}[(X-\mathbb{E}[X|Z])^2] + 2\mathbb{E}[X-\mathbb{E}[X|Z]]\mathbb{E}[\mathbb{E}[X|Z]-g(Z)]}$$

$$+ \mathbb{E}[(\mathbb{E}[X|Z]-g(Z))^2]$$

$$= ① + ② + ③$$

① is unrelated with $g(Z)$

② $= 0$

③ $= 0$ iff $g(Z) = \mathbb{E}[X|Z]$ a.s.

EX26 & EX27

$$X_t = \sum_{i=0}^{\infty} \theta_i \varepsilon_{t-i}$$

$$\Rightarrow X_{t+h} = \sum_{i=0}^{\infty} \theta_i \varepsilon_{t+h-i}$$

$$\Rightarrow \hat{X}_{t+h} = \sum_{i=h}^{\infty} \theta_i \varepsilon_{t+h-i}$$

$$\Rightarrow FE \doteq e_{n+h,n} = \sum_{i=0}^{h-1} \theta_i \varepsilon_{t+h-i}$$

$$\Rightarrow Var(e_{n+h,n}) = \sigma^2\left(\sum_{i=0}^{h-1} \theta_i^2\right)$$

EX26 $\theta_0 = 1$, θ_1, θ_2, $\theta_3 = \theta_4 \cdots = 0$

$$\Rightarrow Var(e_{n+h,n}) = \sigma^2(1+\theta_1^2+\theta_2^2)$$

EX27 $\mathrm{var}(e_{n+h,n}) = \sigma^2 h$

EX29

(i) $\hat\beta_{OLS} - \hat\beta_{GLS}$

$= (X^TX)^{-1}(X^TY) - (X^TV^{-1}X)^{-1}(X^TV^{-1}Y)$

$= (X^TV^{-1}X)^{-1}[(X^TV^{-1}X)(X^TX)^{-1}X^TY - X^TV^{-1}Y]$

$= (X^TV^{-1}X)^{-1}[X^TV^{-1}P_XY - X^TV^{-1}Y)$

$= (X^TV^{-1}X)^{-1}[X^TV^{-1}(P_X - I)Y]$

$= (X^TV^{-1}X)^{-1}\{X^T(V^{-1}(P_X - I))^TY\}$

$= (X^TV^{-1}X)^{-1}\{\underbrace{X^T(P_X - I)}_{X^T - X^T = 0}V^{-1}Y]$

$= 0$

(ii) $\hat\sigma^2 = \frac{1}{n}\sum(y_t - \hat\beta^Tx_t)^2$

$= \frac{1}{n}\sum(\beta^Tx_t + u_t - \hat\beta^Tx_t)^2$

$= \frac{1}{n}\sum[(\beta-\hat\beta)^Tx_t + u_t]^2$

$= \frac{1}{n}\sum((\beta-\hat\beta)^Tx_t)^2 + \frac{2}{n}\sum(\beta-\hat\beta)^Tx_tu_t + \frac{1}{n}\sum u_t^2$

$= (\beta-\hat\beta)^T(\frac{1}{n}\sum x_tx_t^T)(\beta-\hat\beta) + (\beta-\hat\beta)^T\frac{2}{n}\sum x_tu_t + \frac{1}{n}\sum u_t^2$

$= (\frac{1}{n}\sum u_tx_t^T)(\frac{1}{n}\sum x_tx_t^T)^{-1}(\frac{1}{n}\sum x_tu_t)$

$$= 2\left(\frac{1}{n}\sum u_t x_t^T\right)\left(\frac{1}{n}\sum x_t x_t^T\right)^{-1}\left(\frac{1}{n}\sum x_t u_t\right) + \frac{1}{n}\sum u_t^2)$$

$$= \frac{1}{n}\sum u_t^2 - \left(\frac{1}{n}\sum u_t x_t^T\right)\left(\frac{1}{n}\sum x_t x_t^T\right)^{-1}\left(\frac{1}{n}\sum x_t u_t\right)$$

$$\Rightarrow \mathbb{E}[\hat{\sigma}^2] = \mathbb{E}\left[\frac{1}{n}\sum u_t^2\right] - \mathbb{E}\left[\left(\frac{1}{n}\sum u_t x_t^T\right)\left(\frac{1}{n}\sum x_t x_t^T\right)^{-1}\left(\frac{1}{n}\sum x_t u_t\right)\right]$$

