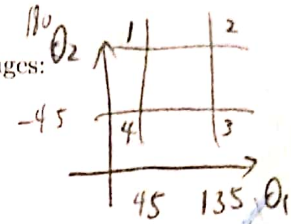
5.1 Inverse Manipulator Kinematics I: Workspace and Plane Geometry

5.1.1

Plot the workspace of the 2-link planar manipulator for $l_1 = 5, l_2 = 3$ for the joint motion ranges:

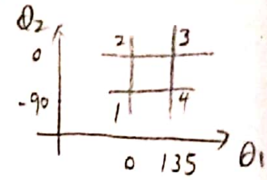
$$45° < \theta_1 < 135° \qquad -45° < \theta_2 < 180°$$



5.1.2

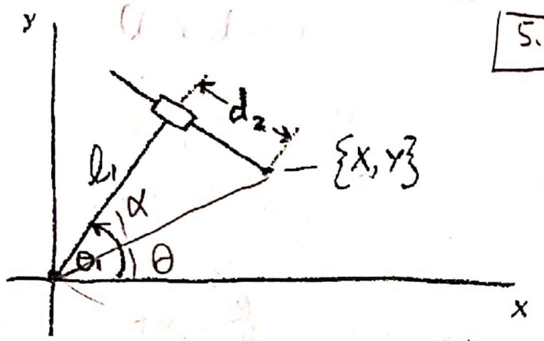
Plot the workspace of the 2-link planar manipulator for $l_1 = 2.5, l_2 = 3.5$ for the joint motion ranges:

$$0° < \theta_1 < 135° \qquad -90° < \theta_2 < 0°$$

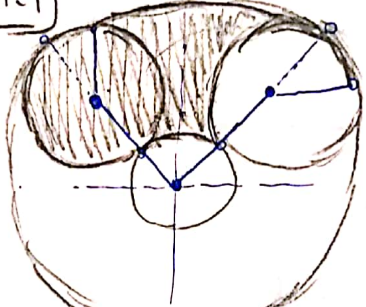


5.1.3

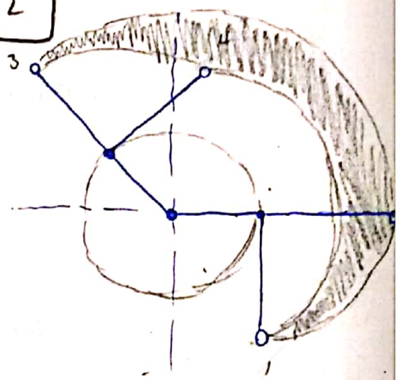
Solve the inverse kinematics of the following planar robot. Find all solutions:

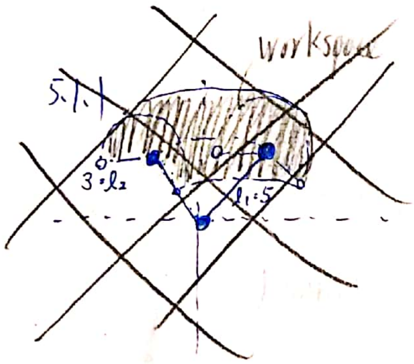


5.1.1

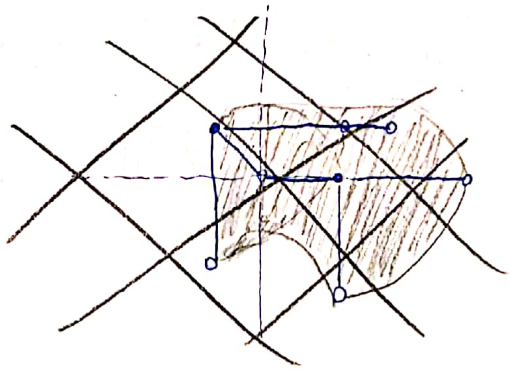


5.1.2





5.1.2



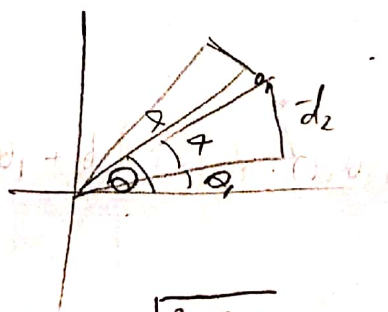
5.1.3.

