The Binomial No-Arbitrage Pricing Model

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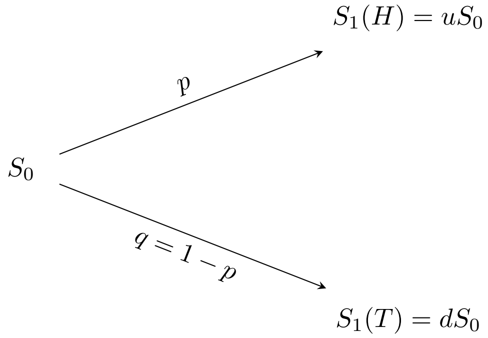
1 One-Period Binomial Model

Stock price per share at t_0: S_0. At t_1, stock price will be one of $S_1(H) = uS_0$ and $S_1(T) = dS_0$, depending on the result of a toss of a coin. Result is measurable at t_1 and unknown at t_0. We do not need $p = \mathbf{P}(H)$ to equal $q = 1 - p = \mathbf{P}(T)$.



Assume $u > r$. Also introduce interest rate r such that 1 dollar at t_0 becomes $1 + r$ dollars at t_1, only $r > -1$ is required. With the assumption that the stock price is always positive, to avoid arbitrage, we requrie

$$0 < d < 1 + r < u$$

Consider a *European Call option*, which gives its owner the right but not the obligation to buy one share of stock at the *strike price* K. i.e. payoff$= (S_1 - K)^+$.
The *arbitrage pricing theory* replicates the option by trading the stock and money markets. The no-arbitrage price of the option should be the time-zero price of the portfolio of stocks and money markets that replicates the time-one payoff of the option. It assumes the following:

- share of stock can be subdivided for sale or purchase
- the interest rate for investing is the same as the interest rate for borrowing
- the purchase price of stock is the same as the selling price
- at any time, the stock can take only two possible values in the next period

In general, suppose a derivative security pays $V_1(H)$ and $V_1(T)$ at t_1, and suppose we can replicate the portfolio with Δ_0 shares of stock and $X_0 - \Delta_0 S_0$ in the money market. Then our t_1 portfolio value is

$$X_1 = \Delta S_1 + (1 + r)(X_0 - \Delta S_0) = (1 + r)X_0 + \Delta_0(S_1 - (1 + r)S_0)$$

We need $X_1(H) = V_1(H)$ and $X_1(T) = V_1(T)$, i.e.

$$X_0 + \Delta_0 \left(\frac{1}{1 + r} S_1(H) - S_0 \right) = \frac{1}{1 + r} V_1(H)$$
$$X_0 + \Delta_0 \left(\frac{1}{1 + r} S_1(T) - S_0 \right) = \frac{1}{1 + r} V_1(T)$$

Introduce artificial quantity $\tilde{p}$ and $\tilde{q} = 1 - \tilde{p}$, such that

$$S_0 = \frac{1}{1+r}\left[\tilde{p}S_1(H) + \tilde{q}S_1(T)\right] \Longrightarrow \tilde{p} = \frac{1+r-d}{u-d}, \tilde{q} = \frac{u-1-r}{u-d}$$

Then multiply the first equation by $\tilde{p}$ plus the second equation multiplied by $\tilde{q}$, we get

$$X_0 = \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)]$$

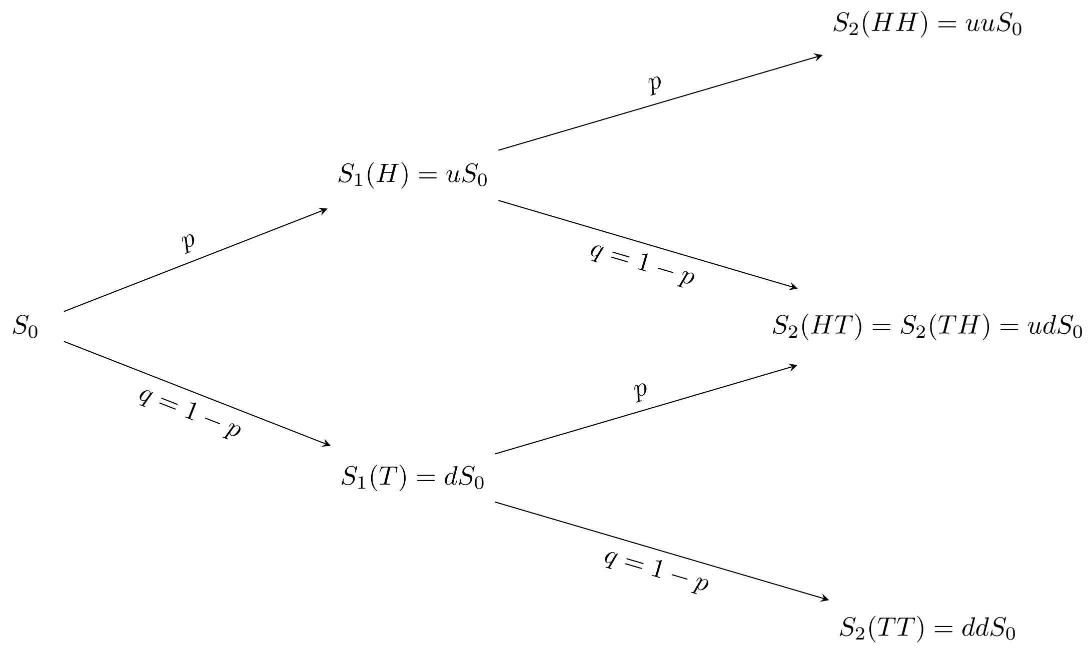
Therefore, by the no-arbitrage argument, the derivative that pays V_1 at t_1 should be priced at

