Sample Question for Time Series

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1. Stationary ARMA Processes

1.1. Stationary ARMA Models

1.2. Causality and Invertibility

Causality and invertibility describes the relationship between the two process X_t (the generated process) and ε_t (the innovation process).

Definition 1 (Causality). *An ARMA process defined by $A(L)X_t = \Theta(L)\varepsilon_t$ is said to be **causal** (or, be a causal function of ε_t) if there exists a sequence of constants $\{c_j\}$ such that $\sum_j |c_j| < \infty$ and $X_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}$.*

The absolute summability of coefficients $\{c_j\}$ ensures the convergence of the moving average representation of infinite order. In particular, under this assumption, $E\left(\sum_{j=0}^{\infty} |c_j| \, |\varepsilon_{t-j}|\right) \leq E\,|\varepsilon_{t-j}| \sum_{j=0}^{\infty} |c_j| < \infty$, consequently, $\sum_{j=0}^{\infty} c_j \varepsilon_{t-j}$ converges absolutely with probability one. (It also converges in mean square, as long as the moment of ε exists).

Similarly, we may consider representing an ARMA process as an autoregressive process of infinite order.

Definition 2 (Invertibility). *An ARMA process defined by $A(L)X_t = \Theta(L)\varepsilon_t$ is said to be **invertible** if there exists a sequence of constants $\{a_j\}$ such that $\sum_j |a_j| < \infty$ and $\sum_{j=0}^{\infty} a_j X_{t-j} = \varepsilon_t$.*

If the polynomial $A(\cdot)$ has all its roots different from 1 in modulus, we can invert the operator $A(L)$ and represent an $AR(p)$ process (**??**) by an infinite moving average

$$X_t = A(L)^{-1}\varepsilon_t = \sum_{j=-\infty}^{\infty} c_j \varepsilon_{t-j}.$$

In particular, if $A(\cdot)$ has all its roots strictly greater than 1 in modulus, we have

$$X_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j},\ c_0 = 1.$$

Just like that the $AR(p)$ process may be represented as an infinite moving average, a $MA(q)$ process also has an infinite autoregressive representation under analogous conditions. In particular, if $\Theta(\cdot)$ has all its roots different from 1 in modulus, we can represent an $MA(q)$ process (**??**) by an infinite autoregression

$$\sum_{j=-\infty}^{\infty} a_j X_{t-j} = \varepsilon_t.$$

And if $\Theta(\cdot)$ has all its roots strictly greater than 1 in modulus, we have

$$\sum_{j=0}^{\infty} a_j X_{t-j} = \varepsilon_t,\ a_0 = 1.$$

Parallel to the simple AR or MA models, an ARMA process also has infinite autoregressive (AR(∞)) or moving average (MA(∞)) representations under similar assumptions.

Exercise 1 Let $\{X_t\}$ be an ARMA process defined by $A(L)X_t = \Theta(L)\varepsilon_t$ and the polynomials $A(\cdot)$ and $\Theta(\cdot)$ have no common zeros. Then, $\{X_t\}$ is causal if and only if $A(z) \neq 0$ for all $|z| \leq 1$. The coefficients $\{c_j\}$ in $X_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}$ are determined by the relation $C(z) = \sum_j c_j z^j = \Theta(z)/A(z)$, $|z| \leq 1$.

Exercise 2 Let $\{X_t\}$ be an ARMA process defined by $A(L)X_t = \Theta(L)\varepsilon_t$ and the polynomials $A(\cdot)$ and $\Theta(\cdot)$ have no common zeros. Then, $\{X_t\}$ is invertible if and only if $\Theta(z) \neq 0$ for all $|z| \leq 1$. The coefficients $\{a_j\}$ in $\sum_{j=0}^{\infty} a_j X_{t-j} = \varepsilon_t$ are determined by the relation $a(z) = \sum_j a_j z^j = A(z)/\Theta(z)$, $|z| \leq 1$.

