



ROBERT SEDGEWICK | KEVIN WAYNE

<http://algs4.cs.princeton.edu>

## 4.4 SHORTEST PATHS

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- ▶ *APIs*
- ▶ *shortest-paths properties*
- ▶ *Dijkstra's algorithm*
- ▶ *edge-weighted DAGs*
- ▶ *negative weights*

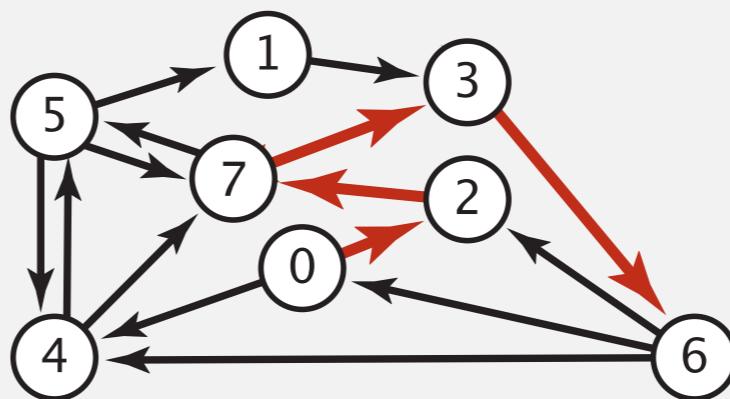
# Shortest paths in an edge-weighted digraph

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Given an edge-weighted digraph, find the shortest path from  $s$  to  $t$ .

edge-weighted digraph

|      |      |
|------|------|
| 4->5 | 0.35 |
| 5->4 | 0.35 |
| 4->7 | 0.37 |
| 5->7 | 0.28 |
| 7->5 | 0.28 |
| 5->1 | 0.32 |
| 0->4 | 0.38 |
| 0->2 | 0.26 |
| 7->3 | 0.39 |
| 1->3 | 0.29 |
| 2->7 | 0.34 |
| 6->2 | 0.40 |
| 3->6 | 0.52 |
| 6->0 | 0.58 |
| 6->4 | 0.93 |

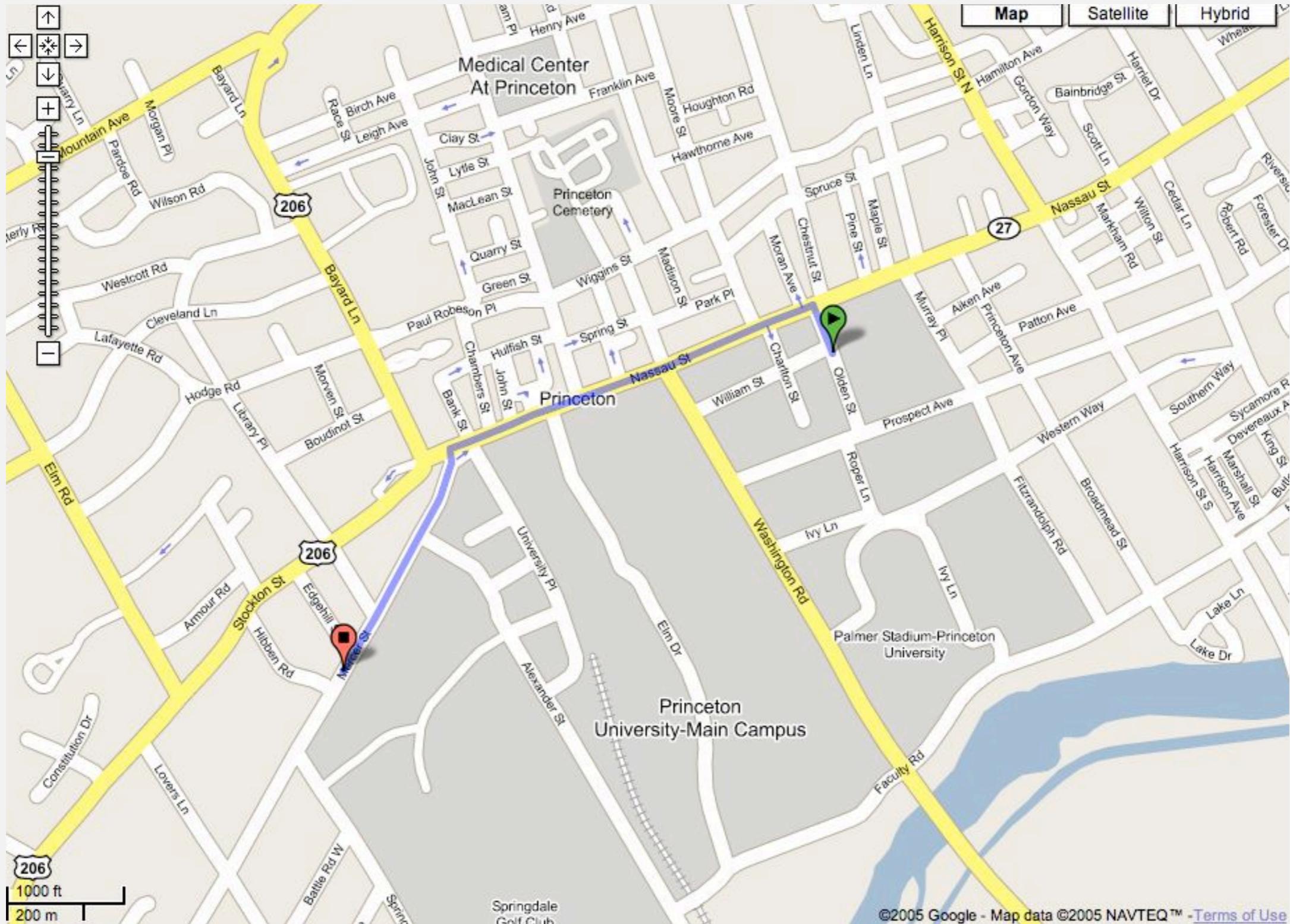


shortest path from 0 to 6

|      |      |
|------|------|
| 0->2 | 0.26 |
| 2->7 | 0.34 |
| 7->3 | 0.39 |
| 3->6 | 0.52 |

# Google maps

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# Car navigation

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# Shortest path applications

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- PERT/CPM.
- Map routing.
- Seam carving.
- Robot navigation.
- Texture mapping.
- Typesetting in TeX.
- Urban traffic planning.
- Optimal pipelining of VLSI chip.
- Telemarketer operator scheduling.
- Routing of telecommunications messages.
- Network routing protocols (OSPF, BGP, RIP).
- Exploiting arbitrage opportunities in currency exchange.
- Optimal truck routing through given traffic congestion pattern.



[http://en.wikipedia.org/wiki/Seam\\_carving](http://en.wikipedia.org/wiki/Seam_carving)



Reference: Network Flows: Theory, Algorithms, and Applications, R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, Prentice Hall, 1993.

# Shortest path variants

---

## Which vertices?

- Single source: from one vertex  $s$  to every other vertex.
- Source-sink: from one vertex  $s$  to another  $t$ .
- All pairs: between all pairs of vertices.

## Restrictions on edge weights?

- Nonnegative weights.
- Euclidean weights.
- Arbitrary weights.

## Cycles?

- No directed cycles.
- No "negative cycles."

**Simplifying assumption.** Shortest paths from  $s$  to each vertex  $v$  exist.

# Algorithms

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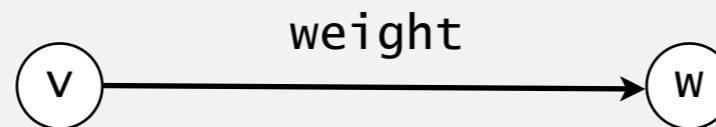
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# Weighted directed edge API

```
public class DirectedEdge  
  
    DirectedEdge(int v, int w, double weight)      weighted edge v→w  
  
    int from()                                     vertex v  
  
    int to()                                       vertex w  
  
    double weight()                                weight of this edge  
  
    String toString()                             string representation
```



Idiom for processing an edge e: `int v = e.from(), w = e.to();`

# Weighted directed edge: implementation in Java

Similar to Edge for undirected graphs, but a bit simpler.

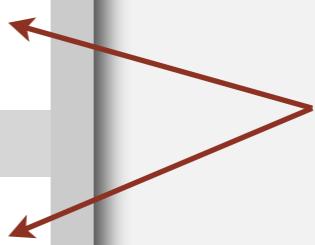
```
public class DirectedEdge
{
    private final int v, w;
    private final double weight;

    public DirectedEdge(int v, int w, double weight)
    {
        this.v = v;
        this.w = w;
        this.weight = weight;
    }

    public int from()
    {   return v;   }

    public int to()
    {   return w;   }

    public int weight()
    {   return weight;   }
}
```



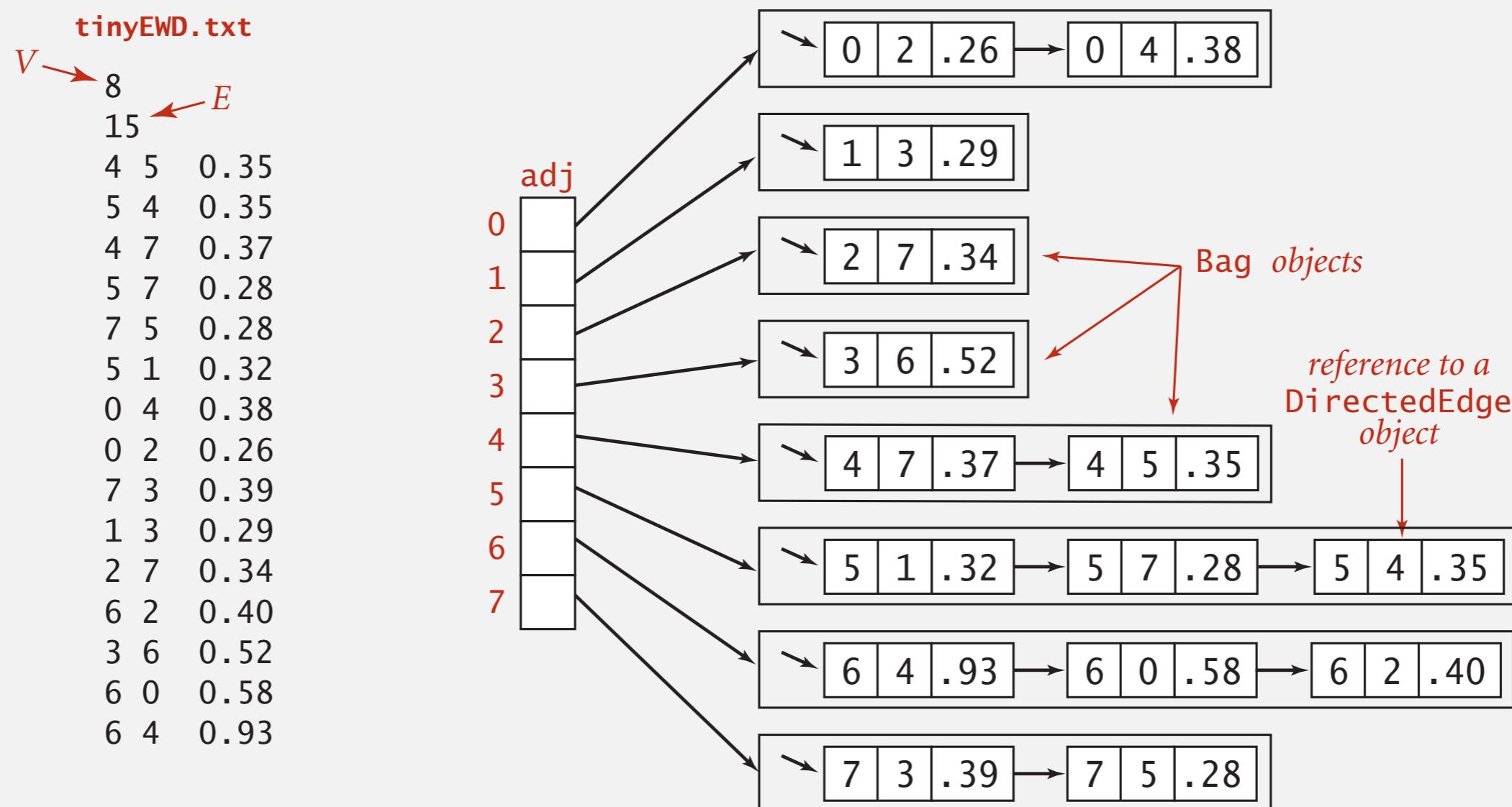
from() and to() replace  
either() and other()

# Edge-weighted digraph API

|                                   |   |
|-----------------------------------|---|
| public class EdgeWeightedDigraph  |   |
| EdgeWeightedDigraph(int V)        | <i>edge-weighted digraph with <math>V</math> vertices</i> |
| EdgeWeightedDigraph(In in)        | <i>edge-weighted digraph from input stream</i>            |
| void addEdge(DirectedEdge e)      | <i>add weighted directed edge e</i>                       |
| Iterable<DirectedEdge> adj(int v) | <i>edges pointing from <math>v</math></i>                 |
| int V()                           | <i>number of vertices</i>                                 |
| int E()                           | <i>number of edges</i>                                    |
| Iterable<DirectedEdge> edges()    | <i>all edges</i>  |
| String toString()                 | <i>string representation</i>                              |

**Conventions.** Allow self-loops and parallel edges.

# Edge-weighted digraph: adjacency-lists representation



# Edge-weighted digraph: adjacency-lists implementation in Java

Same as EdgeWeightedGraph except replace Graph with Digraph.

```
public class EdgeWeightedDigraph
{
    private final int V;
    private final Bag<DirectedEdge>[] adj;

    public EdgeWeightedDigraph(int V)
    {
        this.V = V;
        adj = (Bag<DirectedEdge>[]) new Bag[V];
        for (int v = 0; v < V; v++)
            adj[v] = new Bag<DirectedEdge>();
    }

    public void addEdge(DirectedEdge e)
    {
        int v = e.from();
        adj[v].add(e);
    }

    public Iterable<DirectedEdge> adj(int v)
    { return adj[v]; }
}
```

←  
add edge  $e = v \rightarrow w$  to  
only  $v$ 's adjacency list

# Single-source shortest paths API

Goal. Find the shortest path from  $s$  to every other vertex.

```
public class SP
```

```
    SP(EdgeWeightedDigraph G, int s) shortest paths from s in graph G
```

```
    double distTo(int v) length of shortest path from s to v
```

```
    Iterable <DirectedEdge> pathTo(int v) shortest path from s to v
```

```
    boolean hasPathTo(int v) is there a path from s to v?
```

```
SP sp = new SP(G, s);
for (int v = 0; v < G.V(); v++)
{
    StdOut.printf("%d to %d (%.2f): ", s, v, sp.distTo(v));
    for (DirectedEdge e : sp.pathTo(v))
        StdOut.print(e + " ");
    StdOut.println();
}
```

# Single-source shortest paths API

**Goal.** Find the shortest path from  $s$  to every other vertex.

```
public class SP
```

```
    SP(EdgeWeightedDigraph G, int s) shortest paths from s in graph G
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```
    double distTo(int v) length of shortest path from s to v
```

```
    Iterable <DirectedEdge> pathTo(int v) shortest path from s to v
```

```
    boolean hasPathTo(int v) is there a path from s to v?
```

```
% java SP tinyEWG.txt 0
0 to 0 (0.00):
0 to 1 (1.05): 0->4 0.38  4->5 0.35  5->1 0.32
0 to 2 (0.26): 0->2 0.26
0 to 3 (0.99): 0->2 0.26  2->7 0.34  7->3 0.39
0 to 4 (0.38): 0->4 0.38
0 to 5 (0.73): 0->4 0.38  4->5 0.35
0 to 6 (1.51): 0->2 0.26  2->7 0.34  7->3 0.39  3->6 0.52
0 to 7 (0.60): 0->2 0.26  2->7 0.34
```

# Algorithms

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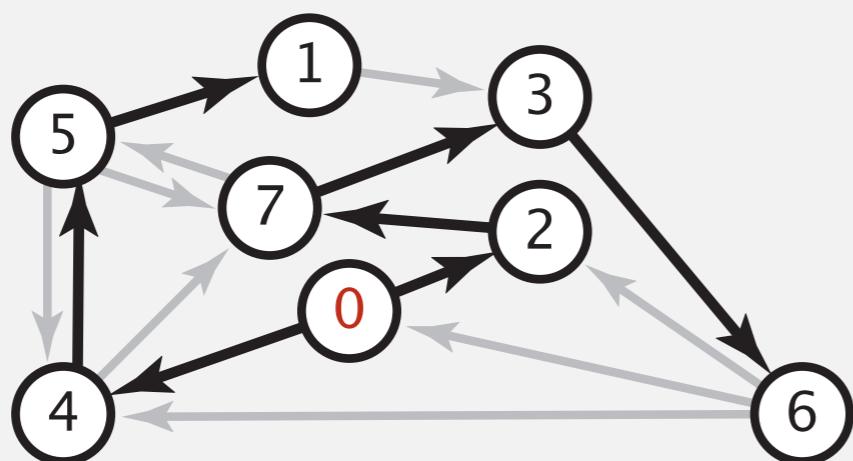
# Data structures for single-source shortest paths

Goal. Find the shortest path from  $s$  to every other vertex.