$$= \sigma^2 - \mathbb{E}\left[\left(\frac{1}{n}\sum u_t x_t^T\right)\left(\frac{1}{n}\sum x_t x_t^T\right)^{-1}\left(\frac{1}{n}\sum x_t u_t\right)\right]$$

where $\mathbb{E}[u^T X (X^T X)^{-1} (X^T u)]$

$$= \text{trace}\ \mathbb{E}[u^T X (X^T X)^{-1} X^T u]$$

$$= \mathbb{E}[\text{trace}(u^T X (X^T X)^{-1} X^T u]$$

$$\cancel{= \mathbb{E}[\text{trace}(u u^T X (X^T X)^{-1} X^T)]}$$

$$= \mathbb{E}[\text{trace}(u u^T P_X)]$$

$$= p\sigma^2$$

$$\Rightarrow \mathbb{E}[\hat{\sigma}^2] = (1-p)\sigma^2$$

$$\frac{\alpha_0}{1-\alpha_1} = \alpha_0 + \alpha_1 \frac{\alpha_0}{1-\alpha_1} = \frac{\alpha_0(1-\alpha_1) + \alpha_1\alpha_0}{1-\alpha_1} = \frac{\alpha_0}{1-\alpha_1}$$

Ex 30 (i) $\mathbb{E}[u_t^2] = \mathbb{E}[\sigma_t^2 \varepsilon_t^2] = \mathbb{E}[\sigma_t^2] = \alpha_0 + \alpha_1 \mathbb{E}[u_{t-1}^2]$

$$\Rightarrow \mathbb{E}[u_t^2] = \frac{\alpha_0}{1-\alpha_1} \quad \Rightarrow \quad \alpha_1 \in (0,1)$$

(ii) OLS: reg u_t^2 on 1 and u_{t-1}^2

$$\underset{(\alpha_0,\alpha_1)^T}{\text{argmin}} \frac{1}{n-2}\sum (u_t^2 - \alpha_0 - \alpha_1 u_{t-1}^2)^2$$

(iii) $\Rightarrow \sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{d} N(0, \mathbb{E}[\sigma_t^4 - 1]\, \mathbb{E}^{-1}[z_t z_t^T]\, \mathbb{E}[z_t \sigma_t^2 z_t^T]\, \mathbb{E}^{-1}[z_t z_t^T])$

where $z_t = (1, u_{t-1}^2)^T$

EX31 (i) $\mathbb{E}[u_t^2] = \mathbb{E}[\sigma_t^2 \varepsilon_t^2]$

$$= \mathbb{E}[\sigma_t^2]$$

$$= \alpha_0 + \alpha_1 \mathbb{E}[u_{t-1}^2] + \gamma_1 \mathbb{E}[\sigma_{t-1}^2]$$

$$= \cdots$$

$$= \alpha_0 (1+\gamma_1+\gamma_1^2+\cdots) + \alpha_1 (1+\gamma_1+\gamma_1^2 \cdots) \mathbb{E}[u_t^2]$$

$$\Rightarrow \mathbb{E}[u_t^2] = \frac{\alpha_0 \frac{1}{1-\gamma_1}}{1-\alpha_1 \frac{1}{1-\gamma_1}} = \frac{\alpha_0}{1-\gamma_1-\alpha_1} \Rightarrow \alpha_1+\gamma_1 \in (0,1)$$

(ii) IGARCH(p, q)

$$\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \gamma_j = 1$$

Under IGARCH(1,1) $\Rightarrow \alpha_1+\gamma_1 = 1$

(iii) Using MLE with setting $\varepsilon_t \overset{iid}{\sim} N(0,1)$

with $\mathcal{L} = \frac{1}{\sqrt{2\pi\sigma_t^2}} \exp\left\{-\frac{u_t^2}{2\sigma_t^2}\right\}$

and $\frac{\partial \sigma_t^2}{\partial \theta} = (1 \;\; u_{t-1}^2, \sigma_{t-1}^2)^T + \gamma_1 \frac{\partial \sigma_{t-1}^2}{\partial \theta}$ $\quad \theta = (\alpha_0, \alpha_1, \gamma_1)^T$

Note: ARCH/GARCH is not coverd this year!