Exercise 3 (Invertibility/Causality Condition)

(1) For a MA(2) process $X_t = \varepsilon_t + \sum_{i=1}^{2} \theta_i \varepsilon_{t-i}$, show that: if X_t is invertible θ_j must satisfy

$$\theta_1 + \theta_2 > -1,\, \theta_2 - \theta_1 > -1,\; |\theta_2| < 1.$$

(2) For an AR(2) process $X_t = \sum_{i=1}^{2} \alpha_i X_{t-i} + \varepsilon_t$, show that: if X_t is causal α_j must satisfy

$$\alpha_1 + \alpha_2 < 1,\, \alpha_1 - \alpha_2 > -1,\; |\alpha_2| < 1.$$

Exercise 4 Given an AR(2) process $X_t + 0.2X_{t-1} - 0.48X_{t-2} = \varepsilon_t$, determine whether it is causal.

Exercise 5 Given an ARMA process $X_t + 0.6X_{t-2} = \varepsilon_t + 1.2\varepsilon_{t-1}$, determine whether it is causal and invertible.

1.3. ARIMA Models

A process $\{X_t\}$ is stationary if its statistical properties do not change over time. This property is stronger than that of identical distribution because it requires that the joint distribution among $\{X_t\}_{t\in I}$ does not change over time, not only the marginal distribution. We sometimes call this strict stationarity.

Definition 3 (Stationarity). *A process $\{X_t\}$ is strict stationary if, for any $t_1, \cdots, t_k$, and any h, the joint distribution of*

$$(X_{t_1}, \cdots, X_{t_k})$$

is the same as the joint distribution of

$$(X_{t_1+h}, \cdots, X_{t_k+h}).$$

The concept of a strict stationary time series requires that its probability structure is invariant under a shift of time. However, strict stationarity is a severe requirement. A relaxed concept is the notion of "*stationarity up to order m*". Under this weaker condition, we do not insist the invariance of probability distribution, but only that the main features (more precisely, moments) of the distribution are invariant under a shift of time.

Definition 4 (m-th Order Stationarity). *A process $\{X_t\}$ is stationary upto order m (or m-th order stationary) if, for any $t_1, \cdots, t_k$, and any h, all the joint moments upto order m of $(X_{t_1}, \cdots, X_{t_k})$ exist and equals to the corresponding joint moments up to order m of $(X_{t_1+h}, \cdots, X_{t_k+h})$.*

A very important special case of m-th order stationarity is the second order stationarity, or covariance stationarity.

Definition 5 (Covariance Stationarity). *A process $\{X_t\}$ is covariance stationary if (i) $E(X_t) = \mu$, for any t; (ii) $E(X_t^2) < \infty$, for any t; (iii) $Cov(X_t, X_{t+h}) = \gamma(h)$, for any h and t.*

Exercise 6 Is a linear combination of two covariance stationary processes still covariance stationary? Prove it if it is true. Give a counterexample if not true.

Exercise 7 Is a linear combination of two jointly covariance stationary processes still covariance stationary? Prove it if it is true. Give a counterexample if not true.

Exercise 8 If $\{X_t\}$ is a covariance stationary process and $\{c_j\}$ is a sequence of real numbers that $\sum_j |c_j| < \infty$, prove that $Y_t = \sum_{j=-\infty}^{\infty} c_j X_{t-j}$ is also a covariance stationary process.

1.4. Autocovariance Function

An important concept that captures the variability and serial correlation in a covariance stationary time series $\{X_t\}$ is its **autocovariance function** defined as $\gamma_X(h) = \mathrm{Cov}(X_t, X_{t+h})$. By definition, $\gamma_X(0) = \mathrm{Var}(X_t)$. In addition, the autocovariance function is (i) even: $\gamma_X(h) = \gamma_X(-h)$ and (ii) $|\gamma_X(h)| \leq \gamma_X(0)$. Standardizing the autocovariance function by the variance, we obtain the **autocorrelation function** of $\{X_t\}$:

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)},$$

which measures the correlation between X_t and X_{t+h}. It is easy to show that $\rho_X(h)$ is also even, and $\rho_X(0) = 1$. For this reason, the autocorrelation function is often represented in a graph for $h \geq 0$ and this graph is called a **correlogram**. The autocorrelation functions of covariance stationary AR and MA processes calculated in the following exercises.