Observation. A shortest-paths tree (SPT) solution exists. Why?

Consequence. Can represent the SPT with two vertex-indexed arrays:

- $\text{distTo}[v]$  is length of shortest path from  $s$  to  $v$ .
- $\text{edgeTo}[v]$  is last edge on shortest path from  $s$  to  $v$ .



shortest-paths tree from 0

|   | edgeTo[] | distTo[] |
|---|----------|----------|
| 0 | null     | 0        |
| 1 | 5->1     | 0.32     |
| 2 | 0->2     | 0.26     |
| 3 | 7->3     | 0.37     |
| 4 | 0->4     | 0.38     |
| 5 | 4->5     | 0.35     |
| 6 | 3->6     | 0.52     |
| 7 | 2->7     | 0.34     |

parent-link representation

# Data structures for single-source shortest paths

---

**Goal.** Find the shortest path from  $s$  to every other vertex.

**Observation.** A **shortest-paths tree** (SPT) solution exists. Why?

**Consequence.** Can represent the SPT with two vertex-indexed arrays:

- $\text{distTo}[v]$  is length of shortest path from  $s$  to  $v$ .
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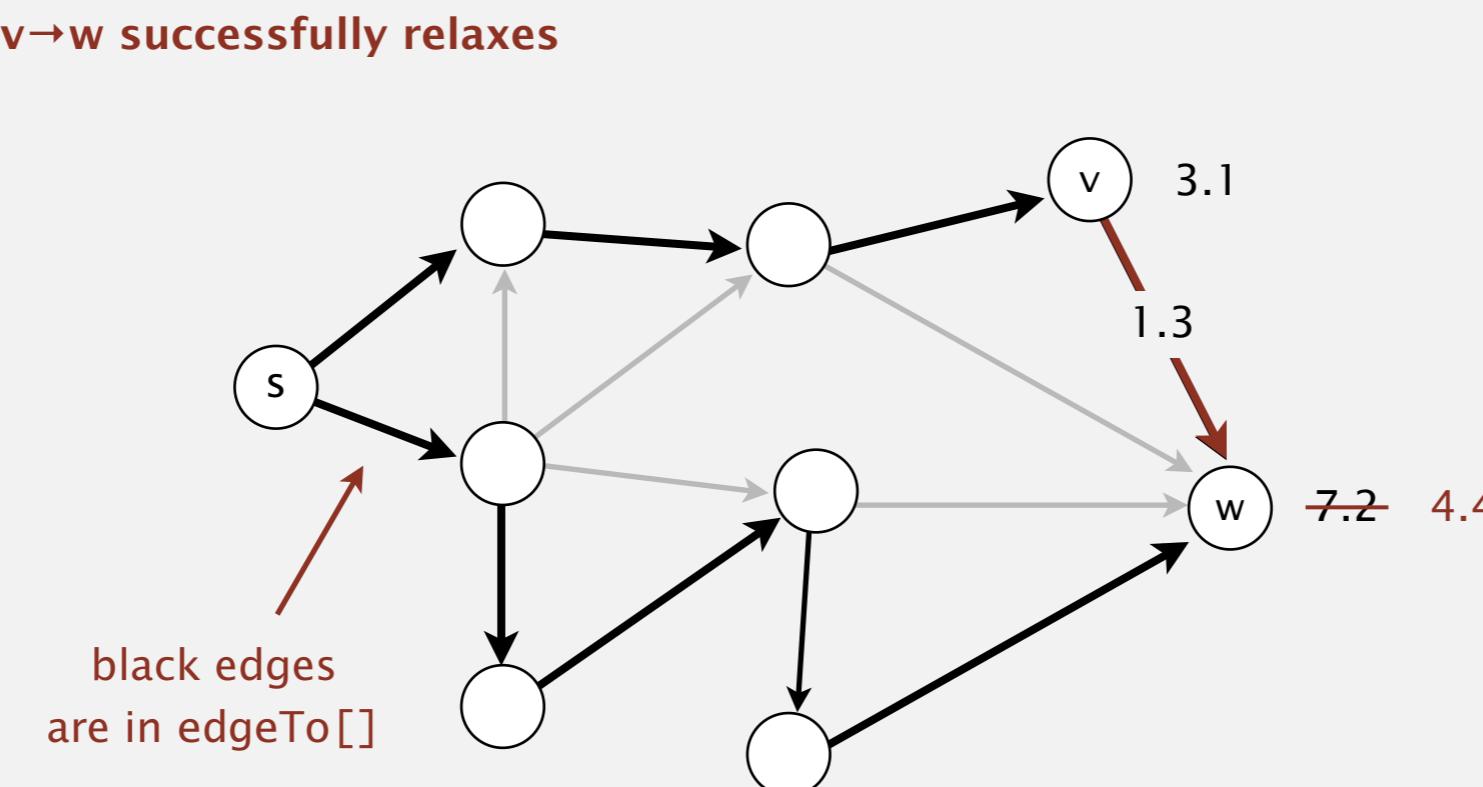
```
public double distTo(int v)
{   return distTo[v];  }

public Iterable<DirectedEdge> pathTo(int v)
{
    Stack<DirectedEdge> path = new Stack<DirectedEdge>();
    for (DirectedEdge e = edgeTo[v]; e != null; e = edgeTo[e.from()])
        path.push(e);
    return path;
}
```

# Edge relaxation

Relax edge  $e = v \rightarrow w$ .

- $\text{distTo}[v]$  is length of shortest **known** path from  $s$  to  $v$ .
- $\text{distTo}[w]$  is length of shortest **known** path from  $s$  to  $w$ .
- $\text{edgeTo}[w]$  is last edge on shortest **known** path from  $s$  to  $w$ .
- If  $e = v \rightarrow w$  gives shorter path to  $w$  through  $v$ ,  
update both  $\text{distTo}[w]$  and  $\text{edgeTo}[w]$ .



# Edge relaxation

---

## Relax edge $e = v \rightarrow w$ .

- $\text{distTo}[v]$  is length of shortest **known** path from  $s$  to  $v$ .
- $\text{distTo}[w]$  is length of shortest **known** path from  $s$  to  $w$ .
- $\text{edgeTo}[w]$  is last edge on shortest **known** path from  $s$  to  $w$ .
- If  $e = v \rightarrow w$  gives shorter path to  $w$  through  $v$ ,  
update both  $\text{distTo}[w]$  and  $\text{edgeTo}[w]$ .

```
private void relax(DirectedEdge e)
{
    int v = e.from(), w = e.to();
    if (distTo[w] > distTo[v] + e.weight())
    {
        distTo[w] = distTo[v] + e.weight();
        edgeTo[w] = e;
    }
}
```

# Shortest-paths optimality conditions

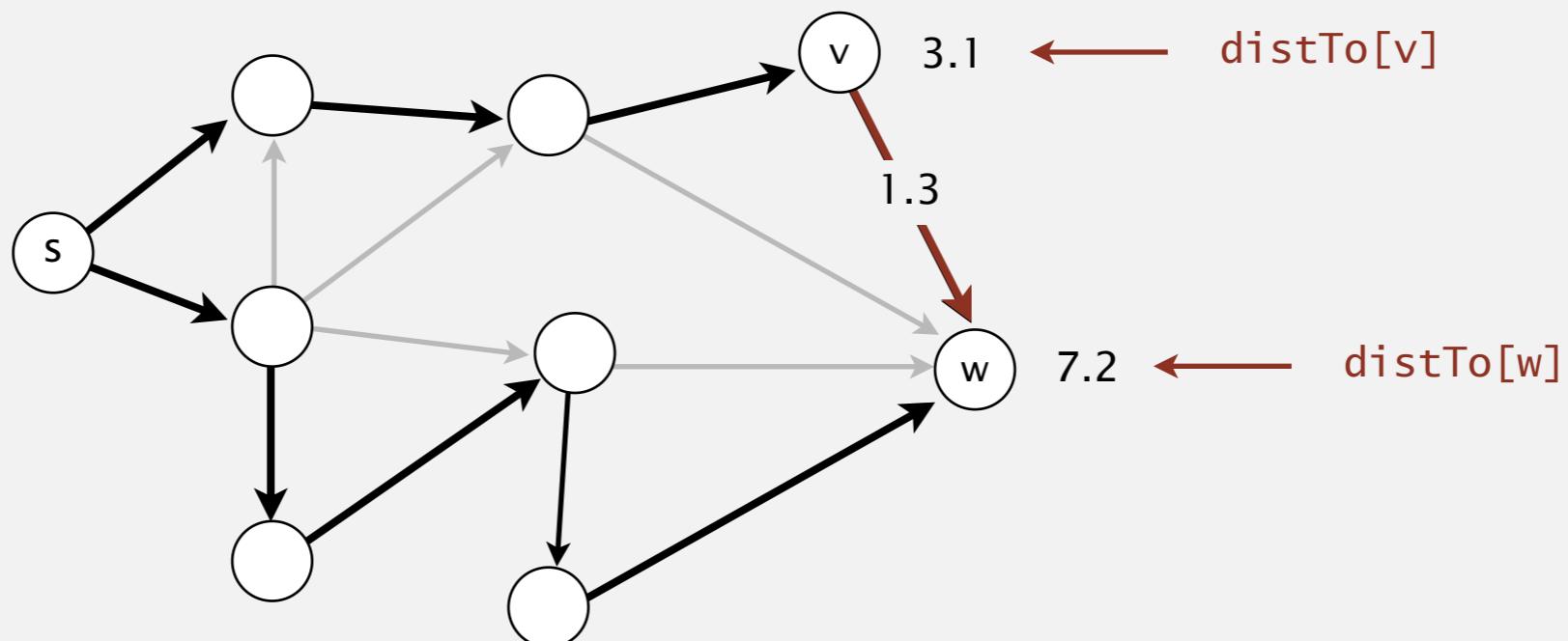
**Proposition.** Let  $G$  be an edge-weighted digraph.

Then  $\text{distTo}[]$  are the shortest path distances from  $s$  iff:

- $\text{distTo}[s] = 0$ .
- For each vertex  $v$ ,  $\text{distTo}[v]$  is the length of some path from  $s$  to  $v$ .
- For each edge  $e = v \rightarrow w$ ,  $\text{distTo}[w] \leq \text{distTo}[v] + e.\text{weight}()$ .

**Pf.**  $\Leftarrow$  [ necessary ]

- Suppose that  $\text{distTo}[w] > \text{distTo}[v] + e.\text{weight}()$  for some edge  $e = v \rightarrow w$ .
- Then,  $e$  gives a path from  $s$  to  $w$  (through  $v$ ) of length less than  $\text{distTo}[w]$ .



## Shortest-paths optimality conditions

**Proposition.** Let  $G$  be an edge-weighted digraph.

Then  $\text{distTo}[]$  are the shortest path distances from  $s$  iff:

- $\text{distTo}[s] = 0$ .
- For each vertex  $v$ ,  $\text{distTo}[v]$  is the length of some path from  $s$  to  $v$ .
- For each edge  $e = v \rightarrow w$ ,  $\text{distTo}[w] \leq \text{distTo}[v] + e.\text{weight}()$ .

**Pf.**  $\Rightarrow$  [ sufficient ]

- Suppose that  $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_k = w$  is a shortest path from  $s$  to  $w$ .
- Then,  $\text{distTo}[v_1] \leq \text{distTo}[v_0] + e_1.\text{weight}()$   
 $\text{distTo}[v_2] \leq \text{distTo}[v_1] + e_2.\text{weight}()$   
 $\dots$   
 $\text{distTo}[v_k] \leq \text{distTo}[v_{k-1}] + e_k.\text{weight}()$

$e_i = i^{\text{th}}$  edge on shortest path from  $s$  to  $w$

- Add inequalities; simplify; and substitute  $\text{distTo}[v_0] = \text{distTo}[s] = 0$ :

$$\text{distTo}[w] = \text{distTo}[v_k] \leq e_1.\text{weight}() + e_2.\text{weight}() + \dots + e_k.\text{weight}()$$

weight of shortest path from  $s$  to  $w$

- Thus,  $\text{distTo}[w]$  is the weight of shortest path to  $w$ . ■

weight of some path from  $s$  to  $w$

# Generic shortest-paths algorithm

---

## Generic algorithm (to compute SPT from $s$ )

---

**Initialize  $\text{distTo}[s] = 0$  and  $\text{distTo}[v] = \infty$  for all other vertices.**

**Repeat until optimality conditions are satisfied:**

- Relax any edge.**
- 

**Proposition.** Generic algorithm computes SPT (if it exists) from  $s$ .

**Pf sketch.**

- Throughout algorithm,  $\text{distTo}[v]$  is the length of a simple path from  $s$  to  $v$  (and  $\text{edgeTo}[v]$  is last edge on path).
- Each successful relaxation decreases  $\text{distTo}[v]$  for some  $v$ .
- The entry  $\text{distTo}[v]$  can decrease at most a finite number of times. ■

# Generic shortest-paths algorithm

---

## Generic algorithm (to compute SPT from s)

---

**Initialize  $\text{distTo}[s] = 0$  and  $\text{distTo}[v] = \infty$  for all other vertices.**

**Repeat until optimality conditions are satisfied:**

- Relax any edge.
- 

**Efficient implementations.** How to choose which edge to relax?

Ex 1. Dijkstra's algorithm (nonnegative weights).

Ex 2. Topological sort algorithm (no directed cycles).

Ex 3. Bellman-Ford algorithm (no negative cycles).

