Basics of Ordinary Differential Equations

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Contents

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1 Introduction

This is my study notes on Ordinary Differential Equations (ODEs), which mainly serves as a summary of key points and methods, thus omitting some proofs which can be found in textbook. The content is mainly based on Prof. Chaoyu Quan's course **MAT2002 Ordinary Differential Equations** at CUHK(SZ), Prof. Jeffrey R. Chasnov's notes, and the textbook [BDM21].

2 First-Order Differential Equations

In this section we study how to solve first-order ODEs (only involving first-order derivatives). We will start from the simplest linear case (Section 2.1), then turn to more general cases.

2.1 Linear Equations

We will first give the formulation of the first-order linear ordinary differential equations.

Definition 1 (First-Order Linear ODE). *The Initial Value Problem (IVP) of the general first-order linear ODE is given by*

$$\begin{cases} \dfrac{dy}{dt} = p(t)y + q(t), \\ y(t_0) = y_0, \end{cases} \tag{1}$$

for some given functions $p(t)$, $q(t)$ and constants t_0 and y_0.

Next, we will introduce the **method of integrating factors** to solve the above ODE. Multiply (1) by a function $\mu(t)$ (a.k.a., the integrating factor), leading to

$$\mu(t)\frac{dy}{dt} - \mu(t)p(t)y(t) = \mu(t)q(t). \tag{2}$$

Suppose that

$$\mu(t)\frac{dy}{dt} - \mu(t)p(t)y(t) = \frac{d}{dt}\left(\mu(t)y(t)\right), \tag{3}$$

then (2) becomes

$$\frac{d}{dt}\left(\mu(t)y(t)\right) = \mu(t)q(t) \Rightarrow \mu(t)y(t) = \int \mu(t)q(t)\,dt + c, \quad c \in \mathbb{R} \tag{4}$$

If $\mu(t)$ is **non-zero**, we can obtain the general solution

$$y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)q(t)\,dt + c\right] \tag{5}$$

The problem becomes how to find such $\mu(t)$? From (3) we have

$$\mu(t)y'(t) - \mu(t)p(t)y(t) = \mu'(t)y(t) + \mu(t)y'(t) \Rightarrow y(t)\left(\frac{d\mu}{dt} + p(t)\mu(t)\right) = 0. \tag{6}$$

The equation is satisfied if $y(t) = 0$ or $\mu'(t) + p(t)\mu(t) = 0$. **The first case** $y(t) = 0$ **is not desirable**, since if the initial condition y_0 is non-zero, we have a contradiction. Therefore, we consider the second case and obtain the equation

$$\frac{d\mu}{dt} = -p(t)\mu \tag{7}$$

as the ODE for μ. This is a **separable equation** which will be detailed in Section 2.2, and revisited in Example 5. $\mu(t) \equiv 0$ is one solution but without any interest. When $\mu(t) \neq 0$, we have

$$\frac{1}{\mu}\frac{d\mu}{dt} = -p(t) \Rightarrow \ln|\mu(t)| = -\int p(t)\,dt + c \tag{8}$$

Choosing the arbitrary constant c to be zero, we obtain a simplest integrating factor

$$\mu(t) = \exp\left(-\int p(t)\,dt\right) \tag{9}$$

Plug in (5) we obtain the final solution. The general solution $y(t)$ to the ODE $y' = p(t)y + q(t)$ is given as

$$y(t) = e^{\int p(t)\,dt}\left[\int e^{-\int p(t)\,dt} q(t)\,dt + c\right]. \tag{10}$$

The particular solution and the constant c can be computed with the initial condition $y(t_0) = y_0$.

Example 2. *Solve the ODE*

$$\begin{cases} t\dfrac{dy}{dt} + 2y = 4t^2, \\ \qquad y(1) = 2, \end{cases} \tag{11}$$

Suppose $t \neq 0$. Write the ODE in the form $y' = p(t)y + q(t)$ and identify p, q

$$t\frac{dy}{dt} + 2y = 4t^2 \Rightarrow \frac{dy}{dt} = -\frac{2}{t}y + 4t \Rightarrow p(t) = -\frac{2}{t}, \quad q(t) = 4t. \tag{12}$$

Compute the integrating factor

$$\mu(t) = \exp\left(-\int p(t)\,dt\right) = t^2 \tag{13}$$

We obtain the general solution

$$y(t) = \frac{1}{t^2}\left[\int t^2 \times 4t\,dt + c\right] = t^2 + \frac{c}{t^2}. \tag{14}$$

From the initial condition we have $c = 1$, thus the particular solution is

$$y(t) = t^2 + \frac{1}{t^2} \tag{15}$$

The general and particular solutions are shown in Figure 1.

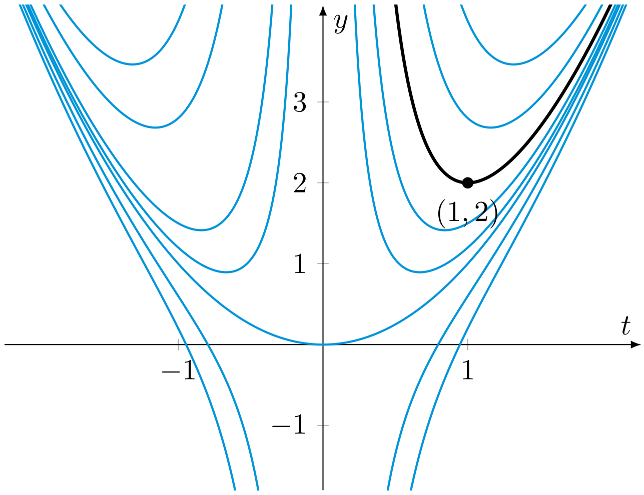


Figure 1: Integral curves of the differential equation $ty' + 2y = 4t^2$; the black curve passes through the point $(1, 2)$.

2.2 Separable Equations

Definition 3 (Separable Equation)**.** *A first-order ODE $y' = f(t, y)$ is separable if it can be written in the form*

$$M(t) + N(y)\frac{dy}{dt} = 0 \tag{16}$$

for some functions M, N.

Let's see an example.

Example 4. *Solve the ODE*

$$\begin{cases} \dfrac{dy}{dt} = \dfrac{\sin(t)}{1-y^2}, \\ y(t_0) = y_0, \end{cases} \tag{17}$$

The key is to **separate y and t, placing them on opposite sides of the equation**. Bring y to the LHS we have

$$(1-y^2)\frac{dy}{dt} = \sin(t) \tag{18}$$

Integrate both sides, we obtain

$$y(t) - \frac{1}{3}y(t)^3 = -\cos(t) + c, \quad c \in \mathbb{R}. \tag{19}$$

Using the initial condition to solve for c

$$\begin{cases} y(t) - \frac{1}{3}y(t)^3 = -\cos(t) + c, \\ \quad y_0 - \frac{1}{3}y_0^3 = -\cos(t_0) + c, \end{cases} \tag{20}$$

The particular solution is given by

$$y(t) - \frac{1}{3}y(t)^3 = \cos(t_0) - \cos(t) + y_0 - \frac{1}{3}y_0^3. \tag{21}$$

Example 5. *Solve the ODE*

$$\frac{dy}{dt} = P(t)y \tag{22}$$

When $y \neq 0$, we have

$$\frac{1}{y}\frac{dy}{dt} = P(t) \Rightarrow \ln|y| = \int P(t)\,dt + \overline{c} \Rightarrow y = \pm e^{\overline{c}} \cdot e^{\int P(t)\,dt} = ce^{\int P(t)\,dt} \tag{23}$$

where $\overline{c} \in \mathbb{R}$ and $c = \pm e^{\overline{c}}$. Clearly $y = 0$ **is also a solution to** (22), so if we allow $c = 0$, then the solution $y = 0$ is also included in (23).

Example 6. *Solve the ODE*

$$\frac{dy}{dt} + \frac{1}{2}y = \frac{3}{2} \tag{24}$$

First write in the form

$$\frac{dy}{dt} = \frac{1}{2}(3 - y) \tag{25}$$

When $y \neq 3$, separate variables, integrate both sides and remove absolute value

$$\frac{1}{3-y}\frac{dy}{dt} = \frac{1}{2} \Rightarrow 3 - y = \pm e^{c} \cdot e^{-\frac{1}{2}t} \tag{26}$$

So the final solution is

$$y = 3 + Ce^{-\frac{1}{2}t} \tag{27}$$

where $C \in \mathbb{R}$, since we included the solution $y = 3$.

2.3 Transformation Methods

There are many transformation methods, we will only discuss two of them.

2.3.1 Bernoulli Equation

Definition 7 (Bernoulli Equation)**.** *Let n be a real number, $n \neq 0, 1$, and $p(t), q(t)$ be given functions. The Bernoulli equation is a first order non-linear ODE of the form*

$$\frac{dy}{dt} + p(t)y = q(t)y^n. \tag{28}$$

Multiply (28) with y^{-n}

$$y^{-n}\frac{dy}{dt} + p(t)y^{1-n} = q(t) \tag{29}$$

Since $\frac{d}{dt}\left(y^{1-n}\right) = (1-n)y^{-n}\frac{dy}{dt}$, (29) simplifies to

$$\frac{d}{dt}y^{1-n} + (1-n)p(t)y^{1-n} = (1-n)q(t) \tag{30}$$

Then, consider a **new variable** $v(t) = y^{1-n}(t)$, (30) becomes

$$\frac{dv}{dt} + P(t)v = Q(t), \quad P(t) = (1-n)p(t), \quad Q(t) = (1-n)q(t) \tag{31}$$

which is a first-order linear ODE for v. Let $\mu(t)$ be the integrating factor for (31), then the general solution is

$$v(t) = \frac{1}{\mu(t)}\left[\int Q(t)\mu(t)\,dt + c\right] \quad \Rightarrow \quad y(t) = \left(\frac{1}{\mu(t)}\left[\int Q(t)\mu(t)\,dt + c\right]\right)^{\frac{1}{1-n}} \tag{32}$$

2.3.2 Homogeneous First-Order Equations

Definition 8 (Homogeneous First-Order Equation)**.** *A first-order ODE $\frac{dy}{dt} = f(t, y)$ is called homogeneous if the function f only depends on the ratio $\frac{y}{t}$. That is, we can express*

$$f(t, y) = F\left(\frac{y}{t}\right) \quad \textit{for some function } F. \tag{33}$$

We will still use a **transformation** method. Define a **new variable** $v = y/t \iff y = vt$. Then, the RHS of the ODE becomes just $F(v)$. For the LHS, by the product rule we have

$$y(t) = tv(t) \Rightarrow \frac{dy}{dt} = t\frac{dv}{dt} + v(t) \Rightarrow t\frac{dv}{dt} + v(t) = F(v). \tag{34}$$

Note that the initial condition $y(t_0) = y_0$ also transforms:

$$y(t_0) = y_0 \Rightarrow t_0 v(t_0) = y_0, \tag{35}$$

and it is important to see that if $y_0 \neq 0$ then we cannot choose $t_0 = 0$, otherwise we get a contradiction. The transformed ODE in the variable v is now

$$\frac{dv}{dt} = \frac{F(v) - v}{t} \Rightarrow \frac{1}{F(v) - v}\frac{dv}{dt} = \frac{1}{t} \tag{36}$$

which is a **separable equation**.

Example 9. *Solve the ODE*

$$\frac{dy}{dt} = \frac{y - 4t}{t - y} = f(t, y) \tag{37}$$

Dividing numerator and denominator by t leads to

$$f(t, y) = \frac{y - 4t}{t - y} = \frac{y/t - 4}{1 - y/t} = F(y/t), \text{ where } F(s) = \frac{s - 4}{1 - s}. \tag{38}$$

Using a transformation $y = tv$ we find that v satisfies

$$\frac{1}{F(v) - v}\frac{dv}{dt} = \frac{1}{t} \Rightarrow \frac{1 - v}{(v - 2)(v + 2)}\frac{dv}{dt} = \frac{1}{t}. \tag{39}$$

Using partial fractions the coefficient can be simplified to

$$\frac{1 - v}{(v - 2)(v + 2)} = -\frac{1}{4(v - 2)} - \frac{3}{4(v + 2)}. \tag{40}$$

Then, integrating gives the general solution

$$-\frac{1}{4}\ln|v - 2| - \frac{3}{4}\ln|v + 2| = \ln|t| + c \Rightarrow -\frac{1}{4}\ln|y(t)/t - 2| - \frac{3}{4}\ln|y(t)/t + 2| = \ln|t| + c. \tag{41}$$

This gives

$$|y(t)/t - 2|^{-1/4}\,|y(t)/t + 2|^{-3/4} = e^c|t|, \quad c \in \mathbb{R}. \tag{42}$$

Which can be rewritten as

$$|t|\,|y(t)/t - 2|^{1/4}\,|y(t)/t + 2|^{3/4} = k, \quad k \geq 0. \tag{43}$$

2.4 Exact Equations

2.4.1 General Method

An ODE of the following form is not separable:

$$M(t,y) + N(t,y)\frac{dy}{dt} = 0 \tag{44}$$

where M, N are some functions. If the LHS of this equation can be written as $\frac{d\Psi(t,y(t))}{dt}$ for some function $\Psi(t,y)$, then integrating gives the general (implicit) solution

$$\Psi(t,y(t)) = c, \quad c \in \mathbb{R} \tag{45}$$

The requirement for

$$\frac{d\Psi(t,y(t))}{dt} = M(t,y) + N(t,y)\frac{dy}{dt} \tag{46}$$

implies

$$\frac{\partial\Psi}{\partial y}(t,y) = N(t,y), \quad \frac{\partial\Psi}{\partial t}(t,y) = M(t,y), \tag{47}$$

since $\frac{d\Psi(t,y(t))}{dt} = \frac{\partial\Psi}{\partial t}(t,y) + \frac{\partial\Psi}{\partial y}(t,y)\frac{dy}{dt}$. This is summarized in the following definition.

Definition 10 (Exact Equation)**.** *A first-order ODE $M(t,y) + N(t,y)\frac{dy}{dt} = 0$ is an exact equation if there exists a function $\Psi(t,y)$ such that*

$$\frac{\partial\Psi}{\partial y}(t,y) = N(t,y), \quad \frac{\partial\Psi}{\partial t}(t,y) = M(t,y). \tag{48}$$

The general solution $y(t)$ to the ODE is given implicitly as $\Psi(t,y(t)) = c, \quad c \in \mathbb{R}$.

Thus, the question becomes:

1. How to determine an ODE of the form $M(t,y) + N(t,y)\frac{dy}{dt} = 0$ is exact?
2. If it is an exact equation, how to find the function $\Psi(t,y)$?

Solution is given by the following theorem.

Theorem 11. *Let $M(t,y)$ and $N(t,y)$ be continuous functions of t and y in some simply connected domain, and have continuous first-order partial derivatives. Then the equation*

$$M(t,y) + N(t,y)\frac{dy}{dt} = 0 \tag{49}$$

is an exact differential equation if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t} \tag{50}$$

We will prove the theorem to gain a better understanding of it.

Proof. "$\Leftarrow$". Given that (49) is an exact differential equation, (48) holds, and by taking partial derivatives on t, y we obtain

$$\frac{\partial M}{\partial y} = \frac{\partial^2 \Psi}{\partial t \partial y}, \quad \frac{\partial N}{\partial t} = \frac{\partial^2 \Psi}{\partial y \partial t} \tag{51}$$

From the continuity of $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial t}$, we know that $\frac{\partial^2 \Psi}{\partial t \partial y}$ and $\frac{\partial^2 \Psi}{\partial y \partial t}$ are continuous. Therefore, we can obtain

$$\frac{\partial^2 \Psi}{\partial t \partial y} = \frac{\partial^2 \Psi}{\partial y \partial t} \tag{52}$$

That is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t} \tag{53}$$

Thus the necessity is proven.

"$\Rightarrow$". We want to show that if (49) satisfies (50), then we can find function $\Psi(t, y)$ satisfying (48). Integrating both sides of $\frac{\partial \Psi}{\partial t} = M(t, y)$ with respect to t, we obtain

$$\int M(t, y) dt + \varphi(y) = \Psi(t, y) \tag{54}$$

Here $\varphi(y)$ is an arbitrary differentiable function of y. We choose a suitable $\varphi(y)$ such that $\Psi(t, y)$ also satisfies $\frac{\partial \Psi}{\partial y} = N(t, y)$, that is, taking the partial derivative with respect to y on both sides of (54), we get

$$\frac{\partial \Psi}{\partial y} = \frac{\partial}{\partial y} \int M(t, y) dt + \frac{d\varphi(y)}{dy} = N(t, y). \tag{55}$$

Therefore

$$\frac{d\varphi(y)}{dy} = N(t, y) - \frac{\partial}{\partial y} \int M(t, y) dt. \tag{56}$$

Note that $\varphi(y)$ is an arbitrary differentiable function of y, so the RHS of (56) must be independent of t, which means the partial derivative of the RHS of (56) with respect to t should be zero. In fact,

$$\frac{\partial}{\partial t} \left[N(t, y) - \frac{\partial}{\partial y} \int M(t, y) dt \right] = \frac{\partial N}{\partial t} - \frac{\partial}{\partial t} \left[\frac{\partial}{\partial y} \int M(t, y) dt \right]. \tag{57}$$

Since $M(t, y)$ and $N(t, y)$ are continuous functions of t, y in some simply connected domain, and have continuous first-order partial derivatives, the order of differentiation with respect to t and y in (57) can be interchanged. Using (50), we get

$$\frac{\partial}{\partial t} \left[N(t, y) - \frac{\partial}{\partial y} \int M(t, y) dt \right] = \frac{\partial N}{\partial t} - \frac{\partial}{\partial y} \left[\frac{\partial}{\partial t} \int M(t, y) dt \right] \tag{58}$$

$$= \frac{\partial N}{\partial t} - \frac{\partial M}{\partial y} = 0. \tag{59}$$

Thus the RHS of (56) is a function of y only. Integrating both sides, we obtain

$$\varphi(y) = \int \left[N(t, y) - \frac{\partial}{\partial y} \int M(t, y) dt \right] dy. \tag{60}$$

Substituting (60) into (54), we can find

$$\Psi(t, y) = \int M(t, y) dt + \int \left[N(t, y) - \frac{\partial}{\partial y} \int M(t, y) dt \right] dy. \tag{61}$$

In this way, we have proved that if (49) satisfies condition (50), then a $\Psi(t, y)$ that satisfies (48) must exist, and its specific expression is (61), thus proving sufficiency. Combining both directions finishes the proof. □

Example 12. *Solve the ODE*

$$3t^2 + 6ty^2 + (6t^2y + 4y^3)\frac{dy}{dt} = 0 \tag{62}$$

Here, $M = 3t^2 + 6ty^2, N = 6t^2y + 4y^3$, easy to verify that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, so the equation is exact. Find Ψ such that it satisfies

$$\frac{\partial \Psi}{\partial t} = M = 3t^2 + 6ty^2, \tag{63}$$

$$\frac{\partial \Psi}{\partial y} = N = 6t^2y + 4y^3 \tag{64}$$

Integrating (63) with respect to t, we get

$$\Psi = t^3 + 3t^2y^2 + \varphi(y). \tag{65}$$

Taking the partial derivative of (65) with respect to y, and using (64), we get

$$\frac{\partial \Psi}{\partial y} = 6t^2y + \frac{d\varphi(y)}{dy} = 6t^2y + 4y^3 \tag{66}$$

Thus

$$\frac{d\varphi(y)}{dy} = 4y^3 \tag{67}$$

Solving this, we get

$$\varphi(y) = y^4. \tag{68}$$

Substituting $\varphi(y)$ into (65), we get

$$\Psi = t^3 + 3t^2y^2 + y^4. \tag{69}$$

Therefore, the general solution of the equation is

$$t^3 + 3t^2y^2 + y^4 = c, \tag{70}$$

where c is an arbitrary constant. Alternatively, we can directly apply (61):

$$\int M(t,y)dt + \int \left[N(t,y) - \frac{\partial}{\partial y}\int M(t,y)dt \right] dy \tag{71}$$

$$= t^3 + 3t^2y^2 + \int (6t^2y + 4y^3 - 6t^2y)dy \tag{72}$$

$$= t^3 + 3t^2y^2 + y^4 = c, \tag{73}$$

where c is arbitrary constant.

2.4.2 Exact Equations with Integrating Factor

How to solve a non-exact ODE? Similar to the way we treated the first-order linear ODEs, consider multiplying with a integrating factor μ and hope things are better. We obtain after multiplying a new ODE

$$\mu M(t,y) + \mu N(t,y)\frac{dy}{dt} = 0 \tag{74}$$

If (74) is an exact equation, then by previous theorem, the following relation must be satisfied:

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial t}(\mu N) \tag{75}$$

Let's first investigate two cases.

Case 1. μ is just a function of t, i.e., $\mu = \mu(t)$. Then (75) simplifies to

$$N(t,y)\frac{d\mu}{dt} + \mu(t)N_t(t,y) = \mu(t)M_y(t,y). \tag{76}$$

If $N(t,y) \neq 0$, then we obtain an ODE for μ:

$$\frac{d\mu}{dt} = \mu(t)\left(\frac{M_y - N_t}{N}\right)(t,y) =: \mu(t)K(t,y). \tag{77}$$

Further suppose the factor $K(t,y)$ **depends only on** t, then (77) is a first-order **linear** ODE in $\mu(t)$ which can be solved by the method of integrating factors.

Case 2. μ is just a function of y, i.e., $\mu = \mu(y)$. Then (75) simplifies to

$$M(t,y)\frac{d\mu}{dy} + \mu(y)M_y(t,y) = \mu(y)N_t(t,y). \tag{78}$$

If $M(t,y) \neq 0$, then we obtain an ODE for μ:

$$\frac{d\mu}{dy} = \mu(y)\left(\frac{N_t - M_y}{M}\right)(t,y) =: \mu(y)H(t,y). \tag{79}$$

Further suppose the factor $H(t,y)$ **depends only on** y, then (79) is a first order **linear** ODE in $\mu(y)$ (where the independent variable is now y), and again can be solved by the method of integrating factors.

After obtaining μ, plug in (74) to obtain an exact equation.

Example 13. *Solve the ODE*

$$3ty + y^2 + (t^2 + ty)\frac{dy}{dt} = 0 \tag{80}$$

Clearly the ODE is not exact. Compute $K = \frac{M_y - N_t}{N} = \frac{t+y}{t^2+ty} = \frac{1}{t}$ and $H = \frac{N_t - M_y}{M} = \frac{-t-y}{3ty+y^2}$. We see that K is only a function of t but H is not just a function of y. So we expect the integrating factor μ to be a function of t only, which solves the ODE

$$\frac{d\mu}{dt} = \frac{\mu(t)}{t} \Rightarrow \mu(t) = ct, \quad c \in \mathbb{R} \tag{81}$$

Multiplying this integrating factor (take $c = 1$) with the ODE yields

$$t(3ty + y^2) + t(t^2 + ty)\frac{dy}{dt} = 0, \tag{82}$$

which is now an exact equation with function $\Psi(t, y)$ given as

$$\Psi(t, y) = t^3 y + \frac{1}{2}t^2 y^2. \tag{83}$$

So the general (implicit) solution to the ODE is

$$t^3 y(t) + \frac{1}{2}t^2 y^2(t) = c, \quad c \in \mathbb{R}. \tag{84}$$

Summary on methods in Section 2:

Type	Method	Explicit/Implicit solution
$y' = p(t)y + q(t)$	Integrating factor	$y(t) = \mu(t)^{-1}(\int \mu(t)q(t)dt + c)$
$M(t) + N(y)y' = 0$	Separable equation	$m(t) + n(y(t)) = c^*$
$y' + p(t)y = q(t)y^n$	$v := y^{1-n}$	$y(t) = (\mu^{-1}(\int Q(t)\mu(t)dt + c))^{1/(1-n)}$
$y' = F(y/t)$	$v = y/t$	$1/(F(v) - v)\frac{dv}{dt} = \frac{1}{t}$
$M(t, y) + N(t, y)y' = 0$	Exact equation	$\Psi(t, y(t)) = c$

*: $m(t) = \int M(t)dt, n(y(t)) = \int N(y)dy$.

2.5 Existence and Uniqueness Theorems

For completeness, we will state the existence and uniqueness theorems for IVP of first-order ODEs. The existence and uniqueness for first-order linear ODEs is characterized by the following theorem.

Theorem 14 (Existence and Uniqueness for First-Order Linear ODE)**.** *Suppose functions p and q are continuous on $(\alpha, \beta) \subset \mathbb{R}$ (where α, β are some real numbers). Then, for any $t_0 \in (\alpha, \beta)$, $y_0 \in \mathbb{R}$, there exists a unique function $y(t)$ satisfying*

$$\begin{cases} \dfrac{dy}{dt} = p(t)y + q(t), \quad \forall t \in (\alpha, \beta), \\ y(t_0) = y_0, \end{cases} \tag{85}$$

And the solution is defined throughout the interval (α, β).

The above theorem states that the unique solution to the IVP exists throughout any interval (α, β) containing $t = t_0$ if the functions p and q are continuous in (α, β). In other words, **the solution globally exists in the interval (α, β) in which p and q are continuous.**

The existence and uniqueness for first-order non-linear ODEs is characterized by the following theorem.

Theorem 15 (Existence and Uniqueness for First-Order Non-Linear ODE)**.** *Consider the IVP*

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0. \tag{86}$$

Let R be a closed rectangle

$$R = \{(t, y) \mid |t - t_0| \leq a,\ |y - y_0| \leq b\} \quad (a > 0, b > 0). \tag{87}$$

Assume that both $f(t, y)$ and $\frac{\partial f}{\partial y}$ are continuous on R. Then the IVP has a unique solution $y = y(t)$ defined on the interval $(t_0 - h, t_0 + h)$, where $h = \min\left(\frac{b}{M}, a\right)$ and $M = \max_{(t,y)\in R} |f(t, y)|$.

Under the assumption of the theorem, the solution only exists in a small interval $(t_0 - h, t_0 + h) \subset [t_0 - a, t_0 + a]$ since $h = \min\left(\frac{b}{M}, a\right)$ depends on the size of the region R. And h also depends on the values of the function $f(t, y)$ in the region R ($M = \max_{(t,y)\in R} |f(t, y)|$). **The solution only locally exists in the interval $[t_0 - a, t_0 + a]$.**

3 Second-Order Linear Differential Equations

In this section we study second-order **linear** ODEs. Section 3.1 introduces general theory of homogeneous equations, Section 3.2 and 3.3 study how to solve them, and Section 3.4 deals with non-homogeneous equations.

3.1 General Theory of Homogeneous Equations

We will first present the existence and uniqueness theorem for the second-order linear equations.

Theorem 16 (Existence and Uniqueness for Second-Order Linear ODE)**.** *Consider the IVP*

$$y'' + p(t)y' + q(t)y = r(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_1. \tag{88}$$

Suppose $I = (\alpha, \beta) \subset \mathbb{R}$ is any open interval such that $t_0 \in I$, and the functions p, q, r are continuous in I. Then, there is exactly one solution $y(t)$ to the IVP for $t \in I$. The solution $y(t)$ is defined throughout the interval where p, q, r are continuous.

Now we introduce the following classification.

Definition 17 (Homogeneous)**.** *A second-order linear ODE*

$$p(t)y'' + q(t)y' + r(t)y = s(t), \quad p(t) \neq 0, \tag{89}$$

is called homogeneous if $s(t) \equiv 0$. Otherwise, if $s(t) \neq 0$, the ODE is called non-homogeneous.