Exercise 9: Autocovariance Function of MA Processes Define the autocovariance function of a process $\{X_t\}$ as $\gamma_X(h) = \mathrm{Cov}(X_t, X_{t+h})$.

(1) Let $\{X_t\}$ be a MA(1) process such that $X_t = \varepsilon_t + \theta\varepsilon_{t-1}$, show that

$$\gamma_X(h) = \begin{cases} (1+\theta^2)\sigma^2, & h = 0 \\ \theta\sigma^2, & |h| = 1 \\ 0, & |h| > 1. \end{cases}$$

(2) More generally, if $\{X_t\}$ is a MA(q) process $X_t = \sum_{i=0}^{q} \theta_i \varepsilon_{t-i}$ $(\theta_0 = 1)$, show that

$$\gamma_X(h) \quad = \quad \begin{cases} \sigma^2 \sum_{j=0}^{q-|h|} \theta_j \theta_{j+|h|}, & |h| \le q \\ 0, & |h| > q. \end{cases}$$

Exercise 10: Autocovariance Function of AR(1) Processes Show that the autocovariance and autocorrelation functions of an AR(1) process:

$$X_t = \alpha X_{t-1} + \varepsilon_t, |\alpha| < 1,$$

are given by

$$\gamma_X(h) = \frac{\sigma^2}{1-\alpha^2}\alpha^{|h|}, \text{ and } \rho_X(h) = \alpha^{|h|}.$$

Correlogram of Some AR Processes.

The calculation of autocovariance function for higher order AR processes or ARMA processes are more complicated. One way is to express the process into a moving average of infinite order and then use the following result that calculates the autocovariance function of an infinite order moving average process as a special case.

Exercise 11 If $\{X_t\}$ is a covariance stationary process with autocovariance function $\gamma(\cdot)$ and if $\sum |c_j| < \infty$, then, for each t, the series $Y_t = \sum_j c_j X_{t-j}$ converges absolutely with probability one and in mean square to the same limit. In addition, the process $\{Y_t\}$ is covariance stationary with autocovariance function

$$\gamma_Y(h) = \sum_{j,k} c_j c_k \gamma(h - j + k).$$

In the special case that $\{X_t\}$ are IID with mean zero and variance σ^2,

$$\gamma_Y(h) = \sigma^2 \sum_{j=0}^{\infty} c_j c_{j+|h|}.$$

Exercise 12 (Brockwell and Davis) Calculate the autocovariance function of a causal and invertible ARMA(p,q) process

$$X_t - \sum_{i=1}^{p} \alpha_i X_{t-i} = \varepsilon_t + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}$$

Exercise 13 (Brockwell and Davis) Consider an ARMA (2,1) process

$$(1 - L + 0.25L^2)X_t = (1 + L)\varepsilon_t,$$

Calculate the autocovariance function.

Exercise 15 Let $\{Z_t\}$ be a stationary zero-mean time series. Define

$$X_t = Z_t - 0.4Z_{t-1}$$

and

$$Y_t = Z_t - 2.5Z_{t-1}$$

(1) Express the autocovariance functions of $\{X_t\}$ and $\{Y_t\}$ in terms of the autocovariance function of Z_t. (2) Show that $\{X_t\}$ and $\{Y_t\}$ have the same autocorrelation functions.

1.5. Estimation

Estimation of AR models can be obtained by regressing X_t on its lagged values X_{t-j} to appropriate order. Or, by equating the sample and theoretical autocovariances at lags 0, 1, ..., p. This is the Yule-Walker estimator.