EX31 (i) $\mathbb{E}[u_t^2] = \mathbb{E}[\sigma_t^2 \varepsilon_t^2]$

$$= \mathbb{E}[\sigma_t^2]$$

$$= \alpha_0 + \alpha_1 \mathbb{E}[u_{t-1}^2] + \gamma_1 \mathbb{E}[\sigma_{t-1}^2]$$

$$= \cdots$$

$$= \alpha_0 (1+\gamma_1+\gamma_1^2+\cdots) + \alpha_1 (1+\gamma_1+\gamma_1^2 \cdots) \mathbb{E}[u_t^2]$$

$$\Rightarrow \mathbb{E}[u_t^2] = \frac{\alpha_0 \cdot \frac{1}{1-\gamma_1}}{1-\alpha_1 \frac{1}{1-\gamma_1}} = \frac{\alpha_0}{1-\gamma_1-\alpha_1} \Rightarrow \alpha_1+\gamma_1 \in (0,1)$$

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Note: ARCH/GARCH is not coverd this year!

2. VAR

EX32 Denote $B = (C, A_1, \cdots A_p)^T$ $(mp+1) \times m$ matrix

$$f_{Y_t|Y_{t-1},\cdots Y_{t-p}}(Y_t|Y_{t-1},\cdots Y_{t-p},\theta) = (2\pi)^{-\frac{m}{2}} |\Omega^{-1}|^{\frac{1}{2}} \exp\left[-\frac{1}{2}(Y_t - B^T X_t)^T \Omega^{-1} (Y_t - B^T X_t)\right]$$

$$\mathcal{L} = \prod_{t=p+1}^{n} f_{Y_t|Y_{t-1}\cdots Y_{t-p}}$$

$$\Rightarrow \ln \mathcal{L} = -\frac{m(n-p)}{2} \ln 2\pi + \frac{n-p}{2} \ln|\Omega^{-1}| - \frac{1}{2}\sum_{t=p+1}^{n} (Y_t - B^T X_t)^T \Omega^{-1} (Y_t - B^T X_t)$$

$$\Rightarrow \hat{B}^T = \left(\sum_{t=p+1}^{n} Y_t X_t^T\right)\left[\sum_{t=p+1}^{n} X_t X_t^T\right]^{-1}$$

EX33 From EX32, we have

$$\hat{b}_j^T = \left(\sum_{t=p+1}^{n} Y_{jt} X_t^T\right)\left(\sum_{t=p+1}^{n} X_t X_t^T\right)^{-1}$$

$\Rightarrow$ equivalent to reg Y_{jt} on X_t.

EX34

$$\cancel{\hat{\Omega} = \frac{1}{n-p}\sum (Y_t - \hat{B}^T X_t)}$$

$$\hat{\Omega} = \frac{1}{n-p}\sum_{t=p+1}^{n} (Y_t - \hat{B}^T X_t)(Y_t - \hat{B}^T X_t)^T$$

$\hat{B}$ is given by EX32.

EX35 Denote $X_t = (Y_{t-1}, Y_{t-2}, \cdots Y_{t-p})$

$$f(Y_t | X_t, \theta) = f(Y_{I,t} | X_t; \theta) \cdot f(Y_{II,t} | Y_{I,t}, X_t, \theta)$$

where $f(Y_{I,t} | X_t, \theta) = (2\pi)^{-\frac{m_1}{2}} |\Omega_{11}|^{-\frac{1}{2}} \exp\{-\frac{1}{2}(Y_{I,t} - C_I - F_I^T X_{I,t} - F_{II}^T X_{II,t})^T$

$\Omega_{11}^{-1}(Y_{I,t} - C_I - F_I^T X_{I,t} - F_{II}^T X_{II,t})\}$ ①

$f(Y_{II,t} | Y_{I,t}, X_t, \theta) = (2\pi)^{-\frac{m_2}{2}} |\Omega_{22} - \Omega_{21}\Omega_{11}^{-1}\Omega_{12}|^{-\frac{1}{2}} \exp\{-\frac{1}{2}(Y_{II,t} - \mu_{II,t})^T$

$(\Omega_{22} - \Omega_{21}\Omega_{11}^{-1}\Omega_{12})^{-1}(Y_{II,t} - \mu_{II,t})\}$

where $\mu_{II,t} = E[Y_{II,t} | X_t] + \Omega_{21}\Omega_{11}^{-1}(Y_{I,t} - E[Y_{I,t} | X_t])$