For second-order homogeneous linear equations we have the following **principle of superposition**.

Theorem 18 (Principle of Superposition)**.** *If y_1 and y_2 are two solutions of the ODE*

$$p(t)y'' + q(t)y' + r(t)y = 0. \tag{90}$$

Then for any constants $c_1, c_2 \in \mathbb{R}$, the function $c_1y_1(t) + c_2y_2(t)$ is also a solution to the ODE.

Clearly the principle of superposition **holds for homogeneous linear equations of any order** due to the linear structure.

Let's return to the second-order case. In other words, from two solutions we can construct infinite solutions to the homogeneous linear ODE. We can define a family of solutions

$$S = \{y = c_1y_1 + c_2y_2 \mid c_1, c_2 \in \mathbb{R}\} \tag{91}$$

to the ODE. The next question is: Given two solutions $y_1(t)$ and $y_2(t)$, can **any** solution to the ODE be expressed as a linear combination of $y_1(t)$ and $y_2(t)$?

Definition 19 (Wronskian)**.** *Given $y_1(t)$ and $y_2(t)$,*

$$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \tag{92}$$

is called the Wronskian for y_1 and y_2.

Indeed, we have the following theorem.

Theorem 20. *Suppose that I is an open interval in which $p(t)$ and $q(t)$ are continuous. Let $y_1(t)$ and $y_2(t)$ be two solutions to the ODE*

$$y'' + p(t)y' + q(t)y = 0 \tag{93}$$

for $t \in I$. Then, any solution $y(t)$ to the ODE can be expressed as

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \tag{94}$$

for constants c_1 and c_2 $\iff \exists\, t_0 \in I$ such that the Wronskian $W(y_1, y_2)[t_0] \neq 0$.

The theorem says that if $y_1(t)$ and $y_2(t)$ are two solutions to the above ODE and $W(y_1, y_2)[t_0] \neq 0$, then the general solution to the above ODE is given by the (94). In this case, we say that (y_1, y_2) form a **fundamental set of solutions (FSS)** to the ODE.

Example 21. *$y_1(t) = \exp(-2t)$ and $y_2(t) = \exp(-3t)$ are solutions to the ODE*

$$y'' + 5y' + 6y = 0. \tag{95}$$

$$W[y_1, y_2](t) = \begin{vmatrix} \exp(-2t) & \exp(-3t) \\ -2\exp(-2t) & -3\exp(-3t) \end{vmatrix} = -\exp(-5t) \neq 0 \tag{96}$$

Any solution to the ODE $y'' + 5y' + 6y = 0$ can be written as the linear combination of $y_1(t) = \exp(-2t)$ and $y_2(t) = \exp(-3t)$, they form a FSS to the ODE.

Example 22. *For the ODE*

$$2t^2 y'' + 3ty' - y = 0, \quad t > 0, \tag{97}$$

the function $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ are solutions. Compute the Wronskian

$$W(y_1, y_2)[t] = -\frac{3}{2} t^{-3/2}, \tag{98}$$

which is non-zero for $t > 0$. Therefore we can deduce that (y_1, y_2) form a FSS for the ODE, and a general solution y to the ODE can be expressed as

$$y(t) = c_1 t^{1/2} + c_2 t^{-1}, \tag{99}$$

for some constants c_1, c_2.

Next we will examine further the properties of the Wronskian of two solutions to the second-order linear homogeneous ODE. We will show an explicit formula for the Wronskian even if the two solutions are unknown.

Theorem 23 (Abel's Identity)**.** *Let I be an open interval in which p and q are continuous. Suppose y_1 and y_2 are two non-zero solutions to the ODE*

$$y'' + p(t)y' + q(t)y = 0. \tag{100}$$

Then, the Wronskian is given as

$$W(y_1, y_2)[t] = c \exp\left(-\int p(t)\, dt\right), \tag{101}$$

*where the constant c depends on y_1 and y_2, but not on t. Consequently, $W(y_1, y_2)[t] = 0$ **if and only if** $c = 0$. In particular, if $W(y_1, y_2)[t_0] \neq 0$ for some $t_0 \in I$, then it holds that $W(y_1, y_2)[t] \neq 0$ for all $t \in I$. And, if $W(y_1, y_2)[t_0] = 0$ for some $t_0 \in I$, then it holds that $W(y_1, y_2)[t] = 0$ for all $t \in I$.*

Proof. The idea is to derive an ODE for the Wronskian W. Going back to the ODE, as y_1 is a solution we have

$$y_1'' + p(t)y_1' + q(t)y_1 = 0 \quad \Rightarrow \quad y_2 y_1'' + y_2 p(t) y_1' + y_2 q(t) y_1 = 0. \tag{102}$$

Similarly, as y_2 is a solution,

$$y_1 y_2'' + y_1 p(t) y_2' + y_1 q(t) y_2 = 0. \tag{103}$$

Subtracting one from another gives

$$(y_1 y_2'' - y_2 y_1'') + p(t)(y_1 y_2' - y_2 y_1') = 0. \tag{104}$$

Noting that

$$W(y_1, y_2)[t] = y_1(t)y_2'(t) - y_2(t)y_1'(t) \quad \Rightarrow \quad W'(y_1, y_2)[t] = y_1(t)y_2''(t) - y_2(t)y_1''(t). \tag{105}$$

From (104) we have

$$W' + p(t)W = 0, \tag{106}$$

which is a linear first-order equation. By integrating factors, we find the general solution

$$W(y_1, y_2)[t] = c \exp\left(-\int p(t)\, dt\right), \tag{107}$$

for some constant $c \in \mathbb{R}$. As a constant of integration, c does not depend on t.

□

An implication of the theorem is that (y_1, y_2) form a FSS to $y'' + p(t)y' + q(t)y = 0$ if and only if $W(y_1, y_2)[t] \neq 0$, $\forall t \in I$.

Does a FSS always exists? This is answered in the next theorem.

Theorem 24. *Let I be an open interval of $\mathbb{R}$, p and q are continuous functions in I. For any $t_0 \in I$, let $y_1(t)$ be the (unique) solution to the IVP*

$$y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = 1, \quad y'(t_0) = 0, \tag{108}$$

and $y_2(t)$ be the (unique) solution to the IVP

$$y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = 0, \quad y'(t_0) = 1. \tag{109}$$

Then, (y_1, y_2) forms a FSS to the ODE.

Proof. Note that the existence of y_1 and y_2 to the corresponding IVPs is guaranteed by the Existence and Uniqueness Theorem. We only need to show that the Wronskian $W(y_1, y_2)[t_0]$ is non-zero. Computing gives

$$W(y_1, y_2)[t_0] = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1. \tag{110}$$

□

Indeed, the FSS are not unique. There are many different choices for $y_1(t_0), y_1'(t_0), y_2(t_0), y_2'(t_0)$ such that the corresponding solutions y_1, y_2 satisfy $W(y_1, y_2)[t_0] \neq 0$.

The FSS is closely related to the concept of linear (in)dependence in linear algebra.

Definition 25 (Linear Dependence)**.** *Consider 2 functions $x_1(t), x_2(t)$ defined on an interval $I \subset \mathbb{R}$. We say that $x_1(t), x_2(t)$ are linearly dependent if there are non-zero constants α_1, α_2, such that*

$$\alpha_1 x_1(t) + \alpha_2 x_2(t) = 0 \quad \forall t \in I. \tag{111}$$

If $x_1(t), x_2(t)$ are not linearly dependent, then they are linearly independent.

Theorem 26. *If $y_1(t)$ and $y_2(t)$ are two solutions to the ODE $y'' + p(t)y' + q(t)y = 0$, $t \in I$, where p, q are given continuous functions in I (some open interval). Then $y_1(t)$ and $y_2(t)$ are linearly independent at each point $t \in I \iff W[y_1, y_2](t) \neq 0$, $\forall t \in I$ ((y_1, y_2) forms a FSS).*

Proof. "$\Leftarrow$".

$$\begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{112}$$

has only the zero solution, since the determinant $W[y_1, y_2](t) \neq 0$. Thus, $c_1 y_1(t) + c_2 y_2(t) = 0$ implies $c_1 = c_2 = 0$, meaning that $y_1(t)$ and $y_2(t)$ are linearly independent.

"$\Rightarrow$". If $W[y_1, y_2](t_0) = 0$ for some $t_0 \in I$. Then the linear system

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{113}$$

has non-zero solution $(c_1^*, c_2^*) \neq (0,0)$. Define $\phi(t) = c_1^* y_1(t) + c_2^* y_2(t), t \in I$, then $\phi(t)$ is the solution to the ODE $y'' + p(t)y' + q(t)y = 0$ with initial conditions $y(t_0) = 0, y'(t_0) = 0$. But $y(t) = 0, t \in I$ is also the solution to the ODE with initial conditions $y(t_0) = 0, y'(t_0) = 0$. By the existence and uniqueness theorem, $\phi(t) = c_1^* y_1(t) + c_2^* y_2(t) = 0$. This implies that $y_1(t)$ and $y_2(t)$ are linearly dependent.

□

Clearly the proof also works for linear homogeneous ODEs of higher order.

This is similar to the steps in linear algebra for solving the homogeneous linear system $A\mathbf{x} = \mathbf{0}$: we need to find a set of linearly independent solutions (the basis of the null space of A), then all solutions can be expressed as a linear combination of these solutions. Another remark is that **a n-th order linear homogeneous ODE has at most n linear independent solutions**.

Based on the above results, the strategy to solve

$$y'' + p(t)y' + q(t)y = 0, \quad t \in I, \tag{114}$$

can be summarized as follows:

1. **Find two solutions y_1, y_2 satisfying the ODE.**
2. Find $t_* \in I$ such that the Wronskian $W(y_1, y_2)[t_*]$ is non-zero. Then, the general solution to the ODE is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \tag{115}$$

   for some constants c_1, c_2.
3. If initial conditions are prescribed at some $t_0 \in I$, compute c_1 and c_2 to determine the particular solution.

Step 1 is highly nontrivial, and is the basis to all methods in Section 3 (and Section 4).

3.2 Homogeneous Equations with Constant Coefficients

3.2.1 General Method

Although the FSS for the second-order linear homogeneous ODE $y'' + p(t)y' + q(t)y = 0$ always exist, but unfortunately, **there is no method to find the FSS explicitly.** However, when p, q are **constants**, we can find the FSS for $y'' + py' + qy = 0$ explicitly.

We will study the solutions to the ODE

$$ay'' + by' + cy = 0 \tag{116}$$

for fixed real constants $a, b, c \in \mathbb{R}$ with $a \neq 0$. Consider substituting a **trial function** $y(t) = \exp(rt)$ for some constant r into (116), which yields

$$\left(ar^2 + br + c\right)\exp(rt) = 0. \tag{117}$$

Since $\exp(rt) > 0$, we have

$$ar^2 + br + c = 0. \tag{118}$$

(118) is known as the **characteristic equation** for the ODE (116). If we can find the roots of the characteristic equation, then we know that $\exp(rt)$, where r is a root, is a solution to (116). By the quadratic formula we obtain

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{119}$$

Three possibilities:

1. Two distinct real roots r_1, r_2 if $b^2 > 4ac$.
2. Two complex roots (complex conjugate pairs) $r_1, \overline{r_1}$ if $b^2 < 4ac$.
3. A repeated real root r if $b^2 = 4ac$.

Discriminant $\Delta := b^2 - 4ac$.

Case 1. Two distinct real roots $\Delta > 0$. In the case $b^2 > 4ac$, we obtain two real roots

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{120}$$

This gives us two solutions

$$y_1(t) = \exp(r_1 t), \quad y_2(t) = \exp(r_2 t). \tag{121}$$

Check the Wronskian:

$$W(y_1, y_2)[t] = y_1(t)y_2'(t) - y_2(t)y_1'(t) \tag{122}$$

$$= r_2 \exp((r_1 + r_2)t) - r_1 \exp((r_1 + r_2)t) \tag{123}$$

$$= (r_2 - r_1)\exp((r_1 + r_2)t). \tag{124}$$

Clearly $W(y_1, y_2)[t] \neq 0$ for all $t \in \mathbb{R}$. Thus, $y_1(t) = \exp(r_1 t)$, $y_2(t) = \exp(r_2 t)$ is the FSS of the ODE (116). From previous theorems, any solution $y(t)$ to the ODE is of the form

$$y(t) = c_1 \exp(r_1 t) + c_2 \exp(r_2 t) \tag{125}$$

for some constants c_1 and c_2.

Example 27. *Solve the ODE*

$$y'' + 9y' + 20y = 0 \tag{126}$$

Consider a trial function $y(t) = \exp(rt)$. The characteristic equation is

$$r^2 + 9r + 20 = (r+4)(r+5) = 0. \tag{127}$$

We have two real roots $r_1 = -4$ and $r_2 = -5$. Hence, the general solution is

$$y(t) = c_1 \exp(-4t) + c_2 \exp(-5t), \quad c_1, c_2 \in \mathbb{R}. \tag{128}$$

Case 2. Complex roots $\Delta < 0$. We now consider the case $\Delta = b^2 - 4ac < 0$. Then, the roots to the characteristic equation $ar^2 + br + c = 0$ is a complex-conjugate pair:

$$r_1 = \lambda + i\mu, \quad \lambda = \frac{-b}{2a}, \quad \mu = \frac{\sqrt{4ac - b^2}}{2a}, \quad i := \sqrt{-1}, \quad r_2 = \overline{r_1} = \lambda - i\mu. \tag{129}$$

We obtain two functions

$$y_1(t) = \exp(r_1 t) = \exp((\lambda + i\mu)t), \quad y_2(t) = \exp(r_2 t) = \exp((\lambda - i\mu)t). \tag{130}$$

Using Euler's formula, we arrive at

$$y_1(t) = \exp(\lambda t)\left(\cos(\mu t) + i \sin(\mu t)\right), \quad y_2(t) = \exp(\lambda t)\left(\cos(\mu t) - i\sin(\mu t)\right) \tag{131}$$

Let's check that y_1 and y_2 are linearly independent. Suppose there are constants $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$\alpha_1 y_1(t) + \alpha_2 y_2(t) = 0 \quad \forall t \in I \Rightarrow e^{\lambda t}\left((\alpha_1 + \alpha_2)\cos(\mu t) + i(\alpha_1 - \alpha_2)\sin(\mu t)\right) = 0. \tag{132}$$

The exponential is non-zero for all $t \in \mathbb{R}$, so to make the above expression zero, we need

$$\alpha_1 + \alpha_2 = 0, \quad \alpha_1 - \alpha_2 = 0 \quad \Rightarrow \quad \alpha_1 = \alpha_2 = 0. \tag{133}$$

So y_1 and y_2 are linearly independent. We can also calculate the Wronskian $W(y_1, y_2)[t] = -2i\mu e^{2\lambda t} \neq 0$, since $\mu \neq 0$ (otherwise we will not have $b^2 - 4ac < 0$). Thus, any solution $y(t)$ to the ODE is of the form

$$y(t) = e^{\lambda t}\left((c_1 + c_2)\cos(\mu t) + i(c_1 - c_2)\sin(\mu t)\right) \tag{134}$$

or

$$y(t) = e^{\lambda t}\left(d_1 \cos(\mu t) + d_2 i \sin(\mu t)\right) \tag{135}$$

for some constants d_1 and d_2. However, the solution is expressed as a complex-valued function. Since the coefficients of the ODE are real numbers, it would be better for us to obtain a real-valued function as a solution.

Theorem 28. *Given an ODE*

$$y'' + p(t)y' + q(t)y = 0 \tag{136}$$

with p and q are continuous real-valued functions. If $y(t) = u(t) + iv(t)$ is a complex-valued solution to the ODE, where u and v are real-valued functions, then its real part $u(t)$ and its imaginary part $v(t)$ are both solutions to the ODE.

Proof. Substituting the complex-valued solution into the ODE gives

$$0 = u''(t) + iv''(t) + p(t)u'(t) + ip(t)v'(t) + q(t)u(t) + iq(t)v(t) \tag{137}$$
$$= (u''(t) + p(t)u'(t) + q(t)u(t)) + i(v''(t) + p(t)v'(t) + q(t)v(t)). \tag{138}$$

A complex number is zero if and only if its real and imaginary parts are both zero, thus we have

$$u'' + p(t)u' + q(t)u = 0, \quad v'' + p(t)v' + q(t)v = 0. \tag{139}$$

□

Clearly the proof also works for linear homogeneous ODEs of higher order, and this result will be utilized again in Section 4.1.

From (135) we get the real-valued functions

$$u(t) = e^{\lambda t}\cos(\mu t), \quad v(t) = e^{\lambda t}\sin(\mu t) \tag{140}$$

Clearly u and v are linearly independent, and the Wronskian can be computed as $W(u, v)[t] = \mu e^{2\lambda t} \neq 0$, since $\mu \neq 0$. Thus, any solution y to the ODE $ay'' + by' + cy = 0$ with $b^2 - 4ac < 0$ can be expressed as

$$y(t) = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t) \tag{141}$$

which is a real-valued function.

Case 3. One repeated root $\Delta = 0$. The last case is when $b^2 - 4ac = 0$ and we have a repeated root to the characteristic equation:

$$r_1 = r_2 = -\frac{b}{2a} \tag{142}$$

The problem is apparent: both roots give the same function

$$y_1(t) = y_2(t) = \exp\left(-\frac{b}{2a}t\right). \tag{143}$$

We will use the Wronskian to find a solution that is linearly independent to $y_1(t)$. By Theorem 23, if $y_1 = \exp\left(-\frac{b}{2a}t\right)$ and y_2 are two solutions to the ODE $ay'' + by' + cy = 0$, then the Wronskian is

$$W(y_1, y_2)[t] = d\exp\left(-\int \frac{b}{a}dt\right) = d\exp\left(-\frac{b}{a}t\right) \tag{144}$$

for some constant $d \in \mathbb{R}$. On the other hand we have

$$W(y_1, y_2)[t] = y_1(t)y_2'(t) - y_1'(t)y_2(t) = e^{-\frac{b}{2a}t}y_2'(t) + \frac{b}{2a}e^{-\frac{b}{2a}t}y_2(t). \tag{145}$$

Choose $d = 1$, we have

$$e^{-\frac{b}{2a}t}y_2'(t) + \frac{b}{2a}e^{-\frac{b}{2a}t}y_2(t) = e^{-\frac{b}{a}t} \quad \Rightarrow \quad y_2'(t) + \frac{b}{2a}y_2(t) = e^{-\frac{b}{2a}t} \tag{146}$$

which is a first-order linear ODE for y_2, the solution is

$$y_2(t) = te^{-\frac{b}{2a}t} \tag{147}$$

where we have neglected any constants of integration. Now check the linear independence for $y_1 = e^{-\frac{b}{2a}t}$ and $y_2 = te^{-\frac{b}{2a}t}$. Suppose α_1 and α_2 are two constants such that

$$\alpha_1 y_1(t) + \alpha_2 y_2(t) = 0 \quad \forall t \in I \Rightarrow e^{-\frac{b}{2a}t}(\alpha_1 + t\alpha_2) = 0. \tag{148}$$

Since the exponential is never zero, for $\alpha_1 + t\alpha_2$ to be zero for all $t \in I$, we must have $\alpha_1 = \alpha_2 = 0$. We can also compute the Wronskian $W(y_1, y_2)[t] = e^{-\frac{b}{a}t} \neq 0$. Thus any solution y to the ODE $ay'' + by' + cy = 0$ with $b^2 - 4ac = 0$ can be expressed as

$$y(t) = c_1 e^{-\frac{b}{2a}t} + c_2 te^{-\frac{b}{2a}t} \tag{149}$$

for constants $c_1, c_2 \in \mathbb{R}$.

Summary of Section 3.2.1:

For the second-order linear ODE

$$ay'' + by' + cy = 0 \tag{150}$$

with constants a, b, c. Let r_1 and r_2 be the roots to the characteristic equation

$$ar^2 + br + c = 0 \tag{151}$$

- If $b^2 > 4ac$, then r_1 and r_2 are real numbers, and the general solution is given as

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \tag{152}$$

- If $b^2 < 4ac$, then r_1 and r_2 are complex numbers such that $r_1 = \lambda + i\mu$ and $r_2 = \overline{r_1} = \lambda - i\mu$ for real numbers λ, μ. Then, the general solution is given as

$$y(t) = e^{\lambda t}\left(c_1 \cos(\mu t) + c_2 \sin(\mu t)\right) \tag{153}$$

- If $b^2 = 4ac$, then $r_1 = r_2 = r$. Then the general solution is given as

$$y(t) = c_1 e^{-\frac{b}{2a}t} + c_2 te^{-\frac{b}{2a}t} \tag{154}$$

3.2.2 Euler Equations

Euler equations (a.k.a Cauchy-Euler equations) are the differential equations of the form

$$x^2 \frac{d^2 y}{dx^2} + Ax\frac{dy}{dx} + By = 0, \quad x > 0 \tag{155}$$

where A and B are constants. This is a second-order homogeneous linear ODE with **non-constant** coefficients, but we will convert it into an ODE with **constant** coefficients. Introducing a new independent variable

$$t = \ln x, \quad \text{or} \quad x = e^t, \tag{156}$$

and let

$$Y(t) = y(e^t) = y(x). \tag{157}$$

Taking the derivative we have

$$\frac{dy(x)}{dx} = \frac{dY(t)}{dx} = \frac{dY}{dt} \cdot \frac{dt}{dx} = Y'(t)\frac{1}{x}. \tag{158}$$

Then,

$$x\frac{dy(x)}{dx} = Y'(t). \tag{159}$$

Taking derivative again we get

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(Y'(t)\frac{1}{x}\right) && (160)\\
&= \frac{1}{x}\frac{d}{dx}Y'(t) + Y'(t)\left(-\frac{1}{x^2}\right) && (161)\\
&= \frac{1}{x}\frac{d}{dt}Y'(t)\frac{dt}{dx} - \frac{1}{x^2}Y'(t) && (162)\\
&= \frac{1}{x^2}(Y''(t) - Y'(t)). && (163)
\end{aligned}$$

Then,

$$x^2\frac{d^2y}{dx^2} = Y''(t) - Y'(t). \tag{164}$$

Substituting $x\frac{dy}{dx}$ and $x^2\frac{d^2y}{dx^2}$ into the Euler equation we get

$$Y''(t) + (A-1)Y'(t) + BY(t) = 0. \tag{165}$$

This is a constant coefficient linear equation, the general solution $Y(t)$ can be obtained. Then, the general solution of the Euler equation is

$$y(x) = Y(\ln x). \tag{166}$$

An alternative method for solving Euler equations is using **trial solution** $y = x^r$ (r is the power to be determined), then $y' = rx^{r-1}, y'' = r(r-1)x^{r-2}$, then

$$r(r-1)x^r + Arx^r + Bx^r = 0. \tag{167}$$

Thus,

$$r^2 + (A-1)r + B = 0. \tag{168}$$

- Case 1: Two distinct real roots r_1, r_2. the general solution is

$$y(t) = c_1x^{r_1} + c_2x^{r_2}. \tag{169}$$

- Case 2: One repeated real root $r_1 = r_2$. The general solution is

$$y(x) = c_1x^{r_1} + c_2\ln(x)x^{r_1}. \tag{170}$$

- Case 3: two distinct complex roots: $\lambda \pm i\mu, \lambda, \mu \in \mathbb{R}$. The general solution is

$$y(x) = c_1x^\lambda\cos(\mu\ln(x)) + c_2x^\lambda\sin(\mu\ln(x)). \tag{171}$$

The Euler equation has a **regular singular point** at $x = 0$, which is related to the series solution of ODEs. Details can be found in Section 5.2 of Prof. Jeffrey R. Chasnov's notes and Section 5.4 of [BDM21].

3.3 Homogeneous Equations with Non-Constant Coefficients

The **reduction of order** method can be applied to a second-order homogeneous ODE with **non-constant** coefficient. Although the general method for finding a FSS for $y'' + p(t)y' + q(t)y = 0$ is not available, but if we can find one nonzero-solution $y_1(t)$ of the ODE, then we can use the reduction of order method to find $y_2(t)$ so that (y_1, y_2) forms a FSS.

Consider the ODE

$$y'' + p(t)y' + q(t)y = 0. \tag{172}$$

Suppose $y_1(t)$ is a non-zero solution to the ODE. To find a second solution, consider the function

$$y(t) = v(t)y_1(t). \tag{173}$$

Then, the product rule gives

$$y'(t) = v'(t)y_1(t) + v(t)y_1'(t), \tag{174}$$

$$y''(t) = v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t). \tag{175}$$

If y is a solution to the ODE, we have

$$0 = y'' + p(t)y' + q(t)y \tag{176}$$

$$= v''y_1 + 2v'y_1' + vy_1'' + p(t)(v'y_1 + vy_1') + q(t)vy_1 \tag{177}$$

$$= y_1v'' + (2y_1' + p(t)y_1)v'. \tag{178}$$

This gives us a second-order ODE for v that only involves v'' and v'. Define a new function $z = v'$, leading to

$$y_1(t)z' + (2y_1'(t) + p(t)y_1(t))z = 0. \tag{179}$$

Here we treat y_1 and y_1' as given functions. Note that this is a first-order linear ODE

$$\frac{dz}{dt} + \frac{2y_1' + py_1}{y_1}z = 0, \tag{180}$$

since $y_1 \neq 0$. In other words, **we have reduced the order of the original ODE by one**. Solving this gives

$$v'(t) = z(t) = \exp\left(-\int \frac{2y_1' + py_1}{y_1}dt\right) \tag{181}$$

$$= \exp\left(-\int p(t)dt - 2\ln(y_1(t))\right) \tag{182}$$

$$= \frac{1}{y_1^2(t)}\exp\left(-\int p(t)dt\right). \tag{183}$$

Integrating once more leads to

$$v(t) = \int (y_1(t))^{-2} e^{-\int p(t)dt}dt \tag{184}$$

and the second solution to the ODE is given as

$$y_2(t) = y_1(t)\int (y_1(t))^{-2} e^{-\int p(t)dt}dt. \tag{185}$$

Example 29. *Given that* $y_1(t) = t^{-1}$ *is a solution of*

$$2t^2y'' + 3ty' - y = 0, \quad t > 0, \tag{186}$$

find a FSS.

The ODE can be written as $y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0$, thus $p(t) = \frac{3}{2t}$. Plug in (185) we obtain

$$y_2 = \frac{1}{t}\int t^2 e^{-\frac{3}{2}\ln(t)+c_1} \tag{187}$$

$$= e^{c_1}\frac{1}{t}\int t^{\frac{1}{2}} \tag{188}$$

$$= e^{c_1}\frac{1}{t}(\frac{2}{3}t^{\frac{3}{2}} + c_2) \tag{189}$$

Take $e^{c_1} = \frac{3}{2}$ and $c_2 = 0$ we obtain $y_2 = t^{\frac{1}{2}}$. We can **verify the Wronskian** $W(y_1, y_2)[t] = \frac{3}{2}t^{-\frac{3}{2}} \neq 0$ for $t > 0$. Consequently, y_1, y_2 form a FSS for the ODE.

Note that this method can be used to find a second solution to the ODE if you **already have one solution**. The difficulty actually lies in finding a first solution to the ODE.