$$V_0 = \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)]$$

at t_0. $\tilde{p}$ and $\tilde{q}$ (together constitutes probability measure $\mathbf{Q}$) are artificial probabilities that makes **all traded assets** to have expected discount r. $\mathbf{Q}$ is called risk-neutral measure, since if $\mathbf{Q}$ is the probability, investors are risk neutral.

2 Multiperiod Binomial Model

Extend the single period model to multi-period.



Suppose an European call option that pays $V_2 = (S_2 - K)^+$, where V_2 and S_2 depend on the first and second coin toss. Suppose an agent sells the option for V_0 dollars, then she buys Δ_0 shares of stock and invests $V_0 - \Delta_0 S_0$ in the money market to hedge her position. At t_1, the agent has a portfolio (not including one unit short in the option) valued at

$$X_1(H) = \Delta_0 S_1(H) + (1+r)(V_0 - \Delta_0 S_0)$$
$$X_1(T) = \Delta_0 S_1(T) + (1+r)(V_0 - \Delta_0 S_0)$$

She readjust her positions to Δ_1 shares of stock, and invests $X_1 - \Delta_1 S_1$ in the money market, then at t_2, she has

$$V_2(HH) = \Delta_1(H)S_2(HH) + (1+r)(X_1(H) - \Delta_1(H)S_1(H))$$
$$V_2(HT) = \Delta_1(H)S_2(HT) + (1+r)(X_1(H) - \Delta_1(H)S_1(H))$$
$$V_2(TH) = \Delta_1(T)S_2(TH) + (1+r)(X_1(T) - \Delta_1(T)S_1(T))$$
$$V_2(TT) = \Delta_1(T)S_2(TT) + (1+r)(X_1(T) - \Delta_1(T)S_1(T))$$

There are six equations, with six unknowns: $V_0, X_1(H), X_1(T), \Delta_0, \Delta_1(H), \Delta_1(T)$. The solution can be solved by breaking the problem into three 1-period models

$$\begin{aligned}
\Delta_1(T) &= \frac{V_2(TH) - V_2(TT)}{S_2(TH) - S_2(TT)} \\
V_1(T) = X_1(T) &= \frac{1}{1+r}[\tilde{p}V_2(TH) + \tilde{q}V_2(TT)] \\
\Delta_1(H) &= \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)} \\
V_1(H) = X_1(H) &= \frac{1}{1+r}[\tilde{p}V_2(HH) + \tilde{q}V_2(HT)] \\
\Delta_0 &= \frac{V_1(H) - V_1(T)}{(S_1(H) - S_1(T)} \\
V_0 = X_0 &= \frac{1}{1+r}[\tilde{p}V_1(H) + \tilde{q}V_1(T)]
\end{aligned}$$

V_n is call the no-arbitrage price of the derivative security at time n.

Theorem 1. ***(Replication in the multiperiod binomial model)*** *Consider an N-period binomial asset-pricing model, with $0 < d < 1 + r < u$, and with*

$$\tilde{p} = \frac{1+r-d}{u-d}, \tilde{q} = \frac{u-1-r}{u-d}$$

Let V_N be a random variable (a derivative security paying off at time N) depending on the first N coin tosses $\omega_1\omega_2...\omega_N$. Define recursively backward in time the sequence of random variables $V_{N-1}, V_{N-2}, ..., V_0$ by

$$V_n(\omega_1\omega_2...\omega_n) = \frac{1}{1+r}[\tilde{p}V_{n+1}(\omega_1\omega_2...\omega_n H) + \tilde{q}V_{n+1}(\omega_1\omega_2...\omega_n T)],$$

so that each V_n depends on the first n coin tosses $\omega_1\omega_2...\omega_n$ where n ranges between $N-1$ and 0. Next define

$$\Delta_n(\omega_1...\omega_n) = \frac{V_{n+1}(\omega_1...\omega_n H) - V_{n+1}(\omega_1...\omega_n T)}{S_{n+1}(\omega_1...\omega_n H) - S_{n+1}(\omega_1...\omega_n T)}$$

where again n ranges between 0 and $N-1$. If we set $X_0 = V_0$ and define recursively forward in time the portfolio values $X_1, ..., X_N$ by

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n)$$

then we will have

$$X_N(\omega_1...\omega_N) = V_N(\omega_1...\omega_N) \textit{ for all } \omega_1...\omega_N.$$

Proof by forward induction. Theorem also applies to path-dependent options. refer to example 1.2.4 in Shreve.