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ME565 - Winter 2020 **Due: 5:00 pm Friday January 17^{th}, 2020**
Homework 1

Exercise 1–1: Solve the following linear system of ODEs:

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x}, \quad \text{for} \quad \mathbf{A} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$$

Please compute this by hand for a generic initial condition $\mathbf{x}(0) = \mathbf{x}_0$. Hint: you will want to compute the matrix exponential $e^{\mathbf{A}t}$. You may use MATLAB to check your work.

Exercise 1-2 Express the following in the form $a + bi$ (for real a and b) and also in the form $Re^{i\theta}$ (for real R and θ):

(a) $\frac{1}{3+2i}$

(b) $\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^3$

(c) $(-i)^2, (-i)^3, (-i)^4, (-i)^5, \ldots$

Exercise 1-3 Find all solutions of

(a) $e^z = 1$

(b) $e^z = -i$

(c) $e^z = -1 + i$

Exercise 1-4 Find all solutions of

(a) $z^3 = i$

(b) $z^3 = -i$

(c) $z^5 = 1$

(d) $z^2 = -1 - i$

Exercise 1-6 Find all analytic functions $f = u + iv$ with $u(x, y) = x^2 - y^2$. Simplify the expression $f(z)$ as much as possible.

Exercise 1–1: Solve the following linear system of ODEs:

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x}, \quad \text{for} \quad \mathbf{A} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$$

Please compute this by hand for a generic initial condition $\mathbf{x}(0) = \mathbf{x}_0$. Hint: you will want to compute the matrix exponential $e^{\mathbf{A}t}$. You may use MATLAB to check your work.

$$\frac{d}{dt}\underline{X} = \underline{\underline{A}}\,\underline{X}$$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$\Rightarrow \underline{X}(t) = e^{\underline{\underline{A}}t}\underline{X}(0)$$

$$\Rightarrow e^{\underline{\underline{A}}t} = \mathbb{I} + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} \lambda+1 & -2 \\ -2 & \lambda+1 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda+1)^2 - 4 = 0$$

$$\Rightarrow \underline{\lambda = -3 \text{ or } 1}$$

$$\Rightarrow e^{At} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{t} \end{bmatrix}$$

$$\Rightarrow \underline{X(t) = \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{t} \end{bmatrix} X_0} \quad \#$$

Exercise 1-2 Express the following in the form $a + bi$ (for real a and b) and also in the form $Re^{i\theta}$ (for real R and θ):

(a) $\frac{1}{3+2i}$

(b) $\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^3$

(c) $(-i)^2, (-i)^3, (-i)^4, (-i)^5, \ldots$

(a)

$$\frac{1}{3+2i} = \frac{3-2i}{(3+2i)(3-2i)} = \frac{3-2i}{13} = \frac{3}{13} - \frac{2}{13}i$$

$$R = \sqrt{\left(\frac{3}{13}\right)^2 + \left(\frac{-2}{13}\right)^2} = \frac{\sqrt{13}}{13}$$

$$\theta = \tan^{-1}\left(\frac{\frac{3}{13}}{\frac{-2}{13}}\right) = \tan^{-1}\left(\frac{3}{-2}\right) \approx -0.98$$

$$\frac{1}{3+2i} = \frac{3}{13} - \frac{2}{13}i = \frac{\sqrt{13}}{13}e^{-0.98i}$$ #

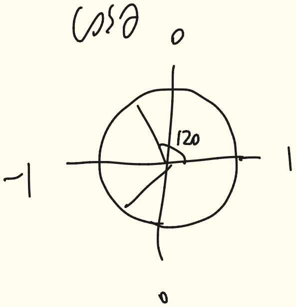
Exercise 1-2 Express the following in the form $a + bi$ (for real a and b) and also in the form $Re^{i\theta}$ (for real R and θ):

(a) $\frac{1}{3+2i}$

(b) $\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^3$

(c) $(-i)^2, (-i)^3, (-i)^4, (-i)^5, \ldots$

(b) $\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^3 = \left(Re^{i\theta}\right)^3$



$$R = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{-\sqrt{3}}{2}\right)^2} = 1$$

$$\theta = \tan^{-1}\left(\frac{\frac{1}{2}}{\frac{-\sqrt{3}}{2}}\right) = -\frac{\pi}{6}$$

$$\left(Re^{i\theta}\right)^4 = R^4 e^{i4\theta} = e^{-\frac{2}{3}\pi i} = \cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)$$

$$= \frac{-1}{2} - \frac{\sqrt{3}}{2}i$$ #

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(a) $\frac{1}{3+2i}$

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(c) $(-i)^2, (-i)^3, (-i)^4, (-i)^5, \ldots$

(c)

$$(-i)^2, (-i)^3, \cdots = Z^n$$

$$Z = -i = Re^{i\theta} = e^{\frac{-\pi}{2}i}$$

$$Z^n = \left(e^{\frac{-\pi}{2}i}\right)^n = e^{i(-1)^n\left(\frac{n\pi}{2}\right)}$$

$$(-i)^n = \underline{\cos\left((-1)^n \frac{n\pi}{2}\right) + i\sin\left((-1)^n \frac{n\pi}{2}\right)} = \underline{e^{i(-1)^n \frac{n\pi}{2}}} \; \#$$