3.4 Non-Homogeneous Equations

We now turn our attention to ODE of the form

$$y'' + p(t)y' + q(t)y = r(t), \tag{190}$$

for given functions p, q, and r that are continuous in an interval I. The corresponding homogeneous equation is

$$y'' + p(t)y' + q(t)y = 0. \tag{191}$$

We have the following observation. Let Z_1 and Z_2 be solutions to the non-homogeneous problem (190). Then, the difference $Z := Z_1 - Z_2$ satisfies

$$Z'' + p(t)Z' + q(t)Z = r - r = 0. \tag{192}$$

That is, the difference Z satisfies the homogeneous equation (191). If (y_1, y_2) are a FSS to (191), then we can write $Z = Z_1 - Z_2$ as

$$Z_1(t) - Z_2(t) = c_1y_1(t) + c_2y_2(t), \tag{193}$$

for some constants c_1, c_2. We have actually derived a general expression for the solution to the non-homogeneous equation (190). Let $Y(t)$ denote a solution to (190), then any solution y to (190) can be expressed as

$$y(t) = Y(t) + c_1y_1(t) + c_2y_2(t), \tag{194}$$

where (y_1, y_2) is a FSS to the homogeneous problem (191). This is similar to the steps in linear algebra for solving the non-homogeneous linear system $A\mathbf{x} = \mathbf{b}$: We need to find $\mathbf{x}_p$ and $\mathbf{x}_n$, where the former is a particular solution to $A\mathbf{x} = \mathbf{b}$, and the latter stands for the linear combination of the basis of Null(A). Then general solution to the system is $\mathbf{x}_p + \mathbf{x}_n$.

Definition 30 (Complementary Solution, Particular Solution)**.** *For a solution expression*

$$y(t) = c_1y_1(t) + c_2y_2(t) + Y(t) \tag{195}$$

to the ODE

$$y'' + p(t)y' + q(t)y = r(t), \tag{196}$$

we call the function

$$y_c(t) := c_1y_1(t) + c_2y_2(t) \tag{197}$$

the complementary solution, which is a solution to the homogeneous equation, and the function $Y(t)$ the particular solution, which is a solution to the non-homogeneous equation.

This gives us the way of solving non-homogeneous second-order linear ODEs:

1. Obtain a FSS (y_1, y_2) to the homogeneous problem (191).
2. Find a solution $Y(t)$ to the non-homogeneous problem (190).
3. The general solution to (191) is then given as

$$y(t) = Y(t) + c_1y_1(t) + c_2y_2(t). \tag{198}$$

So the key questions become:

- How do we find y_1 and y_2?
- How do we find $Y(t)$?

Section 3.4.1 and 3.4.2 will focus on these two questions.

3.4.1 Method of Undetermined Coefficients

The general method for finding the second-order linear ODE with non-constant coefficient $a(t)y''+b(t)y'+c(t)y=r(t)$ is still missing. We will first look at the special cases **when a, b, c are real constants and $r(t)$ is in some particular form**. In other words, we will show how to obtain a solution Y to the ODE

$$ay''+by'+cy=r(t) \tag{199}$$

for some specific forms of $r(t)$.

The method for this case is the **method of undetermined coefficients**, which makes a guess on what the particular solution $Y(t)$ could look like. There are only certain classes of functions for $r(t)$ which $Y(t)$ could be obtained explicitly. We will consider $r(t)$ to be a mixture of **polynomials, exponential, sine and cosine**.

Example 31. *Solve*

$$y''-3y'-4y=3e^{2t}. \tag{200}$$

In the standard form we have

$$r(t)=3e^{2t}. \tag{201}$$

Since the derivative of exponential function is also exponential, **a possible choice for the particular solution Y would involve exponential**. Solving the homogeneous problem $y''-3y'-4y=0$, the complementary solution is obtained as

$$y_c(t)=c_1e^{4t}+c_2e^{-t}. \tag{202}$$

Returning to the non-homogeneous problem, **assume $Y(t)$ is of the form**

$$Y(t)=Ae^{qt} \tag{203}$$

for some coefficients A and q that are not determined yet. Plugging into the non-homogeneous equations gives

$$Y''-3Y'-4Y=Aq^2e^{qt}-3Aqe^{qt}-4Ae^{qt}=A(q^2-3q-4)e^{qt}=3e^{2t}. \tag{204}$$

Therefore, it makes sense to choose

$$q=2,\quad A(q^2-3q-4)=3 \quad\Rightarrow\quad A=-\frac{1}{2} \quad\Rightarrow\quad Y(t)=-\frac{1}{2}e^{2t}. \tag{205}$$

Hence, the general solution y to the ODE $y''-3y'-4=3e^{2t}$ can be expressed as

$$y(t)=c_1e^{4t}+c_2e^{-t}-\frac{1}{2}e^{2t}. \tag{206}$$

Example 32. *Solve*

$$y'' - 3y' - 4y = 2e^{-t}. \tag{207}$$

Since $r(t)$ is an exponential, try $Y(t) = Ae^{-t}$ and determine the value of A. However,

$$Y'' - 3Y' - 4Y = A(1 + 3 - 4)e^{-t} = 0. \tag{208}$$

So no choice of A would satisfy the non-homogeneous ODE. Actually, a FSS to the homogeneous ODE $y'' - 3y' - 4y = 0$ is $y_1 = e^{4t}$ and $y_2 = e^{-t}$. That is, the guess function $Y(t) = Ae^{-t}$ actually is a solution to the homogeneous problem, and consequently, it cannot be a solution to the non-homogeneous problem! In this case, where the assumed form of the particular solution Y is a duplicate of one of the solutions to the homogeneous problem, we can consider a new guess for Y which looks like

$$Y(t) = Ate^{-t}, \tag{209}$$

for undetermined constant A, which is similar to the FSS $(e^{-\frac{b}{2a}t}, te^{-\frac{b}{2a}t})$ for the ODE $ay'' + by' + cy = 0$ when $b^2 = 4ac$. Trying this new guess yields

$$Y'' - 3Y' - 4Y = -5Ae^{-t} = 2e^{-t}. \tag{210}$$

This means that we should take

$$A = -\frac{1}{5} \quad \Rightarrow \quad Y(t) = -\frac{2}{5}te^{-t}. \tag{211}$$

Thus a general solution y to the ODE $y'' - 3y' - 4y = 2e^{-t}$ is

$$y(t) = c_1e^{4t} + c_2e^{-t} - \frac{2}{5}te^{-t}. \tag{212}$$

Example 33. *Solve*

$$y'' - 3y' - 4y = t^2 + t + 1. \tag{213}$$

We know the complementary solution is $y_c = c_1e^{4t} + c_2e^{-t}$. Since $r(t)$ is a polynomial of degree 2, a possible guess is that the particular solution Y is also a polynomial of the same degree, that is $Y(t) = At^2 + Bt + C$ for some undetermined coefficients A, B, C. Then, plugging into the equation gives

$$Y'' - 3Y' - 4Y = 2A - 3(2At + B) - 4(At^2 + Bt + C) \tag{214}$$

$$= -4At^2 - (4B + 6A)t + (2A - 3B - 4C) = t^2 + t + 1. \tag{215}$$

Comparing coefficients immediately gives

$$A = -\frac{1}{4}, \quad B = \frac{1}{8}, \quad C = -\frac{15}{32}, \tag{216}$$

so the general solution y to the ODE $y'' - 3y' - 4y = t^2 + t + 1$ can be expressed as

$$y(t) = c_1e^{4t} + c_2e^{-t} - \frac{1}{4}t^2 + \frac{1}{8}t - \frac{15}{32}. \tag{217}$$

What if $r(t)$ involves the multiplication of exponential function and polynomials? The method is summarized as follows.

Case 1. $r(t) = P_n(t)e^{\alpha t}$**. A possible guess is**

$$Y(t) = t^s Q_n(t)e^{\alpha t}, \tag{218}$$

where $Q_n(t) = A_0 + A_1 t + \cdots + A_n t^n$ is a polynomial with undetermined coefficients $A_0, \ldots, A_n$, and $s \in \{0, 1, 2\}$ is an exponent determined by the following criterion:

$$s = \begin{cases} 0 & \text{if } \alpha \neq r_1, \alpha \neq r_2, \\ 1 & \text{if } \alpha = r_1 \neq r_2, \\ 2 & \text{if } r_1 = r_2 = \alpha. \end{cases} \tag{219}$$

where r_1 and r_2 are the roots to the characteristic equation

$$ar^2 + br + c = 0. \tag{220}$$

I.e., s is the **multiplicity** of α as a root of the characteristic equation. **The guess** (218) **includes Example 31 to Example 33.**

The problem of determining a particular solution to the ODE

$$ay'' + by' + cy = P_n(t)e^{\alpha t} \tag{221}$$

can be done by a substitution. Let

$$Y(t) = e^{\alpha t}u(t), \tag{222}$$

and by substituting this into the ODE we obtain

$$e^{\alpha t}\left[a[u'' + 2\alpha u' + \alpha^2 u] + b[u' + \alpha u] + cu\right] = e^{\alpha t}P_n(t) \tag{223}$$

$$\Rightarrow au'' + (2\alpha a + b)u' + (a\alpha^2 + b\alpha + c)u = P_n(t). \tag{224}$$

To equal polynomial degree on both sides, it is reasonable to take

$$u(t) = \begin{cases} A_n t^n + \cdots + A_0 & \text{if } a\alpha^2 + b\alpha + c \neq 0, \\ t(A_n t^n + \cdots + A_0) & \text{if } a\alpha^2 + b\alpha + c = 0, 2a\alpha + b \neq 0, \\ t^2(A_n t^n + \cdots + A_0) & \text{if } a\alpha^2 + b\alpha + c = 0, 2a\alpha + b = 0. \end{cases} \tag{225}$$

$$= t^s(A_n t^n + \cdots + A_0), \quad s = \begin{cases} 0 & \text{if } \alpha \neq r_1, \alpha \neq r_2, \\ 1 & \text{if } \alpha = r_1 \neq r_2, \\ 2 & \text{if } r_1 = r_2 = \alpha. \end{cases} \tag{226}$$

Example 34. *Find a particular solution of*

$$y'' - 3y' - 4y = te^{-t}. \tag{227}$$

e^{-t} is a solution to the homogeneous problem, and the non-homogeneous term is $r(t) = te^{-t}$. In this case we have $r_2 = \alpha = -1$ and $r_1 = 4$. Taking $s = 1$ we try a particular solution Y of the form

$$Y(t) = t(A_1 t + A_0)e^{-t} = (A_1 t^2 + A_0 t)e^{-t}. \tag{228}$$

Take derivatives

$$Y'(t) = (-A_1t^2 + (2A_1 - A_0)t + A_0)e^{-t}, \quad Y''(t) = (A_1t^2 + (A_0 - 4A_1)t + 2A_1 - 2A_0)e^{-t}. \tag{229}$$

Substituting these into the equation, we get $(-10A_1t + 2A_1 - 5A_0)e^{-t} = te^{-t}$. Thus, $-10A_1 = 1, 2A_1 - 5A_0 = 0$. Therefore, $A_1 = -\frac{1}{10}, A_0 = -\frac{1}{25}$. The particular solution is

$$Y(t) = t\left(-\frac{1}{10}t - \frac{1}{25}\right)e^{-t}. \tag{230}$$

What if $r(t)$ involves the multiplication of exponential function and polynomial as well as sine(cosine) function?

Example 35. *Solve*

$$y'' - 3y' - 4y = 2\sin(t) \tag{231}$$

The complementary solution is $y_c = c_1e^{4t} + c_2e^{-t}$. Since the non-homogeneous term $r(t) = 2\sin(t)$, a possible solution would involve sine and cosine, so consider

$$Y(t) = a\sin(\alpha t) + b\cos(\beta t) \tag{232}$$

for undetermined coefficients a, b, α, β. Plugging into the non-homogeneous equations gives

$$\begin{aligned} Y'' - 3Y' - 4Y &= -a\alpha^2\sin(\alpha t) - b\beta^2\cos(\beta t) - 3(a\alpha\cos(\alpha t) - b\beta\sin(\beta t)) \\ &\quad - 4(a\sin(\alpha t) + b\cos(\beta t)) && (233) \\ &= \sin(\alpha t)[-a\alpha^2 - 4a] + \cos(\beta t)[-b\beta^2 - 4b] \\ &\quad + \cos(\alpha t)[-3a\alpha] + \sin(\beta t)[3b\beta] && (234) \\ &= 2\sin(t) && (235) \end{aligned}$$

Since the RHS only involves $\sin(t)$, we can set

$$\alpha = 1, \quad \beta = 1. \tag{236}$$

This simplifies the above calculation to

$$\sin(t)[-5a + 3b] + \cos(t)[-5b - 3a] = 2\sin(t). \tag{237}$$

Since there is no term involving the cosine on the RHS, we must have

$$-5a + 3b = 2, -5b - 3a = 0 \quad \Rightarrow \quad a = -\frac{5}{17}, \quad b = \frac{3}{17}. \tag{238}$$

Therefore, the general solution y to the ODE can be expressed as

$$y(t) = c_1e^{4t} + c_2e^{-t} - \frac{5}{17}\sin(t) + \frac{3}{17}\cos(t). \tag{239}$$

Case 2. $r(t) = e^{\alpha t}P_n(t)\cos(\beta t)$ **or** $e^{\alpha t}P_n(t)\sin(\beta t)$**.** Using Euler's formula: $\cos(\beta t) = \frac{1}{2}\left(e^{\beta it} + e^{-\beta it}\right), \sin(\beta t) = \frac{1}{2i}\left(e^{\beta it} - e^{-\beta it}\right)$, the ODE becomes

$$ay'' + by' + cy = \frac{1}{2}P_n(t)\left(e^{(\alpha+\beta i)t} + e^{(\alpha-\beta i)t}\right) \quad \text{or} \tag{240}$$

$$ay'' + by' + cy = \frac{1}{2i}P_n(t)\left(e^{(\alpha+\beta i)t} - e^{(\alpha-\beta i)t}\right). \tag{241}$$

A possible guess for the above two ODEs is

$$Y(t) = t^s\left(Q_n(t)\cos(\beta t) + R_n(t)\sin(\beta t)\right)e^{\alpha t}, \tag{242}$$

where $Q_n(t) = A_0 + A_1 t + \cdots + A_n t^n, R_n(t) = B_0 + B_1 t + \cdots + B_n t^n$ are polynomials with undetermined coefficients $A_0, \ldots, A_n, B_0, \ldots, B_n$, and $s \in \{0, 1\}$ is an exponent determined by

$$s = \begin{cases} 0 & \text{if } \alpha + i\beta \text{ is not a root of the characteristic equation,} \\ 1 & \text{if } \alpha + i\beta \text{ is a root of the characteristic equation.} \end{cases} \tag{243}$$

To see the reason behind this, let us consider the case $r(t) = e^{\alpha t}P_n(t)\sin(\beta t)$ since the two cases are similar. We consider

$$Y(t) = e^{\alpha t}\left(Q(t)\cos(\beta t) + R(t)\sin(\beta t)\right), \tag{244}$$

for some functions Q and R, and upon differentiating:

$$\begin{aligned} Y'(t) &= \alpha e^{\alpha t}\left(Q(t)\cos(\beta t) + R(t)\sin(\beta t)\right) + e^{\alpha t}\beta\left(-Q(t)\sin(\beta t) + R(t)\cos(\beta t)\right) \\ &\quad + e^{\alpha t}\left(Q'(t)\cos(\beta t) + R'(t)\sin(\beta t)\right), \end{aligned} \tag{245}$$

$$\begin{aligned} Y''(t) &= \alpha^2 e^{\alpha t}(Q(t)\cos(\beta t) + R(t)\sin(\beta t)) + 2e^{\alpha t}\alpha\beta(-Q(t)\sin(\beta t) + R(t)\cos(\beta t)) \\ &\quad + 2\alpha e^{\alpha t}(Q'(t)\cos(\beta t) + R'(t)\sin(\beta t)) + \beta^2 e^{\alpha t}(-Q(t)\cos(\beta t) - R(t)\sin(\beta t)) \\ &\quad + 2\beta e^{\alpha t}(-Q'(t)\sin(\beta t) + R'(t)\cos(\beta t)) + e^{\alpha t}(Q''(t)\cos(\beta t) + R''(t)\sin(\beta t)). \end{aligned} \tag{246}$$

Plugging the above expression into the ODE yields

$$e^{\alpha t}P_n(t)\sin(\beta t) = aY'' + bY' + cY \tag{247}$$

$$\begin{aligned} &= e^{\alpha t}\cos(\beta t)[(a\alpha^2 - a\beta^2 + b\alpha + c)Q + (2a\alpha + b)(\beta R + Q') + 2a\beta R' + aQ''] \\ &\quad + e^{\alpha t}\sin(\beta t)[(a\alpha^2 - a\beta^2 + b\alpha + c)R + (2a\alpha + b)(-\beta Q + R') - 2a\beta Q' + aR'']. \end{aligned} \tag{248}$$

Equating coefficients means that

$$(a\alpha^2 - a\beta^2 + b\alpha + c)Q + (2a\alpha + b)(\beta R + Q') + 2a\beta R' + aQ'' = 0, \tag{249}$$

$$(a\alpha^2 - a\beta^2 + b\alpha + c)R + (2a\alpha + b)(-\beta Q + R') - 2a\beta Q' + aR'' = P_n. \tag{250}$$

Observe that, $\alpha + i\beta$ is a root of the characteristic equation if and only if

$$a(\alpha + i\beta)^2 + b(\alpha + i\beta) + c = [a\alpha^2 - a\beta^2 + b\alpha + c] + i(2a\alpha + b)\beta = 0. \tag{251}$$

Using the fact that a complex number is zero if and only if the real and imaginary parts are zero, we have

$$\alpha + i\beta \text{ is a root} \iff a(\alpha^2 - \beta^2) + b\alpha + c = 0, \quad (2a\alpha + b)\beta = 0. \tag{252}$$

As the RHS of (250) are polynomials, we may take Q and R to be polynomials. The question is the degree.

- Case 1: $\alpha + i\beta$ is not a root of the characteristic equation. Then, $(a\alpha^2 - a\beta^2 + b\alpha + c)$ and $(2a\alpha + b)\beta$ are not all zeros. We can take Q and R to have the **same degree** as the polynomial P_n, i.e.,

$$Q(t) = A_n t^n + \cdots + A_0, \quad R(t) = B_n t^n + \cdots + B_0$$

- Case 2: $\alpha + i\beta$ is a root of the characteristic equation, then (249), (250) simplifies to

$$(2a\alpha + b)Q' + 2a\beta R' + aQ'' = 0, \tag{253}$$
$$(2a\alpha + b)R' - 2a\beta Q' + aR'' = P_n. \tag{254}$$

and from the second equation, we see that the degree of the LHS would be the degree of R' or Q' (which ever is higher), thus we take

$$Q(t) = t(A_n t^n + \cdots + A_1 t + A_0), \quad R(t) = t(B_n t^n + \cdots + B_1 t + B_0), \tag{255}$$

in order to match the degree with the RHS.

Example 36. *Find a particular solution of*

$$y'' - 3y' - 4y = -8e^t \cos 2t. \tag{256}$$

We guess our particular solution $Y(t)$ is the product of e^t and a linear combination of $\cos 2t$ and $\sin 2t$, i.e.

$$Y(t) = Ae^t \cos 2t + Be^t \sin 2t \tag{257}$$

It follows that

$$Y'(t) = [A\cos 2t - 2A\sin 2t]e^t + [B\sin 2t + 2B\cos 2t]e^t \tag{258}$$
$$= (A + 2B)e^t \cos 2t + (-2A + B)e^t \sin 2t \tag{259}$$

and

$$Y''(t) = [(A + 2B)\cos 2t - 2(A + 2B)\sin 2t]e^t + [(-2A + B)\sin 2t + 2(-2A + B)\cos 2t]e^t \tag{260}$$
$$= (-3A + 4B)e^t \cos 2t + (-4A - 3B)e^t \sin 2t \tag{261}$$

After substituting for y, y' and y'' in (256) we obtain:

$$e^t \cos 2t[(-3A + 4B) - 3(A + 2B) - 4A] + e^t \sin 2t[(-4A - 3B) - 3(-2A + B) - 4B] = -8e^t \cos 2t \tag{262}$$

Hence:

$$\begin{cases} -10A - 2B = -8, \\ 2A - 10B = 0, \end{cases} \Rightarrow \begin{cases} A = \dfrac{10}{13}, \\ B = \dfrac{2}{13}, \end{cases} \tag{263}$$

Hence our particular solution is:

$$Y(t) = \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin 2t. \tag{264}$$

Summary of Section 3.4.1:

For

$$ay'' + by' + cy = r(t) \tag{265}$$

the trial function $Y(t)$ vs. $r(t)$ is listed as follows:

$r(t)$	$Y(t)$	The value for s
$P_n(t)e^{\alpha t}$	$Q_n(t)t^s e^{\alpha t}$	$s = \begin{cases} 0, & \alpha \text{ is not a root.} \\ 1, & \alpha = r_1 \neq r_2 \\ 2, & \alpha = r_1 = r_2 \end{cases}$ r_1, r_2 are roots of $ar^2 + br + c = 0$
$\begin{cases} P_n e^{\alpha t} \sin \beta t \\ P_n e^{\alpha t} \cos \beta t \end{cases}$	$\begin{cases} [Q_n(t) \cos \beta t \\ +R_n(t) \sin \beta t] t^s e^{\alpha t} \end{cases}$	$s = \begin{cases} 0, & \text{if } \alpha + i\beta \text{ is not a root of } ar^2 + br + c = 0. \\ 1, & \text{if } \alpha + i\beta \text{ is a root of } ar^2 + br + c = 0. \end{cases}$

We will conclude this section with another theorem.

Theorem 37. *Suppose Y_1 is a solution to*

$$ay'' + by' + cy = g_1(t), \tag{266}$$

and Y_2 is a solution to

$$ay'' + by' + cy = g_2(t). \tag{267}$$

Then the sum $Y_1 + Y_2$ is a solution to

$$ay'' + by' + cy = g_1(t) + g_2(t). \tag{268}$$

Proof. Since Y_1 is a solution to $ay'' + by' + cy = g_1(t)$. and Y_2 is a solution to $ay'' + by' + cy = g_2(t)$, we have

$$aY_1'' + bY_1' + cY_1 = g_1(t) \tag{269}$$
$$aY_2'' + bY_2' + cY_2 = g_2(t) \tag{270}$$

Sum the two equations, we have

$$[aY_1'' + bY_1' + cY_1] + [aY_2'' + bY_2' + cY_2] \tag{271}$$
$$=a[Y_1'' + Y_2''] + b[Y_1' + Y_2'] + c[Y_1 + Y_2] \tag{272}$$
$$=a[Y_1 + Y_2]'' + b[Y_1 + Y_2]' + c[Y_1 + Y_2] \tag{273}$$
$$=g_1(t) + g_2(t) = g(t). \tag{274}$$

□

Clearly the result also holds when a, b, c are not constants.

Example 38. *Find a particular solution of*

$$y'' - 3y' - 4y = 3e^{2t} + 2e^{-t} + 2\sin t - 8e^{t}\cos 2t. \tag{275}$$

Combining previous results, we have

$$Y(t) = -\frac{1}{2}e^{2t} - \frac{2}{5}te^{-t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t + \frac{10}{13}e^{t}\cos 2t + \frac{2}{13}e^{t}\sin 2t. \tag{276}$$

3.4.2 Variation of Parameters

The method of undetermined coefficients is straightforward, but requires that the non-homogeneous term $r(t)$ to be in a special form. We need a more general method that in principle can be applied to any equation. One such method is the **variation of parameters**.

Consider a general 2nd-order linear ODE

$$y'' + p(t)y' + q(t)y = r(t), \tag{277}$$

and suppose (y_1, y_2) forms a FSS to the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0. \tag{278}$$

How to find a particular solution to the non-homogeneous equation (277)? Consider for some functions $u_1(t)$, $u_2(t)$ such that the new function

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) \tag{279}$$

solves (277). We now determine what equations u_1 and u_2 have to satisfy. Differentiating (279) yields

$$y' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'. \tag{280}$$

In order to simplify the computation, **impose a condition**

$$u_1'y_1 + u_2'y_2 = 0. \tag{281}$$

Then the derivative becomes

$$y' = u_1y_1' + u_2y_2'. \tag{282}$$

Differentiating again leads to

$$y'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'' \tag{283}$$

Substitute into the non-homogeneous ODE gives

$$\begin{aligned} y'' + p(t)y' + q(t)y &= u_1(y_1'' + p(t)y_1' + q(t)y_1) + u_2(y_2'' + p(t)y_2' + q(t)y_2) && (284)\\ &\quad + u_1'y_1' + u_2'y_2' && (285)\\ &= u_1'y_1' + u_2'y_2' = r(t). && (286) \end{aligned}$$

Thus, we obtain two conditions for u_1 and u_2:

$$u_1'y_1 + u_2'y_2 = 0, \quad u_1'y_1' + u_2'y_2' = r(t), \tag{287}$$

which can be summarized as

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix} \tag{288}$$

Since the determinant is the Wronskian $W(y_1, y_2)[t]$ which is non-zero since (y_1, y_2) is a FSS, (u_1', u_2') can be solved. Using Cramer's rule, we have

$$u_1'(t) = -\frac{y_2 r}{W(y_1, y_2)}(t), \quad u_2'(t) = \frac{y_1 r}{W(y_1, y_2)}(t). \tag{289}$$

Integrating gives

$$u_1(t) = -\int \frac{y_2 r}{W(y_1, y_2)}(t)dt + d_1, \quad u_2(t) = \int \frac{y_1 r}{W(y_1, y_2)}(t)dt + d_2, \tag{290}$$

for constants $d_1, d_2 \in \mathbb{R}$, and the general solution to the non-homogeneous equation is

$$y(t) = (c_1 + d_1)y_1(t) + (c_2 + d_2)y_2(t) - y_1 \int \frac{y_2 r}{W(y_1, y_2)}(t)dt + y_2 \int \frac{y_1 r}{W(y_1, y_2)}(t)dt. \tag{291}$$

We can simply take $d_1 = d_2 = 0$, so the final solution becomes

$$y(t) = c_1 y_1(t) + c_2 y_2(t) - y_1 \int \frac{y_2 r}{W(y_1, y_2)}(t)dt + y_2 \int \frac{y_1 r}{W(y_1, y_2)}(t)dt. \tag{292}$$

for constants $c_1, c_2 \in \mathbb{R}$.

This method is able to treat rather general second-order ODEs (since $p(t)$ and $q(t)$ need not be constants). However, **it is not easy to find a FSS** (if $p(t)$ and $q(t)$ are not constants). Another difficulty lies in the evaluation of the integrals:

$$-\int \frac{y_2 r}{W(y_1, y_2)}(t)dt, \quad \int \frac{y_1 r}{W(y_1, y_2)}(t)dt \tag{293}$$

which may not be possible if r, y_1, y_2 are complicated functions.