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# Edsger W. Dijkstra: select quotes

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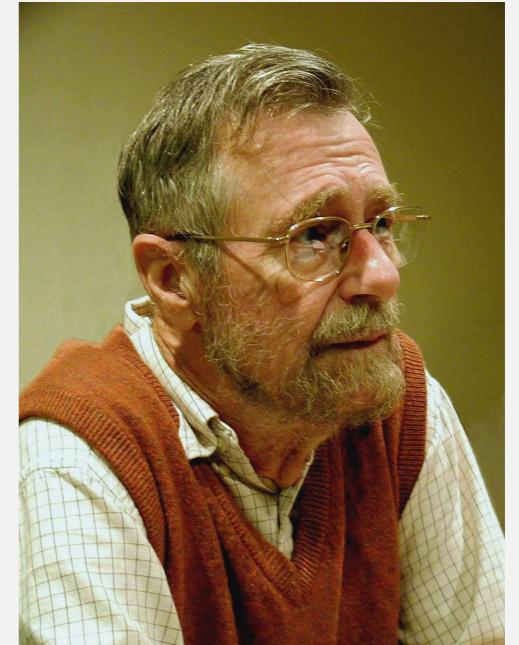
*“Do only what only you can do.”*

*“In their capacity as a tool, computers will be but a ripple on the surface of our culture. In their capacity as intellectual challenge, they are without precedent in the cultural history of mankind.”*

*“The use of COBOL cripples the mind; its teaching should, therefore, be regarded as a criminal offence.”*

*“It is practically impossible to teach good programming to students that have had a prior exposure to BASIC: as potential programmers they are mentally mutilated beyond hope of regeneration.”*

*“APL is a mistake, carried through to perfection. It is the language of the future for the programming techniques of the past: it creates a new generation of coding bums.”*



**Edsger W. Dijkstra**  
**Turing award 1972**

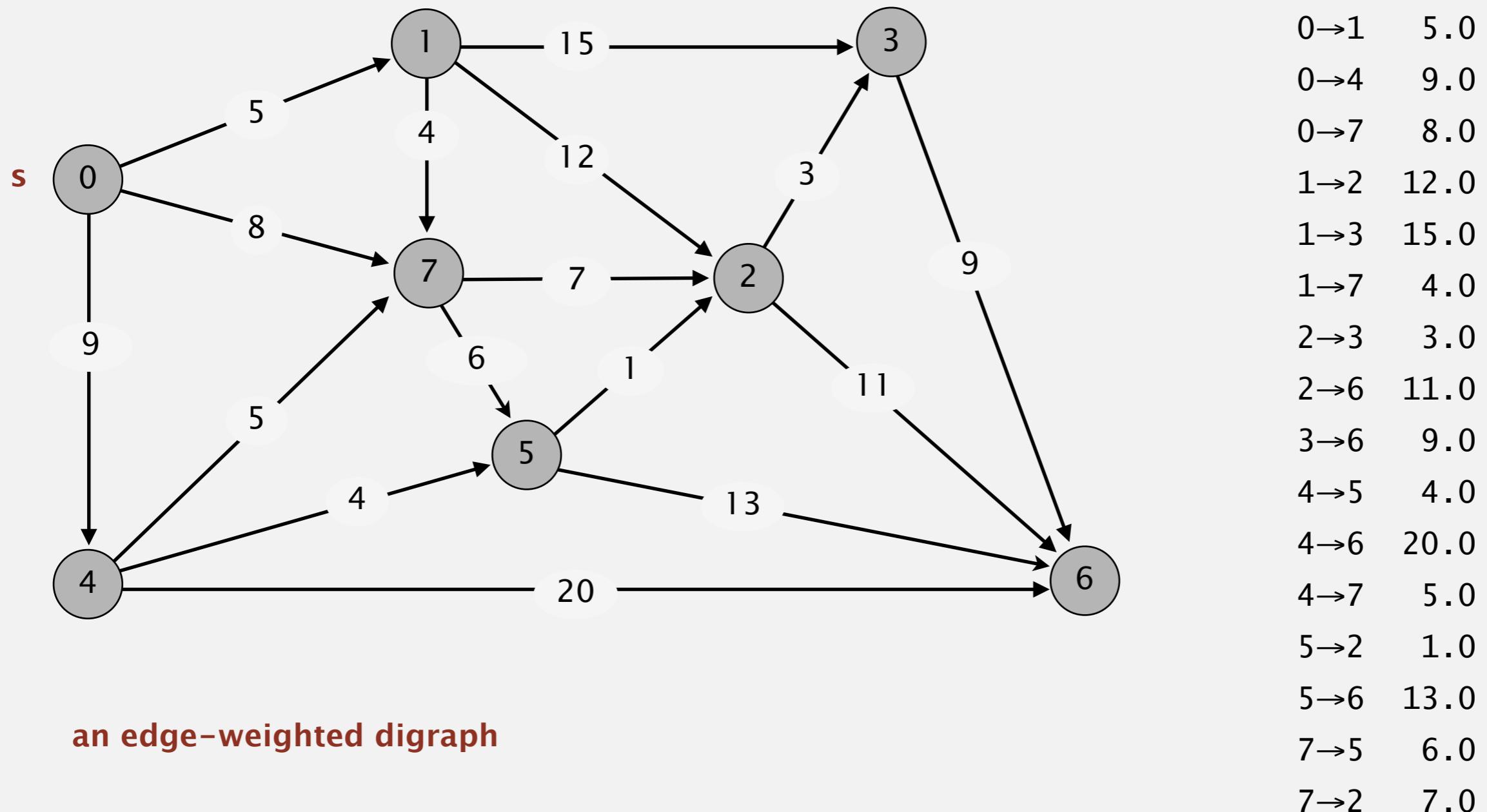
## Edsger W. Dijkstra: select quotes

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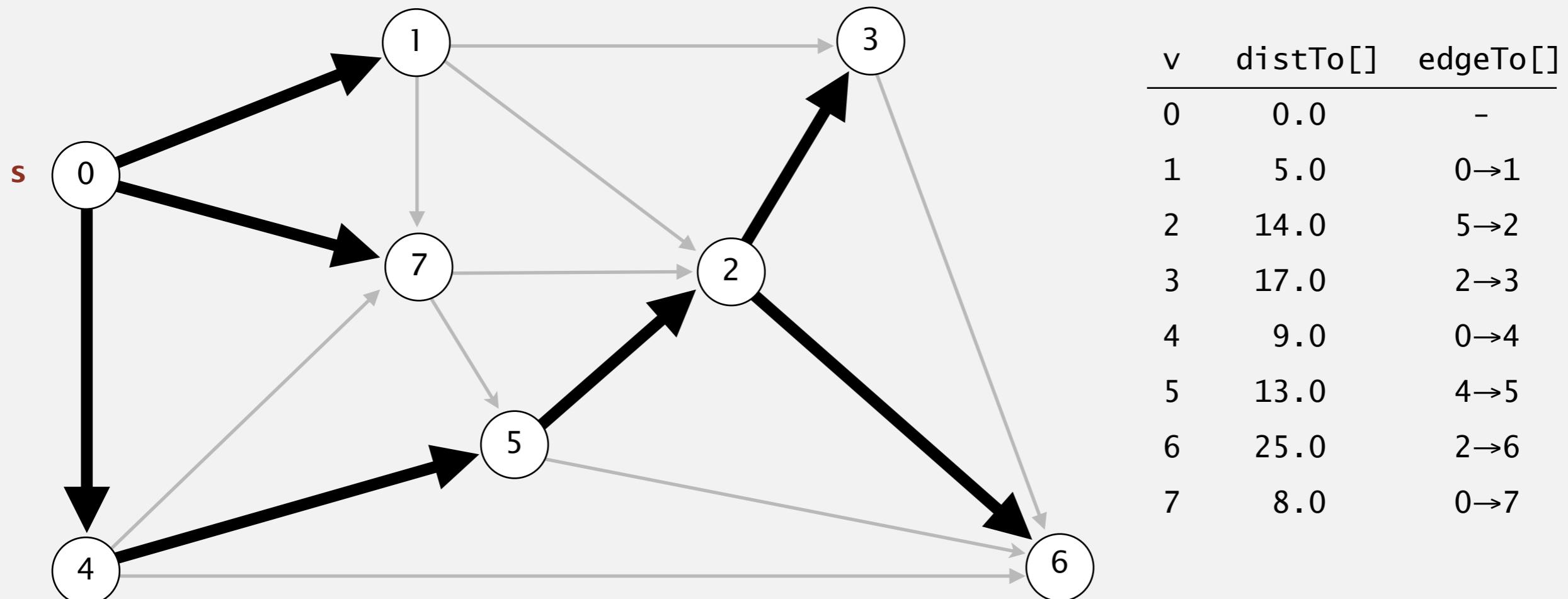
# Dijkstra's algorithm demo

- Consider vertices in increasing order of distance from  $s$  (non-tree vertex with the lowest  $\text{distTo}[]$  value).
- Add vertex to tree and relax all edges pointing from that vertex.



# Dijkstra's algorithm demo

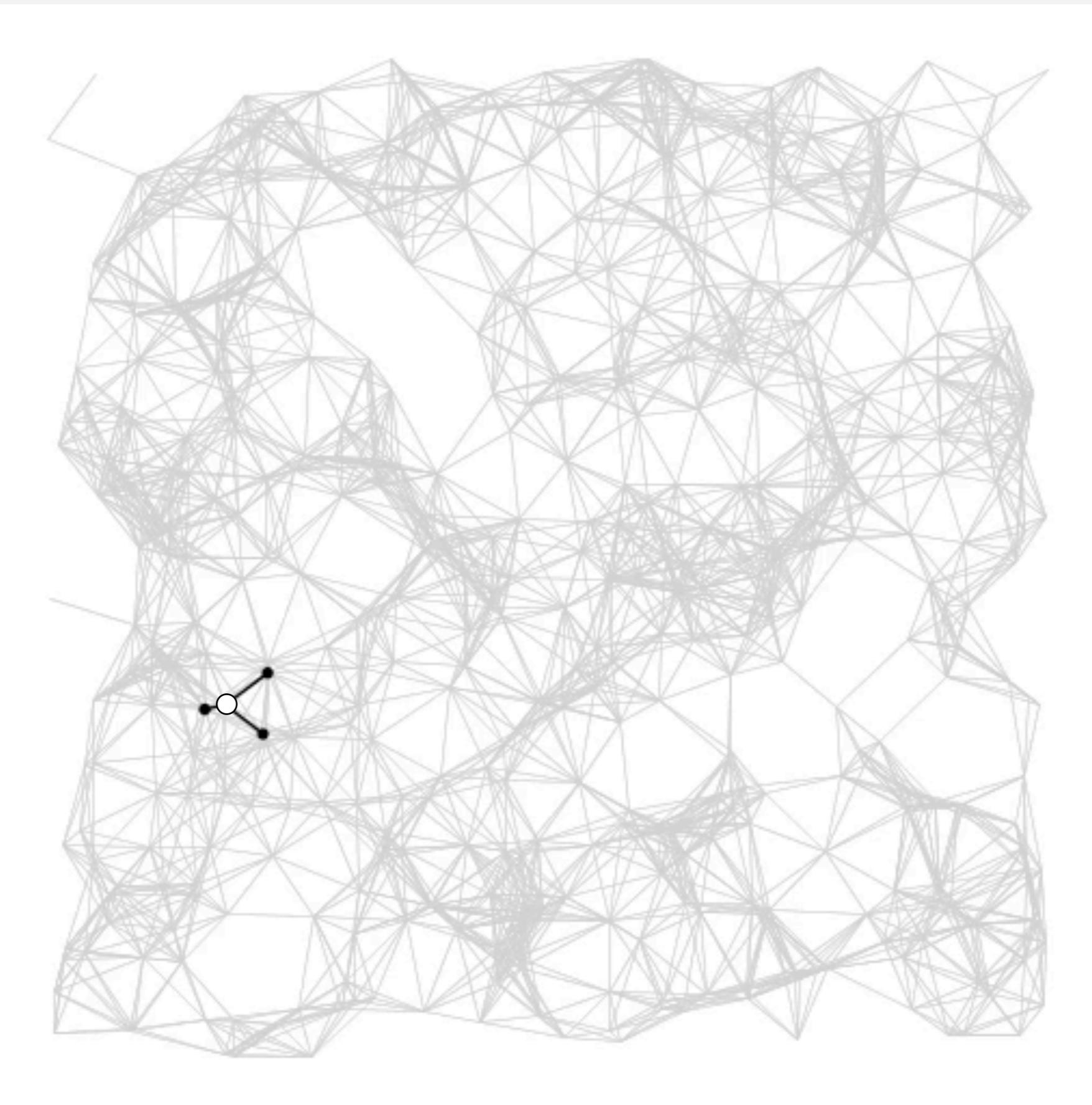
- Consider vertices in increasing order of distance from  $s$  (non-tree vertex with the lowest  $\text{distTo}[]$  value).
- Add vertex to tree and relax all edges pointing from that vertex.



shortest-paths tree from vertex s

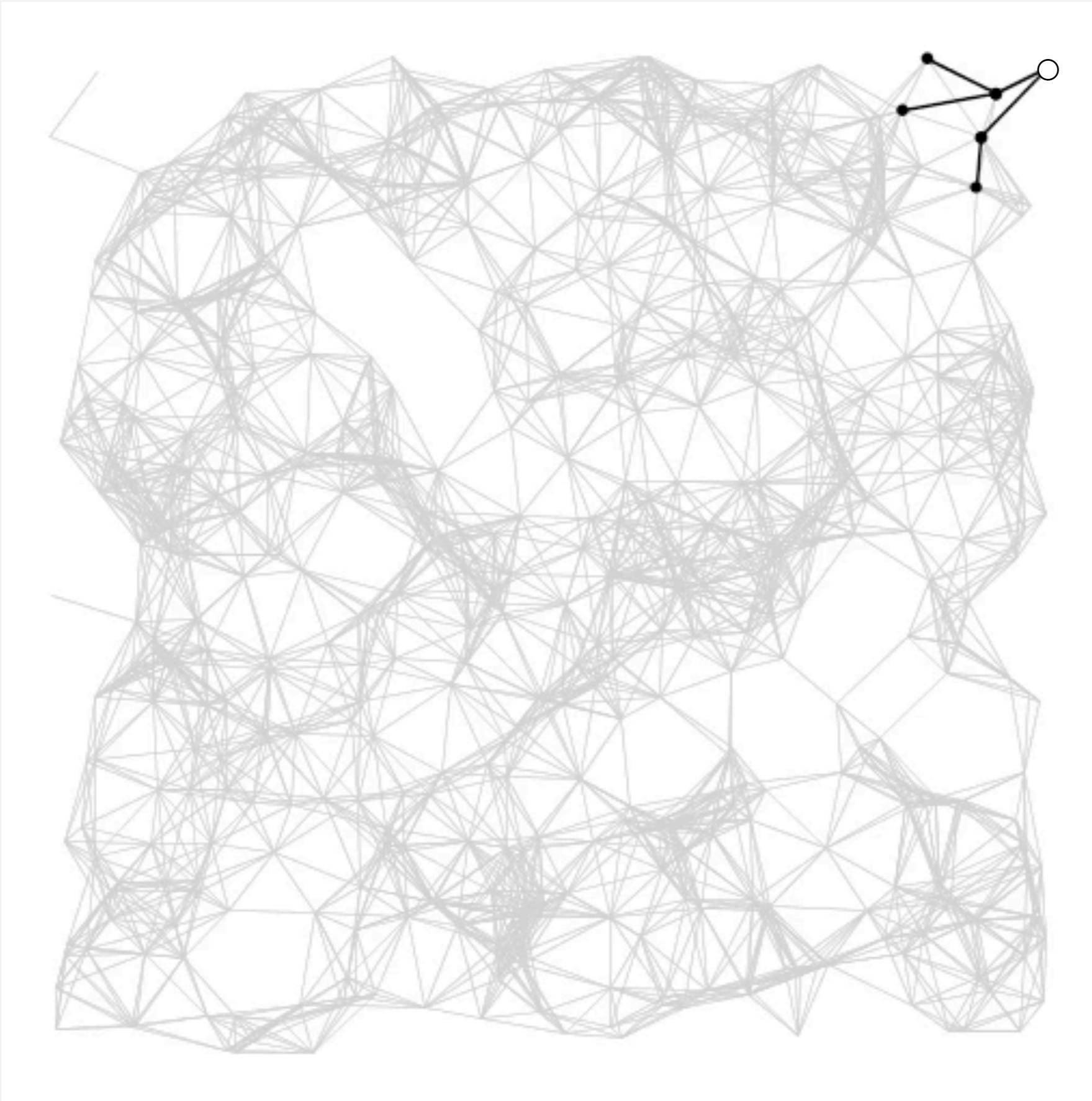
# Dijkstra's algorithm visualization

---



# Dijkstra's algorithm visualization

---



## Dijkstra's algorithm: correctness proof

---

**Proposition.** Dijkstra's algorithm computes a SPT in any edge-weighted digraph with nonnegative weights.

Pf.

