Probability Theory on Coin Toss Space

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Exercise 1.

(i) We first prove disjoint additivity: if E_1 and E_2 are disjoint, then $\mathbb{P}(E_1) + \mathbb{P}(E_2) = \mathbb{P}(E_1 \cup E_2)$.

$$\mathbb{P}(E_1) + \mathbb{P}(E_2) = \sum_{\omega \in E_1} \mathbb{P}(\omega) + \sum_{\omega \in E_2} \mathbb{P}(\omega) = \sum_{\omega \in (E_1 \cup E_2)} \mathbb{P}(\omega) = \mathbb{P}(E_1 \cup E_2)$$

Since A and A^C are disjoint,

$$\mathbb{P}(A^C) + \mathbb{P}(A) = \mathbb{P}(\Omega) = 1 \Rightarrow \mathbb{P}(A^C) = 1 - \mathbb{P}(A)$$

(ii) Note that disjoint additivity holds for more than two sets, so the equality case holds trivially.
Disjoint additivity also shows if $E_1 \subset E_2$, then $\mathbb{P}(E_1) \leq \mathbb{P}(E_2)$, since

$$E_2 = E_1 \cup (E_2 \setminus E_1) \Rightarrow \mathbb{P}(E_2) = \mathbb{P}(E_1) + \mathbb{P}(E_2 \setminus E_1) \Rightarrow \mathbb{P}(E_1) \leq \mathbb{P}(E_2)$$

Now, let $E_0 = A_0$, and let $E_k = A_k \setminus (\cup_{i=1}^{k-1} E_i)$ for $k = 2, 3, ..., N$, then E_i are disjoint and $\cup_{i=1}^{N} E_i = \cup_{i=1}^{N} A_i$. So,

$$\mathbb{P}\left(\bigcup_{n=1}^{N} A_n\right) = \mathbb{P}\left(\bigcup_{n=1}^{N} E_n\right) = \sum_{n=1}^{N} \mathbb{P}(E_n) \leq \sum_{n=1}^{N} \mathbb{P}(A_n)$$

Exercise 2.

(i) The probabilities follow binomial probabilities with $p = \tilde{p} = \frac{1}{2}$. i.e.

$$\begin{aligned}
\tilde{\mathbb{P}}(S_3 = 32) &= \binom{3}{3}\left(\frac{1}{2}\right)^3 = \frac{1}{8} \\
\tilde{\mathbb{P}}(S_3 = 8) &= \binom{3}{2}\left(\frac{1}{2}\right)^3 = \frac{3}{8} \\
\tilde{\mathbb{P}}(S_3 = 2) &= \binom{3}{1}\left(\frac{1}{2}\right)^3 = \frac{3}{8} \\
\tilde{\mathbb{P}}(S_3 = 0.5) &= \binom{3}{0}\left(\frac{1}{2}\right)^3 = \frac{1}{8}
\end{aligned}$$

(ii)

$$
\begin{aligned}
\tilde{\mathbb{E}}S_1 =& \binom{1}{1}\tilde{p}S_1(H) + \tilde{q}S_1(T) = \frac{1}{2}\cdot 8 + \frac{1}{2}\cdot 2 = 5 \\
\tilde{\mathbb{E}}S_2 =& \binom{2}{2}\tilde{p}^2 S_2(HH) + \binom{2}{1}\tilde{p}\tilde{q}S_2(HT) + \binom{2}{0}\tilde{q}^2 S_2(TT) = \frac{1}{4}\cdot 16 + \frac{2}{4}\cdot 4 + \frac{1}{4}\cdot 1 = 6.25 \\
\tilde{\mathbb{E}}S_3 =& \binom{3}{3}\tilde{p}^3 S_3(HHH) + \binom{3}{2}\tilde{p}^2\tilde{q}S_3(HHT) + \binom{3}{1}\tilde{p}\tilde{q}^2 S_3(HTT) + \binom{3}{0}\tilde{q}^3 S_3(TTT) = 7.8125
\end{aligned}
$$

