The Binomial No-Arbitrage Pricing Model

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Exercise 1.

If $X_0 = 0$, $X_1 = \Delta_0(S_1 - (1+r)S_0)$ i.e.

$$\begin{aligned} X_1(H) &= \Delta_0 S_0(u - 1 - r) \\ X_1(T) &= \Delta_0 S_0(d - 1 - r) \end{aligned}$$

Since we need to have $0 < d < 1 + r < u$, if X_1 takes positive value with positive probability (it has to be one of the case), then it will have to be negative in the other.

Exercise 2.

From$X_1 = \Delta_0 S_1 + \Gamma_0(S_1 - 5)^+ - \frac{5}{4}(4\Delta_0 + 1.20\Gamma_0)$, we know that

$$\begin{aligned} X_1(H) &= 8\Delta_0 + 3\Gamma_0 - 5\Delta_0 - 1.5\Gamma_0 = 3\Delta_0 + 1.5\Gamma_0 \\ X_1(T) &= 2\Delta_0 - 5\Delta_0 - 1.5\Gamma_0 = -3\Delta_0 - 1.5\Gamma_0 \end{aligned}$$

i.e. $X_1(H) = -X_1(T)$. Therefore, if we have the value of X_1 being strictly positive in one case, it will have to be strictly negative in the other.

Exercise 3.

$$\begin{aligned} V_0 =& \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)] \\ =& \frac{1}{1+r}[\tilde{p}S_1(H) + \tilde{q}S_1(T)] \\ =& S_0 \end{aligned}$$

The last equality comes from the definition of $\tilde{q}$

Exercise 4.

$$X_{n+1}(\omega_1 ... \omega_n T) = \Delta_n(\omega_1 ... \omega_n) dS_n(\omega_1 ... \omega_n) + (1+r)\left(X_n(\omega_1 ... \omega_n) - \Delta_n(\omega_1 ... \omega_n) S_n(\omega_1 ... \omega_n)\right)$$

Suppress $\omega_1 ... \omega_n$ and write equation as

$$X_{n+1}(T) = \Delta_n dS_n + (1+r)(X_n - \Delta_n S_n)$$

Also, we have

$$\Delta_n = \frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)} = \frac{V_{n+1}(H) - V_{n+1}(T)}{(u-d)S_n}$$

Substituting equation above and $X_n = V_n$ into previous equation to get

$$\begin{aligned}
X_{n+1}(T) =& \frac{d(V_{n+1}(H) - V_{n+1}(T)}{u-d} + (1+r)\left(X_n - \frac{V_{n+1}(H) - V_{n+1}(T)}{u-d}\right) \\
=&(1+r)V_n - \frac{1+r-d}{u-d}(V_{n+1}(H) - V_{n+1}(T)) \\
=&\tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T) - \tilde{p}V_{n+1}(H) + \tilde{p}V_{n+1}(T) \\
=&\tilde{p}V_{n+1}(T) + \tilde{q}V_{n+1}(T) \\
=&V_{n+1}(T)
\end{aligned}$$

Exercise 5.

$$\Delta_1(H) = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)} = \frac{3.2 - 2.4}{16 - 4} = \frac{1}{15}$$

This costs 8/15 dollars and leaves $2.24 - 8/15 = 1.706667$ dollars to invest in the money market. At time two, she will have 2.13333 in the money market. If the stock goes up to 16, the stocks will worth 1.066667, so she will end up with $2.13333 + 1.066667 = 3.2 = V_2(HH)$ in her portfolio. If the stock goes down to 4, the portfolio value will be $2.13333 + 0.26667 = 2.4 = V_2(HT)$.

Now assume if the stock goes down in the second period.

$$\Delta_2(HT) = \frac{V_3(HTH) - V_3(HTT)}{S_3(HTH) - S_3(HTT)} = \frac{0-6}{8-2} = -1$$

Cost of stock at time 2: -4

Investment in the money market: 6.4

Amount in the money market at time 3: 8

Stock value if stock goes up in period 3: $-8 \Rightarrow 8 - 8 = 0 = V_3(HTH)$

Stock value if stock goes down in period 3: $-2 \Rightarrow 8 - 2 = 6 = V_3(HTT)$

Exercise 6.

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \frac{3-0}{8-2} = 0.5$$

Therefore, to gain risk-free on the capital, the trader has to short 0.5 unit of stock and invest $\Delta_0 S_0 = 2$ dollars in the money market, so that at t_1

If the stock goes up to 8, the investor has

$$(8-5)^+ - 0.5(8) + 2(1.25) = 3 - 4 + 2.5 = 1.5$$

If the stock goes down to 2, the investor has

$$(2-5)^+ - 0.5(2) + 2(1.25) = 0 - 1 + 2.5 = 1.5$$

Exercise 7.