Exercise 17 (Yule-Walker Equation) Given a covariance stationary AR(p) process $X_t - \sum_{i=1}^{p} \alpha_i X_{t-i} = \varepsilon_t$, show that

$$\rho_X(h) - \sum_{i=1}^{p} \alpha_i \rho_X(h-i) = 0, \tag{1}$$

where $\rho_X(h)$ is the autocorrelation function of X_t.

Writing equation (1) for $h = 1, ..., p$, and noting the evenness of $\gamma_X(\cdot)$, we obtain the following equations

$$G_p \alpha = g_p$$

where

$$g_p = \begin{bmatrix} \gamma_X(1) \\ \gamma_X(2) \\ \vdots \\ \gamma_X(p) \end{bmatrix}, \ \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_p \end{bmatrix}, \text{ and}$$

$$G_p = \begin{bmatrix} \gamma_X(0) & \gamma_X(1) & \gamma_X(2) & \cdots & \gamma_X(p-1) \\ \gamma_X(1) & \gamma_X(0) & \gamma_X(1) & \cdots & \gamma_X(p-2) \\ \vdots & & & & \vdots \\ \gamma_X(p-1) & \gamma_X(p-2) & & \cdots & \gamma_X(0) \end{bmatrix}.$$

From these equations we can obtain the α's as a function of $\gamma_X(0), \cdots, \gamma_X(p)$ by taking inversion. These equations are called as Yule-Walker equations.

Exercise 18 Given an AR(2) process $X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + u_t$, (i) solve for α_1 and α_2 as a function of the autocorrelation coefficients, and (ii) calculate the autocorrelation function $\rho(k)$ in terms of α_1 and α_2.

Exercise 20 Autocovariance Function of ARMA processes. Let X_t be a covariance stationary ARMA(p, q) process $A(L)X_t = \Theta(L)\varepsilon_t$ and denote its autocovariance function as $\gamma_X(\cdot)$, show that

$$\gamma_X(h) + \sum_{i=1}^{p} \alpha_i \gamma_X(h-i) = 0, \text{ for any } h \geq q+1.$$

Exercise 21: MLE for a Gaussian AR(1) Process Given a stationary AR(1) process $X_t = \alpha X_{t-1} + \varepsilon_t$, $t = 1, ..., n$, with ε_t being IID N(0,σ^2). Write out the log-likelihood function for estimating α and σ^2.

Exercise 22 (MLE for a Gaussian AR(p) Process) Given a stationary AR(p) process $X_t = c + \alpha_1 X_{t-1} + \cdots + \alpha_p X_{t-p} + \varepsilon_t$, $t = 1, ..., n$, with ε_t being IID $N(0, \sigma^2)$. Find the likelihood function for estimating α_j and σ^2.

Exercise 24 Consider regression

$$y_t = \alpha x_t + u_t, t = 1, 2,$$

where the error term is an AR(1) process:

$$u_t = \rho u_{t-1} + \varepsilon_t, \text{ where } \varepsilon_t \sim N(0, (1 - \rho^2)\sigma^2),$$

and $u_t \sim N(0, \sigma^2)$. There are only two observations, and the regressors are nonstochastic. Calculate the MLE of ρ and α.

1.6. Prediction

Exercise 25 Let random variables or vectors X and $Z \in L_2 =$ space of square integrable variables, we are interested in forecasting X based on information in Z. Denote the σ-field generated from Z by $\mathcal{F}_Z$, we look for $\mathcal{F}_Z$-measurable function X^* such that

$$E (X - X^*)^2 = \min_{g(Z)} E (X - g(Z))^2 ,$$

Show that X^* is the expectation of X conditional on Z :

$$X^* = E(X|Z).$$

Exercise 26 Given a MA(2) process

$$X_t = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2}, \text{ where } \varepsilon_t \sim WN(0, \sigma^2),$$

and information upto time n, find the optimal linear forecast of X_{n+h}, $h > 0$, forecast errors, and forecast error variance.