if $d_2 > 0$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$\alpha = \tan^{-1}\left(\frac{d_2}{l_1}\right)$$

if $d_2 < 0$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$\alpha = \tan^{-1}\left(\frac{-d_2}{l_1}\right)$$

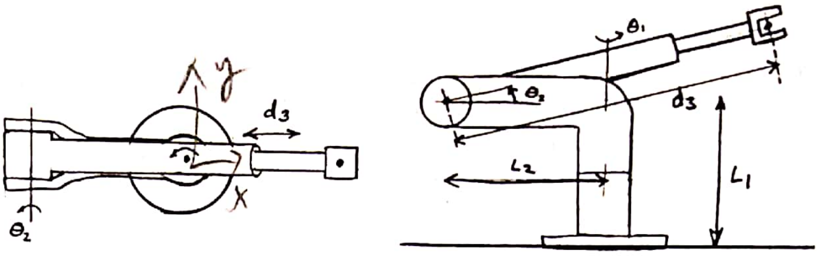


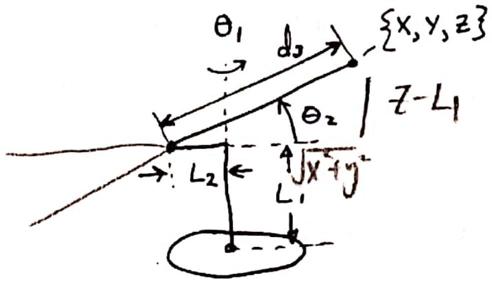
$$\theta + \alpha = \theta_1 = \tan^{-1}\left(\frac{y}{x}\right) + \tan^{-1}\left(\frac{d_2}{l_1}\right) \qquad d_2 = \sqrt{x^2 + y^2 - l_1^2}$$

$$\theta - \alpha = \theta_1 = \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{-d_2}{l_1}\right) \qquad d_2 = -\sqrt{x^2 + y^2 - l_1^2}$$

5.1.4

Geometric Method / Spatial Manipulator





Inverse Kinematics Arm

Please find all solutions for this arm to reach the point X, Y, Z. Geometric method is suggested.

$$d_3 = \pm\sqrt{\left(L_2 + \sqrt{X^2+y^2}\right)^2 + (Z-L_1)^2}$$

$$\theta_1 = \operatorname{atan2}(y, X)$$

$$\begin{cases} \theta_2 = \operatorname{atan2}\left(Z-L_1,\ L_2+\sqrt{X^2+y^2}\right) & \text{if } d_3 \geq 0 \\ \theta_2 = \operatorname{atan2}\left(-(Z-L_1),\ -\left[L_2+\sqrt{X^2+y^2}\right]\right) & \text{if } d_3 < 0 \end{cases}$$

$\boxed{5.2.1}$

$$\begin{cases} \gamma_{23}+\gamma_{32} = 2k_Y k_Z \nabla\theta & \text{— ①} \\ \gamma_{12}+\gamma_{21} = 2k_X k_Y \nabla\theta & \text{— ②} \\ \gamma_{13}+\gamma_{31} = 2k_X k_Z \nabla\theta & \text{— ③} \end{cases}$$

$$①/② = \frac{\gamma_{23}+\gamma_{32}}{\gamma_{12}+\gamma_{21}} = \frac{k_Z}{k_X} \quad \text{— ④}$$

$$\boxed{④^2+⑤^2 = \frac{k_Z^2+k_Y^2}{k_X^2}} + \frac{k_X^2}{k_X^2} - 1$$

$$①/③ = \frac{\gamma_{23}+\gamma_{32}}{\gamma_{13}+\gamma_{31}} = \frac{k_Y}{k_X} \quad \text{— ⑤}$$

$$= \frac{1}{k_X^2} - 1$$

$$\pm\left[\frac{1}{\left(\frac{\gamma_{23}+\gamma_{32}}{\gamma_{12}+\gamma_{21}}\right)^2+\left(\frac{\gamma_{23}+\gamma_{32}}{\gamma_{13}+\gamma_{31}}\right)^2+1}\right]^{\frac{1}{2}} = k_X$$

