State Prices

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Exercise 1.

Proof. (i) Follows from $\mathbb{P}(\omega) > 0$ for all ω
(ii)

$$\tilde{\mathbb{E}}\frac{1}{Z} = \sum_{\omega\in\Omega}\frac{\mathbb{P}(\omega)}{\tilde{\mathbb{P}}(\omega)}\tilde{\mathbb{P}}(\omega) = \sum_{\omega\in\Omega}\mathbb{P}(\omega) = 1$$

(iii)

$$\mathbb{E}Y = \sum_{\omega\in\Omega}Y(\omega)\mathbb{P}(\omega) = \sum_{\omega\in\Omega}Y(\omega)\frac{\mathbb{P}(\omega)}{\tilde{\mathbb{P}}(\omega)}\tilde{\mathbb{P}}(\omega) = \sum_{\omega\in\Omega}Y(\omega)\cdot\frac{1}{Z(\omega)}\tilde{\mathbb{P}}(\omega) = \tilde{\mathbb{E}}\left[\frac{1}{Z}\cdot Y\right]$$

□

Exercise 2.

Proof. (i)

$$\tilde{\mathbb{P}}(\Omega) = \sum_{\omega\in\Omega}\tilde{\mathbb{P}}(\omega) = \sum_{\omega\in\Omega}Z(\omega)\mathbb{P}(\omega) = \mathbb{E}[Z] = 1$$

□

(ii)

$$\tilde{\mathbb{E}}Y = \sum_{\omega\in\Omega}Y(\omega)\tilde{\mathbb{P}}(\omega) = \sum_{\omega\in\Omega}Y(\omega)Z(\omega)\mathbb{P}(\omega) = \mathbb{E}[YZ]$$

(iii) Since $\mathbb{P}(A) = 0$, then $\mathbb{P}(\omega) = 0$ for all $\omega \in A$.

$$\tilde{\mathbb{P}}(A) = \sum_{\omega\in A}\tilde{\mathbb{P}}(\omega) = \sum_{\omega\in A}Z(\omega)\mathbb{P}(\omega) = 0$$

(iv) Since $\mathbb{P}(Z > 0) = 1$, $\mathbb{P}(Z = 0) = 0$, i.e. for all $\omega \in \Omega$ such that $Z(\omega) = 0$, $\mathbb{P}(\omega) = 0$.

$$0 = \tilde{\mathbb{P}}(A) = \sum_{\omega\in A}Z(\omega)\mathbb{P}(\omega) = \sum_{\omega\in A, Z(\omega)=0}Z(\omega)\mathbb{P}(\omega) + \sum_{\omega\in A, Z(\omega)>0}Z(\omega)\mathbb{P}(\omega) = \sum_{\omega\in A, Z(\omega)>0}Z(\omega)\mathbb{P}(\omega)$$

This forces for all $\omega \in A$ such that $Z(\omega) > 0$, $\mathbb{P}(\omega) = 0$. Therefore,

$$\mathbb{P}(A) = \sum_{\omega\in A}\mathbb{P}(\omega) = \sum_{\omega\in A, Z(\omega)>0}\mathbb{P}(\omega) + \sum_{\omega\in A, Z(\omega)=0}\mathbb{P}(\omega) = 0$$

(v)

$$\mathbb{P}(A) = 1 \iff \mathbb{P}(A^c) = 0 \iff \tilde{\mathbb{P}}(A^c) = 0 \iff \tilde{\mathbb{P}}(A) = 1$$

(vi) Suppose we have $\mathbb{P}(H) = \frac{1}{2}$ and $\mathbb{P}(T) = \frac{1}{2}$, and if $Z(H) = 2$ and $Z(T) = 0$, then we have $\mathbb{P}(Z \geq 0) = 1$ and $\tilde{\mathbb{P}}(H) = 2 \cdot \frac{1}{2} = 1$ and $\tilde{\mathbb{P}}(T) = 0 \cdot \frac{1}{2} = 0$ which is not equivalent to $\mathbb{P}$.

Exercise 3.

	A	B	C	D
1	0.66666667			32
2	0.33333333			32
3			16	
4			24	
5		8		8
6		18		8
7	4		4	
8	13.5		6	
9		2		2
10		4.5		2
11			1	
12			1.5	
13				0.5
14				0.5
15				
16	13.5	13.5	13.5	13.5

Exercise 4.

(i)

$$\zeta_3(HHH) = \frac{1}{(1+\frac{1}{4})^3}\frac{27}{64} = \frac{27}{125}$$

$$\zeta_3(HHT) = \zeta_3(HTH) = \zeta_3(THH) = \frac{1}{(1+\frac{1}{4})^3}\frac{27}{32} = \frac{54}{125}$$

$$\zeta_3(HTT) = \zeta_3(THT) = \zeta_3(TTH) = \frac{1}{(1+\frac{1}{4})^3}\frac{27}{16} = \frac{108}{125}$$

$$\zeta_3(TTT) = \frac{1}{(1+\frac{1}{4})^3}\frac{27}{8} = \frac{216}{125}$$

(ii)