Exercise 27 Given a MA(∞) process

$$X_t = \sum_{i=0}^{\infty} \theta_i \varepsilon_{t-i}, \text{ where } \theta_0 = 1,\ \varepsilon_t \sim WN(0, \sigma^2),$$

and information upto time n, find the optimal linear forecast of X_{n+h}, $h > 0$, forecast errors, and forecast error variance.

Exercise 29 Consider a time series regression

$$y_t = \beta' x_t + u_t,$$

assume $\mathrm{E}u_t = 0$, $\mathrm{Var}(u_t) = \sigma^2$, $\mathrm{Cov}(u_t, u_s) = \sigma^2(1-\rho)$, for $t \neq s$, with $-1/(T-1) \leq \rho \leq 1$. The regression has a constant.

(i) Show that be the OLS estimator of β is equivalent to GLS.

Show that $E(\widehat{\sigma}^2) = \sigma^2(1-\rho)$.

Exercise 30 Consider an ARCH(1) model

$$\begin{aligned} u_t &= \sigma_t \varepsilon_t \\ \sigma_t^2 &= \alpha_0 + \alpha_1 u_{t-1}^2, \end{aligned}$$

(i) What is the condition for u_t to be stationary? (2) How to estimate this ARCH model? (3) What is the limiting distribution of the estimator?

Exercise 31 Consider an GARCH(1,1) model

$$\begin{aligned} u_t &= \sigma_t \varepsilon_t \\ \sigma_t^2 &= \alpha_0 + \alpha_1 u_{t-1}^2 + \gamma_1 \sigma_{t-1}^2, \end{aligned}$$

(i) What is the condition for u_t to be stationary? (2) What is the definition of an IGARCH(1,1) ? (3) How to estimate this GARCH model?

2. VAR

Exercise 32 Consider a m-dimensional VAR process $\{Y_t\}$ of order p ($VAR(p)$):

$$Y_t = C + \sum_{i=1}^{p} A_i Y_{t-i} + \varepsilon_t \tag{2}$$

- C is a m-dimensional vector of constants (intercept)
- A_i are $m \times m$ coefficient-matrices of real numbers.

Suppose that ε_t are m-dimensional i.i.d. $N(0,\Omega)$, conditional on the first p observations, write down the conditional log likelihood function for a sample of size n: $(Y_1,\cdots,Y_n)$.

Exercise 33. Consider the VAR process given in Exercise 32 again, denote

$$\begin{aligned} X_t &= \begin{bmatrix} 1, & Y'_{t-1}, & Y'_{t-2}, & \cdots, & Y'_{t-p} \end{bmatrix}' \\ &: \quad (mp+1)\times 1 \text{ vector of regressors,} \end{aligned}$$

$$\begin{aligned} B' &= \begin{bmatrix} C, & A_1, & A_2, & \cdots, & A_p \end{bmatrix} \\ &: \quad m\times(mp+1) \text{ matrix of coefficients,} \end{aligned}$$

then the m equations of the VAR(p) model can be re-written as

$$Y_t = B'X_t + \varepsilon_t.$$

If we denote the j-th row of B' as b'_j, then b'_j is an $1\times(mp+1)$ vector which contains the parameters of the j-th equation of the VAR(p) model. If the conditional MLE of B is $\widehat{B}$, In particular, the j-th row of $\widehat{B}'$ is $\widehat{b}'_j$, **Show that $\widehat{b}'_j$ is simply the estimator from an OLS regression of Y_{jt} on X_t:**

$$Y_{jt} = b'_jX_t + \varepsilon_{jt},\ j = 1,\cdots,m.$$

Exercise 34 Consider the VAR process given in Exercise 32 again, suppose that ε_t are m-dimensional i.i.d. $N(0,\Omega)$, conditional on the first p observations, find out the conditional maximum likelihood estimator of Ω.

