ODE Notes (I)

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Contents

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1 Introduction

This is my study notes on Ordinary Differential Equations (ODEs). This notes studies common methods for solving first, second and higher-order ODEs, without delving into detailed theoretical proofs. The content is mainly based on Prof. Chaoyu Quan's course **MAT2002 Ordinary Differential Equations** at CUHK(SZ), Prof. Jeffrey R. Chasnov's notes, and the textbook [BDM21].

2 First-Order Differential Equations

In this section we study how to solve first-order ODEs (only involving first-order derivatives). We will start from the simplest linear case (Section 2.1), then turn to more general cases.

2.1 Linear Equations

We will first give the formulation of the first-order linear ordinary differential equations.

Definition 1 (First-Order Linear ODE). *The Initial Value Problem (IVP) of the general first-order linear ODE is given by*

$$\begin{cases} \dfrac{dy}{dt} = p(t)y + q(t), \\ y(t_0) = y_0, \end{cases} \tag{1}$$

for some given functions $p(t)$, $q(t)$ and constants t_0 and y_0.

Next, we will introduce the **method of integrating factors** to solve the above ODE. Multiply (1) by a function $\mu(t)$ (a.k.a., the integrating factor), leading to

$$\mu(t)\frac{dy}{dt} - \mu(t)p(t)y(t) = \mu(t)q(t). \tag{2}$$

Suppose that

$$\mu(t)\frac{dy}{dt} - \mu(t)p(t)y(t) = \frac{d}{dt}\left(\mu(t)y(t)\right), \tag{3}$$

then (2) becomes

$$\frac{d}{dt}\left(\mu(t)y(t)\right) = \mu(t)q(t) \Rightarrow \mu(t)y(t) = \int \mu(t)q(t)\,dt + c, \quad c \in \mathbb{R} \tag{4}$$

If $\mu(t)$ is **non-zero**, we can obtain the general solution

$$y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)q(t)\,dt + c\right] \tag{5}$$

The problem becomes how to find such $\mu(t)$? From (3) we have

$$\mu(t)y'(t) - \mu(t)p(t)y(t) = \mu'(t)y(t) + \mu(t)y'(t) \Rightarrow y(t)\left(\frac{d\mu}{dt} + p(t)\mu(t)\right) = 0. \tag{6}$$

The equation is satisfied if $y(t) = 0$ or $\mu'(t) + p(t)\mu(t) = 0$. The first case $y(t) = 0$ is not desirable, since if the initial condition y_0 is non-zero, we have a contradiction. Therefore, we consider the second case and obtain the equation

$$\frac{d\mu}{dt} = -p(t)\mu \tag{7}$$

as the ODE for μ. This is a **separable equation** which will be detailed in Section 2.2, and revisited in Example 6. $\mu(t) \equiv 0$ is one solution but without any interest. When $\mu(t) \neq 0$, we have

$$\frac{1}{\mu}\frac{d\mu}{dt} = -p(t) \Rightarrow \ln|\mu(t)| = -\int p(t)\,dt + c \tag{8}$$

Choosing the arbitrary constant c to be zero, we obtain a simplest integrating factor

$$\mu(t) = \exp\left(-\int p(t)\,dt\right) \tag{9}$$

Plug in (5) we obtain the final solution. The general solution $y(t)$ to the ODE $y' = p(t)y + q(t)$ is given as

$$y(t) = e^{\int p(t)\,dt}\left[\int e^{-\int p(t)\,dt} q(t)\,dt + c\right]. \tag{10}$$

The particular solution and the constant c can be computed with the initial condition $y(t_0) = y_0$.

Example 2. *Solve the ODE*

$$\begin{cases} t\dfrac{dy}{dt} + 2y = 4t^2, \\ \qquad y(1) = 2, \end{cases} \tag{11}$$

Suppose $t \neq 0$. Write the ODE in the form $y' = p(t)y + q(t)$ and identify p, q

$$t\frac{dy}{dt} + 2y = 4t^2 \Rightarrow \frac{dy}{dt} = -\frac{2}{t}y + 4t \Rightarrow p(t) = -\frac{2}{t}, \quad q(t) = 4t. \tag{12}$$

Compute the integrating factor

$$\mu(t) = \exp\left(-\int p(t)\,dt\right) = t^2 \tag{13}$$

We obtain the general solution

$$y(t) = \frac{1}{t^2}\left[\int t^2 \times 4t\,dt + c\right] = t^2 + \frac{c}{t^2}. \tag{14}$$

From the initial condition we have $c = 1$, thus the particular solution is

$$y(t) = t^2 + \frac{1}{t^2} \tag{15}$$

The general and particular solutions are shown in Figure 1.

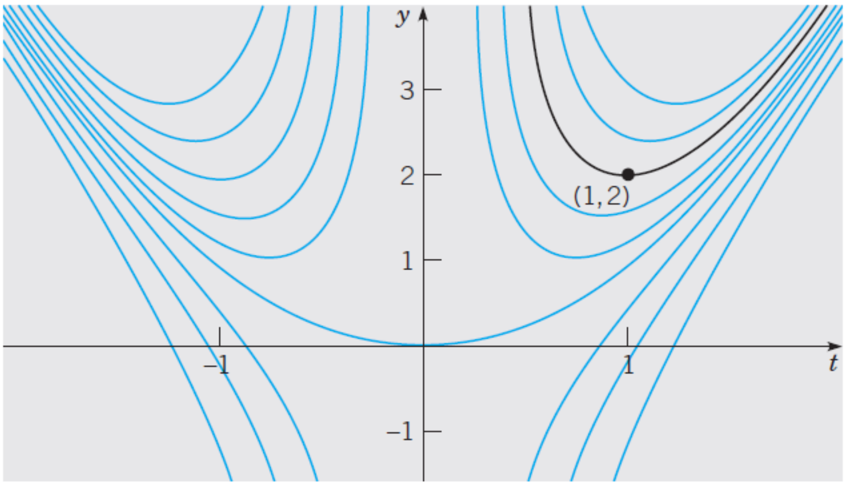

FIGURE 2.1.3 Integral curves of the differential equation $ty' + 2y = 4t^2$; the black curve passes through the point $(1, 2)$.

Figure 1

2.2 Separable Equations

Definition 3 (Separable Equation)**.** *A first order ODE $y' = f(t, y)$ is separable if it can be written in the form*

$$M(t) + N(y)\frac{dy}{dt} = 0 \tag{16}$$

for some functions M, N.

Let's see an example.

Example 4. *Solve the ODE*

$$\begin{cases} \dfrac{dy}{dt} = \dfrac{\sin(t)}{1 - y^2}, \\ y(t_0) = y_0, \end{cases} \tag{17}$$

The key is to **separate y and t, placing them on opposite sides of the equation**. Bring y to the LHS we have

$$(1 - y^2)\frac{dy}{dt} = \sin(t) \tag{18}$$

Integrate both sides, we obtain

$$y(t) - \frac{1}{3}y(t)^3 = -\cos(t) + c, \quad c \in \mathbb{R}. \tag{19}$$

Using the initial condition to solve for c

$$\begin{cases} y(t) - \dfrac{1}{3}y(t)^3 = -\cos(t) + c, \\ \quad y_0 - \dfrac{1}{3}y_0^3 = -\cos(t_0) + c, \end{cases} \tag{20}$$

The particular solution is given by

$$y(t) - \frac{1}{3}y(t)^3 = \cos(t_0) - \cos(t) + y_0 - \frac{1}{3}y_0^3. \tag{21}$$

Example 5. *Solve the ODE*

$$\frac{dy}{dt} = P(t)y \tag{22}$$

When $y \neq 0$, using the separation method we obtain

$$y = \pm e^{\overline{c}} \cdot e^{\int P(t)\,dt} = ce^{\int P(t)\,dt} \tag{23}$$

where $\overline{c} \in \mathbb{R}$ and $c = \pm e^{\overline{c}}$. Clearly $y = 0$ **is also a solution to** (24), so if we allow $c = 0$, then the solution $y = 0$ is also included in (23).

Example 6. *Solve the ODE*

$$\frac{dy}{dt} + \frac{1}{2}y = \frac{3}{2} \tag{24}$$

First write in the form

$$\frac{dy}{dt} = \frac{1}{2}(3 - y) \tag{25}$$

When $y \neq 3$, separate variables

$$\frac{1}{3 - y}\frac{dy}{dt} = \frac{1}{2} \tag{26}$$

After integration and removing the absolute values we obtain

$$3 - y = \pm e^{c} \cdot e^{-\frac{1}{2}t} \tag{27}$$

So the final solution is

$$y = 3 + Ce^{-\frac{1}{2}t} \tag{28}$$

where $C \in \mathbb{R}$, since we included the solution $y = 3$.

2.3 Transformation Methods

There are many transformation methods, we will only discuss two of them.

2.3.1 Bernoulli Equation

Definition 7 (Bernoulli Equation)**.** *Let n be a real number, $n \neq 0, 1$, and $p(t), q(t)$ be given functions. The Bernoulli equation is a first order non-linear ODE of the form*

$$\frac{dy}{dt} + p(t)y = q(t)y^n. \tag{29}$$

Multiply (29) with y^{-n}

$$y^{-n}\frac{dy}{dt} + p(t)y^{1-n} = q(t) \tag{30}$$

Since $\frac{d}{dt}\left(y^{1-n}\right) = (1-n)y^{-n}\frac{dy}{dt}$, (30) simplifies to

$$\frac{d}{dt}y^{1-n} + (1-n)p(t)y^{1-n} = (1-n)q(t) \tag{31}$$

Then, consider a **new variable** $v(t) = y^{1-n}(t)$, (31) becomes

$$\frac{dv}{dt} + P(t)v = Q(t), \quad P(t) = (1-n)p(t), \quad Q(t) = (1-n)q(t) \tag{32}$$

which is a first-order linear ODE for v. Let $\mu(t)$ be the integrating factor for (32), then the general solution is

$$v(t) = \frac{1}{\mu(t)}\left[\int Q(t)\mu(t)\,dt + c\right] \quad \Rightarrow \quad y(t) = \left(\frac{1}{\mu(t)}\left[\int Q(t)\mu(t)\,dt + c\right]\right)^{\frac{1}{1-n}} \tag{33}$$

2.3.2 Homogeneous First-Order Equation

Definition 8 (Homogeneous First-Order Equation)**.** *A first order ODE $\frac{dy}{dt} = f(t, y)$ is called homogeneous if the function f only depends on the ratio $\frac{y}{t}$. That is, we can express*

$$f(t, y) = F\left(\frac{y}{t}\right) \quad \textit{for some function } F. \tag{34}$$

We will still use a **transformation** method. Define a **new variable** $v = y/t \iff y = vt$. Then, the RHS of the ODE becomes just $F(v)$. For the LHS, by the product rule we have

$$y(t) = tv(t) \Rightarrow \frac{dy}{dt} = t\frac{dv}{dt} + v(t) \Rightarrow t\frac{dv}{dt} + v(t) = F(v). \tag{35}$$

Note that the initial condition $y(t_0) = y_0$ also transforms:

$$y(t_0) = y_0 \Rightarrow t_0 v(t_0) = y_0, \tag{36}$$

and it is important to see that if $y_0 \neq 0$ then we cannot choose $t_0 = 0$, otherwise we get a contradiction. The transformed ODE in the variable v is now

$$\frac{dv}{dt} = \frac{F(v) - v}{t} \Rightarrow \frac{1}{F(v) - v}\frac{dv}{dt} = \frac{1}{t} \tag{37}$$

which is a **separable equation**.

Example 9. *Solve the ODE*

$$\frac{dy}{dt} = \frac{y - 4t}{t - y} = f(t, y) \tag{38}$$

Dividing numerator and denominator by t leads to

$$f(t, y) = \frac{y - 4t}{t - y} = \frac{y/t - 4}{1 - y/t} = F(y/t), \text{ where } F(s) = \frac{s - 4}{1 - s}. \tag{39}$$

Using a transformation $y = tv$ we find that v satisfies

$$\frac{1}{F(v) - v}\frac{dv}{dt} = \frac{1}{t} \Rightarrow \frac{1 - v}{(v - 2)(v + 2)}\frac{dv}{dt} = \frac{1}{t}. \tag{40}$$

Using partial fractions the coefficient can be simplified to

$$\frac{1 - v}{(v - 2)(v + 2)} = -\frac{1}{4(v - 2)} - \frac{3}{4(v + 2)}. \tag{41}$$

Then, integrating gives the general solution

$$-\frac{1}{4}\ln|v - 2| - \frac{3}{4}\ln|v + 2| = \ln|t| + c \Rightarrow -\frac{1}{4}\ln|y(t)/t - 2| - \frac{3}{4}\ln|y(t)/t + 2| = \ln|t| + c. \tag{42}$$

This gives

$$|y(t)/t - 2|^{-1/4}\,|y(t)/t + 2|^{-3/4} = e^c|t|, \quad c \in \mathbb{R}. \tag{43}$$

Which can be rewritten as

$$|t|\,|y(t)/t - 2|^{1/4}\,|y(t)/t + 2|^{3/4} = k, \quad k \geq 0. \tag{44}$$

2.4 Exact Equations

2.4.1 General Method

An ODE of the following form is not separable:

$$M(t,y) + N(t,y)\frac{dy}{dt} = 0 \tag{45}$$

where M, N are some functions. If the LHS of this equation can be written as $\frac{d\Psi(t,y(t))}{dt}$ for some function $\Psi(t,y)$, then integrating gives the general (implicit) solution

$$\Psi(t,y(t)) = c, \quad c \in \mathbb{R} \tag{46}$$

The requirement for

$$\frac{d\Psi(t,y(t))}{dt} = M(t,y) + N(t,y)\frac{dy}{dt} \tag{47}$$

implies

$$\frac{\partial \Psi}{\partial y}(t,y) = N(t,y), \quad \frac{\partial \Psi}{\partial t}(t,y) = M(t,y), \tag{48}$$

since $\frac{d\Psi(t,y(t))}{dt} = \frac{\partial \Psi}{\partial t}(t,y) + \frac{\partial \Psi}{\partial y}(t,y)\frac{dy}{dt}$. This is summarized in the following definition.