$= C_{II} + G_I^T X_{I,t} + G_{II}^T X_{II,t} + \Omega_{21}\Omega_{11}^{-1}(Y_{I,t} - C_I - F_I^T X_{I,t} - F_{II}^T X_{II,t})$

$= (C_{II} - \Omega_{21}\Omega_{11}^{-1}C_I) + \Omega_{21}\Omega_{11}^{-1}Y_{I,t} + (G_I^T - \Omega_{21}\Omega_{11}^{-1}F_I^T) X_{I,t} + (G_{II}^T - \Omega_{21}\Omega_{11}^{-1}F_{II}^T) \cdot X_{II,t}$

$\Rightarrow \mathcal{L}(Y_t | X_t, \theta) = \left(-\frac{m_1}{2}\ln 2\pi - \frac{1}{2}\ln|\Omega_{11}| - \frac{1}{2(N-p)}\sum \exp\{(\quad)^T \Omega_{11}^{-1}(\quad)\}\right)$

$+ \left(-\frac{m_2}{2}\ln 2\pi - \frac{1}{2}\ln|\Omega_{22} - \Omega_{21}\Omega_{11}^{-1}\Omega_{12}| - \frac{1}{2(N-p)}\sum \exp\{(\quad)^T (\Omega_{22} - \Omega_{21}\Omega_{11}^{-1}\Omega_{12})^{-1}(\quad)\}\right)$

$\equiv \mathcal{L}_I + \mathcal{L}_{II}$

EX36 $2[\mathcal{L}(\check{\theta}) - \mathcal{L}(\hat{\theta})] = N[\ln|\hat{\Omega}_{11}| - \ln|\check{\Omega}_{11}|] \xrightarrow{d} \chi^2(m_1 m_2 p)$

where $\check{\theta}$ is estimator under alternative (unrestricted)

$\hat{\theta}$ is the estimator under the null (restricted, i.e. $F_{II} = 0$)

Note: $\mathcal{L}(\cdot) = \mathcal{L}_I(\cdot) + \mathcal{L}_{II}(\cdot)$

3. Inference for Stationary Time Series Models

Ex 1)

$$\hat{\alpha} = \frac{\sum Y_{t-1} Y_t}{\sum Y_{t-1}^2} = \alpha + \frac{\sum u_t Y_{t-1}}{\sum Y_{t-1}^2} = \alpha + \frac{\frac{1}{n}\sum u_t Y_{t-1}}{\frac{1}{n}\sum Y_{t-1}^2}$$

where $\frac{1}{n}\sum u_t Y_{t-1} \xrightarrow{a.s.} \mathbb{E}[u_t Y_{t-1}] = \mathbb{E}[Y_{t-1}\mathbb{E}[u_t | Y_{t-1}]] = 0$

$\frac{1}{n}\sum Y_{t-1}^2 \xrightarrow{a.s.} \mathbb{E}[Y_{t-1}^2]$) ergodic.

$$\Rightarrow \hat{\alpha} \xrightarrow{a.s.} \alpha$$

4. Nonstationary Process

EX6

$$\hat{\alpha} = \frac{\sum y_{t-1} y_t}{\sum y_{t-1}^2} = \alpha + \frac{\sum y_{t-1} u_t}{\sum y_{t-1}^2}$$

$$\frac{1}{n^2}\sum_{t=1}^{n} y_{t-1}^2 = \frac{1}{n}\sum_{t=1}^{n}\left(\frac{y_{t-1}}{\sqrt{n}}\right)^2 = \sum_{t=1}^{n}\left(\frac{y_{t-1}}{\sqrt{n}}\right)^2 \frac{1}{n} \simeq \sum_{t=1}^{n}\int_{\frac{t-1}{n}}^{\frac{t}{n}} X_n^2(r)\,dr$$

$$= \int_0^1 X_n^2(r)\,dr \Rightarrow \int_0^1 B^2(r)\,dr \qquad X_n(r) := N(0,r)$$

$$y_t^2 = \left(\sum_{t=1}^{n} u_j\right)^2 = \sum_{j=1}^{n} u_j^2 + 2\sum_{j<i} u_j u_i = \sum_{j=1}^{n} u_j^2 + 2\sum_{i=1}^{n} y_{i-1} u_i$$