The rate of growth of mean stock price under $\tilde{\mathbb{P}}$ is $1 + r = 1.25$.
(iii)

$$
\begin{aligned}
\mathbb{P}(S_3 = 32) =& \binom{3}{3}\left(\frac{2}{3}\right)^3 = \frac{8}{27} \\
\mathbb{P}(S_3 = 8) =& \binom{3}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right) = \frac{12}{27} \\
\mathbb{P}(S_3 = 2) =& \binom{3}{1}\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^2 = \frac{6}{27} \\
\mathbb{P}(S_3 = 0.5) =& \binom{3}{0}\left(\frac{1}{3}\right)^3 = \frac{1}{27}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}S_1 =& \frac{2}{3}\cdot 8 + \frac{1}{3}\cdot 2 = 6 \\
\mathbb{E}S_2 =& \frac{4}{9}\cdot 16 + \frac{4}{9}\cdot 4 + \frac{1}{9}\cdot 1 = 9 \\
\mathbb{E}S_3 =& \frac{8}{27}\cdot 32 + \frac{12}{27}\cdot 8 + \frac{6}{27}\cdot 2 + \frac{1}{27}\cdot 0.5 = 13.5
\end{aligned}
$$

The rate of growth of mean stock price under $\mathbb{P}$ is 1.5.

Exercise 3.

By martingale property, for any $0 \le n \le N-1$,

$$M_n = \mathbb{E}_n M_{n+1} \Rightarrow \varphi(M_n) = \varphi(\mathbb{E}_n M_{n+1})$$

By conditional Jensen's inequality, since φ is convex,

$$\varphi(M_n) = \varphi(\mathbb{E}_n M_{n+1}) \le \mathbb{E}_n[\varphi(M_{n+1})]$$

Thus, $\varphi(M_0), ..., \varphi(M_N)$ is a submartingale.

Exercise 4.

(i) For arbitrary $n \geq 0$, we have

$$\begin{aligned}
\mathbb{E}_n[M_{n+1}] =& \mathbb{E}_n[M_n + X_{n+1}] \\
=& \mathbb{E}_n[M_n \cdot 1] + \mathbb{E}_n[X_{n+1}] && \text{by linearity of C.E.} \\
=& M_n \mathbb{E}_n[1] + \mathbb{E}_n[X_{n+1}] && \text{by taking out what is known} \\
=& M_n + \mathbb{E}[X_{n+1}] && \text{by independence} \\
=& M_n + \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (-1) \\
=& M_n
\end{aligned}$$

(ii) For arbitrary $n \geq 0$, we have

$$\begin{aligned}
\mathbb{E}_n[S_{n+1}] =& \mathbb{E}_n\left[e^{\sigma M_{n+1}} \left(\frac{2}{e^\sigma + e^{-\sigma}}\right)^{n+1}\right] \\
=& \mathbb{E}_n\left[e^{\sigma M_n} \left(\frac{2}{e^\sigma + e^{-\sigma}}\right)^{n+1} e^{\sigma X_{n+1}}\right] \\
=& \left(\frac{2}{e^\sigma + e^{-\sigma}}\right) \mathbb{E}\left[e^{\sigma X_{n+1}}\right] \mathbb{E}_n\left[e^{\sigma M_n} \left(\frac{2}{e^\sigma + e^{-\sigma}}\right)^{n}\right] \\
=& \left(\frac{2}{e^\sigma + e^{-\sigma}}\right) \left(\frac{1}{2}e^\sigma + \frac{1}{2}e^{-\sigma}\right) S_n \\
=& S_n
\end{aligned}$$

Exercise 5.