$$②/① = \frac{k_X}{k_Z} = \frac{\gamma_{12}+\gamma_{21}}{\gamma_{23}+\gamma_{32}} \quad \text{— ⑥}$$

$$⑥^2+⑦^2 = \frac{k_X^2+k_Y^2}{k_Z^2} + \frac{k_Z^2}{k_Z^2} - 1$$

$$②/③ = \frac{k_Y}{k_Z} = \frac{\gamma_{12}+\gamma_{21}}{\gamma_{13}+\gamma_{31}} \quad \text{— ⑦}$$

$$= \frac{1}{k_Z^2} - 1$$

$$\pm\left[\frac{1}{\left(\frac{\gamma_{12}+\gamma_{21}}{\gamma_{23}+\gamma_{32}}\right)^2+\left(\frac{\gamma_{12}+\gamma_{21}}{\gamma_{13}+\gamma_{31}}\right)^2+1}\right]^{\frac{1}{2}} = k_Z$$

$$③/① = \frac{k_X}{k_Y} = \frac{\gamma_{13}+\gamma_{31}}{\gamma_{23}+\gamma_{32}} \quad \text{— ⑧}$$

$$⑧^2+⑨^2 = \frac{k_X^2+k_Z^2}{k_Y^2} + \frac{k_Y^2}{k_Y^2} - 1$$

$$③/② = \frac{k_Z}{k_Y} = \frac{\gamma_{13}+\gamma_{31}}{\gamma_{12}+\gamma_{21}} \quad \text{— ⑨}$$

$$= \frac{1}{k_Y^2} - 1$$

$$\pm\left[\frac{1}{\left(\frac{\gamma_{13}+\gamma_{31}}{\gamma_{23}+\gamma_{32}}\right)^2+\left(\frac{\gamma_{13}+\gamma_{31}}{\gamma_{12}+\gamma_{21}}\right)^2+1}\right]^{\frac{1}{2}} = k_Y$$

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$\boxed{5.2.1}$

$$r_{11}+r_{22}+r_{33}$$
$$=(k_x^2+k_y^2+k_z^2)\cdot V\theta+3c\theta$$
$$=1+2c\theta$$

$$\theta=\cos^{-1}\left(\frac{r_{11}+r_{22}+r_{33}-1}{2}\right)$$

$\boxed{5.2.2}$

$$ {}^{1}_{2}T = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} & \begin{bmatrix} 3 \\ 10 \\ -7 \end{bmatrix} \\ 0 \; 0 \; 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ \left({}^{1}_{2}T\right)^{-1} = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}^T & -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}^T \begin{bmatrix} 3 \\ 10 \\ -7 \end{bmatrix} \\ 0 \; 0 \; 0 & 1 \end{bmatrix} $$

$$ = \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & -1 & 0 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$\boxed{5.2.3}$

$$ {}^{0}_{4}T = \begin{bmatrix} C_4 C_1 S_{23} - S_4 S_1 & -S_4 C_1 S_{23} - S_1 C_4 & C_1 C_{23} & C_1 C_2 a_2 - S_1 d_2 \\ C_4 S_1 S_{23} - S_4 C_1 & -S_4 S_1 S_{23} - C_1 C_4 & S_1 C_{23} & S_1 C_2 a_2 + C_1 d_2 \\ -C_4 C_{23} & S_4 C_{23} & S_{23} & S_2 a_2 + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ = \begin{bmatrix} r_{11} & r_{12} & r_{13} & P_x \\ r_{21} & r_{22} & r_{23} & P_y \\ r_{31} & r_{32} & r_{33} & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ P_x^2 + P_y^2 = C_2^2 a_2^2 + d_2^2 + 2 \cancel{S_1 C_1} C_2 a_2 d_2 - 2 \cancel{C_1 C_2 S_1} a_2 d_2 $$

$$ \begin{cases} \theta_4 = \operatorname{atan2}(r_{32}, -r_{31}) & \text{if } C_{23} \geq 0 \\ \theta_4 = \operatorname{atan2}(-r_{32}, r_{31}) & \text{if } C_{23} < 0 \end{cases} $$