Definition 10 (Exact Equation)**.** *A first order ODE $M(t,y) + N(t,y)\frac{dy}{dt} = 0$ is an exact equation if there exists a function $\Psi(t,y)$ such that*

$$\frac{\partial \Psi}{\partial y}(t,y) = N(t,y), \quad \frac{\partial \Psi}{\partial t}(t,y) = M(t,y). \tag{49}$$

The general solution $y(t)$ to the ODE is given implicitly as $\Psi(t,y(t)) = c, \quad c \in \mathbb{R}$.

Thus, the question becomes:

1. How to determine an ODE of the form $M(t,y) + N(t,y)\frac{dy}{dt} = 0$ is exact?
2. If it is an exact equation, how to find the function $\Psi(t,y)$?

Solution is given by the following theorem.

Theorem 11. *Let $M(t,y)$ and $N(t,y)$ be continuous functions of t and y in some simply connected domain, and have continuous first-order partial derivatives. Then the equation*

$$M(t,y) + N(t,y)\frac{dy}{dt} = 0 \tag{50}$$

is an exact differential equation if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t} \tag{51}$$

We will prove the theorem to gain a better understanding of it.

Proof. "$\Leftarrow$". Given that (50) is an exact differential equation, (49) holds, and by taking partial derivatives on t, y we obtain

$$\frac{\partial M}{\partial y} = \frac{\partial^2 \Psi}{\partial t \partial y}, \quad \frac{\partial N}{\partial t} = \frac{\partial^2 \Psi}{\partial y \partial t} \tag{52}$$

From the continuity of $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial t}$, we know that $\frac{\partial^2 \Psi}{\partial t \partial y}$ and $\frac{\partial^2 \Psi}{\partial y \partial t}$ are continuous. Therefore, we can obtain

$$\frac{\partial^2 \Psi}{\partial t \partial y} = \frac{\partial^2 \Psi}{\partial y \partial t} \tag{53}$$

That is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t} \tag{54}$$

Thus the necessity is proven.

"$\Rightarrow$". We want to show that if (50) satisfies (51), then we can find function $\Psi(t, y)$ satisfying (49). Integrating both sides of $\frac{\partial \Psi}{\partial t} = M(t, y)$ with respect to t, we obtain

$$\int M(t, y)dt + \varphi(y) = \Psi(t, y) \tag{55}$$

Here $\varphi(y)$ is an arbitrary differentiable function of y. We choose a suitable $\varphi(y)$ such that $\Psi(t, y)$ also satisfies $\frac{\partial \Psi}{\partial y} = N(t, y)$, that is, taking the partial derivative with respect to y on both sides of (55), we get

$$\frac{\partial \Psi}{\partial y} = \frac{\partial}{\partial y} \int M(t, y)dt + \frac{d\varphi(y)}{dy} = N(t, y). \tag{56}$$

Therefore

$$\frac{d\varphi(y)}{dy} = N(t, y) - \frac{\partial}{\partial y} \int M(t, y)dt. \tag{57}$$

Note that $\varphi(y)$ is an arbitrary differentiable function of y, so the RHS of (57) must be independent of t, which means the partial derivative of the RHS of (57) with respect to t should be zero. In fact,

$$\frac{\partial}{\partial t}\left[N(t, y) - \frac{\partial}{\partial y} \int M(t, y)dt\right] = \frac{\partial N}{\partial t} - \frac{\partial}{\partial t}\left[\frac{\partial}{\partial y} \int M(t, y)dt\right]. \tag{58}$$

Since $M(t, y)$ and $N(t, y)$ are continuous functions of t, y in some simply connected domain, and have continuous first-order partial derivatives, the order of differentiation with respect to t and y in (58) can be interchanged. Using (51), we get

$$\frac{\partial}{\partial t}\left[N(t, y) - \frac{\partial}{\partial y} \int M(t, y)dt\right] = \frac{\partial N}{\partial t} - \frac{\partial}{\partial y}\left[\frac{\partial}{\partial t} \int M(t, y)dt\right] \tag{59}$$

$$= \frac{\partial N}{\partial t} - \frac{\partial M}{\partial y} = 0. \tag{60}$$

Thus the RHS of (57) is a function of y only. Integrating both sides, we obtain

$$\varphi(y) = \int \left[N(t, y) - \frac{\partial}{\partial y} \int M(t, y)dt\right] dy. \tag{61}$$

Substituting (61) into (55), we can find

$$\Psi(t, y) = \int M(t, y)dt + \int \left[N(t, y) - \frac{\partial}{\partial y} \int M(t, y)dt\right] dy. \tag{62}$$

In this way, we have proved that if (50) satisfies condition (51), then a $\Psi(t, y)$ that satisfies (49) must exist, and its specific expression is (62), thus proving sufficiency. Combining both directions finishes the proof. $\square$

Example 12. *Solve the ODE*

$$3t^2 + 6ty^2 + (6t^2y + 4y^3)\frac{dy}{dt} = 0 \tag{63}$$

Here, $M = 3t^2 + 6ty^2, N = 6t^2y + 4y^3$, easy to verify that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, so the equation is exact. Find Ψ such that it satisfies

$$\frac{\partial \Psi}{\partial t} = M = 3t^2 + 6ty^2, \tag{64}$$

$$\frac{\partial \Psi}{\partial y} = N = 6t^2y + 4y^3 \tag{65}$$

Integrating (64) with respect to t, we get

$$\Psi = t^3 + 3t^2y^2 + \varphi(y). \tag{66}$$

Taking the partial derivative of (66) with respect to y, and using (65), we get

$$\frac{\partial \Psi}{\partial y} = 6t^2y + \frac{d\varphi(y)}{dy} = 6t^2y + 4y^3 \tag{67}$$

Thus

$$\frac{d\varphi(y)}{dy} = 4y^3 \tag{68}$$

Solving this, we get

$$\varphi(y) = y^4. \tag{69}$$

Substituting $\varphi(y)$ into (66), we get

$$\Psi = t^3 + 3t^2y^2 + y^4. \tag{70}$$

Therefore, the general solution of the equation is

$$t^3 + 3t^2y^2 + y^4 = c, \tag{71}$$

where c is an arbitrary constant. Alternatively, we can directly apply (62):

$$\int M(t,y)dt + \int \left[N(t,y) - \frac{\partial}{\partial y}\int M(t,y)dt\right] dy \tag{72}$$

$$= t^3 + 3t^2y^2 + \int (6t^2y + 4y^3 - 6t^2y)dy \tag{73}$$

$$= t^3 + 3t^2y^2 + y^4 = c, \tag{74}$$

where c is arbitrary constant.

2.4.2 Exact Equations with Integrating Factor

How to solve a non-exact ODE? Similar to the way we treated the first-order linear ODEs, consider multiplying with a integrating factor μ and hope things are better. We obtain after multiplying a new ODE

$$\mu M(t,y) + \mu N(t,y)\frac{dy}{dt} = 0 \tag{75}$$

If (75) is an exact equation, then by previous theorem, the following relation must be satisfied:

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial t}(\mu N) \tag{76}$$

Let's first investigate two cases.

Case 1. μ is just a function of t, i.e., $\mu = \mu(t)$. Then (76) simplifies to

$$N(t,y)\frac{d\mu}{dt} + \mu(t)N_t(t,y) = \mu(t)M_y(t,y). \tag{77}$$

If $N(t,y) \neq 0$, then we obtain an ODE for μ:

$$\frac{d\mu}{dt} = \mu(t)\left(\frac{M_y - N_t}{N}\right)(t,y) =: \mu(t)K(t,y). \tag{78}$$

Further suppose the factor $K(t,y)$ **depends only on** t, then (78) is a first-order **linear** ODE in $\mu(t)$ which can be solved by the method of integrating factors.

Case 2. μ is just a function of y, i.e., $\mu = \mu(y)$. Then (76) simplifies to

$$M(t,y)\frac{d\mu}{dy} + \mu(y)M_y(t,y) = \mu(y)N_t(t,y). \tag{79}$$

If $M(t,y) \neq 0$, then we obtain an ODE for μ:

$$\frac{d\mu}{dy} = \mu(y)\left(\frac{N_t - M_y}{M}\right)(t,y) =: \mu(y)H(t,y). \tag{80}$$

Further suppose the factor $H(t,y)$ **depends only on** y, then (80) is a first order **linear** ODE in $\mu(y)$ (where the independent variable is now y), and again can be solved by the method of integrating factors.

After obtaining μ, plug in (75) to obtain an exact equation.

Example 13. *Solve the ODE*

$$3ty + y^2 + (t^2 + ty)\frac{dy}{dt} = 0 \tag{81}$$

Clearly the ODE is not exact. Compute $K = \frac{M_y - N_t}{N} = \frac{t+y}{t^2+ty} = \frac{1}{t}$ and $H = \frac{N_t - M_y}{M} = \frac{-t-y}{3ty+y^2}$. We see that K is only a function of t but H is not just a function of y. So we expect the integrating factor μ to be a function of t only, which solves the ODE

$$\frac{d\mu}{dt} = \frac{\mu(t)}{t} \Rightarrow \mu(t) = ct, \quad c \in \mathbb{R} \tag{82}$$

Multiplying this integrating factor (take $c = 1$) with the ODE yields

$$t(3ty + y^2) + t(t^2 + ty)\frac{dy}{dt} = 0, \tag{83}$$

which is now an exact equation with function $\Psi(t, y)$ given as

$$\Psi(t, y) = t^3 y + \frac{1}{2}t^2 y^2. \tag{84}$$

So the general (implicit) solution to the ODE is

$$t^3 y(t) + \frac{1}{2}t^2 y^2(t) = c, \quad c \in \mathbb{R}. \tag{85}$$

Summary on methods in Section 2:

Type	**Method**	**Explicit/Implicit solution**
$y' = p(t)y + q(t)$	Integrating factor	$y(t) = \mu(t)^{-1}(\int \mu(t)q(t)dt + c)$
$M(t) + N(y)y' = 0$	Separable equation	$m(t) + n(y(t)) = c^*$
$y' + p(t)y = q(t)y^n$	$v := y^{1-n}$	$y(t) = (\mu^{-1}(\int Q(t)\mu(t)dt + c))^{1/(1-n)}$
$y' = F(y/t)$	$v = y/t$	$1/(F(v) - v)\frac{dv}{dt} = \frac{1}{t}$
$M(t, y) + N(t, y)y' = 0$	Exact equation	$\Psi(t, y(t)) = c$

*: $m(t) = \int M(t)dt, n(y(t)) = \int N(y)dy$.

2.5 Existence and Uniqueness Theorems

For completeness, we will state the existence and uniqueness theorems for IVP of first-order ODEs. The existence and uniqueness for first-order linear ODEs is characterized by the following theorem.

Theorem 14 (Existence and Uniqueness for First-Order Linear ODE)**.** *Suppose functions p and q are continuous on $(\alpha, \beta) \subset \mathbb{R}$ (where α, β are some real numbers). Then, for any $t_0 \in (\alpha, \beta)$, $y_0 \in \mathbb{R}$, there exists a unique function $y(t)$ satisfying*

$$\begin{cases} \dfrac{dy}{dt} = p(t)y + q(t), \quad \forall t \in (\alpha, \beta), \\ y(t_0) = y_0, \end{cases} \tag{86}$$

And the solution is defined throughout the interval (α, β).

The above theorem states that the unique solution to the IVP exists throughout any interval (α, β) containing $t = t_0$ if the functions p and q are continuous in (α, β). In other words, **the solution globally exists in the interval (α, β) in which p and q are continuous.**

The existence and uniqueness for first-order non-linear ODEs is characterized by the following theorem.

Theorem 15 (Existence and Uniqueness for First-Order Non-Linear ODE)**.** *Consider the IVP*

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0. \tag{87}$$

Let R be a closed rectangle

$$R = \{(t, y) \mid |t - t_0| \le a,\ |y - y_0| \le b\} \quad (a > 0, b > 0). \tag{88}$$

Assume that both $f(t, y)$ and $\frac{\partial f}{\partial y}$ are continuous on R. Then the IVP has a unique solution $y = y(t)$ defined on the interval $(t_0 - h, t_0 + h)$, where $h = \min\left(\frac{b}{M}, a\right)$ and $M = \max_{(t,y)\in R} |f(t, y)|$.

Under the assumption of the theorem, the solution only exists in a small interval $(t_0 - h, t_0 + h) \subset [t_0 - a, t_0 + a]$ since $h = \min\left(\frac{b}{M}, a\right)$ depends on the size of the region R. And h also depends on the values of the function $f(t, y)$ in the region R ($M = \max_{(t,y)\in R} |f(t, y)|$). **The solution only locally exists in the interval $[t_0 - a, t_0 + a]$.**

3 Second-Order Linear Differential Equations

In this section we study second-order **linear** ODEs. Section 3.1 introduces general theory of homogeneous equations, Section 3.2 and 3.3 study how to solve them, and Section 3.4 deals with non-homogeneous equations.

3.1 General Theory of Homogeneous Equations

We will first present the existence and uniqueness theorem for the second-order linear equations.

Theorem 16 (Existence and Uniqueness for Second-Order Linear ODE)**.** *Consider the IVP*

$$y'' + p(t)y' + q(t)y = r(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_1. \tag{89}$$

Suppose $I = (\alpha, \beta) \subset \mathbb{R}$ is any open interval such that $t_0 \in I$, and the functions p, q, r are continuous in I. Then, there is exactly one solution $y(t)$ to the IVP for $t \in I$. The solution $y(t)$ is defined throughout the interval where p, q, r are continuous.

Now we introduce the following classification.

Definition 17 (Homogeneous)**.** *A second order linear ODE*

$$p(t)y'' + q(t)y' + r(t)y = s(t), \quad p(t) \neq 0, \tag{90}$$

is called homogeneous if $s(t) \equiv 0$. Otherwise, if $s(t) \neq 0$, the ODE is called non-homogeneous.