$$\Rightarrow \frac{1}{n}\sum_{t=1}^{n} y_{t-1}u_t = \frac{1}{2n} y_t^2 - \frac{1}{2n}\sum_{t=1}^{n} u_t^2 = \frac{1}{2}\left(\frac{y_t}{\sqrt{n}}\right)^2 - \frac{1}{2}\left(\frac{1}{n}\sum u_t^2\right)$$

$$= \frac{1}{2}X_n^2(1) - \frac{1}{2}\sum u_t^2 \Rightarrow \frac{1}{2}\left(B^2(1) - \sigma_u^2\right)$$

$$\Rightarrow N(\hat{\alpha}-\alpha) = \frac{\frac{1}{n}\sum y_{t-1}u_t}{\frac{1}{n^2}\sum y_{t-1}^2} \Rightarrow \frac{B^2(1) - \sigma_u^2}{2\int_0^1 B^2(r)\,dr}$$

EX7

$$\hat{\alpha} = \alpha + \frac{\sum y_{t-1}u_t}{\sum y_{t-1}^2}$$

$$\frac{1}{n^2}\sum_{t=1}^{n} y_{t-1}^2 = \sum_{t=1}^{n}\left(\frac{y_{t-1}}{\sqrt{n}}\right)^2\frac{1}{n} = \sum_{t=1}^{n}\int_{\frac{t-1}{n}}^{\frac{t}{n}} X_n^2(r)\,dr$$

$$= \int_0^1 X_n^2(r)\,dr \Rightarrow \int_0^1 B^2(r)\,dr = w^2\int_0^1 W^2(r)\,dr$$

$$\frac{1}{n}\sum y_{t-1}u_t \Rightarrow \frac{1}{2}\left(B^2(1) - \sigma_u^2\right) = \frac{1}{2}\left(B^2(1) - w^2 + w^2 - \sigma_u^2\right)$$

$$= \frac{w^2}{2}\left(W^2(1) - 1\right) + \frac{1}{2}\left(w^2 - \sigma_u^2\right)$$

$$= \frac{w^2}{2}\left(W^2(1) - 1\right) + \lambda\,\mathrm{var}\,u$$

$$\Rightarrow n(\hat{\alpha}-\alpha) \Longrightarrow \frac{\frac{1}{2}\omega^2(W^2(1)-1)+\lambda \, var_u}{\omega^2 \int_0^1 W^2(r)\,dr}$$

Ex8 (spurious)

$$\hat{\beta} \Rightarrow \left(\int B_x(r) B_x^T(r)\,dr\right)^{-1}\left(\int B_x(r) B_y(r)\,dr\right)$$

Ex9 (cointegrating)

$$n(\hat{\beta}-\beta) \Rightarrow \left(\int B_x(r) B_x^T(r)\,dr\right)^{-1}\left(\int B_x(r)\,dB_u(r) + \sum_{t=0}^{\infty} E[v_t u_t]\right)$$

Ex10 (ADF Tests)

$$y_t = \alpha y_{t-1} + u_t \quad , u_t \text{ has } AR(p)$$

$$\Rightarrow \quad u_t = A^{-1}(L)\varepsilon_t \qquad \varepsilon_t \sim iid.$$

$$= \left(\sum_{i=0}^{p} a_i L^i\right)\varepsilon_t$$

$$\Rightarrow \quad y_t = \alpha y_{t-1} + \sum_{j=1}^{p} \psi_j \Delta y_{t-j} + \varepsilon_t$$

$$\Leftrightarrow \quad \Delta y_t = a y_{t-1} + \sum_{j=1}^{p} \psi_j \Delta y_{t-j} + \varepsilon_t$$

$H_0: \; a = 0$

$H_1: \; a < 0$

$$\Rightarrow \quad n\hat{a} \Rightarrow \frac{\sigma \int W(r)\,dW(r)}{\omega \int W^2(r)\,dr}$$

$$\Rightarrow ADF = \frac{n\hat{a}\hat{w}}{\hat{\sigma}} \Rightarrow \frac{\int W(r)\, dW(r)}{\int W^2(r)\, dr}$$

$$= \frac{n\hat{a}}{1-\sum_{j=1}^{p}\hat{a}_j} \qquad \left(\hat{\omega}^2 = \frac{\hat{\sigma}^2}{(1-\sum_{j=1}^{p}\hat{a}_j)^2} \right)$$