$$V_0 = \sum_{\omega \in \Omega} V_N(\omega)\zeta(\omega)\mathbb{P}(\omega)$$

	A	B	C	D	E	F	G	H	I	J	K
1		S0	S1	S2	S3	V3	zeta	P	VzP	p	0.66666667
2	HHH	4	8	16	32	11	0.216	0.2962963	0.704	q	0.33333333
3	HHT	4	8	16	8	5	0.432	0.14814815	0.32	r	0.25
4	HTH	4	8	4	8	2	0.432	0.14814815	0.128		
5	THH	4	2	4	8	0.5	0.432	0.14814815	0.032		
6	HTT	4	8	4	2	0.5	0.864	0.07407407	0.032		
7	THT	4	2	4	2	0	0.864	0.07407407	0		
8	TTH	4	2	1	2	0	0.864	0.07407407	0		
9	TTT	4	2	1	0.5	0	1.728	0.03703704	0		
10									1.216		
11	V2(HT)										
12		V3	zeta3	P	VzP	zeta2					
13	HTH	2	0.432	0.66666667	0.576	0.72					
14	HTT	0.5	0.864	0.33333333	0.144						
15						1					
16	V2(TH)										
17		V3	zeta3	P	VzP	zeta2					
18	THH	0.5	0.432	0.66666667	0.144	0.72					
19	THT	0	0.864	0.33333333	0						
20						0.2					

(iii) $\zeta_2(HT) = \zeta_2(TH) = \frac{1}{(1+r)^2} Z(TH) = \frac{1}{(1+r)^2} Z(HT) = \frac{1}{(1+\frac{1}{4})^2}\frac{9}{8} = \frac{18}{25}$

(iv)

$$V_2(HT) = \frac{1}{\zeta_2(HT)}\mathbb{E}_2[\zeta_3 V_3](HT) = \frac{25}{18}\left(\frac{2}{3}\cdot 0.432\cdot 2 + \frac{1}{3}\cdot 0.864\cdot 0.5\right) = 1$$
$$V_2(TH) = 0.2$$

Exercise 5.

(i)

$$Z(HH) = \frac{\tilde{\mathbb{P}}(HH)}{\mathbb{P}(HH)} = \frac{\frac{1}{4}}{\frac{4}{9}} = \frac{9}{16}$$
$$Z(HT) = \frac{\tilde{\mathbb{P}}(HT)}{\mathbb{P}(HT)} = \frac{\frac{1}{4}}{\frac{2}{9}} = \frac{9}{8}$$
$$Z(TH) = \frac{\tilde{\mathbb{P}}(TH)}{\mathbb{P}(TH)} = \frac{\frac{1}{12}}{\frac{2}{9}} = \frac{3}{8}$$
$$Z(TT) = \frac{\tilde{\mathbb{P}}(TT)}{\mathbb{P}(TT)} = \frac{\frac{5}{12}}{\frac{1}{9}} = \frac{15}{4}$$

(ii)

$$Z_1(H) = \mathbb{E}_1[Z](H) = pZ(HH) + qZ(HT) = \frac{2}{3}\cdot\frac{9}{16} + \frac{1}{3}\cdot\frac{9}{8} = \frac{3}{4}$$
$$Z_1(T) = \mathbb{E}_1[Z](T) = pZ(TH) + qZ(TT) = \frac{2}{3}\cdot\frac{3}{8} + \frac{1}{3}\cdot\frac{15}{4} = \frac{3}{2}$$
$$\text{Note that } Z_0 = pZ(H) + qZ(T) = \frac{2}{3}\cdot\frac{3}{4} + \frac{1}{3}\cdot\frac{3}{2} = 1$$

(iii)

$$\begin{aligned}
V_1(H) &= \frac{1+r_0}{Z_1(H)}\mathbb{E}_1\left[\frac{Z_2}{(1+r_0)(1+r_1)}V_2\right](H) \\
&= \frac{1}{Z_1(H)(1+r_1(H))}\mathbb{E}_1[Z_2V_2](H) \\
&= \frac{1}{\frac{3}{4}(1+\frac{1}{4})}\cdot\left(\frac{2}{3}\frac{9}{16}\cdot 5+\frac{1}{3}\frac{9}{8}\cdot 1\right) \\
&= \frac{12}{5} \\
V_1(T) &= \frac{1}{Z_1(T)(1+r_1(T))}\mathbb{E}_1[Z_2V_2](T) \\
&= \frac{1}{\frac{3}{2}(1+\frac{1}{2})}\cdot\left(\frac{2}{3}\frac{3}{8}\cdot 1\right) \\
&= \frac{1}{9} \\
V_0 &= \mathbb{E}\left[\frac{Z_2}{(1+r_0)(1+r_1)}V_2\right] \\
&= \left(\frac{4}{9}\frac{\frac{9}{16}}{(1+\frac{1}{4})(1+\frac{1}{4})}\cdot 5\right)+\left(\frac{2}{9}\frac{\frac{9}{8}}{(1+\frac{1}{4})(1+\frac{1}{4})}\cdot 1\right)+\left(\frac{2}{9}\frac{\frac{3}{8}}{(1+\frac{1}{4})(1+\frac{1}{2})}\cdot 1\right) \\
&= \frac{236}{225}
\end{aligned}$$