(i)

$$\begin{aligned}
2I_n =& 2\sum_{j=0}^{n-1} M_j(M_{j+1} - M_j) = 2\sum_{j=0}^{n-1}\sum_{i=1}^{j} X_i X_{j+1} = 2\left(\sum_{j=1}^{n}\sum_{i=1}^{j} X_i X_j - \sum_{j=1}^{n} X_j^2\right) \\
=& \left(\sum_{j=1}^{n} X_j\right)^2 + \sum_{j=1}^{n} X_j^2 - 2\sum_{j=1}^{n} X_j^2 = M_n^2 - n
\end{aligned}$$

the last equality holds because $X_j^2 = 1$ in both cases.

(ii)

$$\begin{aligned}
E_n[f(I_{n+1})] =& E_n\left[f\left(\frac{1}{2}M_{n+1}^2 - \frac{n+1}{2}\right)\right] \\
=& E_n\left[f\left(\frac{1}{2}(M_n^2 + 2X_{n+1}M_n + X_{n+1}^2 - n - 1)\right)\right] \\
=& E_n\left[f(I_n + M_n X_{n+1})\right]
\end{aligned}$$

Since I_n and M_n are known at time n, we have

$$
\begin{aligned}
g(I_n) =&E_n[f(I_{n+1})] = E_n[f(I_n + M_n X_{n+1})] \\
=&\frac{1}{2}f(I_n + M_n) + \frac{1}{2}f(I_n - M_n) \\
=&\frac{1}{2}\left(f(I_n + \sqrt{2I_n + n}) + f(I_n - \sqrt{2I_n + n})\right)
\end{aligned}
$$

Exercise 6.

For any $0 \le n \le N-1$,

$$
\begin{aligned}
E_n[I_{n+1}] =&E_n[I_n + \Delta_n(M_{n+1} - M_n)] \\
=&I_n + \Delta_n E_n[M_{n+1} - M_n] \\
=&I_n
\end{aligned}
$$

Exercise 7.

Let $X_0 = 0$, $X_1 = 1$ with probability $\frac{1}{2}$, and $X_1 = -1$ with probability $\frac{1}{2}$. For $n \ge 2$, $X_n = X_{n-1} + \varepsilon_n$ with probability $\frac{1}{2}$, and $X_n = X_{n-1} - \varepsilon_n$ with probability $\frac{1}{2}$, where $\varepsilon_n = 1$ if $X_{n-2} > X_{n-1}$, and $\varepsilon = 2$ if $X_{n-2} < X_{n-1}$.

Exercise 8.

(i) For $n = N-1$, for any $\omega_1 ... \omega_n$,

$$
\begin{aligned}
M'_n(\omega_1...\omega_n) =&\frac{1}{1+r}[\tilde{p}M'_N(\omega_1...\omega_n H) + \tilde{q}M'_N(\omega_1...\omega_n T)] \\
=&\frac{1}{1+r}[\tilde{p}M_N(\omega_1...\omega_n H) + \tilde{q}M_N(\omega_1...\omega_n T)] \\
=&M_N(\omega_1...\omega_n)
\end{aligned}
$$

The argument can be applied recursively until n reaches 0.
(ii) for any $0 \le n \le N-1$, and for every $\omega_1...\omega_n$,

$$
\begin{aligned}
\tilde{\mathbb{E}}_n\left[\frac{V_{n+1}}{(1+r)^{n+1}}\right](\omega_1...\omega_n) &= \frac{1}{(1+r)^{n+1}}[\tilde{p}V_{n+1}(\omega_1...\omega_n H) + \tilde{q}V_{n+1}(\omega_1...\omega_n T)] \\
&= \frac{1}{(1+r)^n}\frac{1}{1+r}[\tilde{p}V_{n+1}(\omega_1...\omega_n H) + \tilde{q}V_{n+1}(\omega_1...\omega_n T)] \\
&= \frac{V_n}{(1+r)^n}
\end{aligned}
$$

