State Prices

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1 Change of Measure

actual probability measure - the model which we seek by empirical estimation of the model parameters.
risk-neutral probability measure - under which the discounted prices of assets are martingales.
The two measures agree on the state space (equivalent measures), but disagree on the probability of each state occurring.
Consider a finite sample space Ω on which we have two probability measures $\mathbb{P}$ and $\tilde{\mathbb{P}}$, both giving positive probability for every $\omega \in \Omega$. Form quotient

$$Z(\omega) = \frac{\tilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)}$$

Z is a random variable because it depends on the outcome ω of a random experiment. Z is called the *Radon-Nikodym derivative of* $\tilde{\mathbb{P}}$ *w.r.t.* $\mathbb{P}$.

Theorem 1. *Let* $\mathbb{P}$ *and* $\tilde{\mathbb{P}}$ *be probability measures on finite sample space* Ω*, assume that* $\mathbb{P}(\omega) > 0$ *and* $\tilde{\mathbb{P}}(\omega) > 0$ *for every* $\omega \in \Omega$*, and define random variable* Z *as* $Z(\omega) = \frac{\tilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)}$*. Then we have the following:*
(i) $\mathbb{P}(Z > 0) = 1$*;*
(ii) $\mathbb{E}Z = 1$*;*
(iii) for any random variable Y*,*

$$\tilde{\mathbb{E}}Y = \mathbb{E}[ZY].$$

Proof. Property (i) follows immediately from the fact that we have assumed $\mathbb{P}(\omega) > 0$ and $\tilde{\mathbb{P}}(\omega) > 0$ for every $\omega \in \Omega$.
Property (ii)

$$\mathbb{E}Z = \sum_{\omega \in \Omega} Z(\omega)\mathbb{P}(\omega) = \sum_{\omega \in \Omega} \frac{\tilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)}\mathbb{P}(\omega) = \sum_{\omega \in \Omega} \tilde{\mathbb{P}}(\omega) = 1$$

Property (iii)

$$\tilde{\mathbb{E}}Y = \sum_{\omega \in \Omega} Y(\omega)\tilde{\mathbb{P}}(\omega) = \sum_{\omega \in \Omega} Y(\omega)\frac{\tilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)}\mathbb{P}(\omega) = \sum_{\omega \in \Omega} Z(\omega)Y(\omega)\mathbb{P}(\omega) = \mathbb{E}[ZY]$$

□

Definition 2. In the N-period binomial model with actual probability measure $\mathbb{P}$ and risk-neutral probability measure $\tilde{\mathbb{P}}$, let Z denote the Radon-Nikodym derivative of $\tilde{\mathbb{P}}$ w.r.t. $\mathbb{P}$; i.e.

$$Z(\omega_1 ... \omega_N) = \frac{\tilde{\mathbb{P}}(\omega_1 ... \omega_N)}{\mathbb{P}(\omega_1 ... \omega_N)} = \left(\frac{\tilde{p}}{p}\right)^{\#H(\omega_1 ... \omega_N)} \left(\frac{\tilde{q}}{q}\right)^{\#T(\omega_1 ... \omega_N)}$$

The *state price density* random variable is

$$\zeta(\omega) = \frac{Z(\omega)}{(1+r)^N}$$

and $\zeta(\omega)\mathbb{P}(\omega)$ is called the *state price* corresponding to ω.

Note that the state price $\zeta(\bar{\omega})\mathbb{P}(\bar{\omega})$ is the price at time zero of a contract that pays 1 at time N if and only if $\bar{\omega}$ occurs.
For contracts that with payoffs not necessarily 1, and with payoffs associated with more than one ω, we can consider the contract as a portfolio of the "simple contracts", each of which pays off 1 if and only if some particular ω occurs. To see this, note that

$$V_0 = \tilde{\mathbb{E}}\frac{V_N}{(1+r)^N} = \mathbb{E}[\zeta V_N] = \sum_{\omega\in\Omega} V_N(\omega)\zeta(\omega)\mathbb{P}(\omega)$$

2 Radon-Nikodym Derivative Process

To estimate Z based on information at time $n < N$, where Z is not required to be a Radon-Kikodym derivative, we develop the following theorem

Theorem 3. *Let Z be a random variable in an N-period binomial model. Define*

$$Z_n = \mathbb{E}_n Z, n = 0, 1, ..., N.$$

Then $Z_n, n = 0, 1, ..., N$ is a martingale under $\mathbb{P}$.

Proof. For $n = 0, 1, ..., N-1$, we use the iterated conditioning property.

$$\mathbb{E}_n[Z_{n+1}] = \mathbb{E}_n[\mathbb{E}_{n+1}[Z]] = \mathbb{E}_n[Z] = Z_n$$

□

Note that the martingale property holds not only under $\mathbb{P}$ but the risk-neutral measure $\tilde{\mathbb{P}}$ as well.