	A	B	C	D	E	F	G	H	I	J	K
1	p	0.5							HHH	32	0
2	q	0.5							2.6875		
3	r	0.25				HH	16	3.2			
4	1+r	1.25				2.15	-0.3333333	-6.3833333			
5									HHT	8	8
6									2.6875		
7			H	8	2.24						
8			1.72	0.06666667	0.01333333						
9									HTH	8	0
10									2.6875		
11						HT	4	2.4			
12						2.15	-1	-4.25			
13									HTT	2	6
14									2.6875		
15	4	1.376									
16	0.17333333	0.69333333									
17									THH	8	0
18									2.6875		
19						TH	4	0.8			
20						2.15	-0.3333333	0.01666667			
21									THT	2	2
22									2.6875		
23			T	2	1.2						
24			1.72	-0.4666667	-0.4133333						
25									TTH	2	2
26									2.6875		
27						TT	1	2.2			
28						2.15	-1	-1.05			
29									TTT	0.5	3.5
30									2.6875		
31											
32											

Exercise 8.

	A	B	C	D	E	F	G	H
1	p	0.5					HHH	32
2	q	0.5					60	11
3	r	0.25			HH	16		
4	1+r	1.25			28	6.4		
5							HHT	8
6							36	5
7			H	8				
8			12	2.96				
9							HTH	8
10							24	2
11					HT	4		
12					16	1		
13							HTT	2
14							18	0.5
15		4						
16	4	1.216						
17							THH	8
18							18	0.5
19					TH	4		
20					10	0.2		
21							THT	2
22							12	0
23			T	2				
24			6	0.08				
25							TTH	2
26							9	0
27					TT	1		
28					7	0		
29							TTT	0.5
30							7.5	0
31								

$$v_3(32,60) = (\frac{1}{4}60 - 4)^+ = 11$$
$$v_3(8,36) = (\frac{1}{4}36 - 4)^+ = 5$$
$$v_3(8,24) = (\frac{1}{4}24 - 4)^+ = 2$$
$$v_3(2,18) = (\frac{1}{4}18 - 4)^+ = 0.5$$
$$v_3(8,18) = (\frac{1}{4}18 - 4)^+ = 0.5$$
$$v_3(2,12) = (\frac{1}{4}12 - 4)^+ = 0$$
$$v_3(2,9) = (\frac{1}{4}9 - 4)^+ = 0$$
$$v_3(0.5,7.5) = (\frac{1}{4}7.5 - 4)^+ = 0$$
$$v_2(16,28) = \frac{1}{1+r}[\tilde{p}v_3(32,60) + \tilde{q}v_3(8,36)] = 6.4$$
$$v_2(4,16) = \frac{1}{1+r}[\tilde{p}v_3(8,24) + \tilde{q}v_3(2,18)] = 1$$
$$v_2(4,10) = \frac{1}{1+r}[\tilde{p}v_3(8,18) + \tilde{q}v_3(2,12)] = 0.2$$
$$v_2(1,7) = \frac{1}{1+r}[\tilde{p}v_3(2,9) + \tilde{q}v_3(0.5,7.5)] = 0$$
$$v_1(8,12) = \frac{1}{1+r}[\tilde{p}v_2(16,28) + \tilde{q}v_2(4,16)] = 2.96$$
$$v_1(2,6) = \frac{1}{1+r}[\tilde{p}v_2(4,10) + \tilde{q}v_2(1,7)] = 0.08$$
$$v_0(4,4) = \frac{1}{1+r}[\tilde{p}v_1(8,12) + \tilde{q}v_1(2,6)] = 1.216$$

Exercise 9.

(i) At each state in time n, define

$$\tilde{p}(\omega_1...\omega_n) = \frac{1 + r(\omega_1...\omega_n) - d(\omega_1...\omega_n)}{u(\omega_1...\omega_n) - d(\omega_1...\omega_n)} \text{ and } \tilde{q}(\omega_1...\omega_n) = 1 - \tilde{p}(\omega_1...\omega_n)$$

Then the price of the derivative paying V_N at time N can be computed recursively, backward in time, by the formula

$$V_n(\omega_1...\omega_n) = \frac{1}{1 + r(\omega_1...\omega_n)}[\tilde{p}(\omega_1...\omega_n)V_{n+1}(\omega_1...\omega_n H) + \tilde{q}(\omega_1...\omega_n)V_{n+1}(\omega_1...\omega_n T)]$$

(ii)

$$\Delta_n(\omega_1...\omega_n) = \frac{V_{n+1}(\omega_1...\omega_n H) - V_{n+1}(\omega_1...\omega_n T)}{S_{n+1}(\omega_1...\omega_n H) - S_{n+1}(\omega_1...\omega_n T)}$$

(iii) First notice that the states are path-independent, so the tree is recombining. Also, since $r = 0$, we have

$$\tilde{p}(\omega_1 ... \omega_n) = \frac{1 - d(\omega_1 ... \omega_n)}{u(\omega_1 ... \omega_n) - d(\omega_1 ... \omega_n)} = \frac{1}{2}$$

i.e. $\tilde{p}$ is constant at 0.5, thus $V_n(\omega_1 ... \omega_n) = \frac{V_{n+1}(\omega_1 ... \omega_n H) + V_{n+1}(\omega_1 ... \omega_n T)}{2}$. Therefore,

$$V_0 = \sum_{i=0}^{5} \binom{5}{i} 2^{-5}(20i - 50)^+ = 2^{-5}(50 + 5(30) + 10(10)) = 9.375$$