(iii) for any $0 \le n \le N-1$, and for every $\omega_1...\omega_n$, (here we suppress $\omega_1...\omega_n$)

$$
\begin{aligned}
\tilde{\mathbb{E}}_n\left[\frac{V'_{n+1}}{(1+r)^{n+1}}\right] &= \frac{1}{(1+r)^{n+1}}\tilde{\mathbb{E}}_n\left[\tilde{\mathbb{E}}_{n+1}\left[\frac{V'_N}{(1+r)^{N-n-1}}\right]\right] \\
&= \frac{1}{(1+r)^N}\tilde{\mathbb{E}}_n\left[V'_N\right] \\
&= \frac{V'_n}{(1+r)^n}
\end{aligned}
$$

(iv) Conclusion can be drawn from part (i), since the terminal payoff are the same $V_N = V'_N$, and $V_0, V_1, ..., V_N$, and $V'_0, V'_1, ..., V'_N$ are martingales.

Exercise 9.

(i) We have

$$\begin{aligned}
\tilde{p}_1(H) &= \frac{1 + r_1(H) - d_1(H)}{u_1(H) - d_1(H)} = \frac{1 + \frac{1}{4} - 1}{\frac{3}{2} - 1} = \frac{1}{2}, \tilde{q}_1(H) = 1 - \tilde{p}_1(H) = \frac{1}{2} \\
\tilde{p}_1(T) &= \frac{1 + r_1(T) - d_1(T)}{u_1(T) - d_1(T)} = \frac{1 + \frac{1}{2} - 1}{4 - 1} = \frac{1}{6}, \tilde{q}_1(T) = 1 - \tilde{p}_1(T) = \frac{5}{6} \\
\tilde{p}_0 &= \frac{1 + r_0 - d_0}{u_0 - d_0} = \frac{1 + \frac{1}{4} - \frac{1}{2}}{2 - \frac{1}{2}} = \frac{1}{2}, \tilde{q}_1(T) = 1 - \tilde{p}_1(T) = \frac{1}{2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\tilde{\mathbb{P}}(HH) &= \tilde{p}_0 \tilde{p}_1(H) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}, \tilde{\mathbb{P}}(HT) = \tilde{p}_0 \tilde{q}_1(H) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \\
\tilde{\mathbb{P}}(TH) &= \tilde{q}_0 \tilde{p}_1(T) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}, \tilde{\mathbb{P}}(HT) = \tilde{q}_0 \tilde{q}_1(T) = \frac{1}{2} \cdot \frac{5}{6} = \frac{5}{12}
\end{aligned}$$

Note that

$$\begin{aligned}
\tilde{\mathbb{E}}\left[\frac{S_2}{(1 + r_0)(1 + r_1)}\right] =& \frac{1}{4} \cdot \frac{12}{(1 + \frac{1}{4})(1 + \frac{1}{4})} + \frac{1}{4} \cdot \frac{8}{(1 + \frac{1}{4})(1 + \frac{1}{4})} + \\
& \frac{1}{12} \cdot \frac{8}{(1 + \frac{1}{4})(1 + \frac{1}{2})} + \frac{5}{12} \cdot \frac{2}{(1 + \frac{1}{4})(1 + \frac{1}{2})} \\
=& 4 = S_0
\end{aligned}$$

(ii) $V_2 = (S_2 - 7)^+$, so $V_2(HH) = 12 - 7 = 5$, $V_2(HT) = 8 - 7 = 1$, $V_2(TH) = 8 - 7 = 1$, $V_2(TT) = 0$. Therefore,

$$\begin{aligned}
V_1(H) &= \tilde{\mathbb{E}}_1\left[\frac{V_2}{1 + r_1}\right](H) = \frac{1}{2} \cdot \frac{5}{1 + \frac{1}{4}} + \frac{1}{2} \cdot \frac{1}{1 + \frac{1}{4}} = \frac{12}{5} \\
V_1(T) &= \tilde{\mathbb{E}}_1\left[\frac{V_2}{1 + r_1}\right](T) = \frac{1}{6} \cdot \frac{1}{1 + \frac{1}{2}} + 0 = \frac{1}{9} \\
V_0 &= \tilde{\mathbb{E}}\left[\frac{V_2}{(1 + r_0)(1 + r_1)}\right] = \frac{1}{4} \cdot \frac{5}{(1 + \frac{1}{4})(1 + \frac{1}{4})} + \frac{1}{4} \cdot \frac{1}{(1 + \frac{1}{4})(1 + \frac{1}{4})} + \frac{1}{12} \cdot \frac{1}{(1 + \frac{1}{4})(1 + \frac{1}{2})} = \frac{236}{225}
\end{aligned}$$