$$ \pm\sqrt{\left(\frac{P_x^2 + P_y^2 - d_2^2}{a_2^2}\right)} = C_2 $$

$$ \begin{cases} \theta_1 = \operatorname{atan2}(r_{23}, r_{13}) & \text{if } C_{23} \geq 0 \\ \theta_1 = \operatorname{atan2}(-r_{23}, -r_{13}) & \text{if } C_{23} < 0 \end{cases} $$

$$ S_2 = \left(\frac{P_z - d_1}{a_2}\right) $$

$$ C_{23} = \pm\sqrt{(r_{31})^2 + (r_{32})^2} $$

$$ \theta_2 + \theta_3 = \operatorname{atan2}\left(r_{33}, \pm\sqrt{(r_{31})^2 + (r_{32})^2}\right) $$

$$ \theta_2 = \operatorname{atan2}\left(\frac{P_z - d_1}{a_2}, \pm\sqrt{\frac{P_x^2 + P_y^2 - d_2^2}{a_2^2}}\right) $$

$$ \theta_3 = \operatorname{atan2}\left(r_{33}, \pm\sqrt{(r_{31})^2 + (r_{32})^2}\right) - \theta_2 $$

$\boxed{5.2.4}$

$r_{13} = C_1 C_3 S_4 - C_4 S_1$

$r_{23} = S_1 C_3 S_4 + C_4 C_1$ $\qquad \theta_4$: known

$r_{33} = S_3 S_4$

$$(r_{13})^2 + (r_{23})^2 = C_3^2 S_4^2 + C_4^2 - \cancel{2 C_1 C_3 S_1 C_4 S_4} + \cancel{2 S_1 C_1 C_3 S_4 C_4}$$

$$(r_{31})^2 + (r_{32})^2 = C_4^2 S_3^2 + C_3^2 - \cancel{2 C_3 C_4 C_5 S_3 S_5} + \cancel{2 C_3 C_4 C_5 S_3 S_5}$$

$$\frac{(r_{13})^2 + (r_{23})^2}{(C_4)^2} - 1 = C_3^2 (\tan\theta_4)^2$$

$$C_3 = \pm\sqrt{\frac{r_{13}^2 + r_{23}^2 - C_4^2}{C_4^2 \cdot \tan\theta_4^2}}$$

$$\pm\sqrt{\frac{(r_{31})^2 + (r_{32})^2 - C_3^2}{C_4^2}} = S_3$$

$$\theta_3 = \operatorname{atan2}(S_3, C_3) = \operatorname{atan2}\left(\pm\sqrt{\frac{r_{31}^2 + r_{32}^2 - C_3^2}{C_4^2}}, \pm\sqrt{\frac{r_{13}^2 + r_{23}^2 - C_4^2}{C_4^2 \cdot (\tan\theta_4)^2}}\right)$$

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5.2.4

$$r_{11}^2 = C_5^2 (S_1 S_4 + C_1 C_3 C_4)^2 + C_1^2 S_3^2 S_5^2 + 2 C_5 S_5 C_1 S_3 (S_1 S_4 + C_1 C_3 C_4)$$

$$r_{12}^2 = S_5^2 (S_1 S_4 + C_1 C_3 S_4)^2 + C_1^2 C_5^2 S_3^2 - 2 C_1 C_5 S_3 S_5 (S_1 S_4 + C_1 C_3 C_4)$$

$$r_{11}^2 + r_{12}^2 = S_1^2 S_4^2 + C_1^2 C_3^2 S_4^2 + 2 S_1 S_4 C_1 C_3 S_4 + C_1^2 S_3^2$$

$$r_{21}^2 = C_5^2 (C_1 S_4 - C_3 C_4 S_1)^2 + S_1^2 S_3^2 S_5^2 - S_1 S_3 S_5 C_5 (C_1 S_4 - C_3 C_4 S_1)$$

$$r_{22}^2 = S_5^2 (C_1 C_4 - C_3 C_4 S_1)^2 + C_5 S_1^2 S_3^2 + S_1 S_3 S_5 C_5 (C_1 C_4 - C_3 C_4 S_1)$$

$\theta_1 =$ too hard

$\wedge_2$