For second-order homogeneous linear equations we have the following **principle of superposition**.

Theorem 18 (Principle of Superposition)**.** *If y_1 and y_2 are two solutions of the ODE*

$$p(t)y'' + q(t)y' + r(t)y = 0. \tag{91}$$

Then for any constants $c_1, c_2 \in \mathbb{R}$, the function $c_1y_1(t) + c_2y_2(t)$ is also a solution to the ODE.

Clearly the principle of superposition **holds for homogeneous linear equations of any order**, which can be easily verified due to the linear structure.

Let's return to the second-order case. In other words, from two solutions we can construct infinite solutions to the homogeneous linear ODE. We can define a family of solutions

$$S = \{y = c_1y_1 + c_2y_2 \mid c_1, c_2 \in \mathbb{R}\} \tag{92}$$

to the ODE. The next question is: Given two solutions $y_1(t)$ and $y_2(t)$, can **any** solution to the ODE be expressed as a linear combination of $y_1(t)$ and $y_2(t)$?

Definition 19 (Wronskian)**.** *Given $y_1(t)$ and $y_2(t)$,*

$$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \tag{93}$$

is called the Wronskian for y_1 and y_2.

Indeed, we have the following theorem.

Theorem 20. *Suppose that I is an open interval in which $p(t)$ and $q(t)$ are continuous. Let $y_1(t)$ and $y_2(t)$ be two solutions to the ODE*

$$y'' + p(t)y' + q(t)y = 0 \tag{94}$$

for $t \in I$. Then, any solution $y(t)$ to the ODE can be expressed as

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \tag{95}$$

for constants c_1 and c_2 $\iff \exists\, t_0 \in I$ such that the Wronskian $W(y_1, y_2)[t_0] \neq 0$.

The theorem says that if $y_1(t)$ and $y_2(t)$ are two solutions to the above ODE and $W(y_1, y_2)[t_0] \neq 0$, then the general solution to the above ODE is given by the (95). In this case, we say that (y_1, y_2) form a **fundamental set of solutions (FSS)** to the ODE.

Example 21. *$y_1(t) = \exp(-2t)$ and $y_2(t) = \exp(-3t)$ are solutions to the ODE*

$$y'' + 5y' + 6y = 0. \tag{7}$$

$$W[y_1, y_2](t) = \begin{vmatrix} \exp(-2t) & \exp(-3t) \\ -2\exp(-2t) & -3\exp(-3t) \end{vmatrix} = -\exp(-5t) \neq 0 \tag{96}$$

Any solution to the ODE $y'' + 5y' + 6y = 0$ can be written as the linear combination of $y_1(t) = \exp(-2t)$ and $y_2(t) = \exp(-3t)$, they form a FSS to the ODE.

Example 22. *For the ODE*

$$2t^2 y'' + 3ty' - y = 0, \quad t > 0, \tag{97}$$

the function $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ are solutions. Let us compute the Wronskian

$$W(y_1, y_2)[t] = -\frac{3}{2} t^{-3/2}, \tag{98}$$

which is non-zero for $t > 0$. Therefore we can deduce that (y_1, y_2) form a FSS for the ODE, and a general solution y to the ODE can be expressed as

$$y(t) = c_1 t^{1/2} + c_2 t^{-1}, \tag{99}$$

for some constants c_1, c_2.

Next we will examine further the properties of the Wronskian of two solutions to the second-order linear homogeneous ODE. We will show an explicit formula for the Wronskian even if the two solutions are unknown.

Theorem 23 (Abel's Identity). *Let I be an open interval in which p and q are continuous. Suppose y_1 and y_2 are two non-zero solutions to the ODE*

$$y'' + p(t)y' + q(t)y = 0. \tag{100}$$

Then, the Wronskian is given as

$$W(y_1, y_2)[t] = c \exp\left(-\int p(t)\,dt\right), \tag{101}$$

*where the constant c depends on y_1 and y_2, but not on t. Consequently, $W(y_1, y_2)[t] = 0$ **if and only if** $c = 0$. In particular, if $W(y_1, y_2)[t_0] \neq 0$ for some $t_0 \in I$, then it holds that $W(y_1, y_2)[t] \neq 0$ for all $t \in I$. And, if $W(y_1, y_2)[t_0] = 0$ for some $t_0 \in I$, then it holds that $W(y_1, y_2)[t] = 0$ for all $t \in I$.*

Proof. The idea is to derive an ODE for the Wronskian W. Going back to the ODE, as y_1 is a solution we have

$$y_1'' + p(t)y_1' + q(t)y_1 = 0 \quad \Rightarrow \quad y_2y_1'' + y_2p(t)y_1' + y_2q(t)y_1 = 0. \tag{102}$$

Similarly, as y_2 is a solution,

$$y_1y_2'' + y_1p(t)y_2' + y_1q(t)y_2 = 0. \tag{103}$$

Subtracting one from another gives

$$(y_1y_2'' - y_2y_1'') + p(t)(y_1y_2' - y_2y_1') = 0. \tag{104}$$

Noting that

$$W(y_1, y_2)[t] = y_1(t)y_2'(t) - y_2(t)y_1'(t) \quad \Rightarrow \quad W'(y_1, y_2)[t] = y_1(t)y_2''(t) - y_2(t)y_1''(t). \tag{105}$$

From (104) we have

$$W' + p(t)W = 0, \tag{106}$$

which is a linear first-order equation. By integrating factors, we find the general solution

$$W(y_1, y_2)[t] = c \exp\left(-\int p(t)\,dt\right), \tag{107}$$

for some constant $c \in \mathbb{R}$. As a constant of integration, c does not depend on t.

□

An implication of the theorem is that (y_1, y_2) form a FSS to $y'' + p(t)y' + q(t)y = 0$ if and only if $W(y_1, y_2)[t] \neq 0$, $\forall t \in I$.

Does a FSS always exists? This is answered in the next theorem.

Theorem 24. *Let I be an open interval of $\mathbb{R}$, p and q are continuous functions in I. For any $t_0 \in I$, let $y_1(t)$ be the (unique) solution to the IVP*

$$y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = 1, \quad y'(t_0) = 0, \tag{108}$$

and $y_2(t)$ be the (unique) solution to the IVP

$$y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = 0, \quad y'(t_0) = 1. \tag{109}$$

Then, (y_1, y_2) forms a FSS to the ODE.

Proof. Note that the existence of y_1 and y_2 to the corresponding IVPs is guaranteed by the Existence and Uniqueness Theorem. We only need to show that the Wronskian $W(y_1, y_2)[t_0]$ is non-zero. Computing gives

$$W(y_1, y_2)[t_0] = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1. \tag{110}$$

□

Indeed, the FSS are not unique. There are many different choices for $y_1(t_0), y_1'(t_0), y_2(t_0), y_2'(t_0)$ such that the corresponding solutions y_1, y_2 satisfy $W(y_1, y_2)[t_0] \neq 0$.

The FSS is closely related to the concept of linear (in)dependence in linear algebra.

Definition 25 (Linear Dependence)**.** *Consider 2 functions $x_1(t), x_2(t)$ defined on an interval $I \subset \mathbb{R}$. We say that $x_1(t), x_2(t)$ are linearly dependent if there are non-zero constants α_1, α_2, such that*

$$\alpha_1 x_1(t) + \alpha_2 x_2(t) = 0 \quad \forall t \in I. \tag{111}$$

Let $x_1(t), x_2(t)$ be defined on an interval $I \subset \mathbb{R}$. If $x_1(t), x_2(t)$ are not linearly dependent, then they are linearly independent.

Theorem 26. *If $y_1(t)$ and $y_2(t)$ are two solutions to the ODE $y'' + p(t)y' + q(t)y = 0$, $t \in I$, where p, q are given continuous functions in I (some open interval). Then $y_1(t)$ and $y_2(t)$ are linearly independent $\iff$ $W[y_1, y_2](t) \neq 0$, $\forall t \in I$ ((y_1, y_2) forms a FSS).*

Proof. "$\Leftarrow$".

$$\begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{112}$$

has only the zero solution, since the determinant $W[y_1, y_2](t) \neq 0$. Thus, $c_1 y_1(t) + c_2 y_2(t) = 0$ implies $c_1 = c_2 = 0$, meaning that $y_1(t)$ and $y_2(t)$ are linearly independent.

"$\Rightarrow$". If $W[y_1, y_2](t_0) = 0$ for some $t_0 \in I$. Then the linear system

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{113}$$

has non-zero solution $(c_1^*, c_2^*) \neq (0, 0)$. Define $\phi(t) = c_1^* y_1(t) + c_2^* y_2(t), t \in I$, then $\phi(t)$ is the solution to the ODE $y'' + p(t)y' + q(t)y = 0$ with initial conditions $y(t_0) = 0, y'(t_0) = 0$. But $y(t) = 0, t \in I$ is also the solution to the ODE with initial conditions $y(t_0) = 0, y'(t_0) = 0$. By the existence and uniqueness theorem, $\phi(t) = c_1^* y_1(t) + c_2^* y_2(t) = 0$. This implies that $y_1(t)$ and $y_2(t)$ are linearly dependent, which is a contradiction.

□

Clearly the proof also works for linear homogeneous ODEs of higher order.

This is similar to the steps in linear algebra for solving the homogeneous linear system $A\mathbf{x} = \mathbf{0}$: we need to find a set of linearly independent solutions (the basis of the null space of A), then all solutions can be expressed as a linear combination of these solutions. Another remark is that **a n-th order linear homogeneous ODE has at most n linear independent solutions**.

Based on the above results, the strategy to solve

$$y'' + p(t)y' + q(t)y = 0, \quad t \in I, \tag{114}$$

can be summarized as follows:

1. **Find two solutions y_1, y_2 satisfying the ODE.**

2. Find $t_* \in I$ such that the Wronskian $W(y_1, y_2)[t_*]$ is non-zero. Then, the general solution to the ODE is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \tag{115}$$

   for some constants c_1, c_2.

3. If initial conditions are prescribed at some $t_0 \in I$, compute c_1 and c_2 to determine the particular solution.

Step 1 is highly nontrivial, and is the basis to all methods in Section 3 (and Section 4).

3.2 Homogeneous Equations with Constant Coefficients

3.2.1 General Method

Although the FSS for the second-order linear homogeneous ODE $y'' + p(t)y' + q(t)y = 0$ always exist, but unfortunately, **there is no method to find the FSS explicitly.** However, when p, q are **constants**, we can find the FSS for $y'' + py' + qy = 0$ explicitly.

We will study the solutions to the ODE

$$ay'' + by' + cy = 0 \tag{116}$$

for fixed real constants $a, b, c \in \mathbb{R}$ with $a \neq 0$. Consider substituting a **trial function** $y(t) = \exp(rt)$ for some constant r into (116), which yields

$$\big(ar^2 + br + c\big)\exp(rt) = 0. \tag{117}$$

Since $\exp(rt) > 0$, we have

$$ar^2 + br + c = 0. \tag{118}$$

(118) is known as the **characteristic equation** for the ODE (116). If we can find the roots of the characteristic equation, then we know that $\exp(rt)$, where r is a root, is a solution to (116). By the quadratic formula we obtain

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{119}$$

Three possibilities:

1. Two distinct real roots r_1, r_2 if $b^2 > 4ac$.
2. Two complex roots (complex conjugate pairs) $r_1, \overline{r_1}$ if $b^2 < 4ac$.
3. A repeated real root r if $b^2 = 4ac$.

Discriminant $\Delta := b^2 - 4ac$.

Case 1. Two distinct real roots $\Delta > 0$. In the case $b^2 > 4ac$, we obtain two real roots

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{120}$$

This gives us two solutions

$$y_1(t) = \exp(r_1 t), \quad y_2(t) = \exp(r_2 t). \tag{121}$$

Check the Wronskian:

$$\begin{aligned} W(y_1, y_2)[t] &= y_1(t)y_2'(t) - y_2(t)y_1'(t) && (122)\\ &= r_2 \exp((r_1 + r_2)t) - r_1 \exp((r_1 + r_2)t) && (123)\\ &= (r_2 - r_1)\exp((r_1 + r_2)t). && (124) \end{aligned}$$

Clearly $W(y_1, y_2)[t] \neq 0$ for all $t \in \mathbb{R}$. Thus, $y_1(t) = \exp(r_1 t)$, $y_2(t) = \exp(r_2 t)$ is the FSS of the ODE (116). From previous theorems, any solution $y(t)$ to the ODE is of the form

$$y(t) = c_1 \exp(r_1 t) + c_2 \exp(r_2 t) \tag{125}$$

for some constants c_1 and c_2.