Exercise 35 Consider an m dimensional time series and partition it as:

$$Y_t = \begin{bmatrix} Y_{I,t} & \cdots & m_1 \text{ dimension} \\ Y_{II,t} & \cdots & m_2 \text{ dimension} \end{bmatrix}$$

- A sub-group of variables, $Y_{I,t}$, is (block) exogenous with respect to another group of variables, $Y_{II,t}$, in the time series sense **if $Y_{II,t}$ can not help forecast $Y_{I,t}$,** we say that $Y_{II,t}$ *does not Granger-cause* $Y_{I,t}$.

If we partition the m-dimensional VAR(p) (given by 2) process Y_t into two sub-groups: $Y_t = (Y_{I,t}, Y_{II,t})$, each has dimension m_1 and m_2 ($m_1 + m_2 = m$), and partition the covariance matrix Ω and the matrix of coefficients B accordingly:

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix}. \tag{3}$$

We also partition the vector of regressors, X_t, as

$$\begin{aligned} X_{I,t} &= \begin{pmatrix} Y'_{I,t-1}, & Y'_{I,t-2}, & \cdots, & Y'_{I,t-p} \end{pmatrix}' \\ &: \quad m_1 p \times 1 \\ \\ X_{II,t} &= \begin{pmatrix} Y'_{II,t-1}, & Y'_{II,t-2}, & \cdots, & Y'_{II,t-p} \end{pmatrix}' \\ &: \quad m_2 p \times 1 \end{aligned}$$

and re-write the VAR model as

$$\begin{aligned} Y_{I,t} &= C_I + F'_I X_{I,t} + F'_{II} X_{II,t} + \varepsilon_{1t}, \\ Y_{II,t} &= C_{II} + G'_I X_{I,t} + G'_{II} X_{II,t} + \varepsilon_{2t} \end{aligned}$$

where F_I, F_{II}, G_I, and G_{II} are the corresponding matrices of coefficients.

- In the partitioned representation:, $Y_{II,t}$ does not Granger-cause $Y_{I,t}$ if

$$F_{II} = 0.$$

If the elements of $Y_{II,t}$ are informative about future $Y_{I,t}$, then $Y_{II,t}$ *Granger-cause* $Y_{I,t}$.

Corresponding to the above partition, show that the sample log likelihood, L, can be decomposed into summation of the following two parts:

- L_I: a sum of log density of $Y_{I,t}$ (conditional on X_t),
- L_{II}: the sum of log density of $Y_{II,t}$ conditional on X_t and $Y_{I,t}$.

Exercise 36 Consider the test of Granger causality or exogeneity in a m-dimensional VAR(p) process $\{Y_t\}$ with the previous partition, assume that ε_t are m-dimensional i.i.d. N(0,Ω), the covariance matrix has partition as described above, write out a **likelihood ratio (LR) test** for the null **hypothesis that $Y_{II,t}$ does not Granger-cause $Y_{I,t}$,** i.e. $F_{II} = 0$ in the partitioned representation.

3. Inference for Stationary Time Series Models

3.1. Basics for Limiting Distributions

Exercise 18 Consider autoregression

$$y_t = \alpha y_{t-1} + u_t;\ \{u_t\} \equiv \text{iid}(0, \sigma^2),\ |\alpha| < 1,$$

show that

$$\hat{\alpha} = \sum y_t y_{t-1} / \sum y_{t-1}^2 \xrightarrow{\text{a.s.}} \alpha.$$

3.2. Distributions of Conventional Estimators and Testing Statistics In Stationary Time Series

1. AR(1)

$$y_t = \alpha y_{t-1} + u_t,\ |\alpha| < 1$$

2. AR(k)

$$y_t = \alpha_1 y_{t-1} + \cdots + \alpha_k y_{t-k} + u_t, \;\; |\alpha| < 1$$

3. Linear Regression with correlated errors: OLS

$$y_t = \beta' x_t + u_t, \; u_t \text{ are serially correlated}$$

Basic Tools
We need **LLN for consistency** and **CLT for the asymptotic normality**.