Example 39. *Solve the ODE*

$$y'' - 3y' + 2y = \frac{e^{3t}}{e^t + 1} \tag{294}$$

First look at the homogeneous problem

$$y'' - 3y' + 2y = 0, \tag{295}$$

the complementary solution is given as

$$y_c(t) = c_1 e^t + c_2 e^{2t}. \tag{296}$$

We now compute u_1 and u_2, where we use

$$y_1 = e^t, \quad y_2 = e^{2t}, \quad r = \frac{e^{3t}}{e^t + 1}, \quad W(y_1, y_2)[t] = e^{3t}. \tag{297}$$

We have

$$u_1'(t) = -\frac{e^{2t}}{e^t + 1}, \quad u_2'(t) = \frac{e^t}{e^t + 1}. \tag{298}$$

Integrating gives

$$u_1(t) = \ln(e^t + 1) - e^t, \quad u_2(t) = \ln(e^t + 1). \tag{299}$$

Hence, a particular solution is

$$Y(t) = u_1 y_1 + u_2 y_2 = e^t \ln(e^t + 1) + e^{2t} \ln(e^t + 1) - e^{2t}. \tag{300}$$

The general solution to the ODE is

$$y(t) = c_1 e^t + c_2 e^{2t} + e^t \ln(e^t + 1) + e^{2t} \ln(e^t + 1) \tag{301}$$

where c_1, c_2 are arbitrary constants.

4 Higher-Order Linear Differential Equations

4.1 General Theory

The general n-th order linear ODE is of the form

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = g(t), \tag{302}$$

and for an IVP we provide initial conditions

$$y(t_0) = x_0, y'(t_0) = x_1, \ldots, y^{(n-1)}(t_0) = x_{n-1}. \tag{303}$$

We first state the existence and uniqueness theorem.

Theorem 40 (Existence and Uniqueness for n-th Order Linear ODE)**.** *Let $I \subset \mathbb{R}$ be an open interval and suppose $g, p_0, p_1, \ldots, p_{n-1}$ are continuous functions in I. For $t_0 \in I$ and $x_0, \ldots, x_{n-1} \in \mathbb{R}$, there is exactly one solution to the IVP*

$$\begin{cases} y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = g(t), \\ y(t_0) = x_0, y'(t_0) = x_1, \ldots, y^{(n-1)}(t_0) = x_{n-1}. \end{cases} \tag{304}$$

Linear (in)dependence is defined in similar way.

Definition 41 (Linear Dependence)**.** *The functions $f_1(t), \ldots, f_n(t)$ are linearly dependent on the interval I if there exists a set of numbers $(\alpha_1, \ldots, \alpha_n) \neq (0, \ldots, 0)$, such that*

$$\alpha_1 f_1(t) + \cdots + \alpha_n f_n(t) = 0, \tag{305}$$

for all $t \in I$. Otherwise, we say that the functions $f_1(t), \ldots, f_n(t)$ are linearly independent.

Similar to Theorem 18, we also have the principle of superposition.

Theorem 42 (Principle of Superposition)**.** *Let $y_1, \ldots, y_n$ be solutions to the homogeneous equation*

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = 0, \tag{306}$$

then, for any constants $c_1, \ldots, c_n \in \mathbb{R}$, the function

$$\phi(t) = c_1 y_1(t) + \cdots + c_n y_n(t) \tag{307}$$

is also a solution to the above homogeneous equation.

We also have the Wronskian.

Definition 43 (Wronskian)**.** *Given functions $f_1, \dots, f_n$ that are differentiable up to order $n-1$, we define the Wronskian W as*

$$W(f_1, \dots, f_n)[t] = \det \begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{pmatrix} [t]. \tag{308}$$

The natural question is: given n solutions $y_1, \dots, y_n$ to the **homogeneous equation**

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = 0. \tag{309}$$

Can **any** solution ϕ to the homogeneous equation be expressed as a linear combination of $y_1, \dots, y_n$? Similarly with the case of the second-order ODE, we have the following theorem.

Theorem 44. *If $p_0, \dots, p_{n-1}$ are continuous functions in I, and $y_1, \dots, y_n$ are solutions to the above homogeneous equation, then every solution ϕ to the homogeneous equation can be expressed as a linear combination of $y_1, \dots, y_n$ if and only if $W(y_1, \dots, y_n)[t_0] \neq 0$ for some $t_0 \in I$. In this case, we call $(y_1, \dots, y_n)$ a* ***fundamental set of solutions (FSS)*** *to the homogeneous equation.*

An analogous result to Abel's identity (Theorem 23):

Theorem 45. *Let $y_1, \dots, y_n$ be solutions to the homogeneous equation*

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = 0, \tag{310}$$

for $t \in I$. Then,

$$W(y_1, \dots, y_n)[t] = ce^{-\int p_{n-1}(t)dt} \tag{311}$$

for a constant c not dependent on $t \in I$.

Proof. The idea is to derive an equation satisfied by the Wronskian. Recall the rule for taking derivatives on determinants: **The derivative of an $n \times n$ determinant is equal to the sum of n determinants, where the k-th determinant is obtained by differentiating the k-th row of the original determinant, and keeping the other rows unchanged.**

For example, denote $D := \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ where $a, b, c, \dots, i$ are functions of t. Then we have

$$\frac{dD}{dt} = \begin{vmatrix} a' & b' & c' \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & c \\ d' & e' & f' \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & c \\ d & e & f \\ g' & h' & i' \end{vmatrix} \tag{312}$$

Thus, we have

$$\frac{d}{dt}W[t] = \begin{vmatrix} y_1' & y_2' & \cdots & y_n' \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} + \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1'' & y_2'' & \cdots & y_n'' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$
$$+ \cdots + \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix} = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}. \tag{313}$$

The first $n-1$ determinants all have two identical rows, thus they are all zero and only the last determinant is nonzero. Using that for each $1 \leq k \leq n$,

$$y_k^{(n)} = -p_{n-1}y_k^{(n-1)} - \cdots - p_1 y_k' - p_0 y_k, \tag{314}$$

then applying elementary row operations we find that

$$\frac{d}{dt}W[t] = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ -p_{n-1}y_1^{(n-1)} & -p_{n-1}y_2^{(n-1)} & \cdots & -p_{n-1}y_n^{(n-1)} \end{vmatrix} = -p_{n-1}W[t]. \tag{315}$$

Thus,

$$W(y_1, \ldots, y_n)[t] = ce^{-\int p_{n-1}(t)dt} \tag{316}$$

for a constant c not dependent on $t \in I$.

□

Finally, the relationship between linear (in)dependence and Wronskian.

Theorem 46. *If $y_1, \ldots, y_n$ are solutions to the ODE $y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = 0, t \in I$, then $y_1, \ldots, y_n$ are linearly independent at each point $t \in I \iff W[y_1, \ldots, y_n](t) \neq 0, \forall t \in I$ ($(y_1, \ldots, y_n)$ forms a FSS).*

4.2 Homogeneous Equations with Constant Coefficients

We will study, for constants $a_n \neq 0, a_{n-1}, \ldots, a_0 \in \mathbb{R}$, the equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0. \tag{317}$$

Still, consider a trial function $\phi = e^{rt}$ for $r \in \mathbb{R}$. Substituting this gives the **characteristic equation**

$$a_n r^n + \cdots + a_1 r + a_0 = 0. \tag{318}$$

The characteristic polynomial is

$$Z(r) = a_n r^n + \cdots + a_1 r + a_0. \tag{319}$$

From the fundamental theorem of algebra, every polynomial with real coefficients of degree n has n complex roots. Hence

$$Z(r) = a_n (r - r_1)(r - r_2) \cdots (r - r_n), \tag{320}$$

where $r_1, \ldots, r_n$ are complex numbers, it is possible that some roots are repeated.

Definition 47 (Multiplicity)**.** *Let $P_k(x)$ be a polynomial of degree k in x. A root r has multiplicity $m \in \mathbb{N}, m \geq 1$, if there is another polynomial $S_{k-m}(x)$ of degree $k - m$ such that $S_{k-m}(r) \neq 0$ and*

$$P_k(x) = S_{k-m}(x)(x - r)^m. \tag{321}$$

Case 1. Real and distinct roots. If the roots of $Z(r) = 0$ are all real and distinct, then we have the solutions

$$y_1(t) = e^{r_1 t}, \ldots, y_n(t) = e^{r_n t}. \tag{322}$$

They are linearly independent solutions and form a FSS. Compute the Wronskian

$$W(e^{r_1 t}, e^{r_2 t}, \ldots, e^{r_n t})(t) = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} & \ldots & e^{r_n t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} & \ldots & r_n e^{r_n t} \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1} e^{r_1 t} & r_2^{n-1} e^{r_2 t} & \ldots & r_n^{n-1} e^{r_n t} \end{vmatrix} \tag{323}$$

$$= e^{(r_1 + \cdots + r_n)t} \begin{vmatrix} 1 & 1 & \ldots & 1 \\ r_1 & r_2 & \ldots & r_n \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1} & r_2^{n-1} & \ldots & r_n^{n-1} \end{vmatrix} \tag{324}$$

$$= e^{(r_1 + \cdots + r_n)t} \prod_{1 \leq i < j \leq n} (r_j - r_i) \neq 0. \tag{325}$$

Example 48. *Solve the ODE*

$$y^{(4)} - 7y''' + 6y'' + 30y' - 36y = 0 \tag{326}$$

The characteristic equation is:

$$r^4 - 7r^3 + 6r^2 + 30r - 36 = 0 \tag{327}$$

which can be factorized as

$$(r-3)(r+2)(r^2 - 6r + 6) = 0 \tag{328}$$

Hence $r_1 = -2, r_2 = 3, r_3 = 3 - \sqrt{3}, r_4 = 3 + \sqrt{3}$. The general solution is given by:

$$y = c_1 e^{-2t} + c_2 e^{3t} + c_3 e^{(3-\sqrt{3})t} + c_4 e^{(3+\sqrt{3})t}. \tag{329}$$

Case 2. Some roots are complex. If some roots are complex, they must appear in pairs, i.e. $\lambda \pm i\mu$ (see the Complex conjugate root theorem). Thus, we could replace the complex-valued solutions $e^{(\lambda+i\mu)t}$ and $e^{(\lambda-i\mu)t}$ by the real-valued solutions $e^{\lambda t}\cos\mu t, e^{\lambda t}\sin\mu t$. (Recall **Case 2** of Section 3.2)

Example 49. *Solve the ODE*

$$y^{(4)} - y = 0 \tag{330}$$

The characteristic equation is:

$$r^4 - 1 = 0. \tag{331}$$

We have $r = 1, -1, \pm i$. Thus $\lambda = 0, \mu = 1$. Hence $\{e^t, e^{-t}, \cos t, \sin t\}$ forms a FSS. The general solution is given by:

$$y = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t. \tag{332}$$

Case 3. Some roots are repeated.

Subcase 1: If one of the real root r_1 is repeated with multiplicity s, then the corresponding linearly independent solutions corresponding to root r_1 are:

$$e^{r_1 t}, te^{r_1 t}, t^2 e^{r_1 t}, \ldots, t^{s-1} e^{r_1 t}. \tag{333}$$

Subcase 2: If the complex root $r_1 = \lambda + i\mu$ is repeated with multiplicity s, then the corresponding conjugate $\overline{r}_1 = \lambda - i\mu$ is also the root with multiplicity s. In this case, we could replace the complex-valued solutions $e^{(\lambda+i\mu)t}, \ldots, t^{s-1}e^{(\lambda+i\mu)t}$ and $e^{(\lambda-i\mu)t}, \ldots, t^{s-1}e^{(\lambda-i\mu)t}$ by the real valued solutions as follows:

$$\begin{cases} e^{\lambda t}\cos\mu t, te^{\lambda t}\cos\mu t, t^2 e^{\lambda t}\cos\mu t, \ldots, t^{s-1}e^{\lambda t}\cos\mu t & \text{– from real parts} \\ e^{\lambda t}\sin\mu t, te^{\lambda t}\sin\mu t, t^2 e^{\lambda t}\sin\mu t, \ldots, t^{s-1}e^{\lambda t}\sin\mu t & \text{– from imaginary parts} \end{cases} \tag{334}$$

These are linearly independent solutions corresponding to the repeated roots $r_1 = \lambda + i\mu$ and $\overline{r}_1 = \lambda - i\mu$.

Example 50. *Solve the ODE*

$$y^{(4)} + 2y'' + y = 0 \tag{335}$$

The characteristic equation is:

$$r^4 + 2r^2 + 1 = (r^2 + 1)(r^2 + 1) = 0. \tag{336}$$

We have $r = i, i, -i, -i$, thus $\lambda = 0, \mu = 1$. The fundamental solution is:

$$e^{it}, te^{it}, e^{-it}, te^{-it}. \tag{337}$$

The general solution is given by:

$$y = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t. \tag{338}$$

4.3 Non-Homogeneous Equations

4.3.1 Method of Undetermined Coefficients

Consider the non-homogeneous equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(t). \tag{339}$$

If Y_1 and Y_2 are both solutions to the non-homogeneous problem, then $Y_1 - Y_2$ is a solution to the corresponding homogeneous equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0. \tag{340}$$

Given a FSS $(y_1, \ldots, y_n)$ to the homogeneous equation, a general solution to the non-homogeneous equation (339) is

$$y(t) = c_1 y_1(t) + \cdots + c_n y_n(t) + Y(t), \tag{341}$$

where $Y(t)$ is a particular solution to the non-homogeneous equation, $c_1 y_1(t) + \cdots + c_n y_n(t)$ is the complementary solution (solution to the homogeneous equation).

Similar to second-order equations, we now find a particular solution Y to the non-homogeneous equation (339) if $g(t)$ is a **sum/product of exponentials, cosine, sine and polynomials.** But the main difference is that the multiplicity of roots to the characteristic equation can be **greater than two.** Thus, higher powers of t need to be multiplied to get the solution to the non-homogeneous equation.

We again investigate the cases:

1. $g(t) = e^{\alpha t} P_m(t)$,
2. $g(t) = e^{\alpha t} P_m(t) \cos(\beta t),$ or $g(t) = e^{\alpha t} P_m(t) \sin(\beta t)$.

Remember the characteristic equation for the homogeneous equation is

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0. \tag{342}$$

The possible particular solutions can be used are

1. $Y(t) = t^s e^{\alpha t} Q_m(t)$, where

   $$Q_m(t) = A_m t^m + \cdots + A_1 t + A_0 \tag{343}$$

   for undetermined coefficients $A_m, \ldots, A_0$, and

   $$s = \begin{cases} 0, & \text{if } \alpha \text{ is not a root of the characteristic equation.} \\ m, & \text{if } \alpha \text{ is a root of the characteristic equation with multiplicity } m \end{cases} \tag{344}$$

2. $Y(t) = t^s e^{\alpha t}[Q_m(t) \cos(\beta t) + R_m(t) \sin(\beta t)]$, where

   $$Q_m = A_m t^m + \cdots + A_1 t + A_0, R_m = B_m t^m + \cdots + B_1 t + B_0 \tag{345}$$

   are polynomials of degree m with undetermined coefficients $A_m, \ldots, A_0, B_m, \ldots, B_0$, and

   $$s = \begin{cases} 0, & \text{if } \alpha + i\beta \text{ is not a root of the characteristic equation.} \\ m, & \text{if } \alpha + i\beta \text{ is a root of the characteristic equation with multiplicity } m. \end{cases} \tag{346}$$

Example 51. *Solve*

$$y''' - 3y'' + 3y' - y = 4e^t \tag{347}$$

For the homogeneous equation, the associated characteristic equation is

$$r^3 - 3r^2 + 3r - 1 = (r-1)^3 = 0, \tag{348}$$

so $r_1 = r_2 = r_3 = 1$, i.e., a repeated eigenvalue of multiplicity three. So set

$$y_1 = e^t, \quad y_2 = te^t, \quad y_3 = t^2e^t, \tag{349}$$

and the complementary solution (to the homogeneous equation) is

$$y_c(t) = c_1e^t + c_2te^t + c_3t^2e^t. \tag{350}$$

Since $g(t) = 4e^t$ and $\alpha = 1$ is a root of the characteristic equation with multiplicity 3, consider $s = 3$ and a trial solution

$$Y(t) = At^3e^t. \tag{351}$$

Computing gives

$$Y''' - 3Y'' + 3Y' - Y = 6Ae^t = 4e^t \Rightarrow A = \frac{2}{3}, \tag{352}$$

so the general solution to the non-homogeneous ODE is

$$y(t) = c_1e^t + c_2te^t + c_3t^2e^t + \frac{2}{3}t^3e^t. \tag{353}$$

Example 52. *Solve*

$$y^{(4)} + 2y'' + y = 3\sin t \tag{354}$$

The characteristic equation corresponding to the homogeneous equation is

$$r^4 + 2r^2 + 1 = (r^2+1)(r^2+1) = 0 \tag{355}$$

so $r_1 = r_3 = i, r_2 = r_4 = -i$, i.e., a repeated pair of complex conjugate roots (multiplicity $= 2$). Thus we have

$$y_1 = \cos t, \quad y_2 = \sin t, \quad y_3 = t\cos t, \quad y_4 = t\sin t, \tag{356}$$

and the complementary solution to the homogeneous equation is

$$y_c(t) = c_1\cos t + c_2\sin t + c_3t\cos t + c_4t\sin t. \tag{357}$$

The non-homogeneous term $g(t) = 3\sin t$, we have $\alpha = 0, \beta = 1, \alpha + i\beta = i$ is the root with multiplicity 2. Thus, $s = 2$. Consider a trial solution

$$Y(t) = At^2 \sin t + Bt^2 \cos t. \tag{358}$$

Then,

$$Y^{(4)} + 2Y'' + Y = -8A\sin t - 8B\cos t = 3\sin t \Rightarrow B = 0, \quad A = -\frac{3}{8}. \tag{359}$$

Hence, the general solution to the non-homogeneous equation is

$$y(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t - \frac{3}{8}t^2 \sin t. \tag{360}$$

4.3.2 Variation of Parameters

Similar to second-order equations, there is also a method to treat rather general high order equations

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = g(t), \quad t \in I. \tag{361}$$

Suppose we have a FSS $y_1, \ldots, y_n$ to the homogeneous equation. Then, the complementary solution is

$$y_c(t) = c_1 y_1(t) + \cdots + c_n y_n(t). \tag{362}$$

Now, we consider a trial solution for the non-homogeneous equation of the form

$$Y(t) = u_1(t)y_1(t) + \cdots + u_n(t)y_n(t) \tag{363}$$

for unknown functions $u_1, \ldots, u_n$. Differentiating gives

$$Y'(t) = u_1(t)y_1'(t) + \cdots + u_n(t)y_n'(t) + u_1'(t)y_1(t) + \cdots + u_n'(t)y_n(t). \tag{364}$$

As before we set the constraint

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) + \cdots + u_n'(t)y_n(t) = 0, \tag{365}$$

so that Y' simplifies to

$$Y'(t) = u_1(t)y_1'(t) + u_2(t)y_2'(t) + \cdots + u_n(t)y_n'(t). \tag{366}$$

Computing Y'' and setting

$$u_1'(t)y_1'(t) + \cdots + u_n'(t)y_n'(t) = 0 \tag{367}$$

leads to the simplified expression for the second derivative

$$Y''(t) = u_1(t)y_1''(t) + \cdots + u_n(t)y_n''(t). \tag{368}$$

Repeat this procedure (differentiating and then setting the sum of terms involving $u_1', \ldots, u_n'$ to zero), we have (after simplification)

$$Y^{(n-1)}(t) = u_1(t)y_1^{(n-1)}(t) + \cdots + u_n(t)y_n^{(n-1)}(t) \tag{369}$$

Thus, the final expression for $Y^{(n)}(t)$ is

$$Y^{(n)}(t) = u_1(t)y_1^{(n)}(t) + \cdots + u_n(t)y_n^{(n)}(t) + u_1'(t)y_1^{(n-1)}(t) + \cdots + u_n'(t)y_n^{(n-1)}(t) \tag{370}$$

In summary, we obtain $n-1$ equations

$$u_1'(t)y_1^{(m)}(t)+\cdots+u_n'(t)y_n^{(m)}(t)=0 \quad \forall\, 0\le m\le n-2, \tag{371}$$

as well as a simplified expression for $Y^{(m)}$:

$$Y^{(m)}(t)=u_1(t)y_1^{(m)}(t)+\cdots+u_n(t)y_n^{(m)}(t), \quad m=0,\dots,n-1, \tag{372}$$

$$Y^{(n)}(t)=u_1(t)y_1^{(n)}(t)+\cdots+u_n(t)y_n^{(n)}(t)+u_1'(t)y_1^{(n-1)}(t)+\cdots+u_n'(t)y_n^{(n-1)}(t). \tag{373}$$

Now, substitute $Y, Y', \dots, Y^{(n-1)}, Y^{(n)}$ into the LHS of the non-homogeneous ODE (361):

$$\text{LHS}=Y^{(n)}+p_{n-1}Y^{(n-1)}+\cdots+p_1Y'+p_0Y \tag{374}$$

Substituting:

$$\begin{aligned}\text{LHS}=&\overbrace{[u_1y_1^{(n)}+\cdots+u_ny_n^{(n)}]+[u_1'y_1^{(n-1)}+\cdots+u_n'y_n^{(n-1)}]}^{\text{from } Y^{(n)}}\\&+p_{n-1}[u_1y_1^{(n-1)}+\cdots+u_ny_n^{(n-1)}]+\cdots+p_1[u_1y_1'+\cdots+u_ny_n']\\&+p_0[u_1y_1+\cdots+u_ny_n]\end{aligned} \tag{375}$$

Regroup all terms by $u_1, u_2, \dots, u_n$:

- Collect all terms of u_1: $u_1y_1^{(n)}+p_{n-1}u_1y_1^{(n-1)}+\cdots+p_1u_1y_1'+p_0u_1y_1=$ $u_1\left[y_1^{(n)}+p_{n-1}y_1^{(n-1)}+\cdots+p_1y_1'+p_0y_1\right]$
- Collect all terms of u_2: $u_2\left[y_2^{(n)}+p_{n-1}y_2^{(n-1)}+\cdots+p_1y_2'+p_0y_2\right]$
- ...
- Collect all terms of u_n: $u_n\left[y_n^{(n)}+p_{n-1}y_n^{(n-1)}+\cdots+p_1y_n'+p_0y_n\right]$
- The remaining terms (only u_i'): $u_1'y_1^{(n-1)}+\cdots+u_n'y_n^{(n-1)}$

Since $y_1,\dots,y_n$ are all solutions to the homogeneous equation, all expressions in the square brackets $[\cdots]$ in the previous step are equal to 0.

$$\text{LHS}=u_1\cdot(0)+u_2\cdot(0)+\cdots+u_n\cdot(0)+u_1'y_1^{(n-1)}+\cdots+u_n'y_n^{(n-1)} \tag{376}$$

Thus, we obtain the last equation:

$$u_1'y_1^{(n-1)}+\cdots+u_n'y_n^{(n-1)}=g(t) \tag{377}$$

Collecting all expressions involving $u_1',\dots,u_n'$, we obtain

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_{n-1} & y_n\\ y_1' & y_2' & \cdots & y_{n-1}' & y_n'\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_{n-1}^{(n-2)} & y_n^{(n-2)}\\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_{n-1}^{(n-1)} & y_n^{(n-1)}\end{pmatrix}\begin{pmatrix}u_1'\\u_2'\\\vdots\\u_{n-1}'\\u_n'\end{pmatrix}=\begin{pmatrix}0\\0\\\vdots\\0\\g(t)\end{pmatrix}. \tag{378}$$

Thus, the derivatives of the unknown functions $u_1, \ldots, u_n$ can be found by inverting the matrix of derivatives, whose determinant is the non-zero Wronskian, since $(y_1, \ldots, y_n)$ forms a FSS. Denote the matrix as $M(t)$. Use Cramer's rule, by setting

$$M_i(t) = \begin{pmatrix} y_1 & \ldots & 0 & \ldots & y_n \\ y_1' & \ldots & 0 & \ldots & y_n' \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & \ldots & 0 & \ldots & y_n^{(n-2)} \\ y_1^{(n-1)} & \ldots & 1 & \ldots & y_n^{(n-1)} \end{pmatrix}, \tag{379}$$

i.e., replace the ith column of $M(t)$ with the vector $(0, \ldots, 0, 1)^T$. Then Cramer's rule gives

$$u_i'(t) = \frac{g(t) \det M_i(t)}{\det M(t)}, \tag{380}$$

and by integrating we get an expression for $u_i(t)$. The particular solution to the non-homogeneous equation is therefore

$$Y(t) = y_1(t) \int \frac{g(t) \det M_1(t)}{\det M(t)} dt + \cdots + y_n(t) \int \frac{g(t) \det M_n(t)}{\det M(t)} dt. \tag{381}$$

However, in general the evaluation of the integrals can be difficult, but we can always use Abel's identity to simplify, since

$$\det M(t) = W(y_1, \ldots, y_n)[t] = ce^{-\int p_{n-1}(t)dt}. \tag{382}$$

Example 53. *Solve*

$$y''' + y' = \sec^2(t) \textit{ for } t \in (-\pi/2, \pi/2). \tag{383}$$

The characteristic equation for the homogeneous problem is $r^3 + r = 0$ and so $r_1 = 0$, $r_2 = i$ and $r_3 = -i$. Hence the complementary solution is

$$y_c(t) = c_1 + c_2 \cos t + c_3 \sin t. \tag{384}$$

By variation of parameters we look for a particular solution of the form

$$Y(t) = u_1 y_1 + u_2 y_2 + u_3 y_3 = u_1(t) + u_2(t) \cos t + u_3(t) \sin t, \tag{385}$$

with

$$M(t) \begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sec^2(t) \end{pmatrix}, \quad M(t) = \begin{pmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{pmatrix}. \tag{386}$$

Define

$$M_1(t) = \begin{pmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 1 & -\cos t & -\sin t \end{pmatrix}, \quad M_2(t) = \begin{pmatrix} 1 & 0 & \sin t \\ 0 & 0 & \cos t \\ 0 & 1 & -\sin t \end{pmatrix}, \quad M_3(t) = \begin{pmatrix} 1 & \cos t & 0 \\ 0 & -\sin t & 0 \\ 0 & -\cos t & 1 \end{pmatrix} \tag{387}$$

We can compute

$$\det M(t) = 1, \quad \det M_1(t) = 1, \quad \det M_2(t) = -\cos t, \quad \det M_3(t) = -\sin t, \tag{388}$$

so

$$u_1 = \int \sec^2(t)dt = \tan(t), \tag{389}$$

$$u_2 = \int -\sec^2(t)\cos(t)dt = -\ln(|\sec(t) + \tan(t)|), \tag{390}$$

$$u_3 = \int -\sec^2(t)\sin(t)dt = -\sec(t). \tag{391}$$

Hence, the particular solution is

$$Y(t) = \tan(t) - \cos(t)\ln(|\sec(t) + \tan(t)|) - \sin(t)\sec(t) \tag{392}$$

$$= -\cos(t)\ln(|\sec(t) + \tan(t)|). \tag{393}$$

5 System of First-Order Linear Equations

Please check Appendix A for linear algebra notations before starting this section.

5.1 Basic Settings

The general system of first-order equations involving 1 independent variable t and n dependent variables $y_1, \ldots, y_n$ is

$$\begin{cases} y_1'(t) = F_1\big(t, y_1, \ldots, y_n\big), \\ y_2'(t) = F_2\big(t, y_1, \ldots, y_n\big), \\ \qquad \vdots \\ y_n'(t) = F_n\big(t, y_1, \ldots, y_n\big), \end{cases} \tag{394}$$

with initial conditions

$$y_1(t_0) = x_1, \ldots, y_n(t_0) = x_n, \tag{395}$$

where $t_0 \in I$, $x_1, \ldots, x_n \in \mathbb{R}$ are given, and all F_i are real functions.