Example 27. *Solve the ODE*

$$y'' + 9y' + 20y = 0 \tag{126}$$

Consider a trial function $y(t) = \exp(rt)$. The characteristic equation is

$$r^2 + 9r + 20 = (r+4)(r+5) = 0. \tag{127}$$

We have two real roots $r_1 = -4$ and $r_2 = -5$. Hence, the general solution is

$$y(t) = c_1 \exp(-4t) + c_2 \exp(-5t), \quad c_1, c_2 \in \mathbb{R}. \tag{128}$$

Case 2. Complex roots $\Delta < 0$. We now consider the case $\Delta = b^2 - 4ac < 0$. Then, the roots to the characteristic equation $ar^2 + br + c = 0$ is a complex-conjugate pair:

$$r_1 = \lambda + i\mu, \quad \lambda = \frac{-b}{2a}, \quad \mu = \frac{\sqrt{4ac - b^2}}{2a}, \quad i := \sqrt{-1}, \quad r_2 = \overline{r_1} = \lambda - i\mu. \tag{129}$$

We obtain two functions

$$y_1(t) = \exp(r_1 t) = \exp((\lambda + i\mu)t), \quad y_2(t) = \exp(r_2 t) = \exp((\lambda - i\mu)t). \tag{130}$$

Using Euler's formula, we arrive at

$$y_1(t) = \exp(\lambda t)\left(\cos(\mu t) + i\sin(\mu t)\right), \quad y_2(t) = \exp(\lambda t)\left(\cos(\mu t) - i\sin(\mu t)\right) \tag{131}$$

Let's check that y_1 and y_2 are linearly independent. Suppose there are constants $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$\alpha_1 y_1(t) + \alpha_2 y_2(t) = 0 \quad \forall t \in I \Rightarrow e^{\lambda t}\left((\alpha_1 + \alpha_2)\cos(\mu t) + i(\alpha_1 - \alpha_2)\sin(\mu t)\right) = 0. \tag{132}$$

The exponential is non-zero for all $t \in \mathbb{R}$, so to make the above expression zero, we need

$$\alpha_1 + \alpha_2 = 0, \quad \alpha_1 - \alpha_2 = 0 \quad \Rightarrow \quad \alpha_1 = \alpha_2 = 0. \tag{133}$$

So y_1 and y_2 are linearly independent. We can also calculate the Wronskian $W(y_1, y_2)[t] = -2i\mu e^{2\lambda t} \neq 0$, since $\mu \neq 0$ (otherwise we will not have $b^2 - 4ac < 0$). Thus, any solution $y(t)$ to the ODE is of the form

$$y(t) = e^{\lambda t}\left((c_1 + c_2)\cos(\mu t) + i(c_1 - c_2)\sin(\mu t)\right) \tag{134}$$

or

$$y(t) = e^{\lambda t}\left(d_1 \cos(\mu t) + d_2 i \sin(\mu t)\right) \tag{135}$$

for some constants d_1 and d_2. However, the solution is expressed as a complex-valued function. Since the coefficients of the ODE are real numbers, it would be better for us to obtain a real-valued function as a solution.

Theorem 28. *Given an ODE*

$$y'' + p(t)y' + q(t)y = 0 \tag{136}$$

with p and q are continuous real-valued functions. If $y(t) = u(t) + iv(t)$ is a complex-valued solution to the ODE, where u and v are real-valued functions, then its real part $u(t)$ and its imaginary part $v(t)$ are both solutions to the ODE.

Proof. Substituting the complex-valued solution into the ODE gives

$$0 = u''(t) + iv''(t) + p(t)u'(t) + ip(t)v'(t) + q(t)u(t) + iq(t)v(t) \tag{137}$$
$$= (u''(t) + p(t)u'(t) + q(t)u(t)) + i(v''(t) + p(t)v'(t) + q(t)v(t)). \tag{138}$$

A complex number is zero if and only if its real and imaginary parts are both zero, thus we have

$$u'' + p(t)u' + q(t)u = 0, \quad v'' + p(t)v' + q(t)v = 0. \tag{139}$$

□

Clearly the proof also works for linear homogeneous ODEs of higher order, and this result will be utilized again in Section 4.1.

From (135) we get the real-valued functions

$$u(t) = e^{\lambda t}\cos(\mu t), \quad v(t) = e^{\lambda t}\sin(\mu t) \tag{140}$$

Clearly u and v are linearly independent, and the Wronskian can be computed as $W(u, v)[t] = \mu e^{2\lambda t} \neq 0$, since $\mu \neq 0$. Thus, any solution y to the ODE $ay'' + by' + cy = 0$ with $b^2 - 4ac < 0$ can be expressed as

$$y(t) = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t) \tag{141}$$

which is a real-valued function.

Case 3. One repeated root $\Delta = 0$. The last case is when $b^2 - 4ac = 0$ and we have a repeated root to the characteristic equation:

$$r_1 = r_2 = -\frac{b}{2a} \tag{142}$$

The problem is apparent: both roots give the same function

$$y_1(t) = y_2(t) = \exp\left(-\frac{b}{2a}t\right). \tag{143}$$

We will use the Wronskian to find a solution that is linearly independent to $y_1(t)$. By Theorem 23, if $y_1 = \exp\left(-\frac{b}{2a}t\right)$ and y_2 are two solutions to the ODE $ay'' + by' + cy = 0$, then the Wronskian is

$$W(y_1, y_2)[t] = d\exp\left(-\int \frac{b}{a}dt\right) = d\exp\left(-\frac{b}{a}t\right) \tag{144}$$

for some constant $d \in \mathbb{R}$. On the other hand we have

$$W(y_1, y_2)[t] = y_1(t)y_2'(t) - y_1'(t)y_2(t) = e^{-\frac{b}{2a}t}y_2'(t) + \frac{b}{2a}e^{-\frac{b}{2a}t}y_2(t). \tag{145}$$

Choose $d = 1$, we have

$$e^{-\frac{b}{2a}t}y_2'(t) + \frac{b}{2a}e^{-\frac{b}{2a}t}y_2(t) = e^{-\frac{b}{a}t} \quad \Rightarrow \quad y_2'(t) + \frac{b}{2a}y_2(t) = e^{-\frac{b}{2a}t} \tag{146}$$

which is a first-order linear ODE for y_2, the solution is

$$y_2(t) = te^{-\frac{b}{2a}t} \tag{147}$$

where we have neglected any constants of integration. Now check the linear independence for $y_1 = e^{-\frac{b}{2a}t}$ and $y_2 = te^{-\frac{b}{2a}t}$. Suppose α_1 and α_2 are two constants such that

$$\alpha_1 y_1(t) + \alpha_2 y_2(t) = 0 \quad \forall t \in I \Rightarrow e^{-\frac{b}{2a}t}(\alpha_1 + t\alpha_2) = 0. \tag{148}$$

Since the exponential is never zero, for $\alpha_1 + t\alpha_2$ to be zero for all $t \in I$, we must have $\alpha_1 = \alpha_2 = 0$. We can also compute the Wronskian $W(y_1, y_2)[t] = e^{-\frac{b}{a}t} \neq 0$. Thus any solution y to the ODE $ay'' + by' + cy = 0$ with $b^2 - 4ac = 0$ can be expressed as

$$y(t) = c_1 e^{-\frac{b}{2a}t} + c_2 t e^{-\frac{b}{2a}t} \tag{149}$$

for constants $c_1, c_2 \in \mathbb{R}$.

Summary of Section 3.2.1:

For the second order linear ODE

$$ay'' + by' + cy = 0 \tag{150}$$

with constants a, b, c. Let r_1 and r_2 be the roots to the characteristic equation

$$ar^2 + br + c = 0 \tag{151}$$

- If $b^2 > 4ac$, then r_1 and r_2 are real numbers, and the general solution is given as

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \tag{152}$$

- If $b^2 < 4ac$, then r_1 and r_2 are complex numbers such that $r_1 = \lambda + i\mu$ and $r_2 = \overline{r_1} = \lambda - i\mu$ for real numbers λ, μ. Then, the general solution is given as

$$y(t) = e^{\lambda t}\left(c_1 \cos(\mu t) + c_2 \sin(\mu t)\right) \tag{153}$$

- If $b^2 = 4ac$, then $r_1 = r_2 = r$. Then the general solution is given as

$$y(t) = c_1 e^{-\frac{b}{2a}t} + c_2 t e^{-\frac{b}{2a}t} \tag{154}$$

3.2.2 Euler Equations

Euler equations (a.k.a Cauchy-Euler equations) are the differential equations of the form

$$x^2 \frac{d^2 y}{dx^2} + Ax\frac{dy}{dx} + By = 0, \quad x > 0 \tag{155}$$

where A and B are constants. This is a second-order homogeneous linear ODE with **non-constant** coefficients, but we will convert it into an ODE with **constant** coefficients. Introducing a new independent variable

$$t = \ln x, \quad \text{or} \quad x = e^t, \tag{156}$$

and let

$$Y(t) = y(e^t) = y(x). \tag{157}$$

Taking the derivative we have

$$\frac{dy(x)}{dx} = \frac{dY(t)}{dx} = \frac{dY}{dt} \cdot \frac{dt}{dx} = Y'(t)\frac{1}{x}. \tag{158}$$

Then,

$$x\frac{dy(x)}{dx} = Y'(t). \tag{159}$$

Taking derivative again we get

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(Y'(t)\frac{1}{x}\right) && (160)\\
&= \frac{1}{x}\frac{d}{dx}Y'(t) + Y'(t)\left(-\frac{1}{x^2}\right) && (161)\\
&= \frac{1}{x}\frac{d}{dt}Y'(t)\frac{dt}{dx} - \frac{1}{x^2}Y'(t) && (162)\\
&= \frac{1}{x^2}(Y''(t) - Y'(t)). && (163)
\end{aligned}$$

Then,

$$x^2\frac{d^2y}{dx^2} = Y''(t) - Y'(t). \tag{164}$$

Substituting $x\frac{dy}{dx}$ and $x^2\frac{d^2y}{dx^2}$ into the Euler equation we get

$$Y''(t) + (A-1)Y'(t) + BY(t) = 0. \tag{165}$$

This is a constant coefficient linear equation, the general solution $Y(t)$ can be obtained. Then, the general solution of the Euler equation is

$$y(x) = Y(\ln x). \tag{166}$$

An alternative method for solving Euler equations is using **trial solution** $y = x^r$ (r is the power to be determined), then $y' = rx^{r-1}, y'' = r(r-1)x^{r-2}$, then

$$r(r-1)x^r + Arx^r + Bx^r = 0. \tag{167}$$

Thus,

$$r^2 + (A-1)r + B = 0. \tag{168}$$

- Case 1: Two distinct real roots r_1, r_2. the general solution is

$$y(t) = c_1x^{r_1} + c_2x^{r_2}. \tag{169}$$

- Case 2: One repeated real root $r_1 = r_2$. The general solution is

$$y(x) = c_1x^{r_1} + c_2\ln(x)x^{r_1}. \tag{170}$$

- Case 3: two distinct complex roots: $\lambda \pm i\mu, \lambda, \mu \in \mathbb{R}$. The general solution is

$$y(x) = c_1x^{\lambda}\cos(\mu\ln(x)) + c_2x^{\lambda}\sin(\mu\ln(x)). \tag{171}$$

The Euler equation has a **regular singular point** at $x = 0$, which is related to the series solution of ODEs. Details can be found in Section 5.2 of Prof. Jeffrey R. Chasnov's notes.

3.3 Homogeneous Equations with Non-Constant Coefficients

The **reduction of order** method can be applied to a second-order homogeneous ODE with **non-constant** coefficient. Although the general method for finding a FSS for $y'' + p(t)y' + q(t)y = 0$ is not available, but if we can find one nonzero-solution $y_1(t)$ of the ODE, then we can use the reduction of order method to find $y_2(t)$ so that (y_1, y_2) forms a FSS.

Consider the ODE

$$y'' + p(t)y' + q(t)y = 0. \tag{172}$$

Suppose $y_1(t)$ is a non-zero solution to the ODE. To find a second solution, consider the function

$$y(t) = v(t)y_1(t). \tag{173}$$

Then, the product rule gives

$$y'(t) = v'(t)y_1(t) + v(t)y_1'(t), \tag{174}$$

$$y''(t) = v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t). \tag{175}$$

If y is a solution to the ODE, we have

$$\begin{aligned} 0 &= y'' + p(t)y' + q(t)y &(176)\\ &= v''y_1 + 2v'y_1' + vy_1'' + p(t)(v'y_1 + vy_1') + q(t)vy_1 &(177)\\ &= y_1v'' + (2y_1' + p(t)y_1)v'. &(178) \end{aligned}$$

This gives us a second-order ODE for v that only involves v'' and v'. Define a new function $z = v'$, leading to

$$y_1(t)z' + (2y_1'(t) + p(t)y_1(t))z = 0. \tag{179}$$

Here we treat y_1 and y_1' as given functions. Note that this is a first-order linear ODE

$$\frac{dz}{dt} + \frac{2y_1' + py_1}{y_1}z = 0, \tag{180}$$

since $y_1 \neq 0$. Solving this gives

$$\begin{aligned} v'(t) = z(t) &= \exp\left(-\int \frac{2y_1' + py_1}{y_1}dt\right) &(181)\\ &= \exp\left(-\int p(t)dt - 2\ln(y_1(t))\right) &(182)\\ &= \frac{1}{y_1^2(t)}\exp\left(-\int p(t)dt\right). &(183) \end{aligned}$$

Integrating once more leads to

$$v(t) = \int (y_1(t))^{-2}\, e^{-\int p(t)dt}dt \tag{184}$$

and the second solution to the ODE is given as

$$y_2(t) = y_1(t)\int (y_1(t))^{-2}\, e^{-\int p(t)dt}dt. \tag{185}$$

Example 29. *Given that $y_1(t) = t^{-1}$ is a solution of*

$$2t^2y'' + 3ty' - y = 0, \quad t > 0, \tag{186}$$

find a FSS.

The ODE can be written as $y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0$, thus $p(t) = \frac{3}{2t}$. Plug in (185) we obtain

$$y_2 = \frac{1}{t}\int t^2 e^{-\frac{3}{2}\ln(t)+c_1} \tag{187}$$

$$= e^{c_1}\frac{1}{t}\int t^{\frac{1}{2}} \tag{188}$$

$$= e^{c_1}\frac{1}{t}(\frac{2}{3}t^{\frac{3}{2}} + c_2) \tag{189}$$

Take $e^{c_1} = \frac{3}{2}$ and $c_2 = 0$ we obtain $y_2 = t^{\frac{1}{2}}$. We can **verify the Wronskian** $W(y_1, y_2)[t] = \frac{3}{2}t^{-\frac{3}{2}} \neq 0$ for $t > 0$. Consequently, y_1, y_2 form a FSS for the ODE.

Note that this method can be used to find a second solution to the ODE if you **already have one solution**. The difficulty actually lies in finding a first solution to the ODE.