(iii)

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \frac{\frac{12}{5} - \frac{1}{9}}{8 - 2} = \frac{103}{270}$$

(iv)

$$\Delta_1(H) = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)} = \frac{5 - 1}{12 - 8} = 1$$

Exercise 10.

(i) for any $0 \le n \le N-1$, we have

$$\begin{aligned}\tilde{\mathbb{E}}_n\left[\frac{X_{n+1}}{(1+r)^{n+1}}\right] &= \tilde{\mathbb{E}}_n\left[\frac{\Delta_n Y_{n+1} S_n + (1+r)(X_n - \Delta_n S_n)}{(1+r)^{n+1}}\right] \\ &= \frac{1}{(1+r)^n}\left[\Delta_n S_n \frac{\tilde{\mathbb{E}}_n[Y_{n+1}]}{1+r} + X_n - \Delta_n S_n\right] \\ &= \frac{X_n}{(1+r)^n} + \frac{1}{(1+r)^{n+1}}\Delta_n S_n\left[\tilde{\mathbb{E}}_n[Y_{n+1}] - 1 - r\right] \\ &= \frac{X_n}{(1+r)^n} + \frac{1}{(1+r)^{n+1}}\Delta_n S_n\left[\frac{1+r-d}{u-d}\cdot u + \frac{u-1-r}{u-d}\cdot d - 1 - r\right] \\ &= \frac{X_n}{(1+r)^n}\end{aligned}$$

(ii) By definition, we have $V_n = \frac{1}{1+r}\tilde{\mathbb{E}}_n[V_{n+1}]$, i.e. $V_0, V_1, ..., V_N$ is a $\tilde{\mathbb{P}}$ martingale. Also, since $X_N = V_N$ for all $\omega_1 ... \omega_N$, we have $X_n = V_n$ for all n in all $\omega_1 ... \omega_n$. Therefore, V_n is indeed the price of the derivative that pays V_N at time N.
(iii) Since $A_n \in (0,1)$,

$$\tilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{1+r}\right] = \tilde{\mathbb{E}}_n\left[\frac{(1-A_{n+1})Y_{n+1}S_n}{1+r}\right] < S_n \tilde{\mathbb{E}}_n\left[\frac{Y_{n+1}}{1+r}\right] = \frac{S_n}{1+r}\left(\frac{1+r-d}{u-d}\cdot u + \frac{u-1-r}{u-d}d\right) = S_n$$

Now, suppose A_n is constant $a \in (0,1)$,

$$\begin{aligned}\tilde{\mathbb{E}}_n\left[\frac{S_{n+1}}{(1-a)^{n+1}(1+r)^{n+1}}\right] &= \frac{S_n}{(1-a)^n(1+r)^n}\tilde{\mathbb{E}}_n\left[\frac{(1-a)Y_{n+1}}{(1-a)(1+r)}\right] \\ &= \frac{S_n}{(1-a)^n(1+r)^n}\left(\frac{1+r-d}{u-d}\cdot u \frac{u-1-d}{u-d}\cdot d\right) \\ &= \frac{S_n}{(1-a)^n(1+r)^n}\end{aligned}$$

Exercise 11.