To ensure that the above IVP has exactly one solution, we state the following existence and uniqueness theorem.

Theorem 54 (Existence and Uniqueness)**.** *Suppose the functions $F_1, \ldots, F_n$ and the partial derivatives $\frac{\partial F_1}{\partial y_1}, \frac{\partial F_1}{\partial y_2}, \ldots, \frac{\partial F_1}{\partial y_n}, \frac{\partial F_2}{\partial y_1}, \ldots, \frac{\partial F_2}{\partial y_n}, \cdots, \frac{\partial F_n}{\partial y_n}$ are all continuous in a region R defined as*

$$R := [t_0 - a, t_0 + a] \times [x_1 - b, x_1 + b] \times \cdots \times [x_n - b, x_n + b] \subset \mathbb{R}^{n+1} \tag{396}$$

where a, b are two positive constants. Then, there is exactly one solution $\mathbf{y}(t)$ $(t \in (t_0 - h, t_0 + h))$ to the above IVP, where $h = \min(a, b/M)$ and $M = \max_{(t,x_1,\cdots,x_n)\in R}\{|F_1(t, x_1, \cdots, x_n)|, \cdots, |F_n(t, x_1, \cdots, x_n)|\}$.

Next, we will formally define linear system.

Definition 55 (Linear System)**.** *If each F_i, $1 \le i \le n$, is linear with respect to $y_1, \ldots, y_n$, then we call the system of ODEs a linear system. Otherwise it is a nonlinear system. The general linear system of first-order ODEs is*

$$\begin{cases} y_1'(t) = p_{11}(t)y_1(t) + p_{12}(t)y_2(t) + \cdots + p_{1n}(t)y_n(t) + g_1(t), \\ y_2'(t) = p_{21}(t)y_1(t) + p_{22}(t)y_2(t) + \cdots + p_{2n}(t)y_n(t) + g_2(t), \\ \qquad \vdots \\ y_n'(t) = p_{n1}(t)y_1(t) + p_{n2}(t)y_2(t) + \cdots + p_{nn}(t)y_n(t) + g_n(t), \end{cases} \tag{397}$$

where $p_{11}(t), \ldots, p_{nn}(t), g_1(t), \ldots, g_n(t)$ are given real functions.

It is more convenient to introduce the matrix form. Denoting vectors

$$\mathbf{y}(t) = (y_1(t), \ldots, y_n(t))^T, \quad \mathbf{g}(t) = (g_1(t), \ldots, g_n(t))^T, \tag{398}$$

and the matrix

$$\mathbf{P}(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) & \dots & p_{1n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1}(t) & p_{n2}(t) & \dots & p_{nn}(t) \end{pmatrix} \tag{399}$$

the general system can be written as

$$\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}(t) + \mathbf{g}(t). \tag{400}$$

Definition 56 (Homogeneous)**.** *A first-order linear system of equations*

$$\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}(t) + \mathbf{g}(t) \tag{401}$$

is called homogeneous if $\mathbf{g}(t) = 0$*, i.e.,* $g_i(t) = 0$ *for* $1 \leq i \leq n$*. Otherwise it is called non-homogeneous.*

The above linear first-order ODE system subject to the initial conditions

$$y_1(t_0) = x_1, \dots, y_n(t_0) = x_n, \tag{402}$$

is the initial value problem (IVP).

Theorem 57 (Existence and Uniqueness)**.** *Let* $I = (\alpha, \beta) \subset \mathbb{R}$ *be an open interval such that functions* $p_{11}(t), \dots, p_{nn}(t), g_1(t), \dots, g_n(t)$ *are continuous in I. For* $t_0 \in I$ *and* $x_1, \dots, x_n \in \mathbb{R}$*, there is exactly one solution* $\mathbf{y} = (y_1(t), \dots, y_n(t))^T$ *to the IVP.*

5.2 General Theory of Homogeneous System

The general first-order linear system is

$$\mathbf{y}' = \mathbf{P}(t)\mathbf{y}(t) + \mathbf{g}(t), \tag{403}$$

for given $\mathbf{g}(t) = (g_1(t), \dots, g_n(t))^T$ and $\mathbf{P}(t)$ is a square matrix of functions

$$\mathbf{P}(t) = \begin{pmatrix} p_{11}(t) & \dots & p_{1n}(t) \\ \vdots & \ddots & \vdots \\ p_{n1}(t) & \dots & p_{nn}(t) \end{pmatrix}. \tag{404}$$

In the following, we assume that all $p_{ij}(t)$ and $g_i(t)$ are continuous in some interval I.

We first look at the corresponding **homogeneous** first-order linear system

$$\mathbf{y}' = \mathbf{P}(t)\mathbf{y}(t). \tag{405}$$

For a system of n first-order equations, we expect n **linearly independent solutions** (we will define linear independence of vector functions soon). We will use the following notation:

$$\mathbf{y}_j(t) = j - \text{th solution}, \quad y_{ij}(t) = i - \text{th component of the } j - \text{th solution} \tag{406}$$

which means

$$\mathbf{y}_j(t) = \begin{pmatrix} y_{1j}(t) \\ y_{2j}(t) \\ \vdots \\ y_{nj}(t) \end{pmatrix}. \tag{407}$$

Theorem 58 (Principle of Superposition)**.** *Let $\mathbf{y}_1, \dots, \mathbf{y}_n$ be n solutions to the homogeneous system*

$$\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}(t) \tag{408}$$

then any linear combination

$$\phi(t) = c_1\mathbf{y}_1(t) + \dots + c_n\mathbf{y}_n(t) \tag{409}$$

is also a solution for any $c_1, \dots, c_n \in \mathbb{R}$.

The natural question is: Can **any** solution to the homogeneous system (408) be written as a linear combination of n solutions $\mathbf{y}_1, \dots, \mathbf{y}_n$? The answer is **yes**, with some analogue of Wronskian for system of equations.

Definition 59 (Wronskian)**.** *Let $\mathbf{y}_1(t), \dots, \mathbf{y}_n(t)$ be n solutions to the homogeneous system (408). We define the matrix*

$$\mathbf{X}(t) := \begin{pmatrix} y_{11}(t) & y_{12}(t) & \dots & y_{1n}(t) \\ y_{21}(t) & y_{22}(t) & \dots & y_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1}(t) & y_{n2}(t) & \dots & y_{nn}(t) \end{pmatrix} = \begin{pmatrix} | & | & \dots & | \\ \mathbf{y}_1(t) & \mathbf{y}_2(t) & \dots & \mathbf{y}_n(t) \\ | & | & \dots & | \end{pmatrix}, \tag{410}$$

where the i-th column of $\mathbf{X}$ is the vector $\mathbf{y}_i(t)$. Then we set the Wronskian $W(\mathbf{y}_1, \dots, \mathbf{y}_n)[t]$ to be

$$W(\mathbf{y}_1, \dots, \mathbf{y}_n)[t] := \det \mathbf{X}(t). \tag{411}$$

Note that this definition of Wronskian **does not involve derivatives**, which is different from previous definitions.

Since we have an analogue of the Wronskian for system of equations, we should expect an analogue of Abel's Identity as well. For systems of equations, this is called **Liouville's formula**.

Theorem 60 (Liouville's Formula / Abel–Jacobi–Liouville Identity). *Let $\mathbf{y}_1, \ldots, \mathbf{y}_n$ be n solutions to the homogeneous system (408) in the open interval I. Then, the Wronskian is given by*

$$W(\mathbf{y}_1, \ldots, \mathbf{y}_n)[t] = c \exp\left(\int \operatorname{tr}(\mathbf{P}(t))dt\right), \tag{412}$$

where the trace of a matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is defined as

$$\operatorname{tr}(\mathbf{A}) := \sum_{i=1}^{n} a_{ii}, \tag{413}$$

and c is a constant not depending on $t \in I$. Consequently, the Wronskian is ***either always zero for*** *$t \in I$* ***or never zero for*** *$t \in I$.*

Proof. We will prove this for the case $n = 2$. Let $\mathbf{y}_1, \mathbf{y}_2$ be two solutions to the homogeneous system, i.e.,

$$\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}(t), \quad \mathbf{P}(t) \in \mathbb{R}^{2\times 2} \text{ for } t \in I. \tag{414}$$

Then, the Wronskian is

$$W(\mathbf{y}_1, \mathbf{y}_2)[t] = \begin{vmatrix} y_{11}(t) & y_{12}(t) \\ y_{21}(t) & y_{22}(t) \end{vmatrix} = y_{11}(t)y_{22}(t) - y_{12}(t)y_{21}(t). \tag{415}$$

Taking the derivative leads to

$$\begin{aligned}
\frac{d}{dt}W[t] &= y_{11}'(t)y_{22}(t) - y_{12}'(t)y_{21}(t) + y_{11}(t)y_{22}'(t) - y_{12}(t)y_{21}'(t) \qquad (416)\\
&= \begin{vmatrix} y_{11}'(t) & y_{12}'(t) \\ y_{21}(t) & y_{22}(t) \end{vmatrix} + \begin{vmatrix} y_{11}(t) & y_{12}(t) \\ y_{21}'(t) & y_{22}'(t) \end{vmatrix} \qquad (417)\\
&= \begin{vmatrix} p_{11}y_{11} + p_{12}y_{21} & p_{11}y_{12} + p_{12}y_{22} \\ y_{21}(t) & y_{22}(t) \end{vmatrix} + \begin{vmatrix} y_{11}(t) & y_{12}(t) \\ p_{21}y_{11} + p_{22}y_{21} & p_{21}y_{12} + p_{22}y_{22} \end{vmatrix} \qquad (418)\\
&= (p_{11} + p_{22})(y_{11}y_{22} - y_{12}y_{21}) = (p_{11} + p_{22})W[t], \qquad (419)
\end{aligned}$$

where we have used from the fact that $\mathbf{y}_1, \mathbf{y}_2$ solve $\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}(t)$ to deduce

$$\begin{pmatrix} y_{11}' \\ y_{21}' \end{pmatrix} = \begin{pmatrix} p_{11}y_{11} + p_{12}y_{21} \\ p_{21}y_{11} + p_{22}y_{21} \end{pmatrix}, \quad \begin{pmatrix} y_{12}' \\ y_{22}' \end{pmatrix} = \begin{pmatrix} p_{11}y_{12} + p_{12}y_{22} \\ p_{21}y_{12} + p_{22}y_{22} \end{pmatrix}. \tag{420}$$

This implies we have

$$\frac{d}{dt}W[t] = (p_{11} + p_{22})W[t] = \operatorname{tr}(\mathbf{P}(t))W[t]. \tag{421}$$

□

Definition 61 (Linear Dependence of Vector Functions)**.** *The vector functions $\mathbf{f}_1(t), \dots, \mathbf{f}_n(t)$ are said to be linearly dependent in the interval I if there exists a set of constants $c_1, \dots, c_n$, not all zero, such that*

$$c_1\mathbf{f}_1(t) + \dots + c_n\mathbf{f}_n(t) = 0, \quad \forall t \in I. \tag{422}$$

If they are not linearly dependent in the interval I, then they are called linearly independent in the interval I. This is equivalent to that the vector functions $\mathbf{f}_1(t), \dots, \mathbf{f}_n(t)$ are linearly independent in the interval I if there ***exists*** *a point $t_0 \in I$ such that $\mathbf{f}_1(t_0), \dots, \mathbf{f}_n(t_0)$ are linearly independent.*

For any point $t \in I$

$$c_1\mathbf{y}_1(t) + \dots + c_n\mathbf{y}_n(t) = \mathbf{0} \Leftrightarrow [\mathbf{y}_1(t), \dots, \mathbf{y}_n(t)]\mathbf{c} = \mathbf{0}, \tag{423}$$

where $\mathbf{c} = (c_1, \dots, c_n)^T$. Since for square matrix A, $Ax = 0$ has only zero solution $\Leftrightarrow \det(A) \neq 0$, it is easy to see that

Theorem 62.

$$W(\mathbf{y}_1, \dots, \mathbf{y}_n)[t] \neq 0, \forall t \in I \Leftrightarrow \det(\mathbf{X}(t)) \neq 0, \forall t \in I \Leftrightarrow \{\mathbf{y}_1, \dots, \mathbf{y}_n\} \text{ are L.I. in } I. \tag{424}$$

Now we can answer the question whether **any** solution to the homogeneous system (408) can be written as a linear combination of $\mathbf{y}_1, \dots, \mathbf{y}_n$.

Theorem 63. *Let $\mathbf{y}_1(t), \dots, \mathbf{y}_n(t)$ be n solutions to the homogeneous system (408) defined on an open interval I. Then, $\mathbf{y}_1(t), \dots, \mathbf{y}_n(t)$ are linearly independent at each point in I* ***if and only if*** *the Wronskian $W(\mathbf{y}_1, \dots, \mathbf{y}_n)[t]$ is non-zero for $t \in I$. In this case, we say that $\{\mathbf{y}_1(t), \dots, \mathbf{y}_n(t)\}$ forms a fundamental set of solutions (FSS), and any solution $\phi(t)$ to the homogeneous system (408) can be expressed as a linear combination:*

$$\phi(t) = c_1\mathbf{y}_1(t) + \dots + c_n\mathbf{y}_n(t), \tag{425}$$

for constants $c_1, \dots, c_n \in \mathbb{R}$ in ***exactly one way****. That is, the constants $c_1, \dots, c_n$ are* ***uniquely determined****.*

Proof. The aim is to show if $\mathbf{y}_1, \dots, \mathbf{y}_n$ are linearly independent at each point in I (or equivalently $W(\mathbf{y}_1, \dots, \mathbf{y}_n) \neq 0$), then any solution can be written as a linear combination of $\mathbf{y}_1, \dots, \mathbf{y}_n$. Let $\boldsymbol{\phi}$ be any solution to be homogeneous system (408) for $t \in I$. Let $t_0 \in I$ and denote the vector

$$\boldsymbol{\xi} := \boldsymbol{\phi}(t_0) = (\xi_1, \dots, \xi_n)^T. \tag{426}$$

Then, we find values $c_1, \dots, c_n \in \mathbb{R}$ that satisfies

$$c_1\mathbf{y}_1(t_0) + \dots + c_n\mathbf{y}_n(t_0) = \boldsymbol{\xi}, \tag{427}$$

or equivalently

$$\begin{pmatrix} y_{11}(t_0) & \dots & y_{1n}(t_0) \\ \vdots & \ddots & \vdots \\ y_{n1}(t_0) & \dots & y_{nn}(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}. \tag{428}$$

As the Wronskian is not zero at t_0, the matrix is invertible and hence there is a unique solution $(c_1^*, \ldots, c_n^*)^T$ to the above problem. Now we define a new function $\boldsymbol{\eta}$ by

$$\boldsymbol{\eta}(t) = c_1^* \mathbf{y}_1(t) + \cdots + c_n^* \mathbf{y}_n(t), \quad \forall t \in I. \tag{429}$$

It is clear that $\boldsymbol{\eta}(t_0) = \boldsymbol{\xi} = \boldsymbol{\phi}(t_0)$. Hence, both $\boldsymbol{\eta}$ and $\boldsymbol{\phi}$ are solutions to the IVP

$$\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}(t), \quad \mathbf{y}(t_0) = \boldsymbol{\xi}. \tag{430}$$

By uniqueness we must have $\boldsymbol{\eta} = \boldsymbol{\phi}$ and thus

$$\boldsymbol{\phi}(t) = c_1^* \mathbf{y}_1(t) + \cdots + c_n^* \mathbf{y}_n(t), \quad \forall t \in I. \tag{431}$$

□

Definition 64 (Fundamental Matrix)**.** *Suppose that $\mathbf{y}_1(t), \ldots, \mathbf{y}_n(t)$ form a FSS for the homogeneous linear system (408). Then the matrix*

$$\boldsymbol{\Psi}(t) = \begin{pmatrix} y_{11}(t) & y_{12}(t) & \ldots & y_{1n}(t) \\ y_{21}(t) & y_{22}(t) & \ldots & y_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1}(t) & y_{n2}(t) & \ldots & y_{nn}(t) \end{pmatrix} = \begin{pmatrix} | & | & \ldots & | \\ \mathbf{y}_1(t) & \mathbf{y}_2(t) & \ldots & \mathbf{y}_n(t) \\ | & | & \ldots & | \end{pmatrix} \tag{432}$$

whose columns are the vectors $\mathbf{y}_1(t), \ldots, \mathbf{y}_n(t)$ is called a fundamental matrix of the system (408).

Next, we will introduce the existence of fundamental set of solutions.

Theorem 65. *Let*

$$\mathbf{e}_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{433}$$

where the entry 1 appears in the $i-$th row, and let $\mathbf{y}_i$ be the unique solution to the IVP

$$\begin{aligned} \mathbf{y}'(t) &= \mathbf{P}(t)\mathbf{y}(t) \quad \textit{for } t \in I, \\ \mathbf{y}(t_0) &= \mathbf{e}_i, \end{aligned} \tag{434}$$

for $t_0 \in I$. Then, the functions $\mathbf{y}_1(t), \ldots, \mathbf{y}_n(t)$ form a FSS to the homogeneous system $\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}(t)$.

Proof. Simply compute the Wronskian at t_0:

$$W(\mathbf{y}_1, \ldots, \mathbf{y}_n)[t_0] = \det \mathbf{I} = 1 \neq 0. \tag{435}$$

□

Note that the FSS is not unique. Let $\mathbf{y}_i$ $(i = 1, \ldots, n)$ be the unique solution to the IVP

$$\begin{aligned} \mathbf{y}'(t) &= \mathbf{P}(t)\mathbf{y}(t) \quad \text{for } t \in I, \\ \mathbf{y}(t_0) &= \mathbf{s}_i, \end{aligned} \tag{436}$$

for $t_0 \in I$. Then, the functions $\mathbf{y}_1(t), \ldots, \mathbf{y}_n(t)$ form a FSS as long as $W(\mathbf{y}_1, \ldots, \mathbf{y}_n)[t_0] = \det[\mathbf{s}_1|\mathbf{s}_2|\ldots|\mathbf{s}_n] \neq 0$.

Just as for second-order equations, a linear ODE system with real-valued coefficients may give rise to complex-valued solutions. But again we have the following theorem.

Theorem 66. *If*

$$\mathbf{y}(t) = \mathbf{u}(t) + i\mathbf{v}(t) \tag{437}$$

is a complex-valued solution to the homogeneous system (408)*, and the entries of* $\mathbf{P}(t), \mathbf{u}(t)$ *and* $\mathbf{v}(t)$ *are real-valued functions, then* $\mathbf{u}(t)$ *and* $\mathbf{v}(t)$ *are both solutions to the homogeneous system* (408).

Summary: The FSS $\mathbf{y}_1, \ldots, \mathbf{y}_n$ to $\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}(t)$ always exists and any solution ϕ to the homogeneous system can be written **uniquely** as a linear combination of $\mathbf{y}_1, \ldots, \mathbf{y}_n$.

Next question: How to find the FSS? Indeed, again, no method can be used for general matrix $\mathbf{P}(t)$. We can only deal with the case when $\mathbf{P}(t)$ is a matrix with constant entries.

5.3 Homogeneous System with Constant Coefficients: 2×2 Matrices

In the following, we focus on $\mathbf{P}(t) = \mathbf{A}$, where $\mathbf{A}$ is a square matrix with real, constant coefficients, and our goal is to derive explicit formula for the FSS $(\mathbf{y}_1, \ldots, \mathbf{y}_n)$. In other words, we focus on systems of the form

$$\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t), \quad t \in I, \tag{438}$$

where $\mathbf{A} \in \mathbb{R}^{n \times n}$.

There is one special case which we can already deal with. When $n = 1$, $\mathbf{A}$ is a scalar, i.e., $\mathbf{A} = a \in \mathbb{R}$, then (438) becomes

$$y'(t) = ay(t) \Rightarrow y(t) = ce^{at}, c \in \mathbb{R}. \tag{439}$$

What about a general matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$? The idea is to try

$$\mathbf{y}(t) = \boldsymbol{\xi} e^{rt}, \tag{440}$$

where $\boldsymbol{\xi}$ is a **constant vector** and $r \in \mathbb{C}$. We have to determine r and $\boldsymbol{\xi}$ to obtain a solution. Substituting this function into the equation yields

$$\mathbf{0} = \mathbf{y}'(t) - \mathbf{A}\mathbf{y}(t) = e^{rt}(r\boldsymbol{\xi} - \mathbf{A}\boldsymbol{\xi}) = e^{rt}(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi}. \tag{441}$$

Since the exponential term is never zero, for $\boldsymbol{\xi} e^{rt}$ to be a solution to the homogeneous system, we require

$$(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}, \tag{442}$$

i.e., the constant r should be an **eigenvalue** of the matrix $\mathbf{A}$ with corresponding **eigenvector** $\boldsymbol{\xi}$.

Let's first discuss the simple cases $\mathbf{A} \in \mathbb{R}^{2 \times 2}$. What are the possibilities for the two eigenvalues r_1 and r_2?

(1) $r_1, r_2 \in \mathbb{R}, r_1 \neq r_2$ - real and distinct;

(2) $r_1, r_2 \in \mathbb{C}, r_1 = \delta + i\mu, \delta, \mu \in \mathbb{R}$ with $r_2 = \delta - i\mu$ - complex conjugate pair;

(3) $r_1 = r_2 \in \mathbb{R}$ - repeated and real.

- (3a) $r_1 = r_2 \in \mathbb{R}$, there are two linearly independent eigenvectors.
- (3b) $r_1 = r_2 \in \mathbb{R}$, there is only one eigenvector.

Case 1. Real Distinct Eigenvalues. Let $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ be the eigenvectors corresponding to r_1 and r_2. Note that $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ are linearly independent, since **eigenvectors belonging to distinct eigenvalues are linearly independent**. Then, compute the Wronskian for the functions $\mathbf{y}_1(t) = \boldsymbol{\xi}_1 e^{r_1 t}$ and $\mathbf{y}_2(t) = \boldsymbol{\xi}_2 e^{r_2 t}$ ($\boldsymbol{\xi}_1 = [\xi_{11}, \xi_{21}]^T, \boldsymbol{\xi}_2 = [\xi_{12}, \xi_{22}]^T$),

$$W(\mathbf{y}_1, \mathbf{y}_2)[t] = \begin{vmatrix} \xi_{11}e^{r_1 t} & \xi_{12}e^{r_2 t} \\ \xi_{21}e^{r_1 t} & \xi_{22}e^{r_2 t} \end{vmatrix} \tag{443}$$

$$= e^{(r_1+r_2)t} \begin{vmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{vmatrix} \neq 0. \tag{444}$$

for any $t \in I$. Hence, by Theorem 63, the general solution to the homogeneous system (438) is

$$\mathbf{y}(t) = c_1 e^{r_1 t}\boldsymbol{\xi}_1 + c_2 e^{r_2 t}\boldsymbol{\xi}_2. \tag{445}$$

Example 67. *For*

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \quad \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \tag{446}$$

with eigenvalues and corresponding eigenvectors

$$r_1 = 3, \quad \boldsymbol{\xi}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad r_2 = -1, \quad \boldsymbol{\xi}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \tag{447}$$

the general solution is

$$\mathbf{y}(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}. \tag{448}$$

Case 2. Complex Conjugate Eigenvalues. Let $r_1 = \delta + i\mu$, with $\delta, \mu \in \mathbb{R}$ and corresponding eigenvector $\boldsymbol{\xi}_1 = \mathbf{u} + i\mathbf{v}$. Then $r_2 = \delta - i\mu$, with $\delta, \mu \in \mathbb{R}$ and corresponding eigenvector $\boldsymbol{\xi}_2 = \mathbf{u} - i\mathbf{v}$ ($\mathbf{u} + i\mathbf{v}$ and $\mathbf{u} - i\mathbf{v}$ are linearly independent). Also, $\mathbf{x}_1(t) = (\mathbf{u} + i\mathbf{v})e^{(\delta+i\mu)t}$, $\mathbf{x}_2(t) = (\mathbf{u} - i\mathbf{v})e^{(\delta-i\mu)t}$ are linearly independent solutions (proof is same as (443)). The general solution of the homogeneous system (438) is

$$\mathbf{y}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t). \tag{449}$$

But the disadvantage of using $\mathbf{x}_1$ and $\mathbf{x}_2$ is that they are complex-valued. Rewrite the above solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ as

$$\mathbf{x}_1 = (\mathbf{u} + i\mathbf{v})e^{\delta t}(\cos(\mu t) + i\sin(\mu t)) \tag{450}$$
$$= e^{\delta t}[\mathbf{u}\cos(\mu t) - \mathbf{v}\sin(\mu t)] + ie^{\delta t}[\mathbf{u}\sin(\mu t) + \mathbf{v}\cos(\mu t)]. \tag{451}$$

$$\mathbf{x}_2 = (\mathbf{u} - i\mathbf{v})e^{\delta t}(\cos(\mu t) - i\sin(\mu t)) \tag{452}$$
$$= e^{\delta t}[\mathbf{u}\cos(\mu t) - \mathbf{v}\sin(\mu t)] - ie^{\delta t}[\mathbf{u}\sin(\mu t) + \mathbf{v}\cos(\mu t)]. \tag{453}$$

Using Theorem 66 we have that the real and imaginary parts of $\mathbf{x}_1$ are also solutions. Hence, we define

$$\mathbf{y}_1(t) = e^{\delta t}[\mathbf{u}\cos(\mu t) - \mathbf{v}\sin(\mu t)], \quad \mathbf{y}_2(t) = e^{\delta t}[\mathbf{u}\sin(\mu t) + \mathbf{v}\cos(\mu t)]. \tag{454}$$

Since $\mathbf{u}+i\mathbf{v}$ and $\mathbf{u}-i\mathbf{v}$ are linearly independent, $0 \neq \det([\mathbf{u}+i\mathbf{v}, \mathbf{u}-i\mathbf{v}]) = \det([2\mathbf{u}, \mathbf{u}-i\mathbf{v}]) = 2\det([\mathbf{u}, \mathbf{u} - i\mathbf{v}]) = 2\det([\mathbf{u}, -i\mathbf{v}]) = -2i\det([\mathbf{u}, \mathbf{v}])$. Thus, $\det([\mathbf{u}, \mathbf{v}]) \neq 0$. We can check that the Wronskian for $\mathbf{y}_1$ and $\mathbf{y}_2$ is non-zero.

$$\det([\mathbf{y}_1, \mathbf{y}_2]) = e^{2\delta t}\det([\mathbf{u}\cos(\mu t) - \mathbf{v}\sin(\mu t), \mathbf{u}\sin(\mu t) + \mathbf{v}\cos(\mu t)]) \tag{455}$$
$$= e^{2\delta t}\det([\mathbf{u}, \mathbf{v}])\det\left(\begin{bmatrix} \cos(\mu t) & \sin(\mu t) \\ -\sin(\mu t) & \cos(\mu t) \end{bmatrix}\right) \tag{456}$$
$$= e^{2\delta t}\det([\mathbf{u}, \mathbf{v}]) \neq 0. \tag{457}$$

Then, by Theorem 63, the general solution to the homogeneous system (438) is

$$\mathbf{y}(t) = c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) \tag{458}$$
$$= e^{\delta t}[c_1(\cos(\mu t)\mathbf{u} - \sin(\mu t)\mathbf{v}) + c_2(\cos(\mu t)\mathbf{v} + \sin(\mu t)\mathbf{u})] \tag{459}$$

Example 68. *For*

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \quad \mathbf{A} = \begin{pmatrix} -3 & -2 \\ 4 & 1 \end{pmatrix} \tag{460}$$

with eigenvalues $r_1 = \bar{r}_2$ *and corresponding eigenvectors* $\xi_1 = \bar{\xi}_2$*:*

$$r_{1,2} = -1 \pm 2i, \quad \boldsymbol{\xi}_{1,2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \pm i \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{461}$$

the general complex solution is

$$\mathbf{y}(t) = c_1 e^{(-1+2i)t} \left[\begin{pmatrix} -1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] + c_2 e^{(-1-2i)t} \left[\begin{pmatrix} -1 \\ 1 \end{pmatrix} - i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]. \tag{462}$$

The general real solution is

$$\mathbf{y}(t) = c_1 e^{-t} \left[\cos(2t) \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \sin(2t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] + c_2 e^{-t} \left[\sin(2t) \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \cos(2t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \tag{463}$$

Case 3a. Repeated Real Eigenvalues, Geometric Multiplicity = Algebraic Multiplicity. If $r_1 = r_2$, then we have an eigenvalue with algebraic multiplicity of two. We need to divide our analysis into two subcases. The first subcase is the geometric multiplicity is also two, which implies there are two linearly independent eigenvectors $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$ corresponding to $r_1 = r_2 =: \lambda$. Then, going back to Case 1, the general solution to the homogeneous system (438) is

$$\mathbf{y}(t) = c_1 e^{\lambda t} \boldsymbol{\xi}_1 + c_2 e^{\lambda t} \boldsymbol{\xi}_2. \tag{464}$$

Example 69. *For*

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \quad \mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \tag{465}$$

with eigenvalues and corresponding eigenvectors

$$r_1 = 2, \quad \boldsymbol{\xi}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad r_2 = 2, \quad \boldsymbol{\xi}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{466}$$

the general solution is

$$\mathbf{y}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{467}$$

In all above cases for the 2×2 matrix $\mathbf{A}$, the number of linearly independent eigenvectors equals $n = 2$. Thus, $\mathbf{A}$ is diagonalizable, the solutions can be found easily. However, if there is a repeated eigenvalue $r_1 = r_2 =: \lambda$ with geometric multiplicity **strictly less** than its algebraic multiplicity, $\mathbf{A}$ is not diagonalizable. This case is more complicated.