3.4 Non-Homogeneous Equations

We now turn our attention to ODE of the form

$$y'' + p(t)y' + q(t)y = r(t), \tag{190}$$

for given functions p, q, and r that are continuous in an interval I. The corresponding homogeneous equation is

$$y'' + p(t)y' + q(t)y = 0. \tag{191}$$

We have the following observation. Let Z_1 and Z_2 be solutions to the non-homogeneous problem (190). Then, the difference $Z := Z_1 - Z_2$ satisfies

$$Z'' + p(t)Z' + q(t)Z = r - r = 0. \tag{192}$$

That is, the difference Z satisfies the homogeneous equation (191). If (y_1, y_2) are a FSS to (191), then we can write $Z = Z_1 - Z_2$ as

$$Z_1(t) - Z_2(t) = c_1y_1(t) + c_2y_2(t), \tag{193}$$

for some constants c_1, c_2. We have actually derived a general expression for the solution to the non-homogeneous equation (190). Let $Y(t)$ denote a solution to (190), then any solution y to (190) can be expressed as

$$y(t) = Y(t) + c_1y_1(t) + c_2y_2(t), \tag{194}$$

where (y_1, y_2) is a FSS to the homogeneous problem (191). This is similar to the steps in linear algebra for solving the non-homogeneous linear system $A\mathbf{x} = \mathbf{b}$: We need to find $\mathbf{x}_p$ and $\mathbf{x}_n$, where the former is a particular solution to $A\mathbf{x} = \mathbf{b}$, and the latter stands for the linear combination of the basis of Null(A). Then general solution to the system is $\mathbf{x}_p + \mathbf{x}_n$.

Definition 30 (Complementary Solution, Particular Solution)**.** *For a solution expression*

$$y(t) = c_1y_1(t) + c_2y_2(t) + Y(t) \tag{195}$$

to the ODE

$$y'' + p(t)y' + q(t)y = r(t), \tag{196}$$

we call the function

$$y_c(t) := c_1y_1(t) + c_2y_2(t) \tag{197}$$

the complementary solution, which is a solution to the homogeneous equation, and the function $Y(t)$ the particular solution, which is a solution to the non-homogeneous equation.

This gives us the way of solving non-homogeneous second-order linear ODEs:

1. Obtain a FSS (y_1, y_2) to the homogeneous problem (191).
2. Find a solution $Y(t)$ to the non-homogeneous problem (190).
3. The general solution to (191) is then given as

$$y(t) = Y(t) + c_1y_1(t) + c_2y_2(t). \tag{198}$$

So the key questions become:

- How do we find y_1 and y_2?
- How do we find $Y(t)$?

Our discussion in Section 3.4.1 and 3.4.2 will focus on these two questions.

3.4.1 Method of Undetermined Coefficients

The general method for finding the second-order linear ODE with non-constant coefficient $a(t)y''+b(t)y'+c(t)y=r(t)$ is still missing. We will first look at the special cases **when a, b, c are real constants and $r(t)$ is in some particular form**. In other words, we will show how to obtain a solution Y to the ODE

$$ay''+by'+cy=r(t) \tag{199}$$

for some specific forms of $r(t)$.

The method for this case is the **method of undetermined coefficients**, which makes a guess on what the particular solution $Y(t)$ could look like. There are only certain classes of functions for $r(t)$ which $Y(t)$ could be obtained explicitly. We will consider $r(t)$ to be a mixture of **polynomials, exponential, sine and cosine**.

Example 31. *Solve*

$$y''-3y'-4y=3e^{2t}. \tag{200}$$

In the standard form we have

$$r(t)=3e^{2t}. \tag{201}$$

Since the derivative of exponential function is also exponential, **a possible choice for the particular solution Y would involve exponential**. Solving the homogeneous problem $y''-3y'-4y=0$, the complementary solution is obtained as

$$y_c(t)=c_1e^{4t}+c_2e^{-t}. \tag{202}$$

Returning to the non-homogeneous problem, **assume $Y(t)$ is of the form**

$$Y(t)=Ae^{qt} \tag{203}$$

for some coefficients A and q that are not determined yet. Plugging into the non-homogeneous equations gives

$$Y''-3Y'-4Y=Aq^2e^{qt}-3Aqe^{qt}-4Ae^{qt}=A(q^2-3q-4)e^{qt}=3e^{2t}. \tag{204}$$

Therefore, it makes sense to choose

$$q=2,\quad A(q^2-3q-4)=3 \quad\Rightarrow\quad A=-\frac{1}{2} \quad\Rightarrow\quad Y(t)=-\frac{1}{2}e^{2t}. \tag{205}$$

Hence, the general solution y to the ODE $y''-3y'-4=3e^{2t}$ can be expressed as

$$y(t)=c_1e^{4t}+c_2e^{-t}-\frac{1}{2}e^{2t}. \tag{206}$$

Example 32. *Solve*

$$y'' - 3y' - 4y = 2e^{-t}. \tag{207}$$

Since $r(t)$ is an exponential, try $Y(t) = Ae^{-t}$ and determine the value of A. However,

$$Y'' - 3Y' - 4Y = A(1 + 3 - 4)e^{-t} = 0. \tag{208}$$

So no choice of A would satisfy the non-homogeneous ODE. Actually, a FSS to the homogeneous ODE $y'' - 3y' - 4y = 0$ is $y_1 = e^{4t}$ and $y_2 = e^{-t}$. That is, the guess function $Y(t) = Ae^{-t}$ actually is a solution to the homogeneous problem, and consequently, it cannot be a solution to the non-homogeneous problem! In this case, where the assumed form of the particular solution Y is a duplicate of one of the solutions to the homogeneous problem, we can consider a new guess for Y which looks like

$$Y(t) = Ate^{-t}, \tag{209}$$

for undetermined constant A, which is similar to the FSS $(e^{-\frac{b}{2a}t}, te^{-\frac{b}{2a}t})$ for the ODE $ay'' + by' + cy = 0$ when $b^2 = 4ac$. Trying this new guess yields

$$Y'' - 3Y' - 4Y = -5Ae^{-t} = 2e^{-t}. \tag{210}$$

This means that we should take

$$A = -\frac{1}{5} \quad \Rightarrow \quad Y(t) = -\frac{2}{5}te^{-t}. \tag{211}$$

Thus a general solution y to the ODE $y'' - 3y' - 4y = 2e^{-t}$ is

$$y(t) = c_1e^{4t} + c_2e^{-t} - \frac{2}{5}te^{-t}. \tag{212}$$

Example 33. *Solve*

$$y'' - 3y' - 4y = t^2 + t + 1. \tag{213}$$

We know the complementary solution is $y_c = c_1e^{4t} + c_2e^{-t}$. Since $r(t)$ is a polynomial of degree 2, a possible guess is that the particular solution Y is also a polynomial of the same degree, that is $Y(t) = At^2 + Bt + C$ for some undetermined coefficients A, B, C. Then, plugging into the equation gives

$$Y'' - 3Y' - 4Y = 2A - 3(2At + B) - 4(At^2 + Bt + C) \tag{214}$$
$$= -4At^2 - (4B + 6A)t + (2A - 3B - 4C) = t^2 + t + 1. \tag{215}$$

Comparing coefficients immediately gives

$$A = -\frac{1}{4}, \quad B = \frac{1}{8}, \quad C = -\frac{15}{32}, \tag{216}$$

so the general solution y to the ODE $y'' - 3y' - 4y = t^2 + t + 1$ can be expressed as

$$y(t) = c_1e^{4t} + c_2e^{-t} - \frac{1}{4}t^2 + \frac{1}{8}t - \frac{15}{32}. \tag{217}$$

What if $r(t)$ involves the multiplication of exponential function and polynomials? The method is summarized as follows.

Case 1. $r(t) = P_n(t)e^{\alpha t}$**. A possible guess is**

$$Y(t) = t^s Q_n(t)e^{\alpha t}, \tag{218}$$

where $Q_n(t) = A_0 + A_1 t + \cdots + A_n t^n$ is a polynomial with undetermined coefficients $A_0, \ldots, A_n$, and $s \in \{0, 1, 2\}$ is an exponent determined by the following criterion:

$$s = \begin{cases} 0 & \text{if } \alpha \neq r_1, \alpha \neq r_2, \\ 1 & \text{if } \alpha = r_1 \neq r_2, \\ 2 & \text{if } r_1 = r_2 = \alpha. \end{cases} \tag{219}$$

where r_1 and r_2 are the roots to the characteristic equation

$$ar^2 + br + c = 0. \tag{220}$$

In fact, s is the **multiplicity** of α as a root of the characteristic equation. **The guess** (218) **includes Example 31 to Example 33!**

The problem of determining a particular solution to the ODE

$$ay'' + by' + cy = P_n(t)e^{\alpha t} \tag{221}$$

can be done by a substitution. Let

$$Y(t) = e^{\alpha t}u(t), \tag{222}$$

and by substituting this into the ODE we obtain

$$e^{\alpha t}\left[a[u'' + 2\alpha u' + \alpha^2 u] + b[u' + \alpha u] + cu\right] = e^{\alpha t}P_n(t) \tag{223}$$

$$\Rightarrow au'' + (2\alpha a + b)u' + (a\alpha^2 + b\alpha + c)u = P_n(t). \tag{224}$$

To equal polynomial degree on both sides, it is reasonable to take

$$u(t) = \begin{cases} A_n t^n + \cdots + A_0 & \text{if } a\alpha^2 + b\alpha + c \neq 0, \\ t(A_n t^n + \cdots + A_0) & \text{if } a\alpha^2 + b\alpha + c = 0, 2a\alpha + b \neq 0, \\ t^2(A_n t^n + \cdots + A_0) & \text{if } a\alpha^2 + b\alpha + c = 0, 2a\alpha + b = 0. \end{cases} \tag{225}$$

$$= t^s(A_n t^n + \cdots + A_0), \quad s = \begin{cases} 0 & \text{if } \alpha \neq r_1, \alpha \neq r_2, \\ 1 & \text{if } \alpha = r_1 \neq r_2, \\ 2 & \text{if } r_1 = r_2 = \alpha. \end{cases} \tag{226}$$

Example 34. *Find a particular solution of*

$$y'' - 3y' - 4y = te^{-t}. \tag{227}$$

e^{-t} is a solution to the homogeneous problem, and the non-homogeneous term is $r(t) = te^{-t}$. In this case we have $r_2 = \alpha = -1$ and $r_1 = 4$. Taking $s = 1$ we try a particular solution Y of the form

$$Y(t) = t(A_1 t + A_0)e^{-t} = (A_1 t^2 + A_0 t)e^{-t}. \tag{228}$$

Take derivatives

$$Y'(t) = (-A_1t^2 + (2A_1 - A_0)t + A_0)e^{-t}, \quad Y''(t) = (A_1t^2 + (A_0 - 4A_1)t + 2A_1 - 2A_0)e^{-t}. \tag{229}$$

Substituting these into the equation, we get $(-10A_1t + 2A_1 - 5A_0)e^{-t} = te^{-t}$. Thus, $-10A_1 = 1, 2A_1 - 5A_0 = 0$. Therefore, $A_1 = -\frac{1}{10}, A_0 = -\frac{1}{25}$. The particular solution is

$$Y(t) = t\left(-\frac{1}{10}t - \frac{1}{25}\right)e^{-t}. \tag{230}$$

What if $r(t)$ involves the multiplication of exponential function and polynomial as well as sine(cosine) function?

Example 35. *Solve*

$$y'' - 3y' - 4y = 2\sin(t) \tag{231}$$

The complementary solution is $y_c = c_1e^{4t} + c_2e^{-t}$. Since the non-homogeneous term $r(t) = 2\sin(t)$, a possible solution would involve sine and cosine, so consider

$$Y(t) = a\sin(\alpha t) + b\cos(\beta t) \tag{232}$$

for undetermined coefficients a, b, α, β. Plugging into the non-homogeneous equations gives

$$\begin{aligned} Y'' - 3Y' - 4Y &= -a\alpha^2\sin(\alpha t) - b\beta^2\cos(\beta t) - 3(a\alpha\cos(\alpha t) - b\beta\sin(\beta t)) \\ &\quad - 4(a\sin(\alpha t) + b\cos(\beta t)) && (233) \\ &= \sin(\alpha t)[-a\alpha^2 - 4a] + \cos(\beta t)[-b\beta^2 - 4b] \\ &\quad + \cos(\alpha t)[-3a\alpha] + \sin(\beta t)[3b\beta] && (234) \\ &= 2\sin(t) && (235) \end{aligned}$$

Since the RHS only involves $\sin(t)$, we can set

$$\alpha = 1, \quad \beta = 1. \tag{236}$$

This simplifies the above calculation to

$$\sin(t)[-5a + 3b] + \cos(t)[-5b - 3a] = 2\sin(t). \tag{237}$$

Since there is no term involving the cosine on the RHS, we must have

$$-5a + 3b = 2, -5b - 3a = 0 \quad \Rightarrow \quad a = -\frac{5}{17}, \quad b = \frac{3}{17}. \tag{238}$$

Therefore, the general solution y to the ODE can be expressed as

$$y(t) = c_1e^{4t} + c_2e^{-t} - \frac{5}{17}\sin(t) + \frac{3}{17}\cos(t). \tag{239}$$

Case 2. $r(t) = e^{\alpha t} P_n(t)\cos(\beta t)$ **or** $e^{\alpha t} P_n(t)\sin(\beta t)$**.** Using Euler's formula: $\cos(\beta t) = \frac{1}{2}\left(e^{\beta i t} + e^{-\beta i t}\right), \sin(\beta t) = \frac{1}{2i}\left(e^{\beta i t} - e^{-\beta i t}\right)$, the ODE becomes

$$ay'' + by' + cy = \frac{1}{2}P_n(t)\left(e^{(\alpha+\beta i)t} + e^{(\alpha-\beta i)t}\right) \tag{240}$$

$$ay'' + by' + cy = \frac{1}{2i}P_n(t)\left(e^{(\alpha+\beta i)t} - e^{(\alpha-\beta i)t}\right). \tag{241}$$

