Jump Modelling

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1 Poisson Processes

1.1 Definition

A process, N_t, is called a Poisson process with intensity λ if (under probability space $(\Omega, \mathcal{F}, \mathbb{P})$)

- $N_0 = 0$ almost surely.
- **Stationary Increment.** $N_{t+s} - N_s \sim H(t)$ for some distribution H.
- **Independent Increment.** $N_s - N_t \perp N_u - N_v$ if $(t, s) \cap (v, u) = \emptyset$.
- $N_t \sim Pois(\lambda t)$ i.e. $\mathbb{P}(N_t = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$
- **Stochastic Continuity.** $\lim_{s \downarrow t} \mathbb{P}(N_s - N_t \geq \epsilon) = 0, \forall t, \epsilon > 0$.
  This can be interpreted as jumps do not happen at pre-determined time.
- **R.C.L.L.** Right Continuous with Left Limit
  i.e. $\lim_{t \downarrow s} N_t = N_s$

1.2 Moment Generating Function

Denote the m.g.f. by

$$g_t(a) = \mathbb{E}[e^{aN_t}],$$

note that $g_t(a)$ has to satisfy the property $g_t(a) = g_{t-s}(a) g_s(a)$ by stationary independent increment. Therefore, $g_t(a)$ has to follow the form $e^{\psi(a,\lambda)t}$.
Note that for random variable X that follows Poisson distribution with mean λt, the m.g.f. can be calculated from

$$\mathbb{E}[e^{aX}] = \sum_{k=0}^{\infty} e^{ka} \frac{(\lambda t)^k e^{-\lambda t}}{k!} = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(e^a \lambda t)^k}{k!} = e^{-\lambda t} e^{e^a \lambda t} = \exp\{(e^a - 1)\lambda t\}$$

Since $X \stackrel{d}{=} N_t$, $g_t(a) = e^{(e^a-1)\lambda t}$.
Exercise: Show that $M_t = e^{aN_t - (e^a-1)\lambda t}$ is a martingale.

$$\mathbb{E}_t[M_{t+s}] = e^{aN_t - (e^a-1)\lambda t} \mathbb{E}[e^{aN_s - (e^a-1)\lambda s}] = M_t \mathbb{E}[e^{aN_t}] e^{-a(e^a-1)s} = M_t$$

1.3 Measure Change

N_t is a $\mathbb{P}$ Poisson process with intensity λ. To make measure change from $\mathbb{P}$ to $\mathbb{P}^*$, introduce quantity

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = e^{bN_T - (e^b-1)\lambda T}$$

We first show that $\frac{d\mathbb{P}^*}{d\mathbb{P}}$ is a valid Radon-Nikodym derivative, and then we will show how the value of b is related to $\mathbb{P}^*$.
To show that $\frac{d\mathbb{P}^*}{d\mathbb{P}}$ is a R.N.D., we note that
i) $\frac{d\mathbb{P}^*}{d\mathbb{P}} \geq 0$ a.s. In fact, $\frac{d\mathbb{P}^*}{d\mathbb{P}} > 0$ a.s. i.e., $\mathbb{P}^*$ is equivalent to $\mathbb{P}$.
ii) $\mathbb{E}_t \left[\frac{d\mathbb{P}^*}{d\mathbb{P}} \right] = e^{bN_t - (e^b-1)\lambda t}$ is a $\mathbb{P}$ Doob-martingale with $\mathbb{E}_0 \left[\frac{d\mathbb{P}^*}{d\mathbb{P}} \right] = 1$, so $\frac{d\mathbb{P}^*}{d\mathbb{P}}$ is a valid Radon-Nikodym

derivative.
To compute the distribution of N_t under $\mathbb{P}^*$, we compute the m.g.f. under $\mathbb{P}^*$

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}^*}\left[e^{aN_t}\right] &= \mathbb{E}^{\mathbb{P}}\left[e^{aN_t}\frac{d\mathbb{P}^*}{d\mathbb{P}}\right] = \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}_t\left[e^{aN_t}\frac{d\mathbb{P}^*}{d\mathbb{P}}\right]\right] = \mathbb{E}^{\mathbb{P}}\left[e^{aN_t}e^{bN_t-(e^b-1)\lambda t}\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[e^{(a+b)N_t-(e^{a+b}-1)\lambda t}\right]e^{(e^{a+b}-1)\lambda t}e^{-(e^b-1)\lambda t} \\
&= e^{(e^a-1)e^b\lambda t}
\end{aligned}$$

Note that this follows exactly the form of the M.G.F. with intensity $e^b\lambda$, so N_t is a $\mathbb{P}^*$ Poisson process with intensity $\lambda^* = e^b\lambda$.