(i)

$$F_N + P_N = S_N - K + (K - S_N)^+ = \begin{cases} S_N - K & , S_N \ge K \\ 0 & , S_N < K \end{cases} = (S_N - K)^+ = C_N$$

(ii)

$$F_n + P_n = \tilde{\mathbb{E}}_n\left[\frac{F_N}{(1+r)^{N-n}}\right] + \tilde{\mathbb{E}}_n\left[\frac{P_N}{(1+r)^{N-n}}\right] = \tilde{\mathbb{E}}_n\left[\frac{F_N + P_N}{(1+r)^{N-n}}\right] = \tilde{\mathbb{E}}_n\left[\frac{C_N}{(1+r)^{N-n}}\right] = C_n$$

(iii)

$$F_0 = \tilde{\mathbb{E}}_0\left[\frac{F_N}{(1+r)^N}\right]\tilde{\mathbb{E}}_0\left[\frac{S_N - K}{(1+r)^N}\right] = \tilde{\mathbb{E}}_0\left[\frac{S_N}{(1+r)^N}\right] - \frac{K}{(1+r)^N} = S_0 - \frac{K}{(1+r)^N}$$

(iv) At time 0: have 1 share of stock and $F_0 - S_0 = -\frac{K}{(1+r)^N}$ in bank account

At time N: $S_N + (1+r)^N \left(-\frac{K}{(1+r)^N}\right) = S_N - K = F_N$

(v)

$$C_0 = F_0 + P_0 = \tilde{\mathbb{E}}_0 \left[\frac{S_N - S_0(1+r)^N}{(1+r)^N}\right] + P_0 = P_0$$

(vi) Put all subscripts (conditional expectations) to n instead of 0

Exercise 12.

At time m, the value V_m of the chooser's option is

$$V_m = \max(C_m, P_m) = P_m + (C_m - P_m)^+ = P_m + F_m^+ = P_m + \left(S_m - \frac{K}{(1+r)^{N-m}}\right)^+$$

Therefore, at time 0, the price of the chooser's option is

$$V_0 = P_0 + C_0'$$

where C' is a call option with strike $\frac{K}{(1+r)^{N-m}}$ and maturity m.

Exercise 13.

(i) For every function g, we can write

$$\tilde{\mathbb{E}}_n[g(S_{n+1}, Y_{n+1})] = \tilde{p}g(uS_n, Y_n + uS_n) + \tilde{q}g(dS_n, Y_n + dS_n) = h(S_n, Y_n)$$

Therefore, $(S_n, Y_n), n = 0, 1, ..., N$ is Markov.

(ii)

$$V_N = f\left(\frac{1}{N+1}\sum_0^N S_n\right) \Rightarrow v_N(s, y) = f\left(\frac{1}{N+1}y\right)$$

Now, for $n = 0, 1, ..., N-1$,

$$v_n(s, y) = \tilde{p}v_{n+1}(us, y + us) + \tilde{q}v_{n+1}(ds, y + ds)$$

Exercise 14.

(i) For every function g, and for $n = M, M+1, ..., N$,

$$\tilde{\mathbb{E}}_n[g(S_{n+1}, Y_{n+1})] = \tilde{p}g(uS_n, Y_n + uS_n) + \tilde{q}g(dS_n, Y_n + dS_n)$$

and for $n = 0, 1, ..., M-2$,

$$\tilde{\mathbb{E}}_n[g(S_{n+1}, Y_{n+1})] = \tilde{p}g(uS_n, 0) + \tilde{q}g(dS_n, 0)$$

(ii) For $n = M+1, ..., N-1$,

$$v_n(s, y) = \frac{1}{1+r}[\tilde{p}v_{n+1}(us, y + us) + \tilde{q}v_{n+1}(ds, y + ds)]$$

For $n = M$,

$$v_n(s) = \frac{1}{1+r}[\tilde{p}v_{n+1}(us, us) + \tilde{q}v_{n+1}(ds, ds)]$$

For $n = 0, 1, ..., M-1$,

$$v_n(s) = \frac{1}{1+r}[\tilde{p}v_{n+1}(us) + \tilde{q}v_{n+1}(ds)]$$