Case 3b. Repeated Real Eigenvalues, Geometric Multiplicity $<$ Algebraic Multiplicity. If the geometric multiplicity of the eigenvalue λ is one, then there is only one eigenvector $\boldsymbol{\xi}$ corresponding to λ. One solution is

$$\mathbf{y}_1 = \boldsymbol{\xi} e^{\lambda t}, \tag{468}$$

what about a second solution that is linearly independent at each point t? As with second-order equations, let's first try

$$\mathbf{z}(t) = t\boldsymbol{\xi} e^{\lambda t}. \tag{469}$$

Differentiating and plugging this into the homogeneous system (438) leads to

$$\mathbf{z}'(t) - \mathbf{A}\mathbf{z}(t) = \boldsymbol{\xi}(\lambda t e^{\lambda t} + e^{\lambda t}) - \mathbf{A}\boldsymbol{\xi} t e^{\lambda t} = (\lambda\boldsymbol{\xi} - \mathbf{A}\boldsymbol{\xi}) t e^{\lambda t} + \boldsymbol{\xi} e^{\lambda t}. \tag{470}$$

There are two terms: one involving the coefficient $te^{\lambda t}$ and the other involving $e^{\lambda t}$. Since we want $\mathbf{z}$ to be a solution, both terms must vanish. Hence, we require

$$\mathbf{A}\boldsymbol{\xi} = \lambda\boldsymbol{\xi}, \quad \boldsymbol{\xi} = \mathbf{0}. \tag{471}$$

The first condition says $\boldsymbol{\xi}$ is an eigenvector for λ, which is true by definition, but the second condition leads to a contradiction. **Therefore, the solution to the homogeneous system (438) cannot be of the form $t\boldsymbol{\xi} e^{\lambda t}$.**

Alternatively, we try

$$\mathbf{w}(t) = (\boldsymbol{\xi} t + \boldsymbol{\eta}) e^{\lambda t}, \tag{472}$$

for some constant vector $\boldsymbol{\eta}$ to be determined. Then, computing $\mathbf{w}'(t) - \mathbf{A}\mathbf{w}(t)$ gives

$$\mathbf{w}'(t) - \mathbf{A}\mathbf{w}(t) = t e^{\lambda t}(\lambda\boldsymbol{\xi} - \mathbf{A}\boldsymbol{\xi}) + e^{\lambda t}(\lambda\boldsymbol{\eta} - \mathbf{A}\boldsymbol{\eta} + \boldsymbol{\xi}). \tag{473}$$

Hence, for $\mathbf{w}$ to be a solution we need

$$\mathbf{A}\boldsymbol{\xi} = \lambda\boldsymbol{\xi}, \quad (\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}. \tag{474}$$

We need to ask two questions:

- Does $\boldsymbol{\eta}$ such that $(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$ exists?
- If such $\boldsymbol{\eta}$ exists, are two solutions $\boldsymbol{\xi} e^{\lambda t}$ and $(\boldsymbol{\xi} t + \boldsymbol{\eta}) e^{\lambda t}$ linearly independent?

For the first question, take another vector $\mathbf{v}$ that is not a constant multiple of the eigenvector $\boldsymbol{\xi}$ ($\mathbf{v} \notin \mathrm{Span}\{\boldsymbol{\xi}\}$ and geometric multiplicity of $\lambda = 1$ implies $\mathbf{v}$ is not an eigenvector). Since $\boldsymbol{\xi} \in \mathbb{R}^2$, $\mathbf{v}$ and $\boldsymbol{\xi}$ are linearly independent, hence they form a basis of $\mathbb{R}^2$. So every vector $\mathbf{x} \in \mathbb{R}^2$ can be written as a linear combination of $\mathbf{v}$ and $\boldsymbol{\xi}$. Define the vector $\mathbf{u} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v}$ ($\mathbf{u} \neq \mathbf{0}$). Then, we can find constants $\alpha, \beta \in \mathbb{R}$ such that

$$\mathbf{u} = \alpha\mathbf{v} + \beta\boldsymbol{\xi}. \tag{475}$$

Multiply $\mathbf{A} - \lambda\mathbf{I}$ to both sides gives

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \alpha(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} + \beta(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\xi} = \alpha(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \alpha\mathbf{u}. \tag{476}$$

Rearranging gives

$$\mathbf{A}\mathbf{u} = (\lambda + \alpha)\mathbf{u}, \tag{477}$$

so $\mathbf{u}$ is an eigenvector corresponding to eigenvalue $\lambda + \alpha$. But, since $\mathbf{A}$ has only one repeated eigenvalue λ, there are no other possible eigenvalues, so α **must be zero**. Thus, we have

$$\mathbf{u} = \beta\boldsymbol{\xi}, \quad \beta \neq 0. \tag{478}$$

that is $\mathbf{u}$ is parallel to $\boldsymbol{\xi}$. Recalle the definition of $\mathbf{u}$, we see that

$$\mathbf{u} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \beta\boldsymbol{\xi}, \tag{479}$$

if we set $\boldsymbol{\eta} = \frac{1}{\beta}\mathbf{v}$, we see that

$$(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}. \tag{480}$$

For the second question, since $\boldsymbol{\xi}$ is an eigenvector corresponding to λ, the vector $\boldsymbol{\eta}$ exists such that

$$\mathbf{A}\boldsymbol{\xi} = \lambda\boldsymbol{\xi}, \quad (\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}. \tag{481}$$

then we have two solutions

$$\mathbf{y}_1(t) = \boldsymbol{\xi}e^{\lambda t}, \quad \mathbf{y}_2(t) = (t\boldsymbol{\xi} + \boldsymbol{\eta})e^{\lambda t}, \tag{482}$$

where $\boldsymbol{\xi} = [\xi_1, \xi_2]^T$, $\boldsymbol{\eta} = [\eta_1, \eta_2]^T$. Computing the Wronskian gives

$$W(\mathbf{y}_1, \mathbf{y}_2)[t] = e^{2\lambda t}\begin{vmatrix} \xi_1 & t\xi_1 + \eta_1 \\ \xi_2 & t\xi_2 + \eta_2 \end{vmatrix} = e^{2\lambda t}\begin{vmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{vmatrix}, \tag{483}$$

so the Wronskian is non-zero if and only if $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are linearly independent. Now suppose there are constants α_1, α_2 such that $\alpha_1\boldsymbol{\xi} + \alpha_2\boldsymbol{\eta} = \mathbf{0}$. Since $\mathbf{A} \neq \lambda\mathbf{I}$ (otherwise $\boldsymbol{\eta}$ would not exist), multiplying $\mathbf{A} - \lambda\mathbf{I}$ leads to

$$\mathbf{0} = \alpha_1(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\xi} + \alpha_2(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta} = \alpha_2\boldsymbol{\xi} \tag{484}$$

This implies that $\alpha_2 = 0$, since $\boldsymbol{\xi}$ is non-zero. Going back we see that

$$\alpha_1\boldsymbol{\xi} = \mathbf{0} \Rightarrow \alpha_1 = 0. \tag{485}$$

Hence, we see that $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are linearly independent, and

$$\mathbf{y}_1(t) = \boldsymbol{\xi}e^{\lambda t}, \quad \mathbf{y}_2(t) = (t\boldsymbol{\xi} + \boldsymbol{\eta})e^{\lambda t} \tag{486}$$

form a FSS. Thus, by Theorem 63, the general solution is

$$\mathbf{y}(t) = c_1\boldsymbol{\xi}e^{\lambda t} + c_2(t\boldsymbol{\xi} + \boldsymbol{\eta})e^{\lambda t}. \tag{487}$$

$\mathbf{A}\boldsymbol{\xi} = \lambda\boldsymbol{\xi}$, $(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}$ gives $(\mathbf{A} - \lambda\mathbf{I})^2\boldsymbol{\eta} = \mathbf{0}$. $\boldsymbol{\eta}$ is called the **generalized eigenvector**.

Definition 70 (Generalized Eigenvector)**.** *Let λ be an eigenvalue of matrix $\mathbf{A}$, a nonzero vector $\boldsymbol{\eta}$ is called a generalized eigenvector if there is a positive integer p such that*

$$(\mathbf{A} - \lambda\mathbf{I})^p\boldsymbol{\eta} = \mathbf{0}. \tag{488}$$

And the generalized eigenvector $\boldsymbol{\eta}$ is called the generalized eigenvector of rank p (p is some positive integer) of matrix $\mathbf{A}$ and corresponding eigenvalue λ if

$$(\mathbf{A} - \lambda\mathbf{I})^p\boldsymbol{\eta} = \mathbf{0}, \quad (\mathbf{A} - \lambda\mathbf{I})^{p-1}\boldsymbol{\eta} \neq \mathbf{0} \tag{489}$$

Here, since $(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi} \neq \mathbf{0}$ and $(\mathbf{A} - \lambda\mathbf{I})^2\boldsymbol{\eta} = \mathbf{0}$, $\boldsymbol{\eta}$ is the **generalized eigenvector** of rank 2. Since $(\mathbf{A} - \lambda\mathbf{I})\boldsymbol{\xi} = \mathbf{0}$ and $(\mathbf{A} - \lambda\mathbf{I})^0\boldsymbol{\xi} \neq \mathbf{0}$, $\boldsymbol{\xi}$ is the **generalized eigenvector** of rank 1 (just usual eigenvector).

Example 71. *For*

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \quad \mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}, \tag{490}$$

the eigenvalues are $r_1 = r_2 = \lambda = 2$, i.e., algebraic multiplicity is two, while the eigenvector corresponding to λ is

$$\boldsymbol{\xi} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \tag{491}$$

so the geometric multiplicity is one. We now need to find a vector $\boldsymbol{\eta}$ such that

$$(\mathbf{A} - 2\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}. \tag{492}$$

Computing $\mathbf{A} - 2\mathbf{I}$ gives

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow \eta_1 + \eta_2 = 1. \tag{493}$$

We can take $\eta_1 = 0$ and $\eta_2 = 1$, leading to the general solution

$$\mathbf{y}(t) = c_1 e^{2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \left[t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] e^{2t} \tag{494}$$

Question: Can we have a systematic way to solve $\mathbf{y}' = \mathbf{A}\mathbf{y}$ with a general matrix $\mathbf{A}_{n\times n}$ with real constant entries?

Answer: Yes. There are two methods that can be used, one is using the matrix exponential (need to compute $e^{\mathbf{A}t}$), the other is using the eigenvalue and eigenvectors.

5.4 Homogeneous System with Constant Coefficients: Matrix Exponential Method

5.4.1 Matrix Exponential and Diagonalizable Matrix

Recall:

$$\mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x}(t) \tag{495}$$

Suppose that $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$ form a FSS for the homogeneous linear system (495). Then the matrix

$$\mathbf{\Psi}(t) = \begin{pmatrix} | & | & \cdots & | \\ \mathbf{x}^{(1)}(t) & \mathbf{x}^{(2)}(t) & \cdots & \mathbf{x}^{(n)}(t) \\ | & | & \cdots & | \end{pmatrix} \tag{496}$$

is called a **fundamental matrix** of the system (495), and the general solution of the system

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t) \tag{497}$$

can be written as

$$\mathbf{x} = \mathbf{\Psi}(t)\mathbf{c}, \tag{498}$$

where $\mathbf{c} = [c_1, \ldots, c_n]^T$ is an arbitrary constant vector. If

$$\mathbf{x}(t_0) = \mathbf{x}_0, \tag{499}$$

then the solution of the corresponding initial value problem is

$$\mathbf{x} = \mathbf{\Psi}(t)\mathbf{\Psi}^{-1}(t_0)\mathbf{x}_0. \tag{500}$$

Since each column of $\mathbf{\Psi}$ is a solution of the system (495), $\mathbf{\Psi}$ satisfies the matrix differential equation

$$\mathbf{\Psi}' = \mathbf{P}(t)\mathbf{\Psi}, \tag{501}$$

where $\mathbf{\Psi}' = \frac{d\mathbf{\Psi}}{dt} = \left(\frac{d\psi_{ij}(t)}{dt}\right)_{n\times n}$, $\psi_{ij}(t)$ is the (i, j)-entry of $\mathbf{\Psi}$.

In Section 5.4, we will use the fundamental matrix and matrix exponential to solve the following IVP

$$\begin{cases} \mathbf{x}' = \mathbf{A}\mathbf{x}, \\ \mathbf{x}(t_0) = \mathbf{x}_0, \end{cases} \tag{502}$$

For a scalar a we have the Taylor series expansion

$$e^{at} = 1 + \sum_{n=1}^{\infty} \frac{a^n t^n}{n!}. \tag{503}$$

For any $n \times n$ constant matrix $\mathbf{A}$, we can show that the following

$$\mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{A}^n}{n!} \tag{504}$$

converges to a matrix. Thus, we can defined the matrix exponential:

$$e^{\mathbf{A}} \triangleq \mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{A}^n}{n!}. \tag{505}$$

And for any $t \in \mathbb{R}$, $\mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{A}^n t^n}{n!}$ also converges, we can define

$$e^{\mathbf{A}t} \triangleq \mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{A}^n t^n}{n!}. \tag{506}$$

To prove this, we will first define convergence for matrix.

Definition 72 (Convergence for Matrix Sequences)**.** *A sequence of $r \times r$ matrices, $\{\mathbf{A}_n\}$, is called convergent if for any given $\varepsilon > 0$ there exists $N > 0$, such that*

$$\|\mathbf{A}_n - \mathbf{A}_m\| < \varepsilon, \quad \forall m, n > N, \tag{507}$$

where the matrix norm $\|\cdot\|$ could be the 1-norm, 2-norm, or ∞-norm for matrices.

The next theorem states the limit of convergent matrix sequences.

Theorem 73. *Every convergent sequence of matrices $\{\mathbf{A}_n\}$ has a limit.*

Proof. Let $a_{ij}^n, 1 \le i, j \le r, n = 1, 2, \cdots$, be the components of $\mathbf{A}_n$. For any given $\varepsilon > 0$ there exists $N > 0$, such that

$$|a_{ij}^n - a_{ij}^m| \le \|\mathbf{A}_n - \mathbf{A}_m\| < \varepsilon, \quad \forall 1 \le i, j \le r, \quad \forall m, n > N, \tag{508}$$

where the matrix norm $\|\cdot\|$ could be the 1-norm, 2-norm, or ∞-norm. Then $\{a_{ij}^n\}$ is a Cauchy sequence, and converges, i.e., $a_{ij}^n \to a_{ij}$ as $n \to \infty$. Let $\mathbf{A} = [a_{ij}]$, then $\mathbf{A}$ is the limit of $\{\mathbf{A}_n\}$. $\square$

Finally we will proof our original claim.

Theorem 74. *The series*

$$\mathbf{I} + \sum_{n=1}^{\infty} \frac{\mathbf{A}^n t^n}{n!} \tag{509}$$

is convergent for any finite number t, the limit matrix is defined as $e^{\mathbf{A}t}$.

Proof. Let

$$\mathbf{S}_n = \sum_{k=0}^{n} \frac{\mathbf{A}^k t^k}{k!}, \tag{510}$$

then for $n > m$ we have

$$\|\mathbf{S}_n - \mathbf{S}_m\| = \left\| \sum_{k=0}^{n} \frac{\mathbf{A}^k t^k}{k!} - \sum_{k=0}^{m} \frac{\mathbf{A}^k t^k}{k!} \right\| = \left\| \sum_{k=m+1}^{n} \frac{\mathbf{A}^k t^k}{k!} \right\| \tag{511}$$

$$\le \sum_{k=m+1}^{n} \frac{\|\mathbf{A}^k\| |t|^k}{k!} \le \sum_{k=m+1}^{n} \frac{\|\mathbf{A}\|^k |t|^k}{k!}. \tag{512}$$

where the matrix norm $\|\cdot\|$ could be the 1-norm, 2-norm, or ∞-norm, and to obtain the two inequalities, we utilized the properties of matrix norm $\|\mathbf{A}+\mathbf{B}\| \leq \|\mathbf{A}\|+\|\mathbf{B}\|$ and $\|\mathbf{AB}\| \leq \|\mathbf{A}\|\|\mathbf{B}\|$ for all $n\times n$ matrices $\mathbf{A}, \mathbf{B}$. Since the series

$$\sum_{k=0}^{\infty} \frac{\|\mathbf{A}\|^k |t|^k}{k!} = e^{\|\mathbf{A}\||t|} \tag{513}$$

converges for any $\|\mathbf{A}\||t|$ (t is finite), then for any given $\varepsilon > 0$ there exists N, such that

$$\sum_{k=m+1}^{n} \frac{\|\mathbf{A}\|^k |t|^k}{k!} < \varepsilon, \quad \forall n > m > N, \tag{514}$$

i.e., $\{\mathbf{S}_n\}$ converges. The limit matrix is defined as $e^{\mathbf{A}t}$. □

Property 75. *We list some important properties of matrix exponential, more can be found in Wikipedia.*

1. $\exp(\mathbf{O}_{n\times n}) = \mathbf{I}_{n\times n}$, *where* $\mathbf{O}$ *is the all zero matrix.*
2. *If* $\mathbf{\Lambda} = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, *then* $e^{\mathbf{\Lambda}} = \operatorname{diag}(e^{\lambda_1}, \ldots, e^{\lambda_n})$.
3. *If* $\mathbf{AB} = \mathbf{BA}$, *then* $e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}}e^{\mathbf{B}}$.
4. $(e^{\mathbf{A}})^{-1} = e^{-\mathbf{A}}$.
5. *For invertible matrix* $\mathbf{P}$, *we have*

$$\exp(\mathbf{PAP}^{-1}) = \mathbf{P}e^{\mathbf{A}}\mathbf{P}^{-1}. \tag{515}$$

In other words, if $\mathbf{A}$ *is diagonalizable, there exists an invertible matrix* $\mathbf{P}$ *such that* $\mathbf{A} = \mathbf{P}^{-1}\operatorname{diag}(\lambda_1, \ldots, \lambda_n)\mathbf{P}$, *then* $e^{\mathbf{A}} = \mathbf{P}^{-1}\operatorname{diag}(e^{\lambda_1}, \ldots, e^{\lambda_n})\mathbf{P}$. *This is our* ***first way of computing matrix exponential****.*

Since $e^{\mathbf{A}t}$ is defined as a convergent matrix power series, it is differentiable and can be differentiated term by term

$$\frac{d}{dt}(e^{\mathbf{A}t}) = \frac{d}{dt}\sum_{p=0}^{\infty}\frac{\mathbf{A}^p t^p}{p!} = \sum_{p=0}^{\infty}\frac{d}{dt}\frac{\mathbf{A}^p t^p}{p!} = \sum_{p=1}^{\infty} p\frac{\mathbf{A}^p t^{p-1}}{p!} \tag{516}$$

$$= \mathbf{A}\sum_{p=1}^{\infty}\frac{\mathbf{A}^{p-1}t^{p-1}}{(p-1)!} = \mathbf{A}e^{\mathbf{A}t} = \sum_{p=1}^{\infty}\frac{\mathbf{A}^{p-1}t^{p-1}}{(p-1)!}\mathbf{A} = e^{\mathbf{A}t}\mathbf{A} \tag{517}$$

Thus, each column of $e^{\mathbf{A}t}$ is a solution for $\mathbf{x}' = \mathbf{Ax}$. Moreover, $W[e^{\mathbf{A}t}] = \det(e^{\mathbf{A}t}) = e^{\operatorname{tr}(\mathbf{A}t)} \neq 0$, which is a corollary of Jacobi's formula. Thus, $e^{\mathbf{A}t}$ is the fundamental matrix for the system

$$\mathbf{x}' = \mathbf{Ax}. \tag{518}$$

The general solution for above ODE is

$$\mathbf{x} = e^{\mathbf{A}t}\mathbf{c}, \tag{519}$$

where $\mathbf{c} = [c_1, \ldots, c_n]^T$ is an arbitary constant vector. Now we look for the solution for IVP

$$\mathbf{x}' = \mathbf{Ax}, \quad \mathbf{x}(t_0) = \mathbf{x}_0. \tag{520}$$

Substituting the initial condition for the above general solution, one has

$$\mathbf{x}_0 = e^{\mathbf{A}t_0}\mathbf{c}. \tag{521}$$

Thus, $\mathbf{c} = (e^{\mathbf{A}t_0})^{-1}\mathbf{x}_0 = e^{-\mathbf{A}t_0}\mathbf{x}_0$. Therefore,

$$\mathbf{x} = e^{\mathbf{A}t}\mathbf{c} = e^{\mathbf{A}t}e^{-\mathbf{A}t_0}\mathbf{x}_0 = e^{\mathbf{A}(t-t_0)}\mathbf{x}_0. \tag{522}$$

Example 76. *Solve*

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \quad \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \tag{523}$$

with eigenvalues and corresponding eigenvectors

$$r_1 = 3, \quad \boldsymbol{\xi}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad r_2 = -1, \quad \boldsymbol{\xi}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{524}$$

$\mathbf{A}$ is diagonalizable,

$$\mathbf{A} = \mathbf{P}\,\mathrm{diag}(3,-1)\mathbf{P}^{-1}, \quad \mathbf{P} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}, \quad \mathbf{P}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{pmatrix} \tag{525}$$

Compute matrix exponential

$$e^{\mathbf{A}t} = \mathbf{P}e^{\mathrm{diag}(3,-1)t}\mathbf{P}^{-1} \tag{526}$$

$$= \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{e^{3t}+e^{-t}}{2} & \frac{e^{3t}-e^{-t}}{4} \\ e^{3t}-e^{-t} & \frac{e^{3t}+e^{-t}}{2} \end{pmatrix} \tag{527}$$

The general solution is

$$\mathbf{y}(t) = e^{\mathbf{A}t}\mathbf{c} \tag{528}$$

$$= \mathbf{P}e^{\mathrm{diag}(3,-1)t}\mathbf{P}^{-1}\mathbf{c} \tag{529}$$

$$= \mathbf{P}e^{\mathrm{diag}(3,-1)t}\mathbf{d} \tag{530}$$

$$= \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = d_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{3t} + d_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{t} \tag{531}$$

Example 77. *Find the fundamental solution matrix* $\Phi(t)$ *satisfying* $\Phi(0) = \mathbf{I}$ *and the general solution for system*

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\mathbf{x} \tag{532}$$

The three eigen-pairs of $\mathbf{A}$ *is* $(r_1, \boldsymbol{\xi}^{(1)}), (r_2, \boldsymbol{\xi}^{(2)}), (r_3, \boldsymbol{\xi}^{(3)})$*, where*

$$r_1 = 2, \boldsymbol{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \quad r_2 = -1, \boldsymbol{\xi}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}; \quad r_3 = -1, \boldsymbol{\xi}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \tag{533}$$

Set

$$\mathbf{\Lambda} = \operatorname{diag}(r_1, r_2, r_3), \quad \mathbf{P} = [\boldsymbol{\xi}^{(1)} \quad \boldsymbol{\xi}^{(2)} \quad \boldsymbol{\xi}^{(3)}] \tag{534}$$

For fundamental solution matrix satisfying $\Phi(0) = \mathbf{I}$, we have:

$$\Phi(t) = e^{\mathbf{A}t} = \mathbf{P}e^{\mathbf{\Lambda}t}\mathbf{P}^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}^{-1} \tag{535}$$

$$= \begin{pmatrix} e^{2t} & e^{-t} & 0 \\ e^{2t} & 0 & e^{-t} \\ e^{2t} & -e^{-t} & -e^{-t} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \tag{536}$$

$$= \begin{pmatrix} \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t} & \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t} & \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t} \\ \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t} & \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t} & \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t} \\ \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t} & \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t} & \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t} \end{pmatrix}. \tag{537}$$

However, if $\mathbf{A}$ is not diagonalizable, computation of $e^{\mathbf{A}}$ is quite involving.

5.4.2 S-N Decomposition

Definition 78 (Semisimple and Nilpotent)**.** *A square matrix $\mathbf{A}$ is called semisimple if it is diagonalizable, and $\mathbf{A}$ is called nilpotent if there is some positive integer k s.t. $\mathbf{A}^k = \mathbf{O}$.*

The S-N decomposition (Semisimple-Nilpotent decomposition, Jordan–Chevalley decomposition) is given by the following theorem.

Theorem 79 (S-N Decomposition)**.** *Let $\mathbf{A}$ be an $n \times n$ matrix. Then, there exist two $n \times n$ matrices $\mathbf{S}$ and $\mathbf{N}$ such that*

(a) $\mathbf{S}$ *is diagonalizable (semisimple),*

(b) $\mathbf{N}$ *is nilpotent and $\mathbf{N}^n = 0$ must hold,*

(c) $\mathbf{A} = \mathbf{S} + \mathbf{N}$*,*

(d) $\mathbf{SN} = \mathbf{NS}$.

The two matrices $\mathbf{S}$ and $\mathbf{N}$ are uniquely determined by these four conditions.