- Each edge  $e = v \rightarrow w$  is relaxed exactly once (when  $v$  is relaxed), leaving  $\text{distTo}[w] \leq \text{distTo}[v] + e.\text{weight}()$ .
- Inequality holds until algorithm terminates because:
  - $\text{distTo}[w]$  cannot increase ← distTo[] values are monotone decreasing
  - $\text{distTo}[v]$  will not change ← we choose lowest distTo[] value at each step  
(and edge weights are nonnegative)
- Thus, upon termination, shortest-paths optimality conditions hold. ■

# Dijkstra's algorithm: Java implementation

```
public class DijkstraSP
{
    private DirectedEdge[] edgeTo;
    private double[] distTo;
    private IndexMinPQ<Double> pq;

    public DijkstraSP(EdgeWeightedDigraph G, int s)
    {
        edgeTo = new DirectedEdge[G.V()];
        distTo = new double[G.V()];
        pq = new IndexMinPQ<Double>(G.V());

        for (int v = 0; v < G.V(); v++)
            distTo[v] = Double.POSITIVE_INFINITY;
        distTo[s] = 0.0;

        pq.insert(s, 0.0);
        while (!pq.isEmpty())
        {
            int v = pq.delMin();
            for (DirectedEdge e : G.adj(v))
                relax(e);
        }
    }
}
```

←  
relax vertices in order  
of distance from s

# Dijkstra's algorithm: Java implementation

---

```
private void relax(DirectedEdge e)
{
    int v = e.from(), w = e.to();
    if (distTo[w] > distTo[v] + e.weight())
    {
        distTo[w] = distTo[v] + e.weight();
        edgeTo[w] = e;
        if (pq.contains(w)) pq.decreaseKey(w, distTo[w]);
        else                  pq.insert      (w, distTo[w]);
    }
}
```



update PQ

# Computing spanning trees in graphs

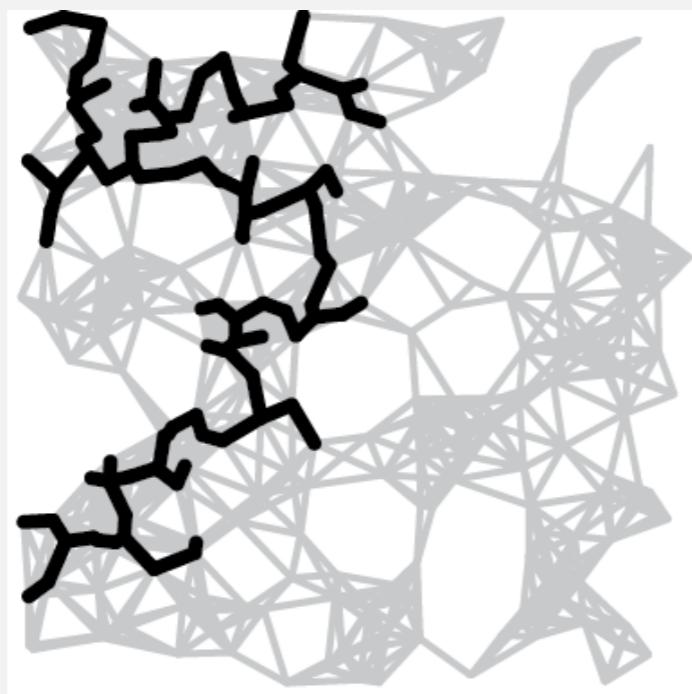
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Dijkstra's algorithm seem familiar?

- Prim's algorithm is essentially the same algorithm.
- Both are in a family of algorithms that compute a graph's spanning tree.

Main distinction: Rule used to choose next vertex for the tree.

- Prim's: Closest vertex to the **tree** (via an undirected edge).
- Dijkstra's: Closest vertex to the **source** (via a directed path).



Note: DFS and BFS are also in this family of algorithms.

# Dijkstra's algorithm: which priority queue?

Depends on PQ implementation:  $V$  insert,  $V$  delete-min,  $E$  decrease-key.

| PQ implementation                               | insert      | delete-min       | decrease-key | total            |
|---|-------------|------------------|--------------|------------------|
| <b>unordered array</b>                          | 1           | $V$              | 1            | $V^2$            |
| <b>binary heap</b>                              | $\log V$    | $\log V$         | $\log V$     | $E \log V$       |
| <b>d-way heap<br/>(Johnson 1975)</b>            | $\log_d V$  | $d \log_d V$     | $\log_d V$   | $E \log_{E/V} V$ |
| <b>Fibonacci heap<br/>(Fredman-Tarjan 1984)</b> | $1^\dagger$ | $\log V^\dagger$ | $1^\dagger$  | $E + V \log V$   |

$\dagger$  amortized

Bottom line.

- Array implementation optimal for dense graphs.
- Binary heap much faster for sparse graphs.
- 4-way heap worth the trouble in performance-critical situations.
- Fibonacci heap best in theory, but not worth implementing.

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## Acyclic edge-weighted digraphs

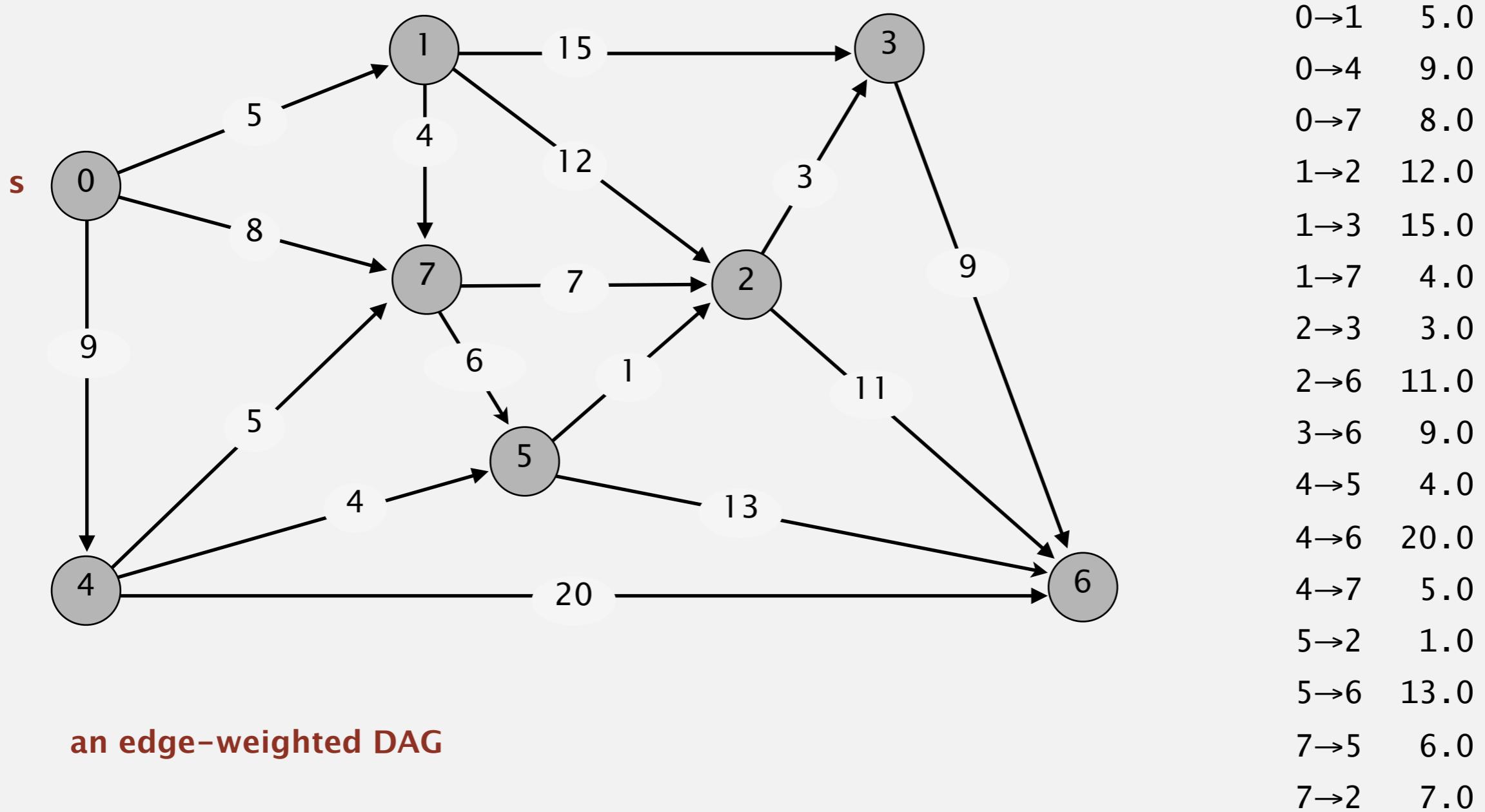
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**Q.** Suppose that an edge-weighted digraph has no directed cycles.  
Is it easier to find shortest paths than in a general digraph?

**A.** Yes!

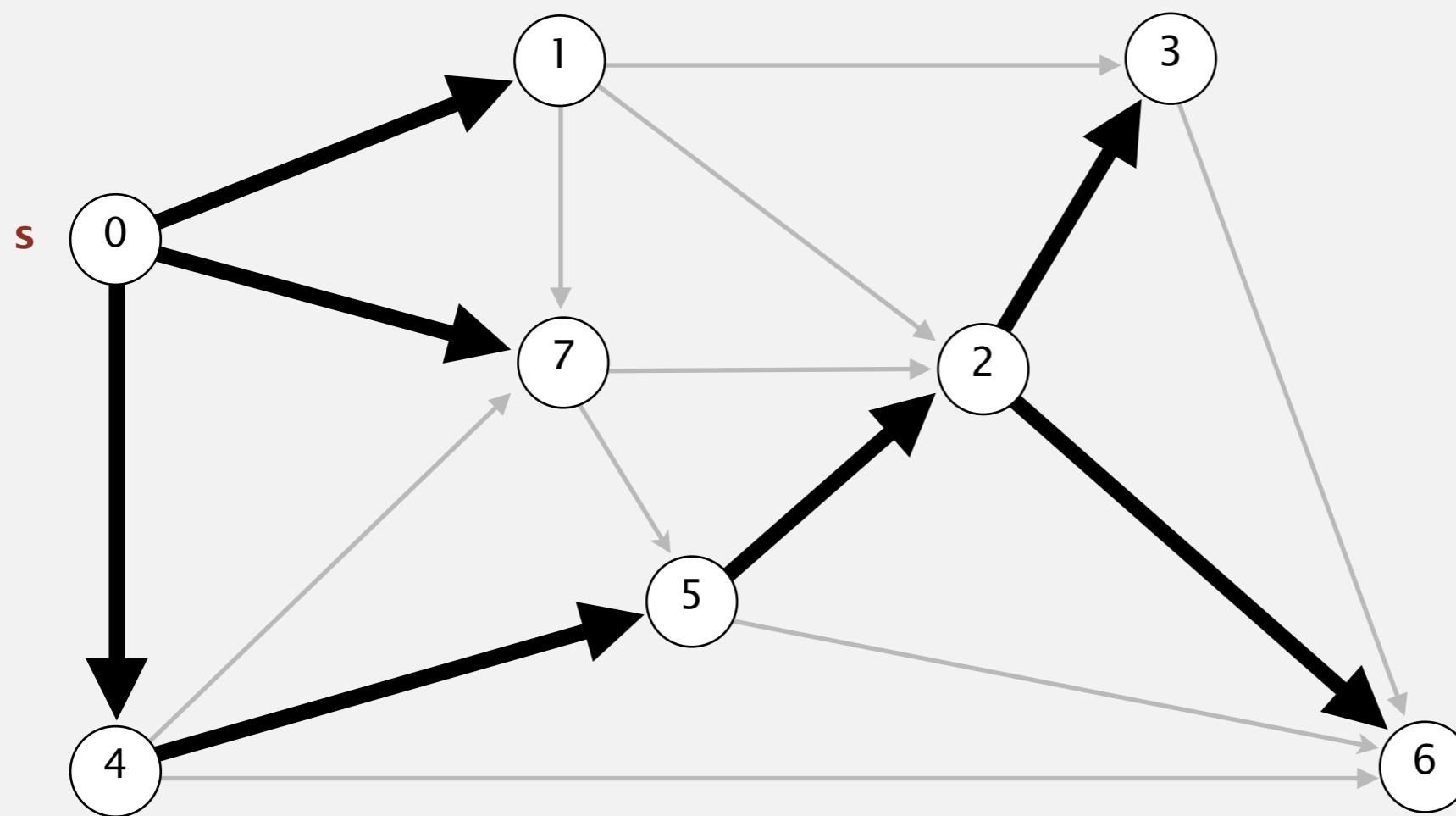
# Acyclic shortest paths demo

- Consider vertices in topological order.
- Relax all edges pointing from that vertex.



# Acyclic shortest paths demo

- Consider vertices in topological order.
- Relax all edges pointing from that vertex.



shortest-paths tree from vertex s

| v | distTo[] | edgeTo[] |
|---|----------|----------|
| 0 | 0.0      | -        |
| 1 | 5.0      | 0→1      |
| 2 | 14.0     | 5→2      |
| 3 | 17.0     | 2→3      |
| 4 | 9.0      | 0→4      |
| 5 | 13.0     | 4→5      |
| 6 | 25.0     | 2→6      |
| 7 | 8.0      | 0→7      |

# Shortest paths in edge-weighted DAGs

---

**Proposition.** Topological sort algorithm computes SPT in any edge-weighted DAG in time proportional to  $E + V$ .

edge weights  
can be negative!

Pf.

- Each edge  $e = v \rightarrow w$  is relaxed exactly once (when  $v$  is relaxed), leaving  $\text{distTo}[w] \leq \text{distTo}[v] + e.\text{weight}()$ .
- Inequality holds until algorithm terminates because:
  - $\text{distTo}[w]$  cannot increase ←  $\text{distTo}[]$  values are monotone decreasing
  - $\text{distTo}[v]$  will not change ← because of topological order, no edge pointing to  $v$  will be relaxed after  $v$  is relaxed
- Thus, upon termination, shortest-paths optimality conditions hold. ■

# Shortest paths in edge-weighted DAGs

```
public class AcyclicSP
{
    private DirectedEdge[] edgeTo;
    private double[] distTo;

    public AcyclicSP(EdgeweightedDigraph G, int s)
    {
        edgeTo = new DirectedEdge[G.V()];
        distTo = new double[G.V()];

        for (int v = 0; v < G.V(); v++)
            distTo[v] = Double.POSITIVE_INFINITY;
        distTo[s] = 0.0;

        Topological topological = new Topological(G); ← topological order
        for (int v : topological.order())
            for (DirectedEdge e : G.adj(v))
                relax(e);
    }
}
```

# Content-aware resizing

---

**Seam carving.** [Avidan and Shamir] Resize an image without distortion for display on cell phones and web browsers.

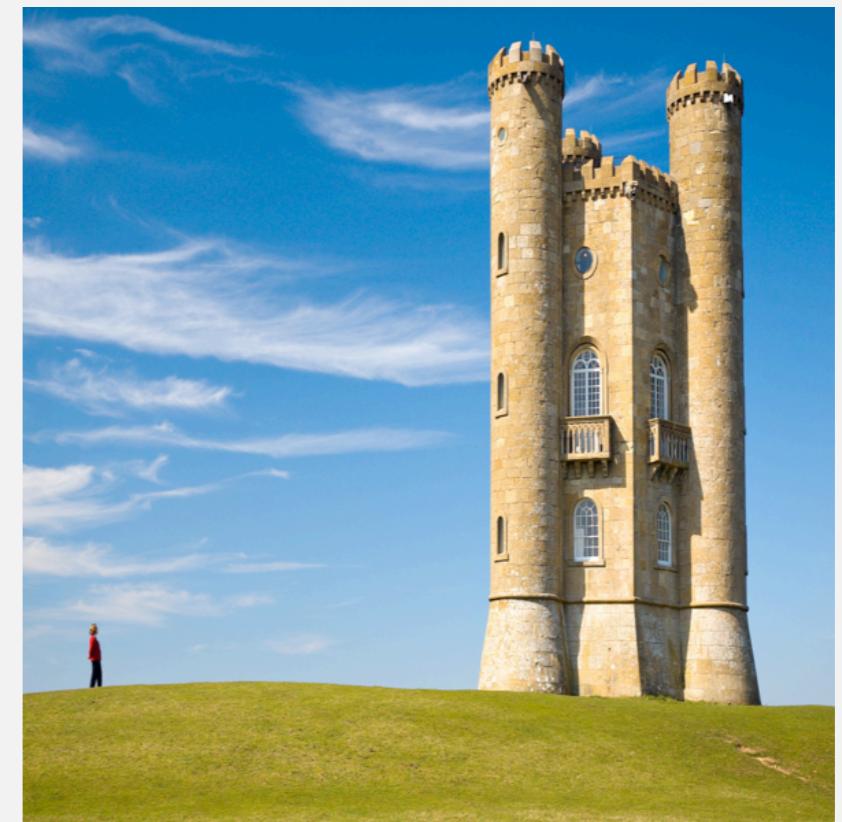


<http://www.youtube.com/watch?v=vIFCV2spKtg>

# Content-aware resizing

---

[Seam carving.](#) [Avidan and Shamir] Resize an image without distortion for display on cell phones and web browsers.