Exercise 1-3 Find all solutions of

(a) $e^z = 1$

(b) $e^z = -i$

(c) $e^z = -1 + i$

(a) $\boxed{e^z = 1}$

$\Rightarrow e^z = e^{a+ib} = e^a e^{ib}$

$1 = Re^{i\theta} = e^{2\pi i}$

$e^a e^{ib} = e^{2\pi i}$

$\Rightarrow a = 0, \ b = 2\pi$

$\Rightarrow z = (2\pi n), \ \forall n \in \mathbb{Z}$

(b) $\boxed{e^z = -i}$

$-i = e^{\frac{3\pi}{2} i}$

$e^z = e^a e^{bi} = e^{\frac{3\pi}{2} i}$

$\Rightarrow a = 0, \ b = \frac{3\pi}{2}$

$\Rightarrow z = (2n\pi + \frac{3\pi}{2}) i, \ \forall n \in \mathbb{Z}$

(c) $\boxed{e^z = -1 + i}$

$-1 + i = Re^{i\theta} = \sqrt{2}(\cos\theta + i\sin\theta) \Rightarrow \begin{cases} \cos\theta = \frac{-1}{\sqrt{2}} \\ \sin\theta = \frac{1}{\sqrt{2}} \end{cases} \Rightarrow \theta = \frac{3}{4}\pi$

$e^z = e^a e^{bi} = \sqrt{2}\, e^{\frac{3\pi}{4} i}$

$\Rightarrow a = \sqrt{2}, \ b = \frac{3\pi}{4}$

$\Rightarrow z = \sqrt{2} + (\frac{3\pi}{4} + 2n\pi) i, \ \forall n \in \mathbb{Z}$

Exercise 1-4 Find all solutions of

(a) $z^3 = i$

(b) $z^3 = -i$

(c) $z^5 = 1$

(d) $z^2 = -1 - i$

(a) $Z^3 = (Re^{i\theta})^3 = R^3 e^{i3\theta}$

$i = Re^{i\theta} = e^{(\frac{\pi}{2}+2n\pi)i}, \ \forall n \in \mathbb{Z}$

$\Rightarrow R^3 e^{i3\theta} = e^{(\frac{\pi}{2}+2n\pi)i}, \ \forall n \in \mathbb{Z}$

$\Rightarrow R=1, \ \theta = \frac{\pi}{6} + \frac{2n\pi}{3}$

$\Rightarrow \underline{Z = e^{(\frac{\pi}{6}+\frac{2n\pi}{3})i} \ \forall n \in \mathbb{Z}}$ #

(b) $Z^3 = (Re^{i\theta})^3 = R^3 e^{i3\theta}$

$-i = Re^{i\theta} = e^{(\frac{3\pi}{2}+2n\pi)i}, \ \forall n \in \mathbb{Z}$

$\Rightarrow R^3 e^{i3\theta} = e^{(\frac{3\pi}{2}+2n\pi)i}, \ \forall n \in \mathbb{Z}$

$\Rightarrow R=1, \ \theta = \frac{\pi}{2} + \frac{2n\pi}{3}$

$\Rightarrow \underline{Z = e^{(\frac{\pi}{2}+\frac{2n\pi}{3})i} \ \forall n \in \mathbb{Z}}$ #

Exercise 1-4 Find all solutions of

(a) $z^3 = i$

(b) $z^3 = -i$

(c) $z^5 = 1$

(d) $z^2 = -1 - i$

(c) $z^5 = (Re^{i\theta})^5 = R^5 e^{i5\theta}$

$1 = Re^{i\theta} = e^{2\pi n i}$, $\forall n \in \mathbb{Z}$

$\Rightarrow R^5 e^{i5\theta} = e^{2\pi n i}$ $\forall n \in \mathbb{Z}$

$\Rightarrow R = 1$, $\theta = \frac{2n\pi}{5}$

$\Rightarrow \underline{z = e^{\frac{2n\pi}{5} i}}$ # $\forall n \in \mathbb{Z}$

(d) $z^2 = (Re^{i\theta})^2 = R^2 e^{i2\theta}$

$-1-i = Re^{i\theta} = \sqrt{2}\, e^{(\frac{5\pi}{4} + 2n\pi) i}$ $\forall n \in \mathbb{Z}$

$\Rightarrow R = \sqrt[4]{2}$, $\theta = \frac{5\pi}{8} + n\pi$ $\forall n \in \mathbb{Z}$

$\Rightarrow \underline{z = \sqrt[4]{2}\, e^{(\frac{5}{8}\pi + n\pi) i}}$ # $\forall n \in \mathbb{Z}$

Exercise 1-6 Find all analytic functions $f = u + iv$ with $u(x, y) = x^2 - y^2$. Simplify the expression $f(z)$ as much as possible.

$$\begin{cases} f = u + iv \\ u(x,y) = x^2 - y^2 \end{cases}$$

Cauchy-Riemann: $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

$$\frac{\partial u}{\partial x} = 2x \to \frac{\partial v}{\partial y} = 2x \to v = 2xy + f(x) + C$$

$$\frac{\partial u}{\partial y} = -2y \to -\frac{\partial v}{\partial x} = -2y \to v = 2xy + f(y) + C$$

$$\underline{v = 2xy + C} \ \#$$

$$f = u + iv = (x^2 - y^2) + i(2xy + C)$$

$$z = x + iy \to x = z - iy$$

$$f(z) = (z - iy)^2 - y^2 + i(2(z - iy)y) + iC$$

$$= z^2 \cancel{-2y^2} \cancel{-2zyi} + \cancel{2yzi} + \cancel{2y^2} + iC$$

$$= z^2 + iC$$

$$\Rightarrow \underline{f(z) = z^2 + iC} \ \#$$