4. Nonstationary Processes

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4.1. Unit Root and Cointegration

Exercise 6. *Unit Root process 1*

$$y_t = \alpha y_{t-1} + u_t,\ \alpha = 1$$

where u_t *are iid* $(0, \sigma^2)$, *and consider the OLS estimation of the AR coefficient*

$$\widehat{\alpha} = \frac{\sum y_{t-1} y_t}{\sum y_{t-1}^2}$$

what's the limiting distribution of $\widehat{\alpha}$ *?*

Exercise 7. *Unit Root process 2*

$$y_t = \alpha y_{t-1} + u_t,\ \alpha = 1$$

where u_t *are stationary but weakly dependent process, say, a mixing process, and consider the OLS estimation of the AR coefficient*

$$\widehat{\alpha} = \frac{\sum y_{t-1} y_t}{\sum y_{t-1}^2}$$

what's the limiting distribution of $\widehat{\alpha}$ *?*

Exercise 8. *Spurious Regression*

$$y_t = \beta x_t + u_t;\ y_t = y_{t-1} + v_{1t}, x_t = x_{t-1} + v_{2t}.$$

where v_{1t} *and* v_{2t} *are stationary but weakly dependent process, say, a mixing process, that follows a multivartae functional central limit theorem, and consider the OLS estimation of the* β *coefficient*

$$\widehat{\beta} = \frac{\sum x_t y_t}{\sum x_t^2}$$

what's the limiting distribution of $\widehat{\beta}$ *?*

Exercise 9. *Cointegrating Regression*

$$y_t = \beta x_t + u_t,\ x_t = x_{t-1} + v_t.$$

where v_t *and* u_t *are stationary but weakly dependent process, say, a mixing process, that follows a multivartae functional central limit theorem, and consider the OLS estimation of the* β *coefficient*

$$\widehat{\beta} = \frac{\sum x_t y_t}{\sum x_t^2}$$

what's the limiting distribution of $\widehat{\beta}$ *?*

Exercise 10. *How to construct the ADF tests for a unit root? What's the limiting distributions?*

Exercise 11. *How to construct the Phillips-Perron tests for a unit root? What's the limiting distributions?*

Exercise 12. *How to test Cointegration ? What is the Residual-Based Test for cointegration? What is the Johansen's test for cointegration?*

Exercise 13. *Trending regression*

$$y_t = \beta x_t + u_t;$$

where

$$x_t = (1, t, \cdots, t^p)',$$

and consider the OLS estimation of the β *coefficient*

$$\widehat{\beta} = \left(\sum x_t^2\right)^{-1}\left(\sum x_t y_t\right)$$

what's the limiting distribution of $\widehat{\beta}$ *?*

Case 1: where u_t *is stationary but weakly dependent process, say, a mixing process, that satisfies LLN and CLT.*

Case 2: where $u_t = u_{t-1} + v_t$, v_t *is a stationary but weakly dependent process, say, a mixing process, so that a partial sum process of* v_t *satisfies FCLT.*

Exercise 14. *Trending Unit Root process 1: Consider a time series*

$$\begin{aligned} y_t &= \mu + y_{t-1}^s, \\ y_t^s &= \alpha y_{t-1}^s + u_t,\ \alpha = 1 \end{aligned}$$

where u_t are iid $(0, \sigma^2)$, and consider the OLS estimation of the AR coefficient α based on the following regression

$$y_t = \widehat{\gamma} + \widehat{\alpha} y_{t-1} + e_t,$$

what's the limiting distribution of $\widehat{\alpha}$?

Exercise 15. *Trending Unit Root process 2: Consider a time series generated by:*

$$y_t = \mu + \alpha y_{t-1} + u_t,\ \alpha = 1$$

where u_t are iid $(0, \sigma^2)$, and consider the OLS estimation of the AR coefficient α based on the following regression

$$y_t = \widehat{\gamma} + \widehat{\alpha} y_{t-1} + e_t,$$

what's the limiting distribution of $\widehat{\alpha}$?

Exercise 16. *Given the results of a unit root or cointegration regression, analyze the empirical results (detect whether or not there is an unit root, or whether or not the time series are cointegrated).*