A possible guess for the above two ODEs is

$$Y(t) = t^s\left(Q_n(t)\cos(\beta t) + R_n(t)\sin(\beta t)\right)e^{\alpha t}, \tag{242}$$

where $Q_n(t) = A_0 + A_1 t + \cdots + A_n t^n, R_n(t) = B_0 + B_1 t + \cdots + B_n t^n$ are polynomials with undetermined coefficients $A_0, \ldots, A_n, B_0, \ldots, B_n$, and $s \in \{0, 1\}$ is an exponent determined by

$$s = \begin{cases} 0 & \text{if } \alpha + i\beta \text{ is not a root of the characteristic equation,} \\ 1 & \text{if } \alpha + i\beta \text{ is a root of the characteristic equation.} \end{cases} \tag{243}$$

To see the reason behind this, let us consider the case $r(t) = e^{\alpha t} P_n(t)\sin(\beta t)$ since the two cases are similar. We consider

$$Y(t) = e^{\alpha t}\left(Q(t)\cos(\beta t) + R(t)\sin(\beta t)\right), \tag{244}$$

for some functions Q and R, and upon differentiating:

$$\begin{aligned} Y'(t) &= \alpha e^{\alpha t}\left(Q(t)\cos(\beta t) + R(t)\sin(\beta t)\right) + e^{\alpha t}\beta\left(-Q(t)\sin(\beta t) + R(t)\cos(\beta t)\right) \\ &\quad + e^{\alpha t}\left(Q'(t)\cos(\beta t) + R'(t)\sin(\beta t)\right), \end{aligned} \tag{245}$$

$$\begin{aligned} Y''(t) &= \alpha^2 e^{\alpha t}(Q(t)\cos(\beta t) + R(t)\sin(\beta t)) + 2e^{\alpha t}\alpha\beta(-Q(t)\sin(\beta t) + R(t)\cos(\beta t)) \\ &\quad + 2\alpha e^{\alpha t}(Q'(t)\cos(\beta t) + R'(t)\sin(\beta t)) + \beta^2 e^{\alpha t}(-Q(t)\cos(\beta t) - R(t)\sin(\beta t)) \\ &\quad + 2\beta e^{\alpha t}(-Q'(t)\sin(\beta t) + R'(t)\cos(\beta t)) + e^{\alpha t}(Q''(t)\cos(\beta t) + R''(t)\sin(\beta t)). \end{aligned} \tag{246}$$

Plugging the above expression into the ODE yields

$$e^{\alpha t}P_n(t)\sin(\beta t) = aY'' + bY' + cY \tag{247}$$

$$\begin{aligned} &= e^{\alpha t}\cos(\beta t)[(a\alpha^2 - a\beta^2 + b\alpha + c)Q + (2a\alpha + b)(\beta R + Q') + 2a\beta R' + aQ''] \\ &\quad + e^{\alpha t}\sin(\beta t)[(a\alpha^2 - a\beta^2 + b\alpha + c)R + (2a\alpha + b)(-\beta Q + R') - 2a\beta Q' + aR'']. \end{aligned} \tag{248}$$

Equating coefficients means that

$$(a\alpha^2 - a\beta^2 + b\alpha + c)Q + (2a\alpha + b)(\beta R + Q') + 2a\beta R' + aQ'' = 0, \tag{249}$$

$$(a\alpha^2 - a\beta^2 + b\alpha + c)R + (2a\alpha + b)(-\beta Q + R') - 2a\beta Q' + aR'' = P_n. \tag{250}$$

Observe that, $\alpha + i\beta$ is a root of the characteristic equation if and only if

$$a(\alpha + i\beta)^2 + b(\alpha + i\beta) + c = [a\alpha^2 - a\beta^2 + b\alpha + c] + i(2a\alpha + b)\beta = 0. \tag{251}$$

Using the fact that a complex number is zero if and only if the real and imaginary parts are zero, we have

$$\alpha + i\beta \text{ is a root} \iff a(\alpha^2 - \beta^2) + b\alpha + c = 0, \quad (2a\alpha + b)\beta = 0. \tag{252}$$

As the RHS of (250) are polynomials, we may take Q and R to be polynomials. The question is the degree.

- Case 1: $\alpha + i\beta$ is not a root of the characteristic equation. Then, $(a\alpha^2 - a\beta^2 + b\alpha + c)$ and $(2a\alpha + b)\beta$ are not all zeros. We can take Q and R to have the **same degree** as the polynomial P_n, i.e.,

$$Q(t) = A_n t^n + \cdots + A_0, \quad R(t) = B_n t^n + \cdots + B_0$$

- Case 2: $\alpha + i\beta$ is a root of the characteristic equation, then (249), (250) simplifies to

$$(2a\alpha + b)Q' + 2a\beta R' + aQ'' = 0, \tag{253}$$
$$(2a\alpha + b)R' - 2a\beta Q' + aR'' = P_n. \tag{254}$$

and from the second equation, we see that the degree of the LHS would be the degree of R' or Q' (which ever is higher), thus we take

$$Q(t) = t(A_n t^n + \cdots + A_1 t + A_0), \quad R(t) = t(B_n t^n + \cdots + B_1 t + B_0), \tag{255}$$

in order to match the degree with the RHS.

Example 36. *Find a particular solution of*

$$y'' - 3y' - 4y = -8e^t \cos 2t. \tag{256}$$

We guess our particular solution $Y(t)$ is the product of e^t and a linear combination of $\cos 2t$ and $\sin 2t$, i.e.

$$Y(t) = Ae^t \cos 2t + Be^t \sin 2t \tag{257}$$

It follows that

$$Y'(t) = [A\cos 2t - 2A\sin 2t]e^t + [B\sin 2t + 2B\cos 2t]e^t \tag{258}$$
$$= (A + 2B)e^t \cos 2t + (-2A + B)e^t \sin 2t \tag{259}$$

and

$$Y''(t) = [(A + 2B)\cos 2t - 2(A + 2B)\sin 2t]e^t + [(-2A + B)\sin 2t + 2(-2A + B)\cos 2t]e^t \tag{260}$$
$$= (-3A + 4B)e^t \cos 2t + (-4A - 3B)e^t \sin 2t \tag{261}$$

After substituting for y, y' and y'' in (256) we obtain:

$$\begin{aligned} &e^t \cos 2t[(-3A + 4B) - 3(A + 2B) - 4A] \\ &+ e^t \sin 2t[(-4A - 3B) - 3(-2A + B) - 4B] = -8e^t \cos 2t \end{aligned} \tag{262}$$

Hence:

$$\begin{cases} -10A - 2B = -8, \\ 2A - 10B = 0, \end{cases} \Rightarrow \begin{cases} A = \dfrac{10}{13}, \\ B = \dfrac{2}{13}, \end{cases} \tag{263}$$

Hence our particular solution is:

$$Y(t) = \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin 2t. \tag{264}$$

Summary of Section 3.4.1:

For

$$ay'' + by' + cy = r(t) \tag{265}$$

the trial function $Y(t)$ vs. $r(t)$ is listed as follows:

$r(t)$	$Y(t)$	The value for s
$P_n(t)e^{\alpha t}$	$Q_n(t)t^s e^{\alpha t}$	$s = \begin{cases} 0, & \alpha \text{ is not a root.} \\ 1, & \alpha = r_1 \neq r_2 \\ 2, & \alpha = r_1 = r_2 \end{cases}$ r_1, r_2 are roots of $ar^2 + br + c = 0$
$\begin{cases} P_n e^{\alpha t}\sin\beta t \\ P_n e^{\alpha t}\cos\beta t \end{cases}$	$\begin{cases} [Q_n(t)\cos\beta t \\ +R_n(t)\sin\beta t]t^s e^{\alpha t} \end{cases}$	$s = \begin{cases} 0, & \text{if } \alpha + i\beta \text{ is not a root of } ar^2 + br + c = 0. \\ 1, & \text{if } \alpha + i\beta \text{ is a root of } ar^2 + br + c = 0. \end{cases}$

We will conclude this section with another theorem.

Theorem 37. *Suppose Y_1 is a solution to*

$$ay'' + by' + cy = g_1(t), \tag{266}$$

and Y_2 is a solution to

$$ay'' + by' + cy = g_2(t). \tag{267}$$

Then the sum $Y_1 + Y_2$ is a solution to

$$ay'' + by' + cy = g_1(t) + g_2(t). \tag{268}$$

Proof. Since Y_1 is a solution to $ay'' + by' + cy = g_1(t)$. and Y_2 is a solution to $ay'' + by' + cy = g_2(t)$, we have

$$aY_1'' + bY_1' + cY_1 = g_1(t) \tag{269}$$

$$aY_2'' + bY_2' + cY_2 = g_2(t) \tag{270}$$

Sum the two equations, we have

$$[aY_1'' + bY_1' + cY_1] + [aY_2'' + bY_2' + cY_2] \tag{271}$$

$$=a[Y_1'' + Y_2''] + b[Y_1' + Y_2'] + c[Y_1 + Y_2] \tag{272}$$

$$=a[Y_1 + Y_2]'' + b[Y_1 + Y_2]' + c[Y_1 + Y_2] \tag{273}$$

$$=g_1(t) + g_2(t) = g(t). \tag{274}$$

□

Clearly the result also holds when a, b, c are not constants.

Example 38. *Find a particular solution of*

$$y'' - 3y' - 4y = 3e^{2t} + 2e^{-t} + 2\sin t - 8e^t \cos 2t. \tag{275}$$

Combining previous results, we have

$$Y(t) = -\frac{1}{2}e^{2t} - \frac{2}{5}te^{-t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t + \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin 2t. \tag{276}$$

3.4.2 Variation of Parameters

The method of undetermined coefficients is straightforward, but requires that the non-homogeneous term $r(t)$ to be in a special form. We need a more general method that in principle can be applied to any equation. One such method is the **variation of parameters**.

Consider a general 2nd-order linear ODE

$$y'' + p(t)y' + q(t)y = r(t), \tag{277}$$

and suppose (y_1, y_2) forms a FSS to the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0. \tag{278}$$

How to find a particular solution to the non-homogeneous equation (277)? Consider for some functions $u_1(t)$, $u_2(t)$ such that the new function

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) \tag{279}$$

solves (277). We now determine what equations u_1 and u_2 have to satisfy. Differentiating (279) yields

$$y' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'. \tag{280}$$

In order to simplify the computation, **impose a condition**

$$u_1'y_1 + u_2'y_2 = 0. \tag{281}$$

Then the derivative becomes

$$y' = u_1y_1' + u_2y_2'. \tag{282}$$

Differentiating again leads to

$$y'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'' \tag{283}$$

Substitute into the non-homogeneous ODE gives

$$\begin{aligned} y'' + p(t)y' + q(t)y &= u_1(y_1'' + p(t)y_1' + q(t)y_1) + u_2(y_2'' + p(t)y_2' + q(t)y_2) && (284)\\ &\quad + u_1'y_1' + u_2'y_2' && (285)\\ &= u_1'y_1' + u_2'y_2' = r(t). && (286)\end{aligned}$$

Thus, we obtain two conditions for u_1 and u_2:

$$u_1'y_1 + u_2'y_2 = 0, \quad u_1'y_1' + u_2'y_2' = r(t), \tag{287}$$

which can be summarized as

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix} \tag{288}$$

Since the determinant is the Wronskian $W(y_1, y_2)[t]$ which is non-zero since (y_1, y_2) is a FSS, (u_1', u_2') can be solved. Using Cramer's rule, we have

$$u_1'(t) = -\frac{y_2 r}{W(y_1, y_2)}(t), \quad u_2'(t) = \frac{y_1 r}{W(y_1, y_2)}(t). \tag{289}$$

Integrating gives

$$u_1(t) = -\int \frac{y_2 r}{W(y_1, y_2)}(t)dt + d_1, \quad u_2(t) = \int \frac{y_1 r}{W(y_1, y_2)}(t)dt + d_2, \tag{290}$$

for constants $d_1, d_2 \in \mathbb{R}$, and the general solution to the non-homogeneous equation is

$$y(t) = (c_1 + d_1)y_1(t) + (c_2 + d_2)y_2(t) - y_1 \int \frac{y_2 r}{W(y_1, y_2)}(t)dt + y_2 \int \frac{y_1 r}{W(y_1, y_2)}(t)dt. \tag{291}$$

We can simply take $d_1 = d_2 = 0$, so the final solution becomes

$$y(t) = c_1 y_1(t) + c_2 y_2(t) - y_1 \int \frac{y_2 r}{W(y_1, y_2)}(t)dt + y_2 \int \frac{y_1 r}{W(y_1, y_2)}(t)dt. \tag{292}$$

for constants $c_1, c_2 \in \mathbb{R}$.

This method is able to treat rather general second-order ODEs (since $p(t)$ and $q(t)$ need not be constants). However, **it is not easy to find a FSS** (if $p(t)$ and $q(t)$ are not constants). Another difficulty lies in the evaluation of the integrals:

$$-\int \frac{y_2 r}{W(y_1, y_2)}(t)dt, \quad \int \frac{y_1 r}{W(y_1, y_2)}(t)dt \tag{293}$$

which may not be possible if r, y_1, y_2 are complicated functions.

Example 39. *Solve the ODE*

$$y'' - 3y' + 2y = \frac{e^{3t}}{e^t + 1} \tag{294}$$

First look at the homogeneous problem

$$y'' - 3y' + 2y = 0, \tag{295}$$

the complementary solution is given as

$$y_c(t) = c_1 e^t + c_2 e^{2t}. \tag{296}$$

We now compute u_1 and u_2, where we use

$$y_1 = e^t, \quad y_2 = e^{2t}, \quad r = \frac{e^{3t}}{e^t + 1}, \quad W(y_1, y_2)[t] = e^{3t}. \tag{297}$$

We have

$$u_1'(t) = -\frac{e^{2t}}{e^t + 1}, \quad u_2'(t) = \frac{e^t}{e^t + 1}. \tag{298}$$

Integrating gives

$$u_1(t) = \ln(e^t + 1) - e^t, \quad u_2(t) = \ln(e^t + 1). \tag{299}$$

Hence, a particular solution is

$$Y(t) = u_1 y_1 + u_2 y_2 = e^t \ln(e^t + 1) + e^{2t} \ln(e^t + 1) - e^{2t}. \tag{300}$$

The general solution to the ODE is

$$y(t) = c_1 e^t + c_2 e^{2t} + e^t \ln(e^t + 1) + e^{2t} \ln(e^t + 1) \tag{301}$$

where c_1, c_2 are arbitrary constants.