[In the wild.](#) Photoshop CS 5, Imagemagick, GIMP, ...

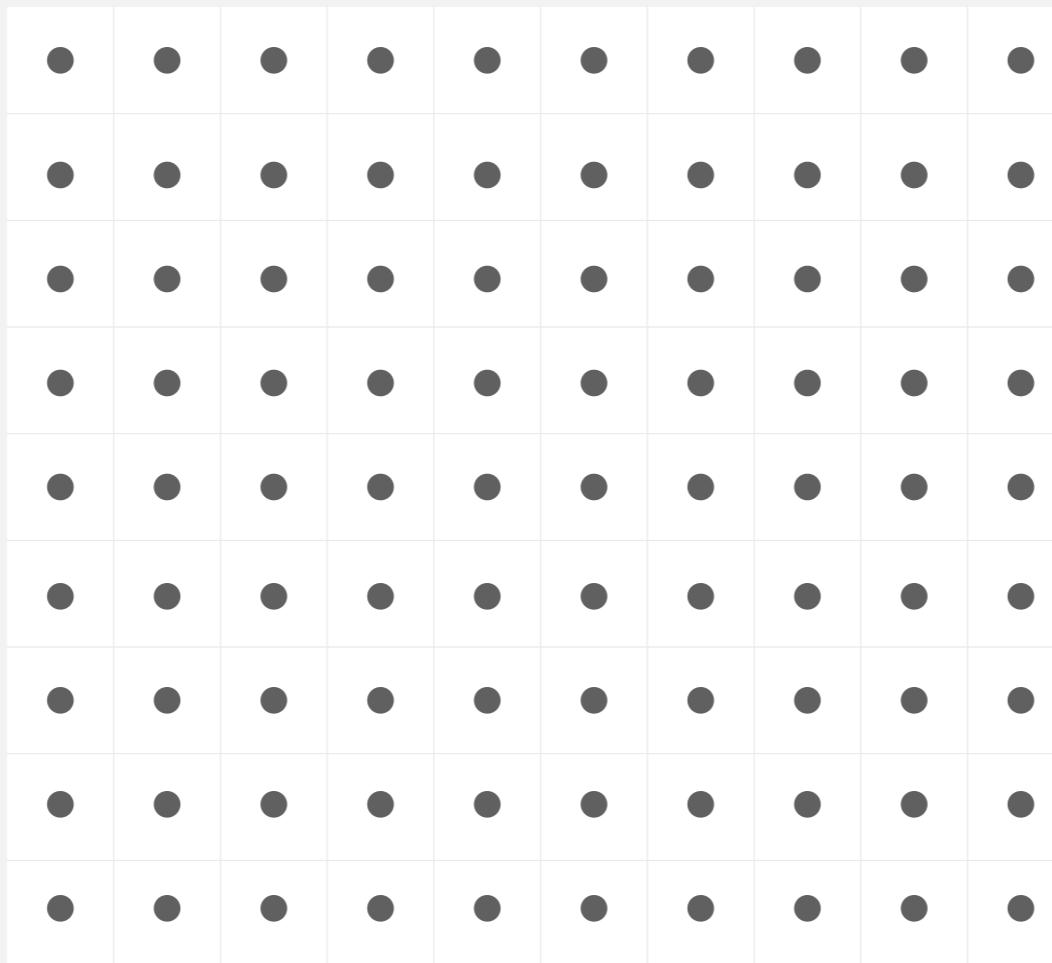


# Content-aware resizing

---

To find vertical seam:

- Grid DAG: vertex = pixel; edge = from pixel to 3 downward neighbors.
- Weight of pixel = energy function of 8 neighboring pixels.
- Seam = shortest path (sum of vertex weights) from top to bottom.

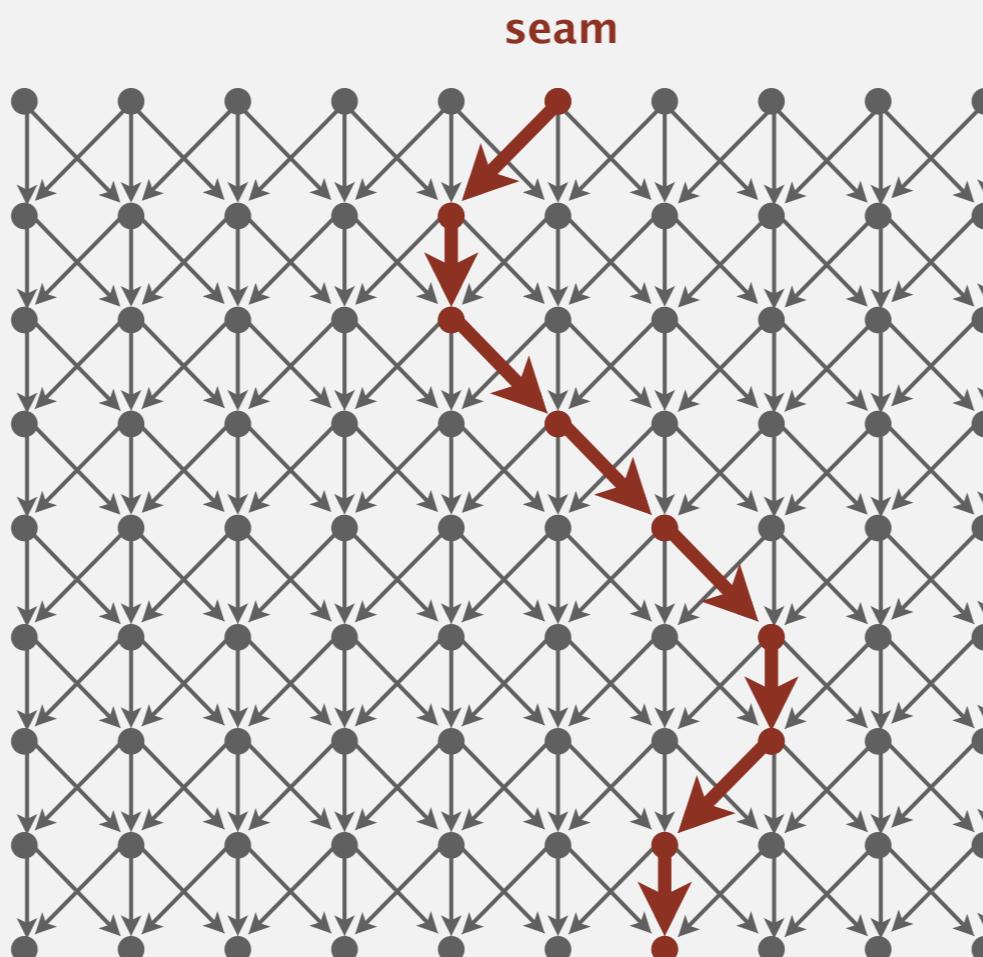


# Content-aware resizing

---

To find vertical seam:

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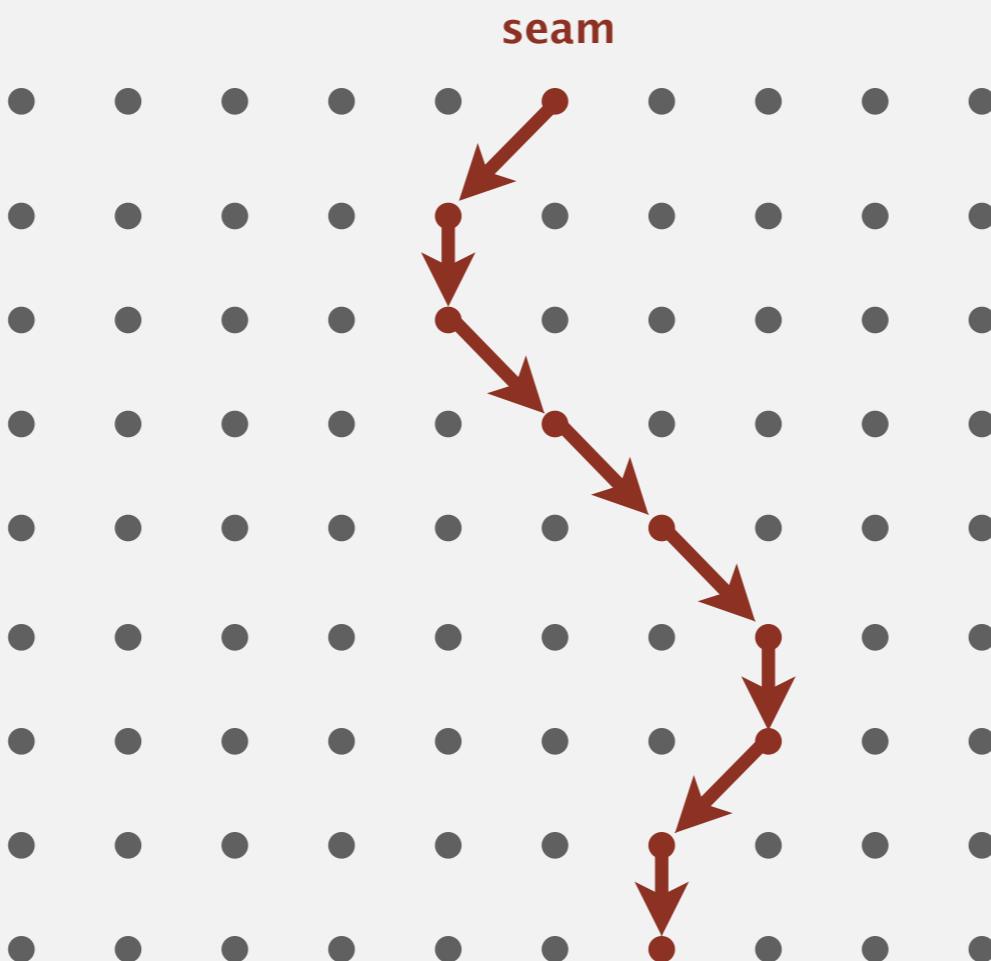


# Content-aware resizing

---

To remove vertical seam:

- Delete pixels on seam (one in each row).

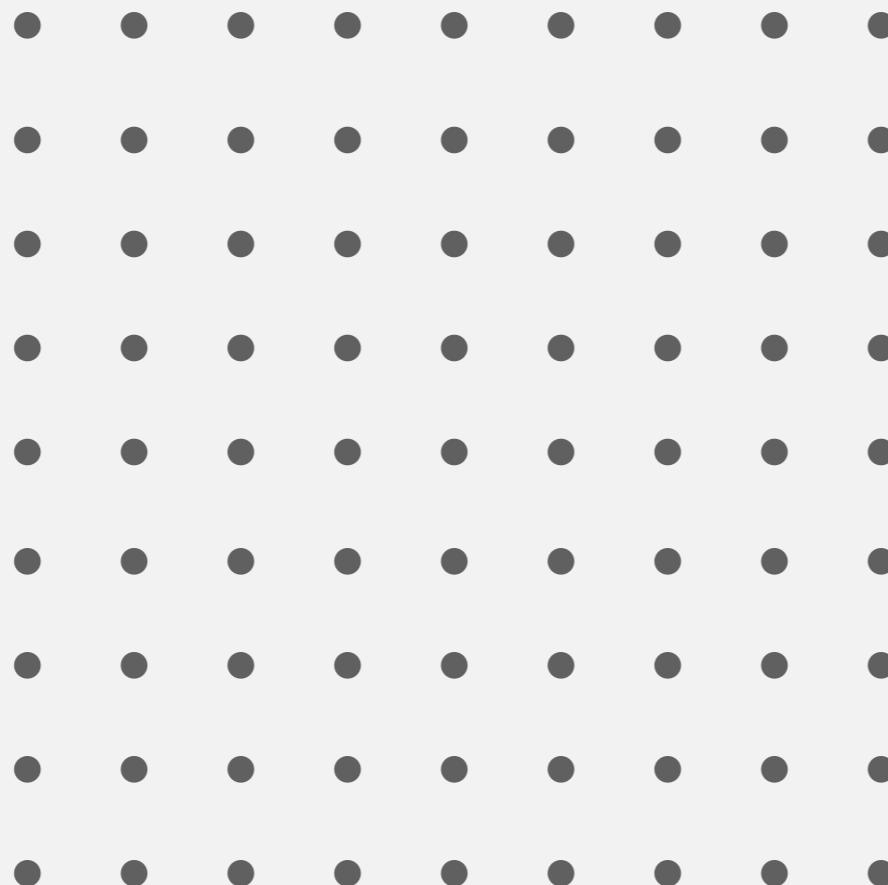


# Content-aware resizing

---

To remove vertical seam:

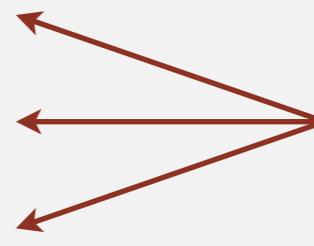
- Delete pixels on seam (one in each row).



# Longest paths in edge-weighted DAGs

Formulate as a shortest paths problem in edge-weighted DAGs.

- Negate all weights.
- Find shortest paths.
- Negate weights in result.