Ex11 (Phillips-Perron Tests)

$$y_t = \alpha y_{t-1} + \beta^T x_t + \varepsilon_t$$

$$H_0 : \alpha = 1$$

$$H_1 : \alpha < 1$$

① $Z_\alpha = n(\hat{\alpha}-1) - \hat{\lambda}\left(\frac{1}{n^2}\sum \tilde{y}_{t-1}^2\right)^{-1} \Rightarrow \left(\int_0^1 W(r)\, dW(r)\right)\left(\int_0^1 W_x^2(r)\, dr\right)^{-1}$

② $Z_t = \hat{\sigma}_u^2 \hat{w}^{-1} t_\alpha - \hat{\lambda}\left\{\hat{\omega}\left(\frac{1}{n^2}\sum \tilde{y}_{t-1}^2\right)^{\frac{1}{2}}\right\}^{-1} \Rightarrow \left(\int_0^1 W(r)\, dW(r)\right)\left(\int_0^1 W^2(r)\, dr\right)^{-1}$

where $W_x(r) = \frac{1}{\omega}B_x(r)$; $W(r) = \frac{1}{\omega}B(r)$

$\tilde{y}_{t-1}$ is the residual from reg y_{t-1} on x_{t-1}

Ex12 Intuition: testing stationarity in the residuals in the preliminary cointegrating regression.

① Residual-Based Test.

$$\hat{u}_t = y_t - \hat{\beta}^T x_t$$

measure of fluctuation in residual

$$\max_k \frac{1}{\sqrt{n}} \left| \sum_{t=1}^{k} \hat{u}_t \right|$$

where $\frac{1}{\sqrt{n}} \sum \hat{u}_t = \frac{1}{\sqrt{n}} \sum u_t - n (\hat{\beta}-\beta)^T \left(\frac{1}{n} \sum \frac{x_t}{\sqrt{n}} \right)$

$$\Rightarrow B_u(r) - \left(\int_0^1 B_x^T dB_u + \sum E[V_t X_t] \right) \left(\int B_x B_x^T \right)^{-1} \int_0^r B_x(v)$$

$$\equiv f(r, B_u, B_x, \Delta_{ux})$$

$$\Rightarrow \max \frac{1}{\sqrt{n}} \left| \sum \hat{u}_t \right| \Rightarrow \sup_{0 \le r \le 1} |f(r, B_u, B_x, \Delta_{ux})|$$

under the null of cointegration

② Johansen's test

A LR-type test for VAR(p)

$$\underset{(m\times 1)}{\Delta Y_t^s} = \Psi Y_{t-1}^s + \sum_{j=1}^{p} \Phi_j \Delta Y_{t-j}^s + \varepsilon_t$$

$H_0: |\Psi| = 0$ (reduced rank)

$H_1: |\Psi| \neq 0$

$\Rightarrow$ If y_t is cointegrated of order r $\Rightarrow$ $\text{rank}(\Psi) = r$

$\Rightarrow \underset{(m\times m)}{\Psi} = \underset{(m\times r)}{\alpha} \underset{(r\times m)}{\beta^T} \Rightarrow \text{rank}(\Psi) = r \Rightarrow H_0:$ $m-r$ unit roots

$H_1: \text{rank}(\Psi) = m$ (full rank)

R_{0t}: reg of Δy_t on $\Delta Y_{t-1}, \cdots \Delta Y_{t-p+1}$ } DWL-decomposition

R_{1t}: reg of Y_{t-1} ------------

$\Rightarrow R_{0t} = \alpha \beta^T R_{1t} + \varepsilon_t$

calculate eigenvalues $\lambda_1 > \lambda_2 > \cdots \lambda_m$ by

$$|\lambda S_{11} - S_{10} S_{00}^{-1} S_{01}| = 0$$

where $S_{11} = \frac{1}{T} \sum R_{1t} R_{1t}^T$ $S_{10} = \frac{1}{T} \sum R_{1t} R_{0t}^T$ $S_{01} = \frac{1}{T} \sum R_{0t} R_{1t}^T$ $S_{00} = \frac{1}{T} \sum R_{0t} R_{0t}^T$

$$\Rightarrow LR = -T \sum_{i=r+1}^{m} \ln(1-\lambda_i)$$

$$\Rightarrow tr \left\{ \left(\int_0^1 \underbrace{W(r)^T}_{\text{Brownian motion}} dW(r) \right) \left[\int_0^1 W(r) W(r)^T dr \right]^{-1} \left[\int_0^1 W(r) dW(r) \right] \right\}$$