Clearly, if we obtain a S-N decomposition for $\mathbf{A}$, then $e^{\mathbf{A}} = e^{\mathbf{S}+\mathbf{N}} = e^{\mathbf{S}}e^{\mathbf{N}}$, which can be computed easily since $\mathbf{S}$ is diagonalizable, and $e^{\mathbf{N}} = \sum_{k=0}^{\infty} \frac{\mathbf{N}^k}{k!}$ has finite terms since $\mathbf{N}$ is nilpotent.

We will show how to construct S-N decomposition for 2×2 and 3×3 matrices with one single eigenvalue. **Construction method for general $n \times n$ matrices** can be found in Section 1 of [HKS96].

Theorem 80. *Let $\mathbf{A}$ be a 2×2 or 3×3 matrix which has only one distinct eigenvalue r. Then $\mathbf{A}$ could be decomposed as $\mathbf{A} = \mathbf{S} + \mathbf{N}$ such that*

1. $\mathbf{S} = r\mathbf{I}$.
2. $\mathbf{N} = \mathbf{A} - \mathbf{S}$.
3. $\mathbf{N}^2 = \mathbf{O}$ *or* $\mathbf{N}^3 = \mathbf{O}$.
4. $\mathbf{SN} = \mathbf{NS}$

We will demonstrate several examples.

Example 81. *Find a fundamental solution matrix of*

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x}. \tag{538}$$

The eigenvalues of $\mathbf{A}$ satisfy:

$$\det(\mathbf{A} - r\mathbf{I}) = \begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix} = (r-2)^2 \implies r = 2. \tag{539}$$

Perform the S-N decomposition for $\mathbf{A}$:

$$\mathbf{A} = \mathbf{S} + \mathbf{N} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \tag{540}$$

And we have

$$e^{\mathbf{S}t} = e^{2t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{541}$$

and

$$\mathbf{N}^2 = \mathbf{O} \implies e^{\mathbf{N}t} = \mathbf{I} + \mathbf{N}t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -t & -t \\ t & t \end{pmatrix} = \begin{pmatrix} 1-t & -t \\ t & 1+t \end{pmatrix} \tag{542}$$

The fundamental matrix is given by:

$$e^{\mathbf{A}t} = e^{\mathbf{S}t} e^{\mathbf{N}t} = e^{2t} \begin{pmatrix} 1-t & -t \\ t & 1+t \end{pmatrix}. \tag{543}$$

The general solution is given by:

$$\mathbf{x} = c_1 \begin{pmatrix} 1-t \\ t \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -t \\ 1+t \end{pmatrix} e^{2t}. \tag{544}$$

Example 82. *Find a fundamental solution matrix of*

$$\mathbf{x}' = \begin{pmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{pmatrix} \mathbf{x} \tag{545}$$

The eigenvalues of $\mathbf{A}$ satisfy:

$$\det(\mathbf{A} - r\mathbf{I}) = \begin{vmatrix} 5-r & -3 & -2 \\ 8 & -5-r & -4 \\ -4 & 3 & 3-r \end{vmatrix} = -(r-1)^3 \implies r = 1. \tag{546}$$

Perform the S-N decomposition for $\mathbf{A}$:

$$\mathbf{A} = \mathbf{S} + \mathbf{N} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 4 & -3 & -2 \\ 8 & -6 & -4 \\ -4 & 3 & 2 \end{pmatrix} \tag{547}$$

And we have

$$e^{\mathbf{S}t} = e^t \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{548}$$

and

$$\mathbf{N}^2 = \mathbf{O} \implies e^{\mathbf{N}t} = \mathbf{I} + \mathbf{N}t = \begin{pmatrix} 4t+1 & -3t & -2t \\ 8t & -6t+1 & -4t \\ -4t & 3t & 2t+1 \end{pmatrix} \tag{549}$$

The fundamental matrix is given by:

$$e^{\mathbf{A}t} = e^{\mathbf{S}t}e^{\mathbf{N}t} = e^t \begin{pmatrix} 4t+1 & -3t & -2t \\ 8t & -6t+1 & -4t \\ -4t & 3t & 2t+1 \end{pmatrix} \tag{550}$$

Thus the fundamental solution matrix is given by:

$$\mathbf{x} = c_1 \begin{pmatrix} 4t+1 \\ 8t \\ -4t \end{pmatrix} e^t + c_2 \begin{pmatrix} -3t \\ -6t+1 \\ 3t \end{pmatrix} e^t + c_3 \begin{pmatrix} -2t \\ -4t \\ 2t+1 \end{pmatrix} e^t. \tag{551}$$

Example 83. *Find a fundamental solution matrix of*

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -3 & 2 & 4 \end{pmatrix} \mathbf{x} \tag{552}$$

The eigenvalues of $\mathbf{A}$ satisfy:

$$\det(\mathbf{A} - r\mathbf{I}) = \begin{vmatrix} 1-r & 1 & 1 \\ 2 & 1-r & -1 \\ -3 & 2 & 4-r \end{vmatrix} = -(r-2)^3 \implies r = 2. \tag{553}$$

Perform the S-N decomposition for $\mathbf{A}$:

$$\mathbf{A} = \mathbf{S} + \mathbf{N} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix} \tag{554}$$

And we have

$$e^{\mathbf{S}t} = e^{2t}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{555}$$

and

$$\mathbf{N}^2 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix}, \quad \mathbf{N}^3 = \mathbf{O} \tag{556}$$

It follows that

$$e^{\mathbf{N}t} = \mathbf{I} + \mathbf{N}t + \frac{1}{2}\mathbf{N}^2 t^2 = \begin{pmatrix} 1-t & t & t \\ 2t - \frac{t^2}{2} & 1 - t + \frac{t^2}{2} & -t + \frac{t^2}{2} \\ -3t + \frac{t^2}{2} & 2t - \frac{t^2}{2} & 1 + 2t - \frac{t^2}{2} \end{pmatrix}. \tag{557}$$

The fundamental matrix is given by:

$$e^{\mathbf{A}t} = e^{\mathbf{S}t}e^{\mathbf{N}t} = e^{2t}\begin{pmatrix} 1-t & t & t \\ 2t - \frac{t^2}{2} & 1 - t + \frac{t^2}{2} & -t + \frac{t^2}{2} \\ -3t + \frac{t^2}{2} & 2t - \frac{t^2}{2} & 1 + 2t - \frac{t^2}{2} \end{pmatrix} \tag{558}$$

Thus the general solution is given by:

$$\mathbf{x} = c_1\begin{pmatrix} 1-t \\ 2t - \frac{t^2}{2} \\ -3t + \frac{t^2}{2} \end{pmatrix} e^{2t} + c_2\begin{pmatrix} t \\ 1 - t + \frac{t^2}{2} \\ 2t - \frac{t^2}{2} \end{pmatrix} e^{2t} + c_3\begin{pmatrix} t \\ -t + \frac{t^2}{2} \\ 1 + 2t - \frac{t^2}{2} \end{pmatrix} e^{2t}. \tag{559}$$

Since we did not detail the method of computing $e^{\mathbf{A}t}$ for a general $n \times n$ matrix $\mathbf{A}$, we will provide another method to solve the general $n \times n$ linear ODE system.

5.5 Homogeneous System with Constant Coefficients: Eigenvalue Method

Consider the system of the form

$$\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t), \quad t \in I, \tag{560}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a constant matrix. Let the eigenvalues of $\mathbf{A}$ be $\lambda_1, \dots, \lambda_k$ where $k \in \mathbb{N}$, and each eigenvalue λ_i has an algebraic multiplicity of $m_i \in \mathbb{N}$. Thus, the characteristic equation is

$$P_{\mathbf{A}}(\lambda) = \det(\lambda\mathbf{I} - \mathbf{A}) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k}, \tag{561}$$

where $m_1 + \cdots + m_k = n$. Now suppose each eigenvalue λ_i has a geometric multiplicity of q_i, where for each $1 \le i \le k$, $1 \le q_i \le m_i$ (recall $1 \le$ geo. mult. $\le$ alg. mult).

Our goal is to find m_i **linearly independent solutions** corresponding to the eigenvalue λ_i (for $i = 1, \dots, k$).

5.5.1 Diagonalizable Matrix

Recall the theorem from linear algebra:

Theorem 84. *Let $\mathbf{A}$ be a square matrix with size n, then $\mathbf{A}$ is diagonalizable if and only if the algebraic multiplicity and geometric multiplicity are the same for each eigenvalue.*

If $q_i = m_i$ for all $i = 1, \dots, k$, then $\mathbf{A}$ is diagonalizable. There are n linearly independent eigenvectors $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_n$ corresponding to eigenvalues $r_1, \dots, r_n$, where $r_1, \dots, r_n \in \{\lambda_1, \dots, \lambda_k\}$, and $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{\Lambda} = \mathrm{diag}(r_1, \dots, r_n)$, $\mathbf{P} = [\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_n]$. Define the new vector $\mathbf{x} := \mathbf{P}^{-1}\mathbf{y}$. Then,

$$\mathbf{x}'(t) = \mathbf{P}^{-1}\mathbf{y}'(t) = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\mathbf{x}(t) \Rightarrow \mathbf{x}'(t) = \mathbf{\Lambda}\mathbf{x}(t). \tag{562}$$

The solution for $\mathbf{x}$ is given by:

$$\mathbf{x}(t) = [c_1 e^{r_1 t}, \dots, c_n e^{r_n t}]^T. \tag{563}$$

and $\mathbf{y}(t) = \mathbf{P}\mathbf{x}(t) = c_1 e^{r_1 t}\boldsymbol{\xi}_1 + \cdots + c_n e^{r_n t}\boldsymbol{\xi}_n$. Indeed, $e^{r_1 t}\boldsymbol{\xi}_1, \dots, e^{r_n t}\boldsymbol{\xi}_n$ form a FSS.

5.5.2 Non-Diagonalizable Matrix

However, if there is a repeated eigenvalue λ with geometric multiplicity **strictly less** than its algebraic multiplicity, $\mathbf{A}$ is not diagonalizable. In the following, we carry out a systematic way to find m_i linearly independent solutions corresponding to the eigenvalue λ_i.

For all distinct eigenvalues $(\lambda_1, \cdots, \lambda_k)$, we will carry out the following process.

For $i = 1, \cdots, k$, do the following.

Let $\lambda = \lambda_i$ be the eigenvalue of $\mathbf{A}$, $m = m_i$ be algebraic multiplicity of λ, and $q = q_i$ be geometric multiplicity of λ. Then we want to find m linearly independent solutions corresponding to λ.

Case 1. If $q = m$ for λ, then suppose $\mathbf{r}_1, \cdots, \mathbf{r}_m$ are m linearly independent eigenvectors w.r.t λ, then we already have m linearly independent solutions $\mathbf{r}_1 e^{\lambda t}, \cdots, \mathbf{r}_m e^{\lambda t}$ (they also satisfy $(\mathbf{r}_i e^{\lambda t})' = \mathbf{A}\mathbf{r}_i e^{\lambda t}$).

Case 2. If $q < m$ for λ, then we will need to construct m linearly independent solutions. We look for the solutions of the following form:

$$\mathbf{y}(t) = \left(\mathbf{r}_0 + \mathbf{r}_1 t + \mathbf{r}_2 \frac{t^2}{2} + \cdots + \mathbf{r}_{m-1} \frac{t^{m-1}}{(m-1)!} \right) e^{\lambda t} \tag{564}$$

Then

$$\begin{aligned}
&\mathbf{y}'(t) - \mathbf{A}\mathbf{y}(t) &(565)\\
&= \left(\mathbf{r}_1 + \mathbf{r}_2 t + \cdots + \mathbf{r}_{m-1} \frac{t^{m-2}}{(m-2)!} \right) e^{\lambda t} + \left(\lambda\mathbf{r}_0 + \lambda\mathbf{r}_1 t + \cdots + \lambda\mathbf{r}_{m-1} \frac{t^{m-1}}{(m-1)!} \right) e^{\lambda t} \\
&\quad - \left(\mathbf{A}\mathbf{r}_0 + \mathbf{A}\mathbf{r}_1 t + \cdots + \mathbf{A}\mathbf{r}_{m-1} \frac{t^{m-1}}{(m-1)!} \right) e^{\lambda t} &(566)\\
&= (\mathbf{r}_1 - (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_0) e^{\lambda t} + (\mathbf{r}_2 - (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_1) t e^{\lambda t} + (\mathbf{r}_3 - (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_2) \frac{t^2}{2} e^{\lambda t} + \cdots \\
&\quad + (\mathbf{r}_{m-1} - (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_{m-2}) \frac{t^{m-2}}{(m-2)!} e^{\lambda t} - (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_{m-1} \frac{t^{m-1}}{(m-1)!} e^{\lambda t} &(567)
\end{aligned}$$

In order to get $\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t)$, we need

$$\begin{cases}
\mathbf{r}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_0 \\
\mathbf{r}_2 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_1 \\
\mathbf{r}_3 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_2 \\
\vdots \\
\mathbf{r}_{m-1} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_{m-2} \\
(\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_{m-1} = \mathbf{0}.
\end{cases} \tag{568}$$

Substituting the first $m-1$ equations into the last equation gives

$$\begin{cases}
(\mathbf{A} - \lambda\mathbf{I})^m \mathbf{r}_0 = \mathbf{0} \\
\mathbf{r}_1 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_0 \\
\mathbf{r}_2 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_1 \\
\mathbf{r}_3 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_2 \\
\vdots \\
\mathbf{r}_{m-1} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_{m-2}.
\end{cases} \tag{569}$$

Theorem 85 (Generalized Eigenspace Decomposition)**.** *If λ is an eigenvalue of $\mathbf{A}$ with algebraic multiplicity m, then*

$$(\mathbf{A} - \lambda\mathbf{I})^m \mathbf{r} = \mathbf{0} \tag{570}$$

will have m linearly independent solutions $\mathbf{r}_0^{(1)}, \ldots, \mathbf{r}_0^{(m)}$.

$Null((\mathbf{A} - \lambda\mathbf{I})^m)$ is called the **generalized eigenspace** for eigenvalue λ with algebraic multiplicity m, $\dim(Null((\mathbf{A} - \lambda\mathbf{I})^m)) = m$. Proof can be found in Chapter 8 of [Axl24].

Start from $\mathbf{r}_0^{(1)}$, we have

$$\begin{cases} \mathbf{r}_1^{(1)} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_0^{(1)} \\ \mathbf{r}_2^{(1)} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_1^{(1)} \\ \mathbf{r}_3^{(1)} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_2^{(1)} \\ \vdots \\ \mathbf{r}_{m-1}^{(1)} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_{m-2}^{(1)}. \end{cases} \tag{571}$$

We can construct the first solution

$$\mathbf{y}^{(1)}(t) = \left(\mathbf{r}_0^{(1)} + \mathbf{r}_1^{(1)}t + \mathbf{r}_2^{(1)}\frac{t^2}{2} + \cdots + \mathbf{r}_{m-1}^{(1)}\frac{t^{m-1}}{(m-1)!}\right)e^{\lambda t} \tag{572}$$

Start from $\mathbf{r}_0^{(2)}$, we have

$$\begin{cases} \mathbf{r}_1^{(2)} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_0^{(2)} \\ \mathbf{r}_2^{(2)} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_1^{(2)} \\ \mathbf{r}_3^{(2)} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_2^{(2)} \\ \vdots \\ \mathbf{r}_{m-1}^{(2)} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_{m-2}^{(2)}. \end{cases} \tag{573}$$

We can construct the second solution

$$\mathbf{y}^{(2)}(t) = \left(\mathbf{r}_0^{(2)} + \mathbf{r}_1^{(2)}t + \mathbf{r}_2^{(2)}\frac{t^2}{2} + \cdots + \mathbf{r}_{m-1}^{(2)}\frac{t^{m-1}}{(m-1)!}\right)e^{\lambda t} \tag{574}$$

The process continue until we start from $\mathbf{r}_0^{(m)}$, then we have

$$\begin{cases} \mathbf{r}_1^{(m)} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_0^{(m)} \\ \mathbf{r}_2^{(m)} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_1^{(m)} \\ \mathbf{r}_3^{(m)} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_2^{(m)} \\ \vdots \\ \mathbf{r}_{m-1}^{(m)} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{r}_{m-2}^{(m)}. \end{cases} \tag{575}$$

We can construct the mth solution

$$\mathbf{y}^{(m)}(t) = \left(\mathbf{r}_0^{(m)} + \mathbf{r}_1^{(m)}t + \mathbf{r}_2^{(m)}\frac{t^2}{2} + \cdots + \mathbf{r}_{m-1}^{(m)}\frac{t^{m-1}}{(m-1)!}\right)e^{\lambda t} \tag{576}$$

Theorem 86.

$$\mathbf{y}^{(1)}(t), \cdots, \mathbf{y}^{(m)}(t) \tag{577}$$

are m linearly independent solutions corresponding to the eigenvalue λ.

Proof. If they are linearly dependent, then there exist a set of numbers $(\alpha_1, \cdots, \alpha_m) \neq (0, \cdots, 0)$ such that

$$\alpha_1 \left(\mathbf{r}_0^{(1)} + \cdots + \mathbf{r}_{m-1}^{(1)} \frac{t^{m-1}}{(m-1)!} \right) e^{\lambda t} + \cdots + \alpha_m \left(\mathbf{r}_0^{(m)} + \cdots + \mathbf{r}_{m-1}^{(m)} \frac{t^{m-1}}{(m-1)!} \right) e^{\lambda t} = \mathbf{0}, \forall t \in I \tag{578}$$

Then $\alpha_1 \mathbf{r}_0^{(1)} + \cdots + \alpha_m \mathbf{r}_0^{(m)} = \mathbf{0}$. Thus $\alpha_1 = \cdots = \alpha_m = 0$ since $\mathbf{r}_0^{(1)}, \cdots, \mathbf{r}_0^{(m)}$ are linearly independent. Contradiction. □

Example 87. *Solve the system*

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \quad \mathbf{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix} \tag{579}$$

The characteristic polynomial is

$$P_{\mathbf{A}}(\lambda) = \begin{vmatrix} \lambda - 1 & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & 1 & \lambda - 2 \end{vmatrix} = (\lambda - 1)^3 \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 1. \tag{580}$$

Furthermore,

$$\mathbf{A} - \mathbf{I} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix}, \quad (\mathbf{A} - \mathbf{I})^2 = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (\mathbf{A} - \mathbf{I})^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{581}$$

The system

$$(\mathbf{A} - \mathbf{I})^3 \mathbf{r}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{r}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{582}$$

has three linearly independent solutions:

$$\mathbf{r}_0^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{r}_0^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{r}_0^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{583}$$

Start from $\mathbf{r}_0^{(1)}$, we have

$$\mathbf{r}_1^{(1)} = (\mathbf{A} - \mathbf{I}) \mathbf{r}_0^{(1)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{584}$$

$$\mathbf{r}_2^{(1)} = (\mathbf{A} - \mathbf{I}) \mathbf{r}_1^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{585}$$

The first solution is

$$\mathbf{y}_1(t) = e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{586}$$

Start from $\mathbf{r}_0^{(2)}$, we have

$$\mathbf{r}_1^{(2)} = (\mathbf{A}-\mathbf{I})\mathbf{r}_0^{(2)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \tag{587}$$

$$\mathbf{r}_2^{(2)} = (\mathbf{A}-\mathbf{I})\mathbf{r}_1^{(2)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}. \tag{588}$$

The second solution is

$$\mathbf{y}_2(t) = e^t \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + \frac{t^2}{2} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} -\frac{t^2}{2} + t \\ 1-t \\ -t \end{pmatrix} e^t. \tag{589}$$

Start from $\mathbf{r}_0^{(3)}$, we have

$$\mathbf{r}_1^{(3)} = (\mathbf{A}-\mathbf{I})\mathbf{r}_0^{(3)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \tag{590}$$

$$\mathbf{r}_2^{(3)} = (\mathbf{A}-\mathbf{I})\mathbf{r}_1^{(3)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{591}$$

The third solution is

$$\mathbf{y}_3(t) = e^t \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \frac{t^2}{2} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} \frac{t^2}{2} \\ t \\ 1+t \end{pmatrix} e^t. \tag{592}$$

The Wronskian is

$$W(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3)[t] = \begin{vmatrix} 1 & t-\frac{t^2}{2} & \frac{t^2}{2} \\ 0 & 1-t & t \\ 0 & -t & 1+t \end{vmatrix} e^{3t} = e^{3t} \neq 0 \tag{593}$$

$\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is a FSS. The general solution is

$$\mathbf{y}(t) = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + c_3\mathbf{y}_3 = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} -\frac{t^2}{2} + t \\ 1-t \\ -t \end{pmatrix} e^t + c_3 \begin{pmatrix} \frac{t^2}{2} \\ t \\ 1+t \end{pmatrix} e^t. \tag{594}$$

Example 88. *Solve the system*

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \quad \mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 1 \\ -1 & 0 & 2 \end{pmatrix} \tag{595}$$

The characteristic polynomial is

$$P_{\mathbf{A}}(\lambda) = \begin{vmatrix} \lambda-1 & 0 & 0 \\ 2 & \lambda-2 & -1 \\ 1 & 0 & \lambda-2 \end{vmatrix} = (\lambda-1)(\lambda-2)^2 \Rightarrow \lambda_1 = 1, \lambda_2 = \lambda_3 = 2. \tag{596}$$

For $\lambda_1 = 1$

$$\mathbf{A} - \lambda_1 \mathbf{I} = \begin{pmatrix} 0 & 0 & 0 \\ -2 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \tag{597}$$

Choose the eigenvector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, we get one solution

$$\mathbf{y}_1(t) = e^t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{598}$$

For $\lambda_1 = 2$

$$\mathbf{A} - \lambda_2 \mathbf{I} = \begin{pmatrix} -1 & 0 & 0 \\ -2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad (\mathbf{A} - \lambda_2 \mathbf{I})^2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{599}$$

The system

$$(\mathbf{A} - \lambda_2 \mathbf{I})^2 \mathbf{r}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \mathbf{r}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{600}$$

has two linearly independent solutions:

$$\mathbf{r}_0^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{r}_0^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{601}$$

Start from $\mathbf{r}_0^{(1)}$, we have

$$\mathbf{r}_1^{(1)} = (\mathbf{A} - \lambda_2 \mathbf{I}) \mathbf{r}_0^{(1)} = \begin{pmatrix} -1 & 0 & 0 \\ -2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{602}$$

We obtain one solution corresponding to λ_2:

$$\mathbf{y}_2(t) = e^{2t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{603}$$

Start from $\mathbf{r}_0^{(2)}$, we have

$$\mathbf{r}_1^{(2)} = (\mathbf{A} - \lambda_2 \mathbf{I}) \mathbf{r}_0^{(2)} = \begin{pmatrix} -1 & 0 & 0 \\ -2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \tag{604}$$

Another solution corresponding to λ_2 is

$$\mathbf{y}_3(t) = e^{2t} \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right] = e^{2t} \begin{pmatrix} 0 \\ t \\ 1 \end{pmatrix}. \tag{605}$$

The Wronskian is

$$W(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3)[t] = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & t \\ 1 & 0 & 1 \end{vmatrix} e^{5t} = e^{5t} \neq 0 \tag{606}$$

$\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is a FSS. The general solution is

$$\mathbf{y}(t) = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + c_3\mathbf{y}_3 = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 0 \\ t \\ 1 \end{pmatrix} e^{2t}. \tag{607}$$

5.6 Non-Homogeneous System

5.6.1 Method of Undetermined Coefficients

We now study for $\mathbf{A} \in \mathbb{R}^{n\times n}$ the non-homogeneous system

$$\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{g}(t). \tag{608}$$

If $\mathbf{y}_1, \ldots, \mathbf{y}_n$ are n linearly independent solutions to the homogeneous system $\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t)$, and $\mathbf{Y}(t)$ is a particular solution to the non-homogeneous system, then the general solution is

$$\mathbf{y}(t) = c_1\mathbf{y}_1(t) + \cdots + c_n\mathbf{y}_n(t) + \mathbf{Y}(t), \tag{609}$$

where $\mathbf{y}_c(t) = c_1\mathbf{y}_1(t) + \cdots + c_n\mathbf{y}_n(t)$ is the complementary solution.

If each component of the non-homogeneous term $\mathbf{g}(t)$ has a sum or product of exponentials, cosine, sine and polynomials, then we can use the method of undetermined coefficients to obtain a particular solution to the non-homogeneous system. Trial solutions for specific examples of $\mathbf{g}$ are listed here:

$\mathbf{g}(t)$	Particular solution form $\mathbf{Y}(t)$	value of s
$\mathbf{P}_m(t)e^{\alpha t}$	$\mathbf{Q}_{m+s}(t)e^{\alpha t}$	alg. mult. of α
$\mathbf{P}_m(t)e^{\alpha t}\cos(\beta t)$	$\mathbf{Q}_{m+s}(t)e^{\alpha t}\cos(\beta t) + \mathbf{R}_{m+s}(t)e^{\alpha t}\sin(\beta t)$	alg. mult. of $\alpha + i\beta$
$\mathbf{P}_m(t)e^{\alpha t}\sin(\beta t)$	$\mathbf{Q}_{m+s}(t)e^{\alpha t}\cos(\beta t) + \mathbf{R}_{m+s}(t)e^{\alpha t}\sin(\beta t)$	alg. mult. of $\alpha + i\beta$

Where, alg. mult. of α / $\alpha + i\beta$ means the multiplicity of α / $\alpha + i\beta$ as the eigenvalue of $\mathbf{A}$, and we use the notation

$$\mathbf{P}_m(t) = \mathbf{a}_m t^m + \mathbf{a}_{m-1}t^{m-1} + \cdots + \mathbf{a}_1 t + \mathbf{a}_0, \tag{610}$$

where $\mathbf{a}_0, \ldots, \mathbf{a}_m$ are constant vectors, so that $\mathbf{P}_m(t)$ is a vector-valued polynomial of degree m.