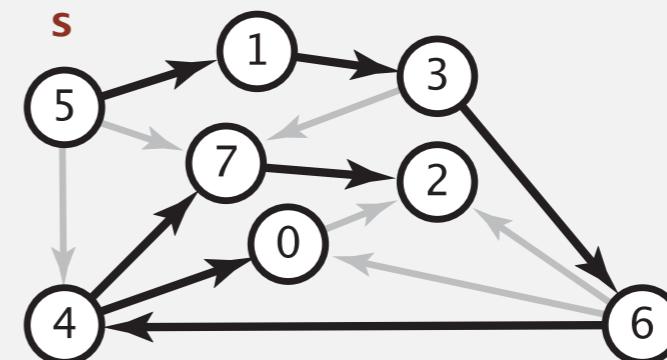
equivalent: reverse sense of equality in `relax()`

longest paths input

|      |      |
|------|------|
| 5->4 | 0.35 |
| 4->7 | 0.37 |
| 5->7 | 0.28 |
| 5->1 | 0.32 |
| 4->0 | 0.38 |
| 0->2 | 0.26 |
| 3->7 | 0.39 |
| 1->3 | 0.29 |
| 7->2 | 0.34 |
| 6->2 | 0.40 |
| 3->6 | 0.52 |
| 6->0 | 0.58 |
| 6->4 | 0.93 |

shortest paths input

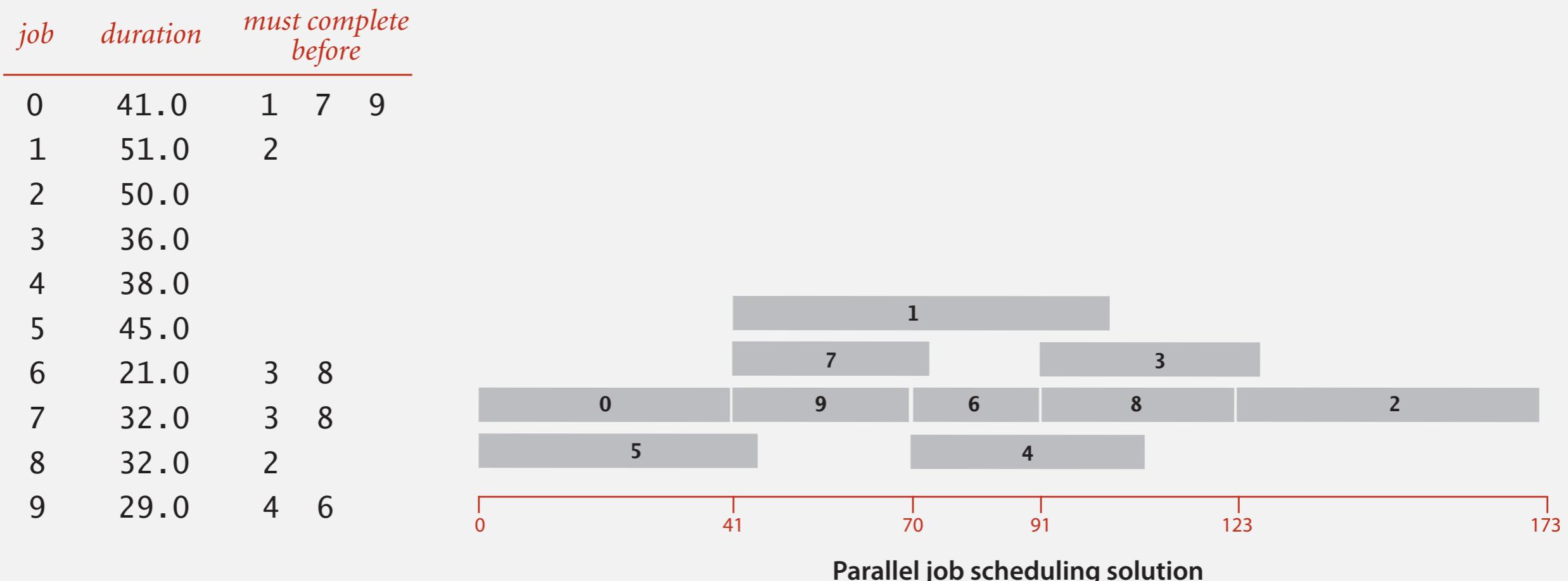
|      |       |
|------|-------|
| 5->4 | -0.35 |
| 4->7 | -0.37 |
| 5->7 | -0.28 |
| 5->1 | -0.32 |
| 4->0 | -0.38 |
| 0->2 | -0.26 |
| 3->7 | -0.39 |
| 1->3 | -0.29 |
| 7->2 | -0.34 |
| 6->2 | -0.40 |
| 3->6 | -0.52 |
| 6->0 | -0.58 |
| 6->4 | -0.93 |



Key point. Topological sort algorithm works even with negative weights.

# Longest paths in edge-weighted DAGs: application

**Parallel job scheduling.** Given a set of jobs with durations and precedence constraints, schedule the jobs (by finding a start time for each) so as to achieve the minimum completion time, while respecting the constraints.

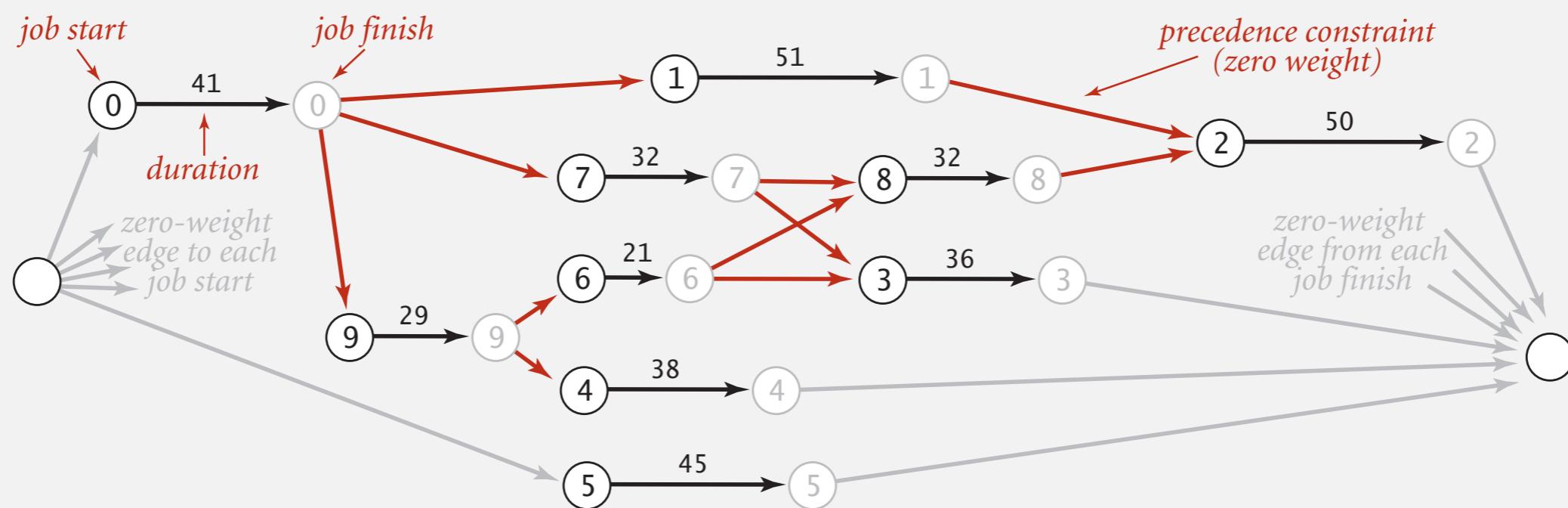


# Critical path method

CPM. To solve a parallel job-scheduling problem, create edge-weighted DAG:

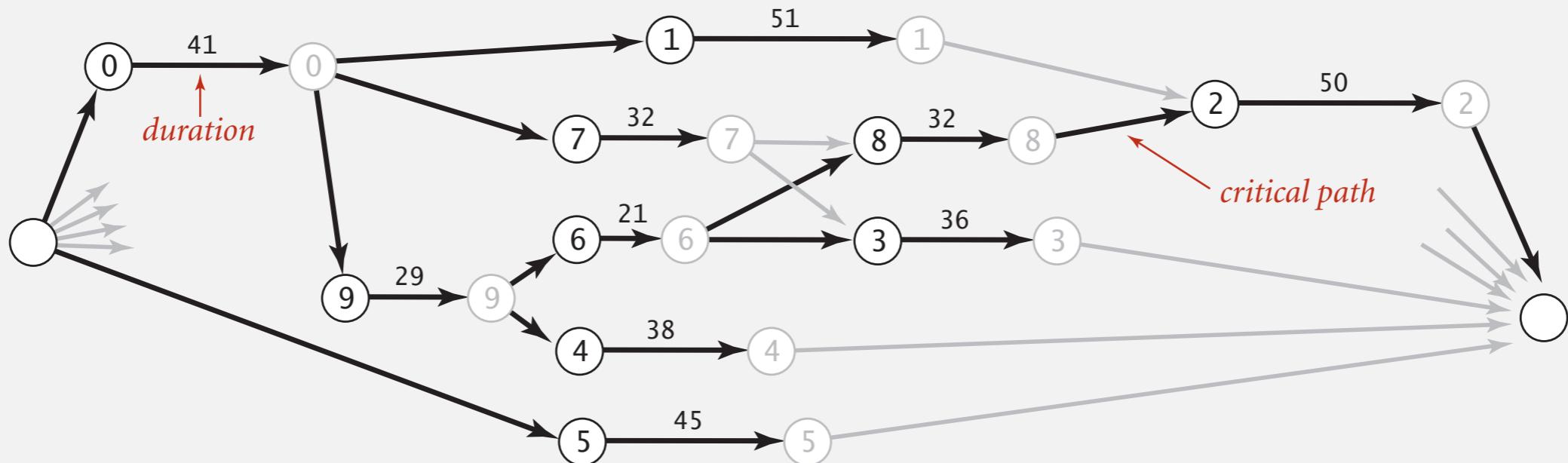
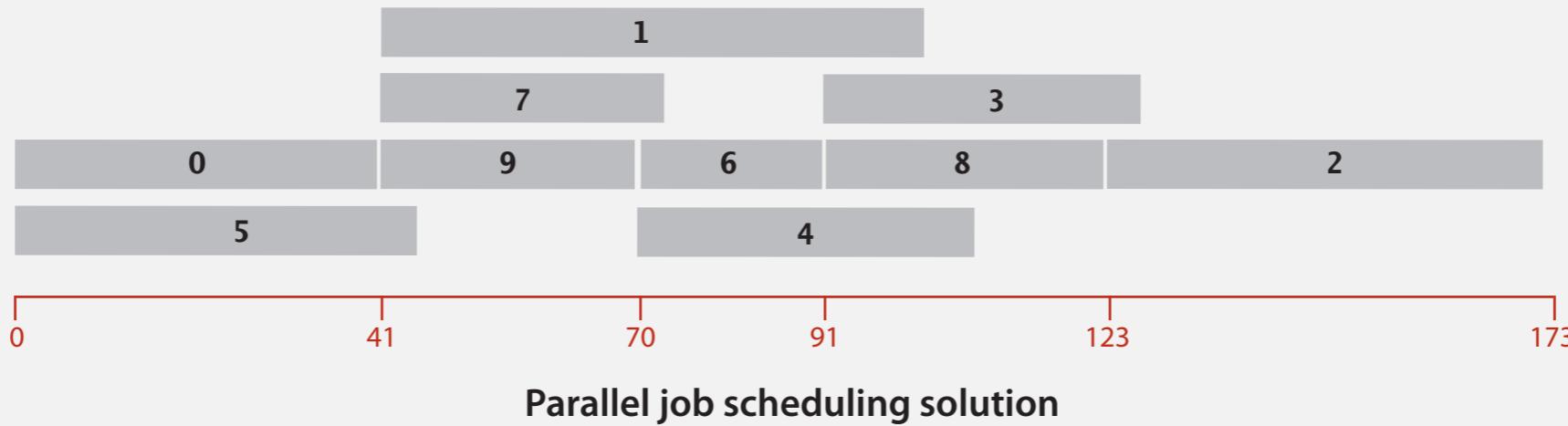
- Source and sink vertices.
- Two vertices (begin and end) for each job.
- Three edges for each job.
  - begin to end (weighted by duration)
  - source to begin (0 weight)
  - end to sink (0 weight)
- One edge for each precedence constraint (0 weight).

| <i>job</i> | <i>duration</i> | <i>must complete before</i> |   |   |
|------------|-----------------|-----------------------------|---|---|
| 0          | 41.0            | 1                           | 7 | 9 |
| 1          | 51.0            | 2                           |   |   |
| 2          | 50.0            |                             |   |   |
| 3          | 36.0            |                             |   |   |
| 4          | 38.0            |                             |   |   |
| 5          | 45.0            |                             |   |   |
| 6          | 21.0            | 3                           | 8 |   |
| 7          | 32.0            | 3                           | 8 |   |
| 8          | 32.0            | 2                           |   |   |
| 9          | 29.0            | 4                           | 6 |   |



# Critical path method

CPM. Use **longest path** from the source to schedule each job.



# Algorithms

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## 4.4 SHORTEST PATHS

---

- ▶ APIs
- ▶ *shortest-paths properties*
- ▶ *Dijkstra's algorithm*
- ▶ **edge-weighted DAGs**
- ▶ *negative weights*

# Algorithms

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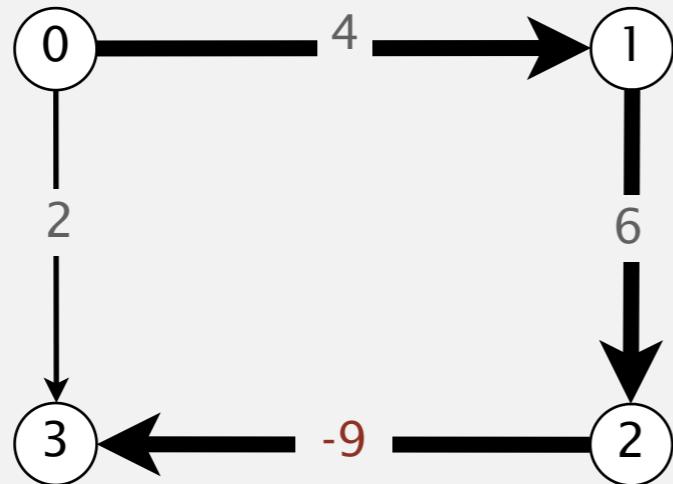
## 4.4 SHORTEST PATHS

---

- ▶ *APIs*
- ▶ *shortest-paths properties*
- ▶ *Dijkstra's algorithm*
- ▶ *edge-weighted DAGs*
- ▶ ***negative weights***

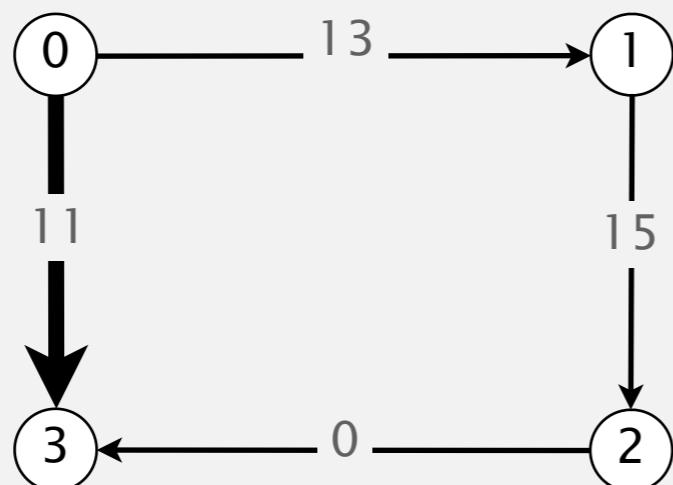
## Shortest paths with negative weights: failed attempts

Dijkstra. Doesn't work with negative edge weights.



Dijkstra selects vertex 3 immediately after 0.  
But shortest path from 0 to 3 is  $0 \rightarrow 1 \rightarrow 2 \rightarrow 3$ .

Re-weighting. Add a constant to every edge weight doesn't work.