1.4 Ito's Lemma with Jump Processes

For $\mathbb{P}$ Poisson process N_t with intensity λ, let

$$X_t \equiv g(N_t) - g(N_0),$$

then for any partition Π of $[0,t]$, $\Pi = \{t_0, t_1, ..., t_m\}$, where $t_0 = 0$ and $t_m = t$, we can use telescoping sum to get

$$X_t = \sum_k g(N_{t_k}) - g(N_{t_k-1})$$

If we take the limit as $\|\Pi\| \downarrow 0$,

$$X_t = \lim_{\|\Pi\|\downarrow 0}\sum_k g(N_{t_k}) - g(N_{t_k-1}) = \int_0^t \underbrace{(g(N_{u^-}+1) - g(N_{u^-}))}_{\text{L.C.R.L}}\;\underbrace{dN_u}_{\text{R.C.L.L}}$$

Therefore,

$$dX_t = (g(N_{t^-}+1) - g(N_{t^-}))dN_t$$

Now, suppose $X_t = g(t, N_t)$, we can argue similarly that

$$dX_t = \partial_t g(t, N_{t^-})dt + (g(t, N_{t^-}+1) - g(t, N_{t^-}))dN_t$$

Ito's Lemma

Suppose

$$\begin{aligned}
dY_t &= \mu(t, Y_{t^-})dt + \sigma(t, Y_{t^-})dN_t \\
X_t &= g(t, Y_t)
\end{aligned}$$

then,

$$dX_t = \partial_t g(t, Y_{t^-})dt + \mu(t, Y_{t^-})\partial_y g(t, Y_{t^-})dt + \left[g(t, Y_{t^-} + \sigma(t, Y_{t^-})) - g(t, Y_{t^-})\right]dN_t$$

Now, suppose the price of underlying asset can be modelled through Geometric Poisson process

$$dS_t = S_{t^-}(\mu dt + \sigma dN_t)$$

To solve the SDE, we make usuall transformation of $X_t = \log(S_t)$, by Ito's lemma above, we have

$$\begin{aligned}
dX_t &= \mu S_{t^-}\cdot\frac{1}{S_t}dt + (\log(S_{t^-} + S_{t^-}\sigma) - \log(S_{t^-}))dN_t \\
&= \mu dt + \log(1+\sigma)dN_t
\end{aligned}$$

Then, $X_t - X_0 = \mu t + \log(1+\sigma)N_t$, so

$$S_t = S_0 e^{\mu t}(1+\sigma)^{N_t}.$$

Note that the expected return of S_t under $\mathbb{P}$ is

$$\mathbb{E}^{\mathbb{P}}\left[\frac{S_t}{S_0}\right] = e^{\mu t}\mathbb{E}\left[(1+\sigma)^{N_t}\right] = e^{\mu t}e^{(e^{\log(1+\sigma)}-1)\lambda t} = e^{(\mu+\sigma\lambda)t}$$

i.e., the jump term has contribution $\sigma\lambda$ for the expected return of S_t.

1.5 Dynamic Hedging Argument with Jump Process

Suppose we have the underlying asset S_t and money market M_t modeled through the following SDEs

$$\begin{aligned} dS_t &= S_{t^-}(\mu dt + \sigma dN_t) \\ dM_t &= rM_t dt \end{aligned}$$

Then suppose we have a derivative valued $(g_t)_{t\geq 0}$ written on S_t with maturity T with final payoff structure $g_T = G(S_T)$. Note that the value of the derivative security g_t is Markov in t and S_t, so we write $g_t = g(t, S_t)$. We are interested in finding the price g_t, in terms of the function g.
Suppose we have a portfolio consisting of $(\alpha_t, \beta_t, -1)$ in S_t, M_t, and g_t

$$V_t = \alpha_t S_t + \beta_t M_t - g_t$$

with $V_0 = 0$. By self-financing constraint,

$$dV_t = \alpha_{t^-} dS_t + \beta_{t^-} dM_t - dg_t$$

By Ito's lemma and by definition,

$$\begin{aligned} dV_t &= \alpha_{t^-} S_{t^-}(\mu dt + \sigma dN_t) + \beta_{t^-} rM_t dt - \{[\partial_t g(t, S_t) + \mu S_t \partial_s g(t, S_{t^-})]dt + [g(t, S_{t^-} + \sigma S_{t^-}) - g(t, S_{t^-})]dN_t\} \\ &= [\alpha_{t^-} S_{t^-}\mu + \beta_{t^-} rM_t - \partial_t g - \mu S_{t^-}\partial_s g]dt + [\alpha_{t^-} S_{t^-}\sigma - g(t, S_{t^-} + \sigma S_{t^-}) + g(t, S_{t^-})]dN_t \end{aligned}$$

To remove local risk, set the risk dN_t term to 0,

$$\alpha_{t^-} S_{t^-}\sigma - g(t, S_{t^-} + \sigma S_{t^-}) + g(t, S_{t^-}) = 0 \Longrightarrow \alpha_{t^-} = \frac{g(t, (1+\sigma))S_{t^-}) - g(t, S_{t^-})}{\sigma S_{t^-}}$$

Since $V_0 = 0$, a drift with non-zero value at any time will result in arbitrage. So we need to set the drift term equal to 0 as well. i.e.,

$$\alpha_{t^-} S_{t^-}\mu + \beta_{t^-} rM_t - \partial_t g - \mu S_{t^-}\partial_s g = 0$$

Since both drift and risk are adjusted to 0, we have $V_t = 0$ for all t, thus it is required that

$$\beta_t M_t = g_t - \alpha_t S_t$$

Substituting into previous result to get the final PIDE representation of the derivative price function

$$\begin{aligned} &\alpha_{t^-} S_{t^-}(\mu - r) + rg_t - \partial_t g - \mu S_{t^-}\partial_s g = 0 \\ \Rightarrow &\partial_t g(t, s) + \mu s \partial_s g(t, s) + \frac{r-\mu}{\sigma}[g(t, (1+\sigma)s) - g(t, s)] = rg(t, s) \end{aligned}$$