4 Higher-Order Linear Differential Equations

4.1 General Theory

The general n-th order linear ODE is of the form

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = g(t), \tag{302}$$

and for an IVP we provide initial conditions

$$y(t_0) = x_0, y'(t_0) = x_1, \ldots, y^{(n-1)}(t_0) = x_{n-1}. \tag{303}$$

We first state the existence and uniqueness theorem.

Theorem 40 (Existence and Uniqueness for n-th Order Linear ODE)**.** *Let $I \subset \mathbb{R}$ be an open interval and suppose $g, p_0, p_1, \ldots, p_{n-1}$ are continuous functions in I. For $t_0 \in I$ and $x_0, \ldots, x_{n-1} \in \mathbb{R}$, there is exactly one solution to the IVP*

$$\begin{cases} y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = g(t), \\ y(t_0) = x_0, y'(t_0) = x_1, \ldots, y^{(n-1)}(t_0) = x_{n-1}. \end{cases} \tag{304}$$

Linear (in)dependence is defined in similar way.

Definition 41 (Linear Dependence)**.** *The functions $f_1(t), \ldots, f_n(t)$ are linearly dependent on the interval I if there exists a set of numbers $(\alpha_1, \ldots, \alpha_n) \neq (0, \ldots, 0)$, such that*

$$\alpha_1 f_1(t) + \cdots + \alpha_n f_n(t) = 0, \tag{305}$$

for all $t \in I$. Otherwise, we say that the functions $f_1(t), \ldots, f_n(t)$ are linearly independent.

Similar to Theorem 18, we also have the principle of superposition.

Theorem 42 (Principle of Superposition)**.** *Let $y_1, \ldots, y_n$ be solutions to the homogeneous equation*

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = 0, \tag{306}$$

then, for any constants $c_1, \ldots, c_n \in \mathbb{R}$, the function

$$\phi(t) = c_1 y_1(t) + \cdots + c_n y_n(t) \tag{307}$$

is also a solution to the above homogeneous equation.

We also have the Wronskian.

Definition 43 (Wronskian)**.** *Given functions $f_1, \dots, f_n$ that are differentiable up to order $n-1$, we define the Wronskian W as*

$$W(f_1, \dots, f_n)[t] = \det \begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{pmatrix} [t]. \tag{308}$$

The natural question is: given n solutions $y_1, \dots, y_n$ to the **homogeneous equation**

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = 0. \tag{309}$$

Can **any** solution ϕ to the homogeneous equation be expressed as a linear combination of $y_1, \dots, y_n$? Similarly with the case of the second-order ODE, we have the following theorem.

Theorem 44. *If $p_0, \dots, p_{n-1}$ are continuous functions in I, and $y_1, \dots, y_n$ are solutions to the above homogeneous equation, then every solution ϕ to the homogeneous equation can be expressed as a linear combination of $y_1, \dots, y_n$ if and only if $W(y_1, \dots, y_n)[t_0] \neq 0$ for some $t_0 \in I$. In this case, we call $(y_1, \dots, y_n)$ a* ***fundamental set of solutions (FSS)*** *to the homogeneous equation.*

An analogous result to Abel's identity (Theorem 23):

Theorem 45. *Let $y_1, \dots, y_n$ be solutions to the homogeneous equation*

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = 0, \tag{310}$$

for $t \in I$. Then,

$$W(y_1, \dots, y_n)[t] = ce^{-\int p_{n-1}(t)dt} \tag{311}$$

for a constant c not dependent on $t \in I$.

Proof. The idea is to derive an equation satisfied by the Wronskian. Recall the rule for taking derivatives on determinants: **The derivative of an $n \times n$ determinant is equal to the sum of n determinants, where the k-th determinant is obtained by differentiating the k-th row of the original determinant, and keeping the other rows unchanged.**

For example, denote $D := \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ where $a, b, c, \dots, i$ are functions of t. Then we have

$$\frac{dD}{dt} = \begin{vmatrix} a' & b' & c' \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & c \\ d' & e' & f' \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & c \\ d & e & f \\ g' & h' & i' \end{vmatrix} \tag{312}$$

Thus, we have

$$\frac{d}{dt}W[t] = \begin{vmatrix} y_1' & y_2' & \cdots & y_n' \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} + \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1'' & y_2'' & \cdots & y_n'' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$
$$+ \cdots + \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix} = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}. \tag{313}$$

The first $n-1$ determinants all have two identical rows, thus they are all zero and only the last determinant is nonzero. Using that for each $1 \leq k \leq n$,

$$y_k^{(n)} = -p_{n-1}y_k^{(n-1)} - \cdots - p_1 y_k' - p_0 y_k, \tag{314}$$

then applying elementary row operations we find that

$$\frac{d}{dt}W[t] = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ -p_{n-1}y_1^{(n-1)} & -p_{n-1}y_2^{(n-1)} & \cdots & -p_{n-1}y_n^{(n-1)} \end{vmatrix} = -p_{n-1}W[t]. \tag{315}$$

Thus,

$$W(y_1, \ldots, y_n)[t] = ce^{-\int p_{n-1}(t)dt} \tag{316}$$

for a constant c not dependent on $t \in I$.

□

Finally, the relationship between linear (in)dependence and Wronskian.

Theorem 46. *If $y_1, \ldots, y_n$ are solutions to the ODE $y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = 0$, $t \in I$, then $y_1, \ldots, y_n$ are linearly independent $\iff W[y_1, \ldots, y_n](t) \neq 0$, $\forall t \in I$ ($(y_1, \ldots, y_n)$ forms a FSS).*

4.2 Homogeneous Equations with Constant Coefficients

We will study, for constants $a_n \neq 0, a_{n-1}, \ldots, a_0 \in \mathbb{R}$, the equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0. \tag{317}$$

Still, consider a trial function $\phi = e^{rt}$ for $r \in \mathbb{R}$. Substituting this gives the **characteristic equation**

$$a_n r^n + \cdots + a_1 r + a_0 = 0. \tag{318}$$

The characteristic polynomial is

$$Z(r) = a_n r^n + \cdots + a_1 r + a_0. \tag{319}$$

From the fundamental theorem of algebra, every polynomial with real coefficients of degree n has n complex roots. Hence

$$Z(r) = a_n (r - r_1)(r - r_2) \cdots (r - r_n), \tag{320}$$

where $r_1, \ldots, r_n$ are complex numbers, it is possible that some roots are repeated.

Definition 47 (Multiplicity)**.** *Let $P_k(x)$ be a polynomial of degree k in x. A root r has multiplicity $m \in \mathbb{N}, m \geq 1$, if there is another polynomial $S_{k-m}(x)$ of degree $k - m$ such that $S_{k-m}(r) \neq 0$ and*

$$P_k(x) = S_{k-m}(x)(x - r)^m. \tag{321}$$

Case 1. Real and distinct roots. If the roots of $Z(r) = 0$ are all real and distinct, then we have the solutions

$$y_1(t) = e^{r_1 t}, \ldots, y_n(t) = e^{r_n t}. \tag{322}$$

They are linearly independent solutions and form a FSS. Compute the Wronskian

$$W(e^{r_1 t}, e^{r_2 t}, \ldots, e^{r_n t})(t) = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} & \ldots & e^{r_n t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} & \ldots & r_n e^{r_n t} \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1} e^{r_1 t} & r_2^{n-1} e^{r_2 t} & \ldots & r_n^{n-1} e^{r_n t} \end{vmatrix} \tag{323}$$

$$= e^{(r_1 + \cdots + r_n)t} \begin{vmatrix} 1 & 1 & \ldots & 1 \\ r_1 & r_2 & \ldots & r_n \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1} & r_2^{n-1} & \ldots & r_n^{n-1} \end{vmatrix} \tag{324}$$

$$= e^{(r_1 + \cdots + r_n)t} \prod_{1 \leq i < j \leq n} (r_j - r_i) \neq 0. \tag{325}$$

Example 48. *Solve the ODE*

$$y^{(4)} - 7y''' + 6y'' + 30y' - 36y = 0 \tag{326}$$

The characteristic equation is:

$$r^4 - 7r^3 + 6r^2 + 30r - 36 = 0 \tag{327}$$

which can be factorized as

$$(r-3)(r+2)(r^2 - 6r + 6) = 0 \tag{328}$$

Hence $r_1 = -2, r_2 = 3, r_3 = 3 - \sqrt{3}, r_4 = 3 + \sqrt{3}$. The general solution is given by:

$$y = c_1 e^{-2t} + c_2 e^{3t} + c_3 e^{(3-\sqrt{3})t} + c_4 e^{(3+\sqrt{3})t}. \tag{329}$$

Case 2. Some roots are complex. If some roots are complex, they must appear in pairs, i.e. $\lambda \pm i\mu$ (see the Complex conjugate root theorem). Thus, we could replace the complex-valued solutions $e^{(\lambda+i\mu)t}$ and $e^{(\lambda-i\mu)t}$ by the real-valued solutions $e^{\lambda t}\cos\mu t, e^{\lambda t}\sin\mu t$. (Recall **Case 2** of Section 3.2)

Example 49. *Solve the ODE*

$$y^{(4)} - y = 0 \tag{330}$$

The characteristic equation is:

$$r^4 - 1 = 0. \tag{331}$$

We have $r = 1, -1, \pm i$. Thus $\lambda = 0, \mu = 1$. Hence $\{e^t, e^{-t}, \cos t, \sin t\}$ forms a FSS. The general solution is given by:

$$y = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t. \tag{332}$$

Case 3. Some roots are repeated.

Subcase 1: If one of the real root r_1 is repeated with multiplicity s, then the corresponding linearly independent solutions corresponding to root r_1 are:

$$e^{r_1 t}, te^{r_1 t}, t^2 e^{r_1 t}, \ldots, t^{s-1} e^{r_1 t}. \tag{333}$$

Subcase 2: If the complex root $r_1 = \lambda + i\mu$ is repeated with multiplicity s, then the corresponding conjugate $\overline{r}_1 = \lambda - i\mu$ is also the root with multiplicity s. In this case, we could replace the complex-valued solutions $e^{(\lambda+i\mu)t}, \ldots, t^{s-1}e^{(\lambda+i\mu)t}$ and $e^{(\lambda-i\mu)t}, \ldots, t^{s-1}e^{(\lambda-i\mu)t}$ by the real valued solutions as follows:

$$e^{\lambda t}\cos\mu t, te^{\lambda t}\cos\mu t, t^2 e^{\lambda t}\cos\mu t, \ldots, t^{s-1}e^{\lambda t}\cos\mu t \quad \text{– from real parts} \tag{334}$$

$$e^{\lambda t}\sin\mu t, te^{\lambda t}\sin\mu t, t^2 e^{\lambda t}\sin\mu t, \ldots, t^{s-1}e^{\lambda t}\sin\mu t \quad \text{– from imaginary parts} \tag{335}$$

These are linearly independent solutions corresponding to the repeated roots $r_1 = \lambda + i\mu$ and $\overline{r}_1 = \lambda - i\mu$.

Example 50. *Solve the ODE*

$$y^{(4)} + 2y'' + y = 0 \tag{336}$$

The characteristic equation is:

$$r^4 + 2r^2 + 1 = (r^2+1)(r^2+1) = 0. \tag{337}$$

We have $r = i, i, -i, -i$, thus $\lambda = 0, \mu = 1$. The fundamental solution is:

$$e^{it}, te^{it}, e^{-it}, te^{-it}. \tag{338}$$

The general solution is given by:

$$y = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t. \tag{339}$$

4.3 Non-Homogeneous Equations

4.3.1 Method of Undetermined Coefficients

Consider the non-homogeneous equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(t). \tag{340}$$

If Y_1 and Y_2 are both solutions to the non-homogeneous problem, then $Y_1 - Y_2$ is a solution to the corresponding homogeneous equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0. \tag{341}$$

Given a FSS $(y_1, \dots, y_n)$ to the homogeneous equation, a general solution to the non-homogeneous equation (340) is

$$y(t) = c_1 y_1(t) + \cdots + c_n y_n(t) + Y(t), \tag{342}$$

where $Y(t)$ is a particular solution to the non-homogeneous equation, $c_1 y_1(t) + \cdots + c_n y_n(t)$ is the complementary solution (solution to the homogeneous equation).

Similar to second-order equations, we now find a particular solution Y to the non-homogeneous equation (340) if $g(t)$ is a **sum/product of exponentials, cosine, sine and polynomials.** But the main difference is that the multiplicity of roots to the characteristic equation can be **greater than two.** Thus, higher powers of t need to be multiplied to get the solution to the non-homogeneous equation.

We again investigate the cases:

1. $g(t) = e^{\alpha t} P_m(t)$,
2. $g(t) = e^{\alpha t} P_m(t) \cos(\beta t),$ or $g(t) = e^{\alpha t} P_m(t) \sin(\beta t)$.