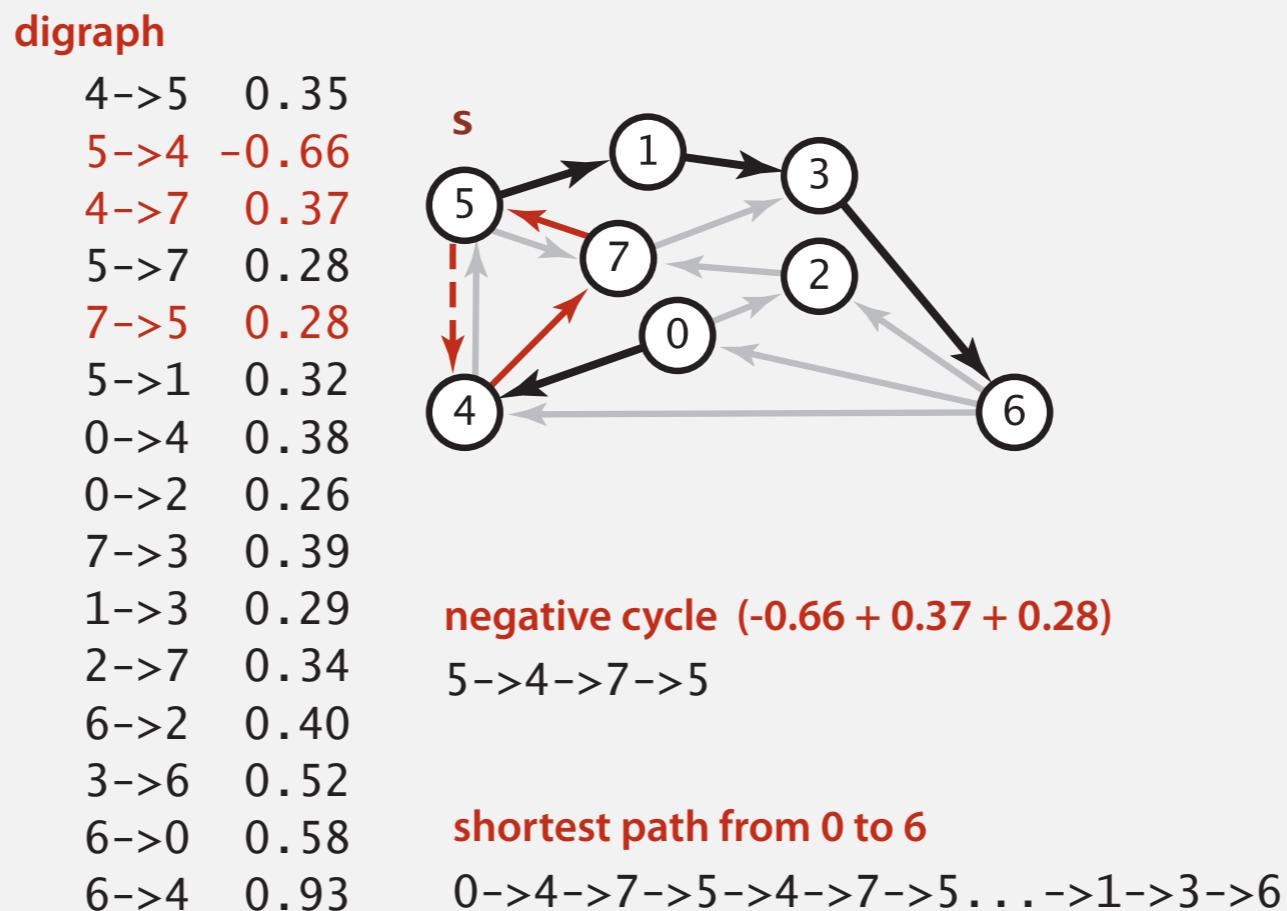


Adding 9 to each edge weight changes the  
shortest path from  $0 \rightarrow 1 \rightarrow 2 \rightarrow 3$  to  $0 \rightarrow 3$ .

Conclusion. Need a different algorithm.

## Negative cycles

**Def.** A **negative cycle** is a directed cycle whose sum of edge weights is negative.



**Proposition.** A SPT exists iff no negative cycles.

assuming all vertices reachable from  $s$

# Bellman-Ford algorithm

## Bellman-Ford algorithm

**Initialize  $\text{distTo}[s] = 0$  and  $\text{distTo}[v] = \infty$  for all other vertices.**

**Repeat  $V$  times:**

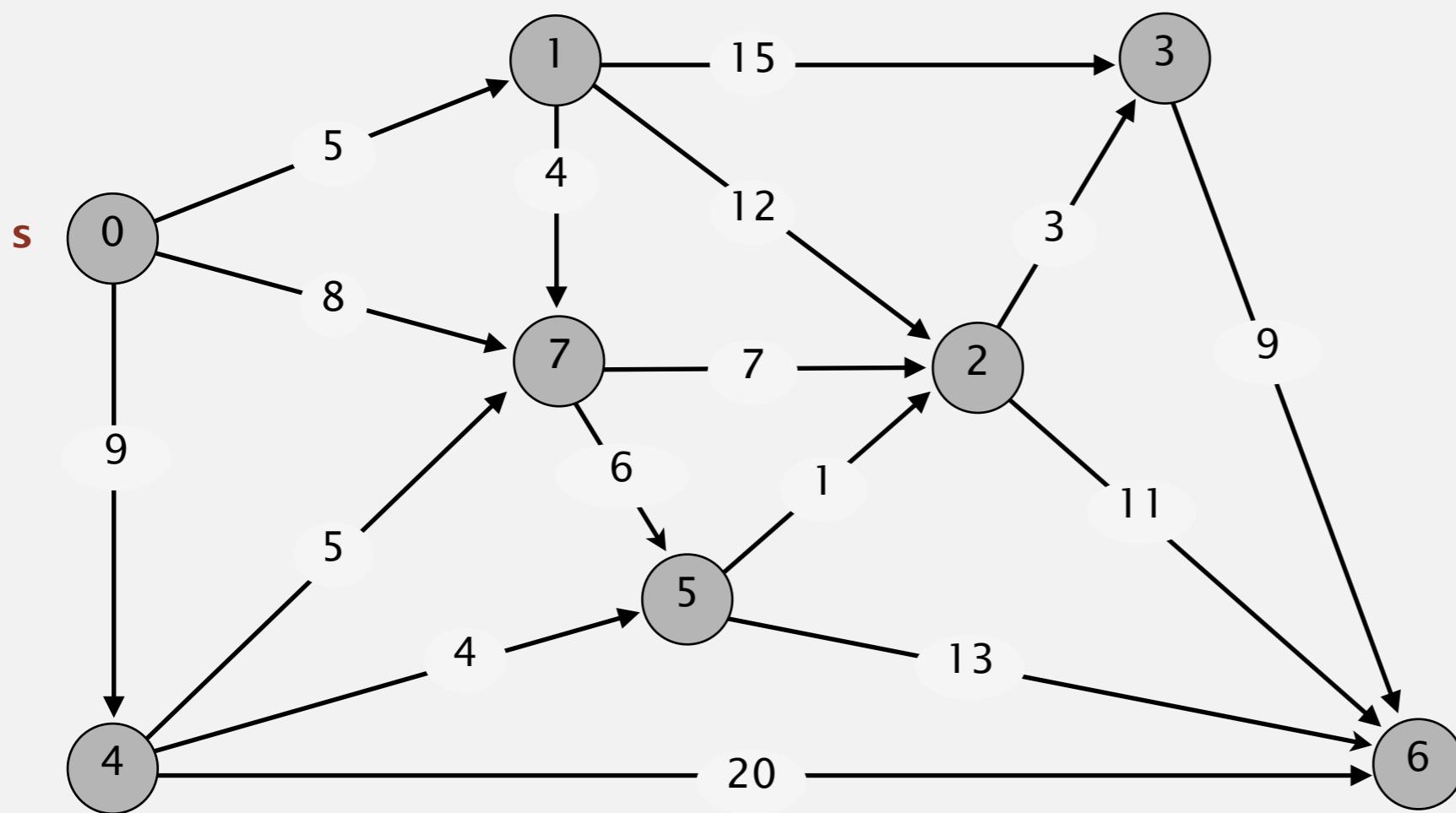
- Relax each edge.

```
for (int i = 0; i < G.V(); i++)
    for (int v = 0; v < G.V(); v++)
        for (DirectedEdge e : G.adj(v))
            relax(e);
```

← pass i (relax each edge)

# Bellman-Ford algorithm demo

Repeat  $V$  times: relax all  $E$  edges.

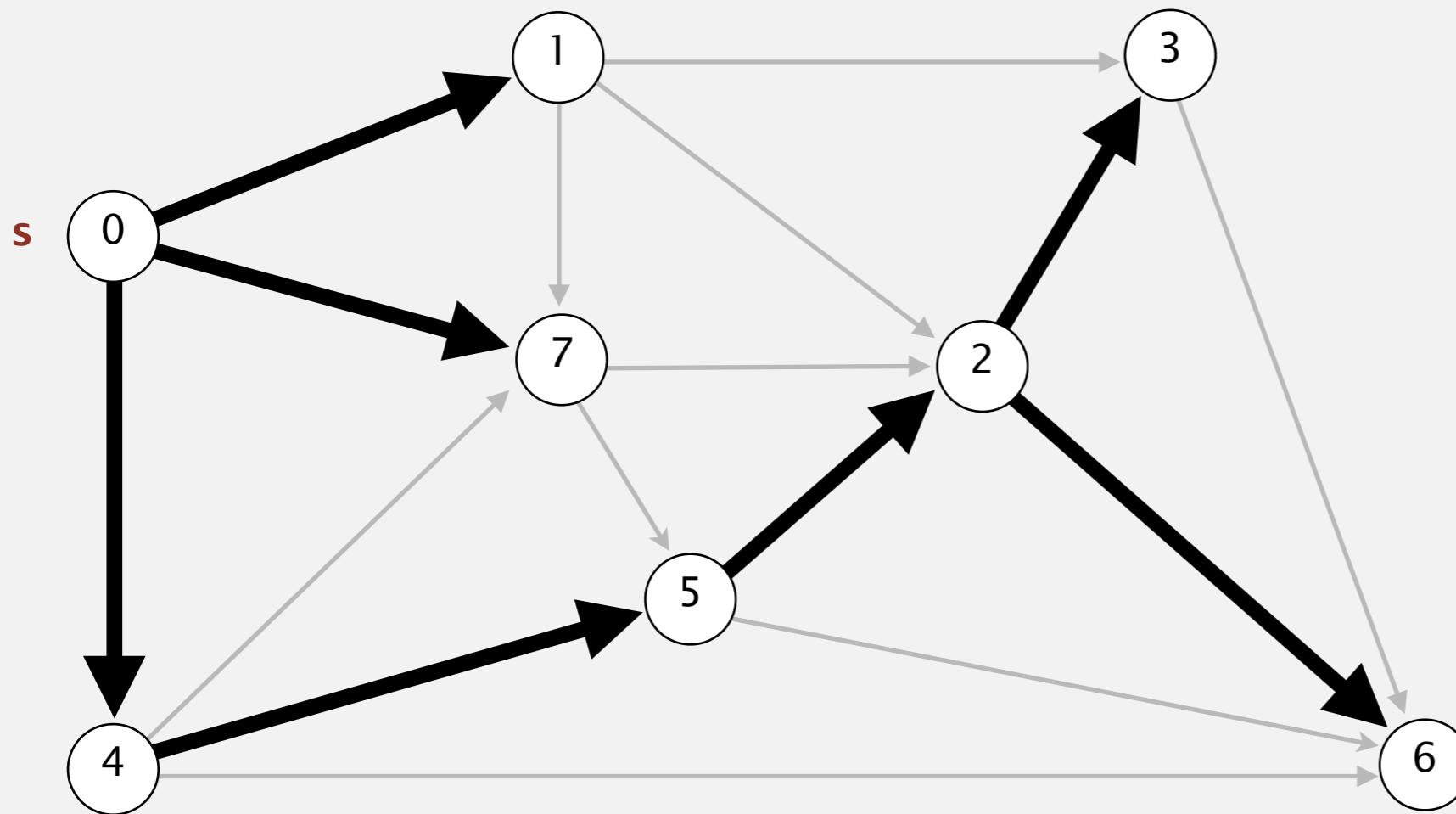


|     |      |
|-----|------|
| 0→1 | 5.0  |
| 0→4 | 9.0  |
| 0→7 | 8.0  |
| 1→2 | 12.0 |
| 1→3 | 15.0 |
| 1→7 | 4.0  |
| 2→3 | 3.0  |
| 2→6 | 11.0 |
| 3→6 | 9.0  |
| 4→5 | 4.0  |
| 4→6 | 20.0 |
| 4→7 | 5.0  |
| 5→2 | 1.0  |
| 5→6 | 13.0 |
| 7→5 | 6.0  |
| 7→2 | 7.0  |

an edge-weighted digraph

# Bellman-Ford algorithm demo

Repeat  $V$  times: relax all  $E$  edges.



| v | distTo[] | edgeTo[] |
|---|----------|----------|
| 0 | 0.0      | -        |
| 1 | 5.0      | 0→1      |
| 2 | 14.0     | 5→2      |
| 3 | 17.0     | 2→3      |
| 4 | 9.0      | 0→4      |
| 5 | 13.0     | 4→5      |
| 6 | 25.0     | 2→6      |
| 7 | 8.0      | 0→7      |

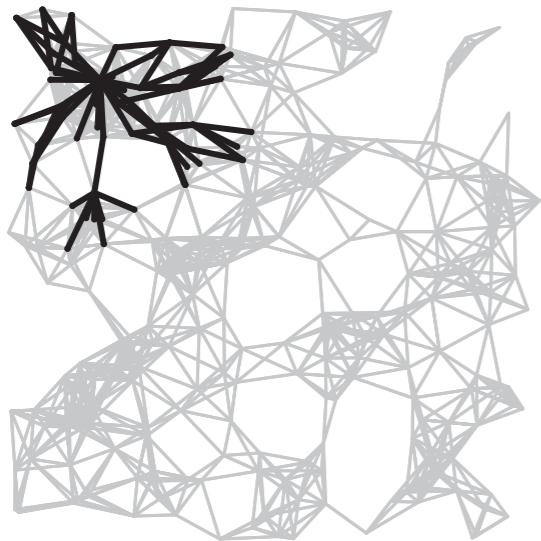
shortest-paths tree from vertex s

# Bellman-Ford algorithm visualization

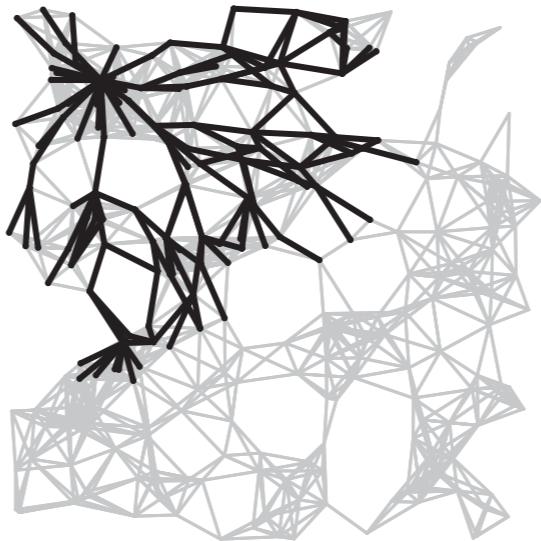
---

passes

4



7



10



13



SPT



# Bellman-Ford algorithm: analysis

---

## Bellman-Ford algorithm

---

**Initialize  $\text{distTo}[s] = 0$  and  $\text{distTo}[v] = \infty$  for all other vertices.**

**Repeat  $V$  times:**

- Relax each edge.
- 

**Proposition.** Dynamic programming algorithm computes SPT in any edge-weighted digraph with no negative cycles in time proportional to  $E \times V$ .

**Pf idea.** After pass  $i$ , found shortest path containing at most  $i$  edges.

## Bellman-Ford algorithm: practical improvement

---

**Observation.** If  $\text{distTo}[v]$  does not change during pass  $i$ , no need to relax any edge pointing from  $v$  in pass  $i+1$ .

**FIFO implementation.** Maintain **queue** of vertices whose  $\text{distTo}[]$  changed.



be careful to keep at most one copy  
of each vertex on queue (why?)

**Overall effect.**

- The running time is still proportional to  $E \times V$  in worst case.
- But much faster than that in practice.