Definition 4. In an N-period binomial model, let $\mathbb{P}$ be the actual probability measure, $\tilde{\mathbb{P}}$ the risk-neutral probability measure, and assume that $\mathbb{P}(\omega) > 0$ and $\tilde{\mathbb{P}}(\omega) > 0$ for every sequence of coin tosses ω. Define the Randon-Nikodym derivative (random variable) $Z(\omega) = \frac{\tilde{\mathbb{P}}(\omega)>0}{\mathbb{P}(\omega)>0}$ for every ω. The Radon-Nikodym derivative process is

$$Z_n = \mathbb{E}_n[Z], n = 0, 1, ..., N$$

In particular, $Z_N = Z$ and $Z_0 = 1$.

Theorem 5. *Assume the conditions of above definition. Let n be a positive integer between 0 and N, and let Y be a random variable depending only on the first n coin tosses. Then*

$$\tilde{\mathbb{E}}Y = \mathbb{E}[Z_n Y]$$

Proof.

$$\tilde{\mathbb{E}}Y = \mathbb{E}[ZY] = \mathbb{E}[\mathbb{E}_n[ZY]] = \mathbb{E}[Y\mathbb{E}_n[Z]] = \mathbb{E}[YZ_n]$$

□

Theorem 6. *Consider an N-period binomial model with $0 < d < 1 + r < u$. Let $\mathbb{P}$ and $\tilde{\mathbb{P}}$ denote the corresponding actual and risk-neutral probability measures, respectively. Let Z be the Radon-Nikodym derivative of $\tilde{\mathbb{P}}$ w.r.t. $\mathbb{P}$, and let $Z_n, n = 0, 1, ..., N$, be the Radon-Nikodym derivative process. Consider a derivative security whose payoff V_N may depend on all N coin tosses. For $n = 0, 1, .., N$, the price at time n of the derivative security is*

$$V_n = \tilde{\mathbb{E}}_n\frac{V_N}{(1+r)^{N-n}} = \frac{(1+r)^n}{Z_n}\mathbb{E}_n\frac{Z_N V_N}{(1+r)^N} = \frac{1}{\zeta_n}\mathbb{E}_n[\zeta_N V_N],$$

where the state price density process ζ_n is defined by

$$\zeta_n = \frac{Z_n}{(1+r)^n}, n = 0, 1, ..., N.$$

3 Capital Asset Pricing Model

Optimal Investment Problem

Given X_0, find an adaped portfolio process $\Delta_0, \Delta_1, ..., \Delta_{N-1}$ that maximizes

$$\mathbb{E}U(X_N)$$

subject to the wealth equation

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n), n = 0, 1, ..., N-1.$$

Note that the agent uses the acutal probability measure $\mathbb{P}$ and uses uitility function U to capture the trade-off between risk and return. Using risk neutral probability does not make sense because under the risk-neutral measure, every asset will have the same expected return; an agent seeking to maximize $\tilde{\mathbb{E}}U(X)$ would invest only in the money market. However, as the number of periods increases, the number of varialbes $\Delta_n(\omega)$ grows exponentially. So we can also formulate the problem as follows:

Same Investment Problem Revisited

Given X_0, find a random variable X_N (without regard to a portfolio process) that maximizes

$$\mathbb{E}U(X_N)$$

subject to

$$\tilde{\mathbb{E}}\frac{X_N}{(1+r)^N} = X_0$$

If we use superscripts to indicate that ω^m isa fulll sequence of coin tosses, and define $\zeta_m = \zeta(\omega^m)$, $p_m = \mathbb{P}(\omega^m)$, and $x_m = X_N(\omega^m)$. Then the problem can be reformulated as follows:
Given X_0, find a vector $(x_1, x_2, ..., x_M)$ that maximizes

$$\sum_{m=1}^{M} p_m U(x_m)$$

subject to

$$\sum_{m=1}^{M} p_m x_m \zeta_m = X_0.$$

The Lagrange multiplier equations are

$$\frac{\partial}{\partial x_m} L = p_m U'(x_m) - \lambda p_m \zeta_m = 0, m = 1, 2, ..., M$$

So

$$U'(x_m) = \lambda \zeta_m \Longrightarrow U'(X_N) = \frac{\lambda Z}{(1+r)^N} \Longrightarrow X_N = (U')^{-1}\left(\frac{\lambda Z}{(1+r)^N}\right)$$

Therefore, we can solve for λ by

$$X_0 = \mathbb{E}\left[\frac{Z}{(1+r)^N}(U')^{-1}\left(\frac{\lambda Z}{(1+r)^N}\right)\right]$$