Remember the characteristic equation for the homogeneous equation is

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0. \tag{343}$$

The possible particular solutions can be used are

1. $Y(t) = t^s e^{\alpha t} Q_m(t)$, where

$$Q_m(t) = A_m t^m + \cdots + A_1 t + A_0 \tag{344}$$

for undetermined coefficients $A_m, \dots, A_0$, and

$$s = \begin{cases} 0, & \text{if } \alpha \text{ is not a root of the characteristic equation.} \\ m, & \text{if } \alpha \text{ is a root of the characteristic equation with multiplicity } m \end{cases} \tag{345}$$

2. $Y(t) = t^s e^{\alpha t} [Q_m(t) \cos(\beta t) + R_m(t) \sin(\beta t)]$, where

$$Q_m = A_m t^m + \cdots + A_1 t + A_0, R_m = B_m t^m + \cdots + B_1 t + B_0 \tag{346}$$

are polynomials of degree m with undetermined coefficients $A_m, \dots, A_0, B_m, \dots, B_0$, and

$$s = \begin{cases} 0, & \text{if } \alpha + i\beta \text{ is not a root of the characteristic equation.} \\ m, & \text{if } \alpha + i\beta \text{ is a root of the characteristic equation with multiplicity } m. \end{cases} \tag{347}$$

Example 51. *Solve*

$$y''' - 3y'' + 3y' - y = 4e^t \tag{348}$$

For the homogeneous equation, the associated characteristic equation is

$$r^3 - 3r^2 + 3r - 1 = (r-1)^3 = 0, \tag{349}$$

so $r_1 = r_2 = r_3 = 1$, i.e., a repeated eigenvalue of multiplicity three. So set

$$y_1 = e^t, \quad y_2 = te^t, \quad y_3 = t^2e^t, \tag{350}$$

and the complementary solution (to the homogeneous equation) is

$$y_c(t) = c_1e^t + c_2te^t + c_3t^2e^t. \tag{351}$$

Since $g(t) = 4e^t$ and $\alpha = 1$ is a root of the characteristic equation with multiplicity 3, consider $s = 3$ and a trial solution

$$Y(t) = At^3e^t. \tag{352}$$

Computing gives

$$Y''' - 3Y'' + 3Y' - Y = 6Ae^t = 4e^t \Rightarrow A = \frac{2}{3}, \tag{353}$$

so the general solution to the non-homogeneous ODE is

$$y(t) = c_1e^t + c_2te^t + c_3t^2e^t + \frac{2}{3}t^3e^t. \tag{354}$$

Example 52. *Solve*

$$y^{(4)} + 2y'' + y = 3\sin t \tag{355}$$

The characteristic equation corresponding to the homogeneous equation is

$$r^4 + 2r^2 + 1 = (r^2+1)(r^2+1) = 0 \tag{356}$$

so $r_1 = r_3 = i, r_2 = r_4 = -i$, i.e., a repeated pair of complex conjugate roots (multiplicity $= 2$). Thus we have

$$y_1 = \cos t, \quad y_2 = \sin t, \quad y_3 = t\cos t, \quad y_4 = t\sin t, \tag{357}$$

and the complementary solution to the homogeneous equation is

$$y_c(t) = c_1\cos t + c_2\sin t + c_3t\cos t + c_4t\sin t. \tag{358}$$

The non-homogeneous term $g(t) = 3\sin t$, we have $\alpha = 0, \beta = 1, \alpha + i\beta = i$ is the root with multiplicity 2. Thus, $s = 2$. Consider a trial solution

$$Y(t) = At^2 \sin t + Bt^2 \cos t. \tag{359}$$

Then,

$$Y^{(4)} + 2Y'' + Y = -8A\sin t - 8B\cos t = 3\sin t \Rightarrow B = 0, \quad A = -\frac{3}{8}. \tag{360}$$

Hence, the general solution to the non-homogeneous equation is

$$y(t) = c_1 \cos t + c_2 \sin t + c_3 t\cos t + c_4 t\sin t - \frac{3}{8}t^2 \sin t. \tag{361}$$

4.3.2 Variation of Parameters

Similar to second-order equations, there is also a method to treat rather general high order equations

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = g(t), \quad t \in I. \tag{362}$$

Suppose we have a FSS $y_1, \ldots, y_n$ to the homogeneous equation. Then, the complementary solution is

$$y_c(t) = c_1 y_1(t) + \cdots + c_n y_n(t). \tag{363}$$

Now, we consider a trial solution for the non-homogeneous equation of the form

$$Y(t) = u_1(t)y_1(t) + \cdots + u_n(t)y_n(t) \tag{364}$$

for unknown functions $u_1, \ldots, u_n$. Differentiating gives

$$Y'(t) = u_1(t)y_1'(t) + \cdots + u_n(t)y_n'(t) + u_1'(t)y_1(t) + \cdots + u_n'(t)y_n(t). \tag{365}$$

As before we set the constraint

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) + \cdots + u_n'(t)y_n(t) = 0, \tag{366}$$

so that Y' simplifies to

$$Y'(t) = u_1(t)y_1'(t) + u_2(t)y_2'(t) + \cdots + u_n(t)y_n'(t). \tag{367}$$

Computing Y'' and setting

$$u_1'(t)y_1'(t) + \cdots + u_n'(t)y_n'(t) = 0 \tag{368}$$

leads to the simplified expression for the second derivative

$$Y''(t) = u_1(t)y_1''(t) + \cdots + u_n(t)y_n''(t). \tag{369}$$

Repeat this procedure (differentiating and then setting the sum of terms involving $u_1', \ldots, u_n'$ to zero), we have (after simplification)

$$Y^{(n-1)}(t) = u_1(t)y_1^{(n-1)}(t) + \cdots + u_n(t)y_n^{(n-1)}(t) \tag{370}$$

Thus, the final expression for $Y^{(n)}(t)$ is

$$Y^{(n)}(t) = u_1(t)y_1^{(n)}(t) + \cdots + u_n(t)y_n^{(n)}(t) + u_1'(t)y_1^{(n-1)}(t) + \cdots + u_n'(t)y_n^{(n-1)}(t) \tag{371}$$

In summary, we obtain $n-1$ equations

$$u_1'(t)y_1^{(m)}(t)+\cdots+u_n'(t)y_n^{(m)}(t)=0 \quad \forall\, 0\le m\le n-2, \tag{372}$$

as well as a simplified expression for $Y^{(m)}$:

$$Y^{(m)}(t)=u_1(t)y_1^{(m)}(t)+\cdots+u_n(t)y_n^{(m)}(t), \quad m=0,\ldots,n-1, \tag{373}$$

$$Y^{(n)}(t)=u_1(t)y_1^{(n)}(t)+\cdots+u_n(t)y_n^{(n)}(t)+u_1'(t)y_1^{(n-1)}(t)+\cdots+u_n'(t)y_n^{(n-1)}(t). \tag{374}$$

Now, substitute $Y, Y', \ldots, Y^{(n-1)}, Y^{(n)}$ into the LHS of the non-homogeneous ODE (362):

$$\text{LHS}=Y^{(n)}+p_{n-1}Y^{(n-1)}+\cdots+p_1Y'+p_0Y \tag{375}$$

Substituting:

$$\begin{aligned}\text{LHS}=&\overbrace{[u_1y_1^{(n)}+\cdots+u_ny_n^{(n)}]+[u_1'y_1^{(n-1)}+\cdots+u_n'y_n^{(n-1)}]}^{\text{from } Y^{(n)}}\\&+p_{n-1}[u_1y_1^{(n-1)}+\cdots+u_ny_n^{(n-1)}]+\cdots+p_1[u_1y_1'+\cdots+u_ny_n']\\&+p_0[u_1y_1+\cdots+u_ny_n]\end{aligned} \tag{376}$$

Regroup all terms by $u_1, u_2, \ldots, u_n$:

- Collect all terms of u_1: $u_1y_1^{(n)}+p_{n-1}u_1y_1^{(n-1)}+\cdots+p_1u_1y_1'+p_0u_1y_1 = u_1\left[y_1^{(n)}+p_{n-1}y_1^{(n-1)}+\cdots+p_1y_1'+p_0y_1\right]$
- Collect all terms of u_2: $u_2\left[y_2^{(n)}+p_{n-1}y_2^{(n-1)}+\cdots+p_1y_2'+p_0y_2\right]$
- ...
- Collect all terms of u_n: $u_n\left[y_n^{(n)}+p_{n-1}y_n^{(n-1)}+\cdots+p_1y_n'+p_0y_n\right]$
- The remaining terms (only u_i'): $u_1'y_1^{(n-1)}+\cdots+u_n'y_n^{(n-1)}$

Since $y_1, \ldots, y_n$ are all solutions to the homogeneous equation, all expressions in the square brackets $[\cdots]$ in the previous step are equal to 0.

$$\text{LHS}=u_1\cdot(0)+u_2\cdot(0)+\cdots+u_n\cdot(0)+u_1'y_1^{(n-1)}+\cdots+u_n'y_n^{(n-1)} \tag{377}$$

Thus, we obtain the last equation:

$$u_1'y_1^{(n-1)}+\cdots+u_n'y_n^{(n-1)}=g(t) \tag{378}$$

Collecting all expressions involving $u_1', \ldots, u_n'$, we obtain

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_{n-1} & y_n \\ y_1' & y_2' & \cdots & y_{n-1}' & y_n' \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_{n-1}^{(n-2)} & y_n^{(n-2)} \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_{n-1}^{(n-1)} & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_{n-1}' \\ u_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(t) \end{pmatrix}. \tag{379}$$

Thus, the derivatives of the unknown functions $u_1, \dots, u_n$ can be found by inverting the matrix of derivatives, whose determinant is the non-zero Wronskian, since $(y_1, \dots, y_n)$ forms a FSS. Denote the matrix as $M(t)$. Use Cramer's rule, by setting

$$M_i(t) = \begin{pmatrix} y_1 & \dots & 0 & \dots & y_n \\ y_1' & \dots & 0 & \dots & y_n' \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & \dots & 0 & \dots & y_n^{(n-2)} \\ y_1^{(n-1)} & \dots & 1 & \dots & y_n^{(n-1)} \end{pmatrix}, \tag{380}$$

i.e., replace the ith column of $M(t)$ with the vector $(0, \dots, 0, 1)^T$. Then Cramer's rule gives

$$u_i'(t) = \frac{g(t) \det M_i(t)}{\det M(t)}, \tag{381}$$

and by integrating we get an expression for $u_i(t)$. The particular solution to the non-homogeneous equation is therefore

$$Y(t) = y_1(t) \int \frac{g(t) \det M_1(t)}{\det M(t)} dt + \cdots + y_n(t) \int \frac{g(t) \det M_n(t)}{\det M(t)} dt. \tag{382}$$

However, in general the evaluation of the integrals can be difficult, but we can always use Abel's identity to simplify, since

$$\det M(t) = W(y_1, \dots, y_n)[t] = ce^{-\int p_{n-1}(t)dt}. \tag{383}$$

Example 53. *Solve*

$$y''' + y' = \sec^2(t) \textit{ for } t \in (-\pi/2, \pi/2). \tag{384}$$

The characteristic equation for the homogeneous problem is $r^3 + r = 0$ and so $r_1 = 0$, $r_2 = i$ and $r_3 = -i$. Hence the complementary solution is

$$y_c(t) = c_1 + c_2 \cos t + c_3 \sin t. \tag{385}$$

By variation of parameters we look for a particular solution of the form

$$Y(t) = u_1 y_1 + u_2 y_2 + u_3 y_3 = u_1(t) + u_2(t) \cos t + u_3(t) \sin t, \tag{386}$$

with

$$M(t) \begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sec^2(t) \end{pmatrix}, \quad M(t) = \begin{pmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{pmatrix}. \tag{387}$$

Define

$$M_1(t) = \begin{pmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 1 & -\cos t & -\sin t \end{pmatrix}, \quad M_2(t) = \begin{pmatrix} 1 & 0 & \sin t \\ 0 & 0 & \cos t \\ 0 & 1 & -\sin t \end{pmatrix}, \quad M_3(t) = \begin{pmatrix} 1 & \cos t & 0 \\ 0 & -\sin t & 0 \\ 0 & -\cos t & 1 \end{pmatrix} \tag{388}$$

We can compute

$$\det M(t) = 1, \quad \det M_1(t) = 1, \quad \det M_2(t) = -\cos t, \quad \det M_3(t) = -\sin t, \tag{389}$$

so

$$u_1 = \int \sec^2(t)dt = \tan(t), \tag{390}$$

$$u_2 = \int -\sec^2(t)\cos(t)dt = -\ln(|\sec(t) + \tan(t)|), \tag{391}$$

$$u_3 = \int -\sec^2(t)\sin(t)dt = -\sec(t). \tag{392}$$

Hence, the particular solution is

$$Y(t) = \tan(t) - \cos(t)\ln(|\sec(t) + \tan(t)|) - \sin(t)\sec(t) \tag{393}$$

$$= -\cos(t)\ln(|\sec(t) + \tan(t)|). \tag{394}$$

References

[BDM21] W. E. Boyce, R. C. DiPrima, and D. B. Meade. *Elementary differential equations and boundary value problems*. John Wiley & Sons, 2021.

A Linear Algebra Notations

In our study of first-order systems, we will deal with the case where the entries of the matrix $\mathbf{A}$ are functions of the independent variable t, hence we can define a matrix function of t as $\mathbf{A}(t)$ where

$$\mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}(t) & a_{m2}(t) & \dots & a_{mn}(t) \end{pmatrix}. \tag{395}$$

We say that $\mathbf{A}(t)$ is **continuous** if all the entries $a_{11}(t), \dots, a_{mn}(t)$ are continuous functions of t. Similarly, we say $\mathbf{A}(t)$ is **differentiable** if all its entries are differentiable functions. Then

$$\frac{d}{dt}\mathbf{A}(t) = \begin{pmatrix} a'_{11}(t) & a'_{12}(t) & \dots & a'_{1n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a'_{m1}(t) & a'_{m2}(t) & \dots & a'_{mn}(t) \end{pmatrix}. \tag{396}$$

We can also define the (indefinite) integral of $\mathbf{A}(t)$ as

$$\int \mathbf{A}(t)dt = \left(\int a_{ij}(t)dt\right)_{1\le i\le m, 1\le j\le n}. \tag{397}$$

We also have the chain rule

$$\frac{d(\mathbf{A}(t)\mathbf{B}(t))}{dt} = \frac{d\mathbf{A}(t)}{dt}\mathbf{B}(t) + \mathbf{A}(t)\frac{d\mathbf{B}(t)}{dt}. \tag{398}$$