# Single source shortest-paths implementation: cost summary

| algorithm                             | restriction         | typical case | worst case | extra space |
|---------------------------------------|---------------------|--------------|------------|-------------|
| <b>topological sort</b>               | no directed cycles  | $E + V$      | $E + V$    | $V$         |
| <b>Dijkstra<br/>(binary heap)</b>     | no negative weights | $E \log V$   | $E \log V$ | $V$         |
| <b>Bellman-Ford</b>                   | no negative cycles  | $E V$        | $E V$      | $V$         |
| <b>Bellman-Ford<br/>(queue-based)</b> |                     | $E + V$      | $E V$      | $V$         |

**Remark 1.** Directed cycles make the problem harder.

**Remark 2.** Negative weights make the problem harder.

**Remark 3.** Negative cycles makes the problem intractable.

# Finding a negative cycle

Negative cycle. Add two method to the API for SP.

boolean hasNegativeCycle()

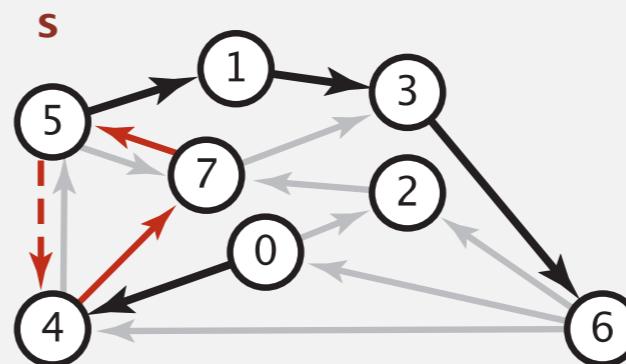
*is there a negative cycle?*

Iterable <DirectedEdge> negativeCycle()

*negative cycle reachable from s*

**digraph**

```
4->5  0.35
5->4 -0.66
4->7  0.37
5->7  0.28
7->5  0.28
5->1  0.32
0->4  0.38
0->2  0.26
7->3  0.39
1->3  0.29
2->7  0.34
6->2  0.40
3->6  0.52
6->0  0.58
6->4  0.93
```



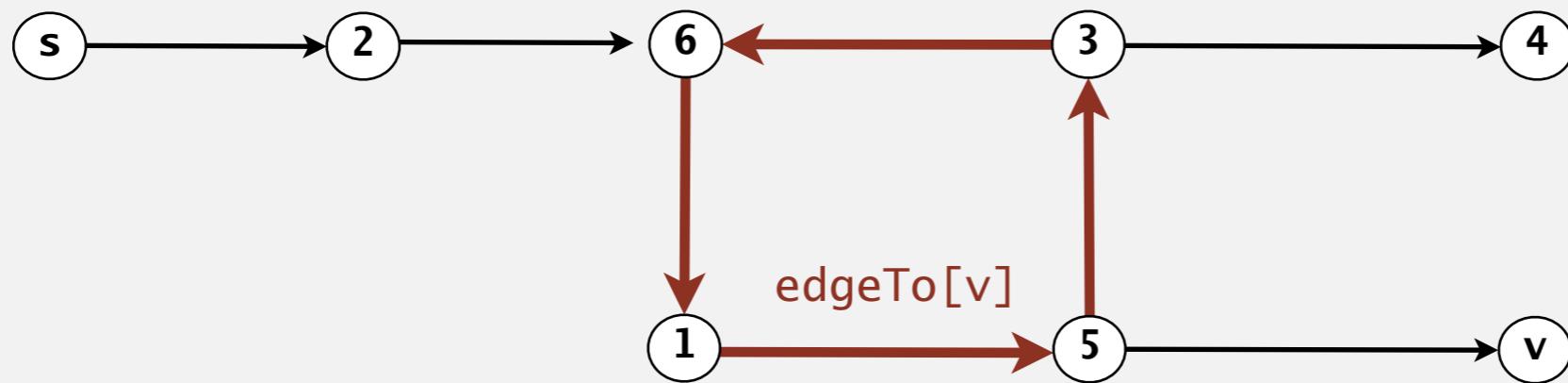
**negative cycle (-0.66 + 0.37 + 0.28)**

5->4->7->5

## Finding a negative cycle

---

**Observation.** If there is a negative cycle, Bellman-Ford gets stuck in loop, updating `distTo[]` and `edgeTo[]` entries of vertices in the cycle.



**Proposition.** If any vertex  $v$  is updated in phase  $v$ , there exists a negative cycle (and can trace back `edgeTo[v]` entries to find it).

**In practice.** Check for negative cycles more frequently.

## Negative cycle application: arbitrage detection

Problem. Given table of exchange rates, is there an arbitrage opportunity?

|     | USD   | EUR   | GBP   | CHF   | CAD   |
|-----|-------|-------|-------|-------|-------|
| USD | 1     | 0.741 | 0.657 | 1.061 | 1.011 |
| EUR | 1.350 | 1     | 0.888 | 1.433 | 1.366 |
| GBP | 1.521 | 1.126 | 1     | 1.614 | 1.538 |
| CHF | 0.943 | 0.698 | 0.620 | 1     | 0.953 |
| CAD | 0.995 | 0.732 | 0.650 | 1.049 | 1     |

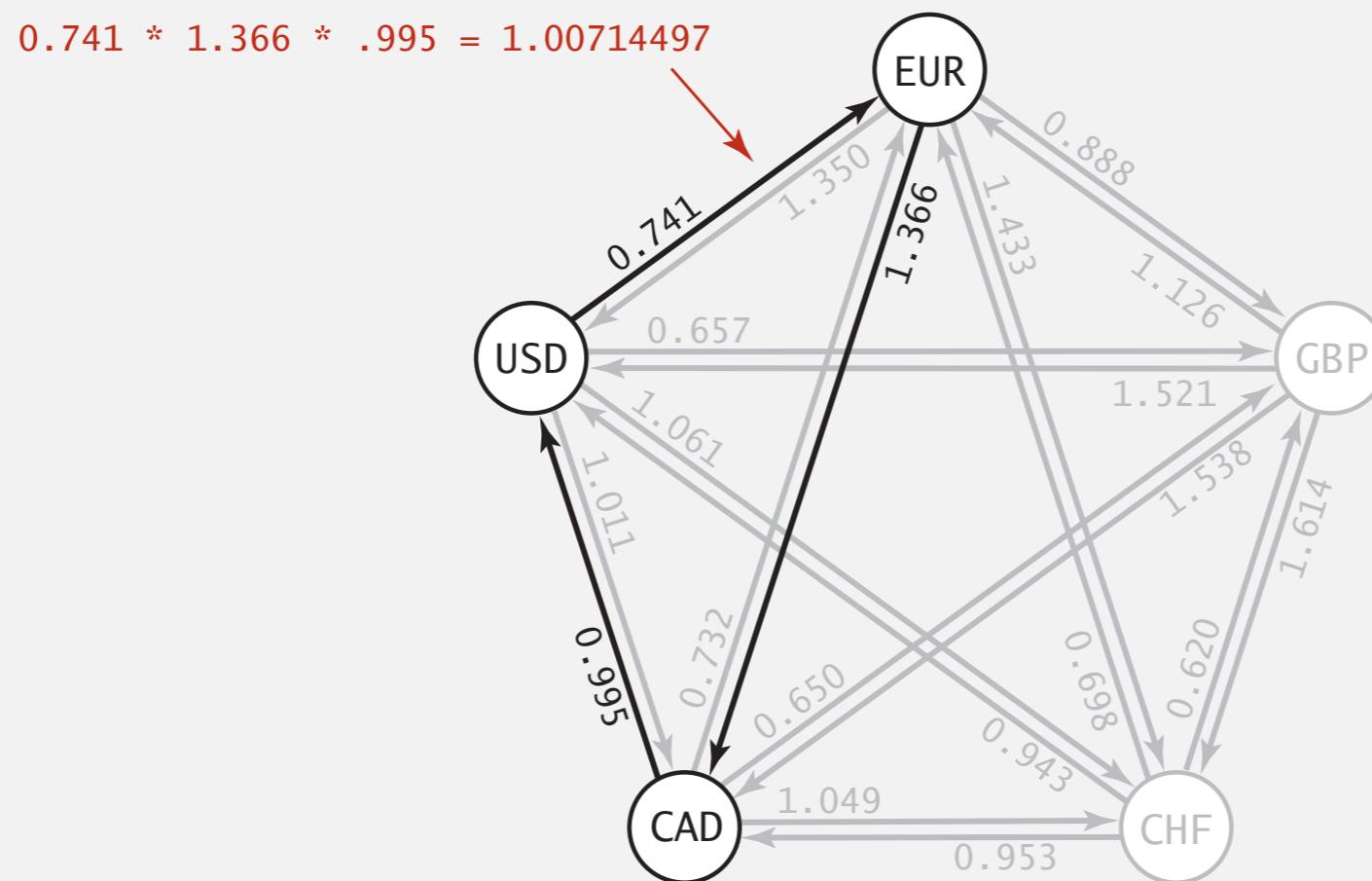
Ex. \$1,000  $\Rightarrow$  741 Euros  $\Rightarrow$  1,012.206 Canadian dollars  $\Rightarrow$  \$1,007.14497.

$$1000 \times 0.741 \times 1.366 \times 0.995 = 1007.14497$$

# Negative cycle application: arbitrage detection

Currency exchange graph.

- Vertex = currency.
- Edge = transaction, with weight equal to exchange rate.
- Find a directed cycle whose product of edge weights is  $> 1$ .

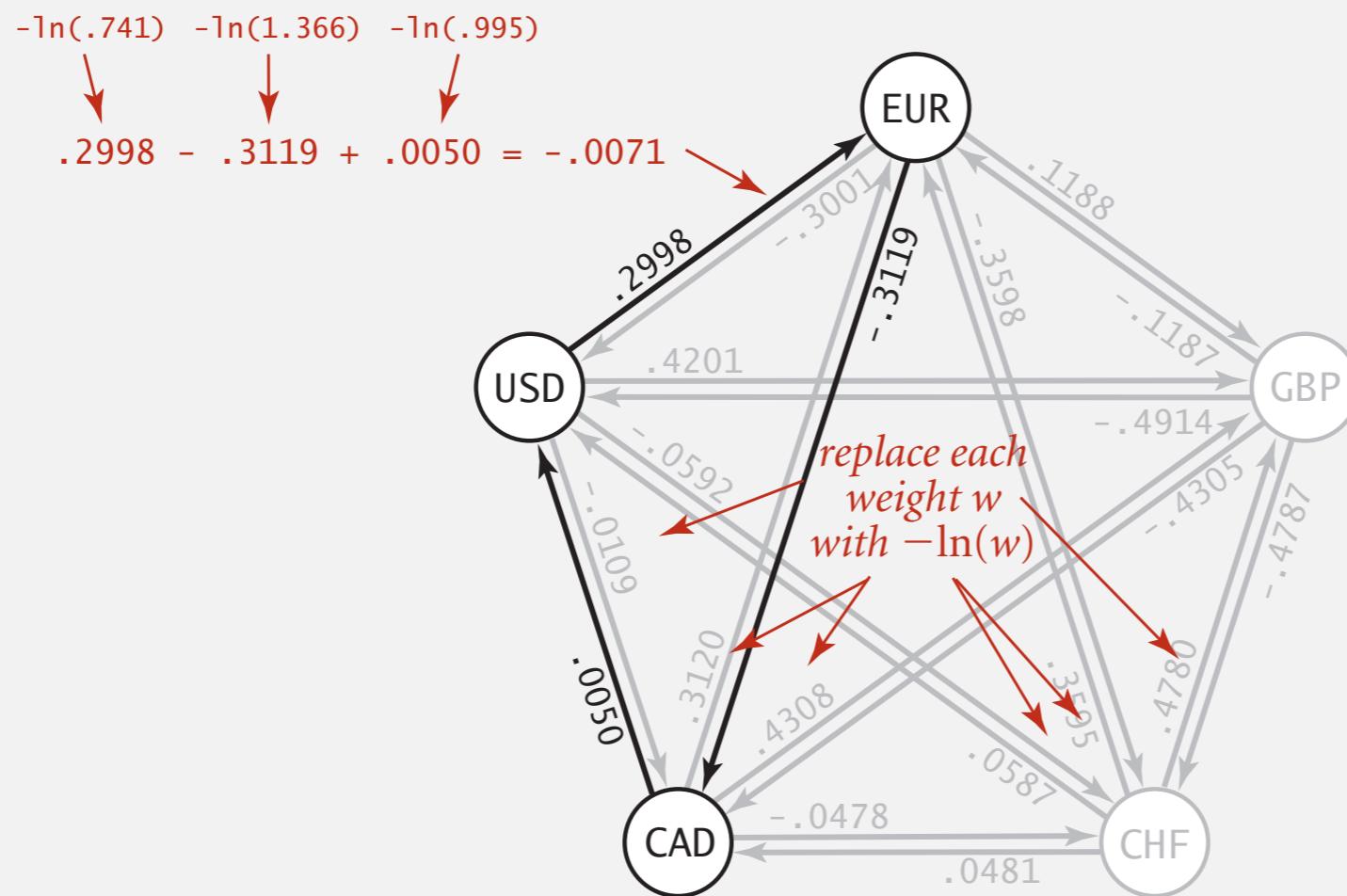


Challenge. Express as a negative cycle detection problem.

# Negative cycle application: arbitrage detection

Model as a negative cycle detection problem by taking logs.

- Let weight of edge  $v \rightarrow w$  be  $-\ln$  (exchange rate from currency  $v$  to  $w$ ).
- Multiplication turns to addition;  $> 1$  turns to  $< 0$ .
- Find a directed cycle whose sum of edge weights is  $< 0$  (negative cycle).



Remark. Fastest algorithm is extraordinarily valuable!

# Shortest paths summary

---

## Dijkstra's algorithm.

- Nearly linear-time when weights are nonnegative.
- Generalization encompasses DFS, BFS, and Prim.

## Acyclic edge-weighted digraphs.

- Arise in applications.
- Faster than Dijkstra's algorithm.
- Negative weights are no problem.

## Negative weights and negative cycles.

- Arise in applications.
- If no negative cycles, can find shortest paths via Bellman-Ford.
- If negative cycles, can find one via Bellman-Ford.

Shortest-paths is a broadly useful problem-solving model.

# Algorithms

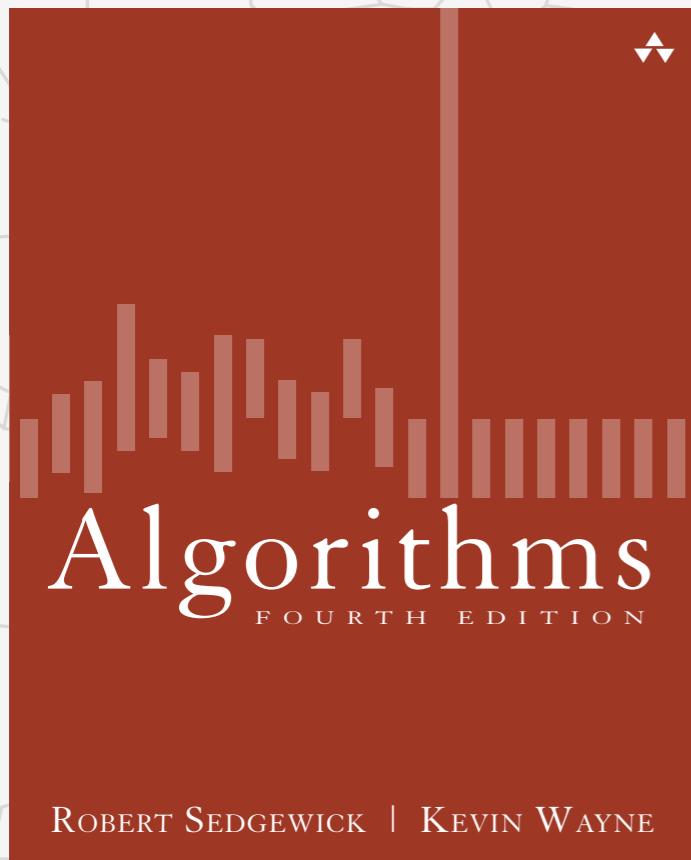
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## 4.4 SHORTEST PATHS

---

- ▶ *APIs*
- ▶ *shortest-paths properties*
- ▶ *Dijkstra's algorithm*
- ▶ *edge-weighted DAGs*
- ▶ ***negative weights***



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## 4.4 SHORTEST PATHS

---

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- ▶ *shortest-paths properties*
- ▶ *Dijkstra's algorithm*
- ▶ *edge-weighted DAGs*
- ▶ *negative weights*