Example 89. *Find a particular solution to*

$$\mathbf{y}'(t) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{y}(t) + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}. \tag{611}$$

Set

$$\mathbf{g}(t) = \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}, \tag{612}$$

and first find the solution to the homogeneous system. The eigenvalues of $\mathbf{A}$ are $r_1 = -3, r_2 = -1$ with corresponding eigenvectors

$$\xi_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{613}$$

Therefore, the complementary solution to the homogeneous system is

$$\mathbf{y}_c(t) = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-3t} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-t}. \tag{614}$$

Next, observe that

$$\mathbf{g}(t) = \underbrace{\begin{pmatrix}2\\0\end{pmatrix} e^{-t}}_{\mathbf{g}_1(t)} + \underbrace{\begin{pmatrix}0\\3\end{pmatrix} t}_{\mathbf{g}_2(t)}. \tag{615}$$

Let us first try a trial solution to the **non-homogeneous system with** $\mathbf{g}_1(t)$ of the form

$$\mathbf{x}(t) = \mathbf{a}te^{-t} \tag{616}$$

for some undetermined vector $\mathbf{a}$. Substituting this into the equation gives

$$\mathbf{x}'(t) - \mathbf{A}\mathbf{x}(t) = -te^{-t}(\mathbf{A}\mathbf{a} + \mathbf{a}) + \mathbf{a}e^{-t} = \begin{pmatrix}2\\0\end{pmatrix} e^{-t}. \tag{617}$$

Comparing the coefficients, naturally we choose $\mathbf{a} = \begin{pmatrix}2\\0\end{pmatrix}$. But we also need to ensure that $\mathbf{A}\mathbf{a} + \mathbf{a} = \mathbf{0}$, which reaches a contradiction since $\mathbf{A}\mathbf{a} + \mathbf{a} = \begin{pmatrix}-2\\2\end{pmatrix} \neq \mathbf{0}$. Therefore, the solution cannot be of the form $\mathbf{a}te^{-t}$. Instead, try

$$\mathbf{x}(t) = \mathbf{a}te^{-t} + \mathbf{b}e^{-t}, \tag{618}$$

then

$$\mathbf{x}'(t) - \mathbf{A}\mathbf{x}(t) = -te^{-t}(\mathbf{A}\mathbf{a} + \mathbf{a}) - e^{-t}(\mathbf{b} + \mathbf{A}\mathbf{b} - \mathbf{a}) = \mathbf{g}_1(t). \tag{619}$$

Thus, we should have

$$\mathbf{A}\mathbf{a} + \mathbf{a} = \mathbf{0}, \quad \mathbf{b} + \mathbf{A}\mathbf{b} - \mathbf{a} = \begin{pmatrix}-2\\0\end{pmatrix}. \tag{620}$$

That is, $\mathbf{a}$ should be an eigenvector to the eigenvalue $r = -1$, so we take $\mathbf{a} = \begin{pmatrix}1\\1\end{pmatrix}$. Then

$$\mathbf{A}\mathbf{b} + \mathbf{b} = \begin{pmatrix}-1\\1\end{pmatrix} \Rightarrow -b_1 + b_2 = -1. \tag{621}$$

We can take $b_1 = 0, b_2 = -1$, so a particular solution to $\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{g}_1(t)$ is

$$\mathbf{x}(t) = \begin{pmatrix}1\\1\end{pmatrix} te^{-t} + \begin{pmatrix}0\\-1\end{pmatrix} e^{-t}. \tag{622}$$

For a particular solution to $\mathbf{y}'(t) = \mathbf{A}\mathbf{y}(t) + \mathbf{g}_2(t)$, try a trial solution of the form

$$\mathbf{z}(t) = \mathbf{c}t + \mathbf{d}. \tag{623}$$

Then,

$$\mathbf{z}'(t) - \mathbf{A}\mathbf{z}(t) = (\mathbf{c} - \mathbf{A}\mathbf{d}) - t\mathbf{A}\mathbf{c} = \begin{pmatrix}0\\3\end{pmatrix} t. \tag{624}$$

Hence, we require

$$\mathbf{A}\mathbf{c} = \begin{pmatrix}0\\-3\end{pmatrix}, \quad \mathbf{A}\mathbf{d} = \mathbf{c}. \tag{625}$$

Solving these equations gives

$$\mathbf{c} = \begin{pmatrix}1\\2\end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix}-4/3\\-5/3\end{pmatrix} \Rightarrow \mathbf{z}(t) = \begin{pmatrix}1\\2\end{pmatrix} t + \begin{pmatrix}-4/3\\-5/3\end{pmatrix}. \tag{626}$$

According to extension of Theorem 37, $\mathbf{x}(t) + \mathbf{z}(t)$ is a particular solution to the original system.

Example 90. *Find a particular solution to*

$$\mathbf{y}'(t) = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix} \mathbf{y}(t) + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}. \tag{627}$$

The eigenvalues of matrix $\mathbf{A}$ are $r_1 = -3, r_2 = 2$ with corresponding eigenvectors

$$\xi_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}. \tag{628}$$

So the general solution to the homogeneous system is

$$\mathbf{y}_c(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2t}. \tag{629}$$

Write the term $\mathbf{g}(t)$ as

$$\mathbf{g}(t) = e^{-2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{t} \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \tag{630}$$

and since neither -2 nor 1 are eigenvalues of $\mathbf{A}$, we try a trial solution of the form

$$\mathbf{z}(t) = \mathbf{a}e^{-2t} + \mathbf{b}e^{t}. \tag{631}$$

Then, computing

$$\mathbf{z}'(t) - \mathbf{A}\mathbf{z}(t) = e^{-2t}(-2\mathbf{a} - \mathbf{A}\mathbf{a}) + e^{t}(\mathbf{b} - \mathbf{A}\mathbf{b}) = e^{-2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{t} \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \tag{632}$$

and comparing coefficients, we need

$$(-2\mathbf{I} - \mathbf{A})\mathbf{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (\mathbf{I} - \mathbf{A})\mathbf{b} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}. \tag{633}$$

Solving these equations gives

$$\mathbf{a} = \begin{pmatrix} 0 \\ -0.25 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \tag{634}$$

so a particular solution is

$$\mathbf{Y}(t) = \begin{pmatrix} 0 \\ -0.25 \end{pmatrix} e^{-2t} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{t}. \tag{635}$$

Example 91. *Find a particular solution to*

$$\mathbf{y}'(t) = \begin{pmatrix} 1 & 5 \\ -1 & 1 \end{pmatrix} \mathbf{y}(t) + \begin{pmatrix} e^{2t} \\ \sin(2t) \end{pmatrix}. \tag{636}$$

The eigenvalues of matrix $\mathbf{A}$ are $r = 1 \pm i\sqrt{5}$, so no additional term is needed in the trial solution. Write the term $\mathbf{g}(t)$ as

$$\mathbf{g}(t) = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}}_{\mathbf{g}_1(t)} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2t)}_{\mathbf{g}_2(t)}. \tag{637}$$

First, corresponding to $\mathbf{g}_1(t)$, try a trial solution of the form

$$\mathbf{y}_1(t) = \mathbf{a}e^{2t} \tag{638}$$

Plug in $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}_1$ we obtain

$$2\mathbf{a}e^{2t} = \mathbf{A}\mathbf{a}e^{2t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} \Rightarrow (2\mathbf{I} - \mathbf{A})\mathbf{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{639}$$

Solve for $\mathbf{a}$

$$\mathbf{a} = \begin{pmatrix} 1/6 \\ -1/6 \end{pmatrix} \tag{640}$$

So the particular solution corresponding to the first part is

$$\mathbf{y}_1(t) = \begin{pmatrix} 1/6 \\ -1/6 \end{pmatrix} e^{2t} \tag{641}$$

Next, corresponding to $\mathbf{g}_2(t)$, try a trial solution of the form

$$\mathbf{y}_2(t) = \mathbf{b}\cos(2t) + \mathbf{c}\sin(2t) \tag{642}$$

Plug in $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}_2$ we obtain

$$-2\mathbf{b}\sin(2t) + 2\mathbf{c}\cos(2t) = \mathbf{A}(\mathbf{b}\cos(2t) + \mathbf{c}\sin(2t)) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin(2t) \tag{643}$$

Compare the coefficients:

- **Coefficient of** $\cos(2t)$**:**

$$2\mathbf{c} = \mathbf{A}\mathbf{b} \Rightarrow \mathbf{c} = \frac{1}{2}\mathbf{A}\mathbf{b} \tag{644}$$

- **Coefficient of** $\sin(2t)$**:**

$$-2\mathbf{b} = \mathbf{A}\mathbf{c} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{645}$$

Substituting $\mathbf{c} = \frac{1}{2}\mathbf{A}\mathbf{b}$ into the second equation:

$$-2\mathbf{b} = \mathbf{A}\left(\frac{1}{2}\mathbf{A}\mathbf{b}\right) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow (\mathbf{A}^2 + 4\mathbf{I})\mathbf{b} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \tag{646}$$

Solve for $\mathbf{b}$ and $\mathbf{c}$

$$\mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} \tag{647}$$

So the particular solution corresponding to the second part is

$$\mathbf{y}_2(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(2t) + \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} \sin(2t) \tag{648}$$

$\mathbf{y}_1(t) + \mathbf{y}_2(t)$ is a particular solution to the original system.

5.6.2 Variation of Parameters

We now consider more general non-homogeneous first-order systems of the form

$$\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}(t) + \mathbf{g}(t), \tag{649}$$

where the matrix $\mathbf{P}(t) = (p_{ij}(t))_{n\times n}$, and $p_{ij}(t), i, j = 1, \cdots, n$ are continuous on the interval I. First, we neglect the non-homogeneous term and study the homogeneous system $\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}(t)$. Recall, if $\mathbf{y}_1(t), \ldots, \mathbf{y}_n(t)$ form a FSS to the homogeneous system $\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}(t)$, then the fudamental matrix $\mathbf{F}$ is defined as

$$\mathbf{F}(t) = \begin{pmatrix} | & | & \cdots & | \\ \mathbf{y}_1(t) & \mathbf{y}_2(t) & \cdots & \mathbf{y}_n(t) \\ | & | & \cdots & | \end{pmatrix} = \begin{pmatrix} y_{11}(t) & y_{12}(t) & \cdots & y_{1n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1}(t) & y_{n2}(t) & \cdots & y_{nn}(t) \end{pmatrix} \tag{650}$$

Property 92. *The fundamental matrix* $\mathbf{F}$ *satisfies*

$$\frac{d\mathbf{F}(t)}{dt} = \mathbf{P}(t)\mathbf{F}(t) \tag{651}$$

Proof.

$$\frac{d}{dt}\mathbf{F}(t) = \frac{d}{dt}\begin{pmatrix} | & | & \cdots & | \\ \mathbf{y}_1(t) & \mathbf{y}_2(t) & \cdots & \mathbf{y}_n(t) \\ | & | & \cdots & | \end{pmatrix} = \begin{pmatrix} | & | & \cdots & | \\ \frac{d}{dt}\mathbf{y}_1(t) & \frac{d}{dt}\mathbf{y}_2(t) & \cdots & \frac{d}{dt}\mathbf{y}_n(t) \\ | & | & \cdots & | \end{pmatrix} \tag{652}$$

$$= \begin{pmatrix} | & | & \cdots & | \\ \mathbf{P}(t)\mathbf{y}_1(t) & \mathbf{P}(t)\mathbf{y}_2(t) & \cdots & \mathbf{P}(t)\mathbf{y}_n(t) \\ | & | & \cdots & | \end{pmatrix} = \mathbf{P}(t)\mathbf{F}(t). \tag{653}$$

□

Return to the non-homogeneous system $\mathbf{y}'(t) = \mathbf{P}(t)\mathbf{y}(t) + \mathbf{g}(t)$. Assume we have a fundamental matrix $\mathbf{F}(t)$ to the homogeneous system with complementary solution

$$\mathbf{y}_c(t) = \mathbf{F}(t)\mathbf{c}, \tag{654}$$

where $\mathbf{c}$ is a constant vector. The method of variation of parameters is to consider a trial solution

$$\mathbf{z}(t) = \mathbf{F}(t)\mathbf{u}(t), \tag{655}$$

where $\mathbf{u}(t)$ is a vector of functions. Then, if $\mathbf{z}$ is a solution to the non-homogeneous system, we have

$$\mathbf{P}(t)\mathbf{F}(t)\mathbf{u}(t) + \mathbf{g}(t) = \mathbf{z}'(t) = \mathbf{F}(t)\mathbf{u}'(t) + \mathbf{F}'(t)\mathbf{u}(t). \tag{656}$$

Since $\mathbf{F}(t)$ is a fundamental matrix, i.e., $\mathbf{F}'(t) = \mathbf{P}(t)\mathbf{F}(t)$, we see that

$$\mathbf{F}(t)\mathbf{u}'(t) = \mathbf{g}(t) \Rightarrow \mathbf{u}'(t) = \mathbf{F}^{-1}(t)\mathbf{g}(t). \tag{657}$$

($\mathbf{F}(t)$ is invertible for any $t \in I$.) Integration gives

$$\mathbf{u}(t) = \int \mathbf{F}^{-1}(t)\mathbf{g}(t)dt. \tag{658}$$

The particular solution is

$$\mathbf{z}(t) = \mathbf{F}(t) \int \mathbf{F}^{-1}(t)\mathbf{g}(t)dt. \tag{659}$$

Therefore the general solution to the non-homogeneous system is

$$\mathbf{y}(t) = \mathbf{y}_c(t) + \mathbf{z}(t) = \mathbf{F}(t)\mathbf{c} + \mathbf{F}(t) \int \mathbf{F}^{-1}(t)\mathbf{g}(t)dt. \tag{660}$$

If we are also given initial conditions $\mathbf{y}(t_0) = \mathbf{v}$, then in the integral we write

$$\int_{t_0}^{t} \mathbf{F}^{-1}(s)\mathbf{g}(s)ds \tag{661}$$

and

$$\mathbf{v} = \mathbf{y}(t_0) = \mathbf{F}(t_0)\mathbf{c} \Rightarrow \mathbf{c} = \mathbf{F}^{-1}(t_0)\mathbf{v}. \tag{662}$$

Hence, the unique solution to the IVP in the interval I is

$$\mathbf{y}(t) = \mathbf{F}(t)\mathbf{F}^{-1}(t_0)\mathbf{v} + \mathbf{F}(t)\left[\int_{t_0}^{t} \mathbf{F}^{-1}(s)\mathbf{g}(s)ds\right]. \tag{663}$$

Example 93. *Find a particular solution to*

$$\mathbf{y}'(t) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{y}(t) + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}. \tag{664}$$

Recall, the complementary solution to the homogeneous system is

$$\mathbf{y}_c(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}. \tag{665}$$

And we used the method of undetermined coefficients to find one particular solution

$$\mathbf{Y}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3}\begin{pmatrix} 4 \\ 5 \end{pmatrix}. \tag{666}$$

Compute the fundamental matrix and its inverse

$$\mathbf{F}(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix}, \quad \mathbf{F}^{-1}(t) = \frac{1}{2}\begin{pmatrix} e^{3t} & -e^{3t} \\ e^{t} & e^{t} \end{pmatrix} \tag{667}$$

then find the unknown coefficients by solving

$$\mathbf{u}'(t) = \mathbf{F}^{-1}(t)\mathbf{g}(t) \Rightarrow \begin{cases} u_1'(t) = e^{2t} - \frac{3}{2}te^{3t}, \\ u_2'(t) = 1 + \frac{3}{2}te^{t}. \end{cases} \tag{668}$$

This gives

$$u_1(t) = \frac{1}{2}e^{2t} - \frac{1}{2}te^{3t} + \frac{1}{6}e^{3t}, \quad u_2(t) = t + \frac{3}{2}te^{t} - \frac{3}{2}e^{t}, \tag{669}$$

where we used

$$\int te^{\alpha t}dt = \frac{\alpha t - 1}{\alpha^2}e^{\alpha t}. \tag{670}$$

Hence, a particular solution is

$$\mathbf{Z}(t) = \mathbf{F}(t)\mathbf{u}(t) = \begin{pmatrix}1\\1\end{pmatrix} te^{-t} + \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix} e^{-t} + \begin{pmatrix}1\\2\end{pmatrix} t - \frac{1}{3}\begin{pmatrix}4\\5\end{pmatrix}. \tag{671}$$

Note that $\mathbf{Y}(t)$ obtained from the method of undetermined coefficients is different from $\mathbf{Z}(t)$ obtained from the variation of parameters. In fact, both $\mathbf{Y}$ and $\mathbf{Z}$ are particular solutions,and the corresponding general solutions to the non-homogeneous system are equivalent:

$$\mathbf{y}(t) = c_1\begin{pmatrix}1\\-1\end{pmatrix} e^{3t} + c_2\begin{pmatrix}1\\1\end{pmatrix} e^{-t} + \mathbf{Y}(t) = d_1\begin{pmatrix}1\\-1\end{pmatrix} e^{3t} + d_2\begin{pmatrix}1\\1\end{pmatrix} e^{-t} + \mathbf{Z}(t) \tag{672}$$

if we choose

$$c_1 = d_1, \quad c_2 = d_2 + \frac{1}{2}. \tag{673}$$

Example 94. *Find a particular solution to*

$$\mathbf{y}'(t) = \begin{pmatrix}1 & 4\\1 & -2\end{pmatrix}\mathbf{y}(t) + \begin{pmatrix}e^{-2t}\\-2e^{t}\end{pmatrix}. \tag{674}$$

Recall, the complementary solution to the homogeneous system is

$$\mathbf{y}_c(t) = c_1\begin{pmatrix}1\\-1\end{pmatrix} e^{-3t} + c_2\begin{pmatrix}4\\1\end{pmatrix} e^{2t}. \tag{675}$$

And we used the method of undetermined coefficients to find one particular solution

$$\mathbf{Y}(t) = \begin{pmatrix}0\\-0.25\end{pmatrix} e^{-2t} + \begin{pmatrix}2\\0\end{pmatrix} e^{t}. \tag{676}$$

Compute the fundamental matrix and its inverse

$$\mathbf{F}(t) = \begin{pmatrix} e^{-3t} & 4e^{2t}\\ -e^{-3t} & e^{2t}\end{pmatrix}, \quad \mathbf{F}^{-1}(t) = \frac{1}{5}\begin{pmatrix} e^{3t} & -4e^{3t}\\ e^{-2t} & e^{-2t}\end{pmatrix} \tag{677}$$

Then solve for the unknown coefficients

$$\mathbf{u}'(t) = \mathbf{F}^{-1}(t)\mathbf{g}(t) \Rightarrow \begin{cases} u_1'(t) = \frac{1}{5}(e^{t} + 8e^{4t}),\\ u_2'(t) = \frac{1}{5}(e^{-4t} - 2e^{-t}).\end{cases} \tag{678}$$

This gives

$$u_1(t) = \frac{1}{5}e^{t} + \frac{2}{5}e^{4t}, \quad u_2(t) = -\frac{1}{20}e^{-4t} + \frac{2}{5}e^{-t}, \tag{679}$$

and the particular solution is

$$\mathbf{Z}(t) = \mathbf{F}(t)\mathbf{u}(t) = \begin{pmatrix} 2e^{t}\\ -0.25e^{-2t}\end{pmatrix}, \tag{680}$$

which coincides with particular solution obtained from the method of undetermined coefficients.

6 Laplace Transform

6.1 Review of Improper Integrals

An improper integral over an unbounded interval is defined as a limit of integrals over finite intervals:

$$\int_a^\infty f(t)dt = \lim_{A\to\infty} \int_a^A f(t)dt, \tag{681}$$

where A is a positive real number. If the integral from a to A exists for each $A > a$, and if the limit as $A \to \infty$ exists, then the improper integral **converges**. Otherwise it **diverges**.

Example 95. *Let $f(t) = e^{ct}, t \geq 0$, where c is a real nonzero constant. Then*

$$\int_0^\infty e^{ct}dt = \lim_{A\to\infty} \int_0^A e^{ct}dt = \lim_{A\to\infty} \frac{e^{ct}}{c}\Big|_0^A = \lim_{A\to\infty} \frac{1}{c}(e^{cA} - 1). \tag{682}$$

If $c < 0$, the improper integral converges to $-1/c$ and it diverges if $c > 0$. If $c = 0$, the integrand $f(t) = 1$. In this case

$$\lim_{A\to\infty} \int_0^A 1dt = \lim_{A\to\infty} (A - 0) = \infty, \tag{683}$$

so the integral diverges.

A function f is said to be **piecewise continuous** if it is continuous except for a finite number of jump discontinuities. If f is piecewise continuous on $\alpha \leq t \leq \beta$ for every $\beta > \alpha$, then f is said to be piecewise continuous on $t \geq \alpha$.

Example 96. *The function below is piecewise continuous on $t \geq 0$:*

$$f(t) = \begin{cases} t^2 & \text{if } 0 \leq t < 2 \\ 5 & \text{if } 2 \leq t < 4 \\ e^{-t} & \text{if } t \geq 4 \end{cases} \tag{684}$$

The integral of a piecewise continuous function on a finite interval is just the sum of the integrals on the subintervals created by the partition points. For instance, for the function $f(t)$ shown in Figure 2, we have

$$\int_\alpha^\beta f(t)\,dt = \int_\alpha^{t_1} f(t)\,dt + \int_{t_1}^{t_2} f(t)\,dt + \int_{t_2}^\beta f(t)\,dt. \tag{685}$$

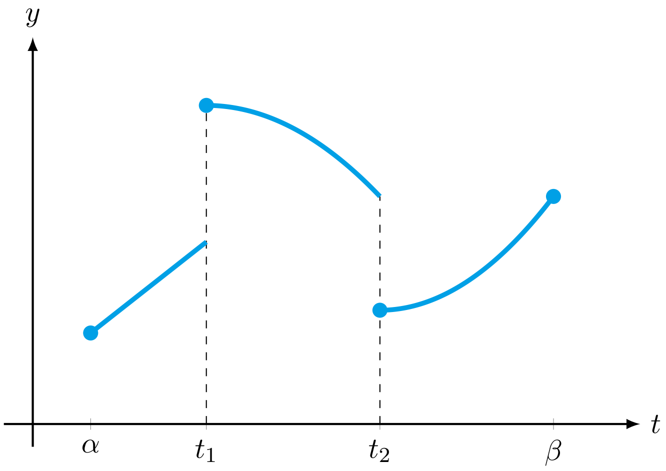


Figure 2: A piecewise continuous function $y = f(t)$.

We will conclude this section with an important theorem.

Theorem 97 (Comparison Theorem)**.** *If f is piecewise continuous for $t \geq a$, if $|f(t)| \leq g(t)$ when $t \geq M$ for some positive constant M, and if $\int_M^\infty g(t)dt$ converges, then $\int_a^\infty f(t)dt$ also converges. On the other hand, if $f(t) \geq g(t) \geq 0$ for $t \geq M$, and if $\int_M^\infty g(t)dt$ diverges, then $\int_a^\infty f(t)dt$ also diverges.*

6.2 Laplace Transform

6.2.1 Definition

Definition 98 (Laplace Transform)**.** *Suppose that $f(t)$ is a real function defined on $[0, +\infty)$ and*

1. *f is piecewise continuous on the interval $0 \le t \le A$ for any positive A.*
2. *$|f(t)| \le Ke^{at}$ when $t \ge M$. In this inequality, K, a, and M are real constants ($K > 0, M > 0$). ($f(t)$ is of **exponential order**.)*

Then the Laplace transform $\mathcal{L}\{f(t)\}(s) = F(s)$ is defined as

$$\mathcal{L}\{f(t)\}(s) = F(s) = \int_0^{\infty} e^{-st} f(t)dt, \tag{686}$$

exists for $s > a$.

Note that there is no restriction on the range of values of a. In fact, a represents the growth rate of the function $f(t)$.

- If $a > 0$, it indicates that the function grows exponentially with time (e.g., $f(t) = e^{2t}$).
- If $a = 0$, it indicates that the function is bounded (e.g., $f(t) = \sin t$ or a constant function).
- If $a < 0$, it indicates that the function decays with time (e.g., $f(t) = e^{-t}$).

The definition points out that the condition for the existence of the Laplace transform is $s > a$. This implies that regardless of whether a is positive, negative, or zero, as long as $s > a$, the integral is convergent.

Next, we will prove that this definition is valid.

Proof. To establish this definition, we must show that the integral in (686) converges for $s > a$. Splitting the improper integral into two parts, we have

$$\int_0^{\infty} e^{-st} f(t)dt = \int_0^{M} e^{-st} f(t)dt + \int_M^{\infty} e^{-st} f(t)dt. \tag{687}$$

The first integral on the right side of (687) exists by Hypothesis 1 of the definition; hence the existence of $F(s)$ depends on the convergence of the second integral. By Hypothesis 2 we have, for $t \ge M$,

$$|e^{-st} f(t)| \le Ke^{-st}e^{at} = Ke^{(a-s)t}. \tag{688}$$

By the comparison theorem, $F(s)$ exists provided that $\int_M^{\infty} e^{(a-s)t}dt$ converges, which is true since $a - s < 0$. □

Example 99. *We will compute Laplace transform for several functions.*

1. *Let $f(t) = 1$, $t \ge 0$. Then,*

$$\mathcal{L}\{1\} = \int_0^{\infty} e^{-st}dt = -\lim_{A\to\infty} \left.\frac{e^{-st}}{s}\right|_0^A = \frac{1}{s}, \quad s > 0. \tag{689}$$

2. *Let* $f(t) = e^{at}$, $t \geq 0$. *Then,*

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}e^{at}dt = \int_0^\infty e^{-(s-a)t}dt = \frac{1}{s-a}, \quad s > a. \tag{690}$$

3. *Let* $f(t) = \sin at$, $t \geq 0$. *Then*

$$\mathcal{L}\{\sin at\} = F(s) = \int_0^\infty e^{-st}\sin atdt, \quad s > 0. \tag{691}$$

Since $F(s) = \lim_{A\to\infty}\int_0^A e^{-st}\sin atdt$, *upon integrating by parts, we obtain*

$$F(s) = \lim_{A\to\infty}\left[-\frac{e^{-st}\cos at}{a}\bigg|_0^A - \frac{s}{a}\int_0^A e^{-st}\cos atdt\right] = \frac{1}{a} - \frac{s}{a}\int_0^\infty e^{-st}\cos atdt. \tag{692}$$

A second integration by parts then yields

$$F(s) = \frac{1}{a} - \frac{s^2}{a^2}\int_0^\infty e^{-st}\sin atdt = \frac{1}{a} - \frac{s^2}{a^2}F(s). \tag{693}$$

Hence, solving for $F(s)$, *we have*

$$F(s) = \frac{a}{s^2+a^2} \quad s > 0. \tag{694}$$

Next, we will introduce inverse Laplace transform.

Definition 100 (Inverse Laplace Transform)**.** *If the Laplace transform* $\mathcal{L}\{f(t)\} = F(s)$ *is defined as*

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}f(t)dt, \tag{695}$$

then the inverse Laplace transform for $F(s) = \mathcal{L}\{f(t)\}$ *is defined as*

$$\mathcal{L}^{-1}\{F(s)\} = f(t), \tag{696}$$

Thus,

$$\mathcal{L}^{-1}\{F(s)\} = f(t) \Leftrightarrow \mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}f(t)dt. \tag{697}$$

Example 101. *We will compute inverse Laplace transform for several functions.*

1. *Let* $f(t) = 1$, $t \geq 0$. *Then,*

$$\mathcal{L}\{1\} = \frac{1}{s}, \quad s > 0. \quad \mathcal{L}^{-1}\{\frac{1}{s}\} = 1. \tag{698}$$

2. *Let* $f(t) = e^{at}$, $t \geq 0$. *Then,*

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a. \quad \mathcal{L}^{-1}\{\frac{1}{s-a}\} = e^{at}. \tag{699}$$

3. *Let* $f(t) = \sin at$, $t \geq 0$. *Then*

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2} \quad s > 0. \quad \mathcal{L}^{-1}\{\frac{a}{s^2 + a^2}\} = \sin at. \tag{700}$$

We conclude this section with a table for elementary Laplace transforms.

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$		$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	Constant c	$\frac{c}{s}$, $s > 0$	10.	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2 + b^2}$, $s > a$
2.	e^{at}	$\frac{1}{s-a}$, $s > a$	11.	$t^n e^{at}$, $n \in \mathbb{Z}^+$	$\frac{n!}{(s-a)^{n+1}}$, $s > a$
3.	t^n, $n \in \mathbb{Z}^+$	$\frac{n!}{s^{n+1}}$, $s > 0$	12.	$u_c(t)$*	$\frac{e^{-cs}}{s}$, $s > 0$
4.	t^p, $p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$, $s > 0$	13.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
5.	$\sin at$	$\frac{a}{s^2 + a^2}$, $s > 0$	14.	$e^{ct}f(t)$	$F(s-c)$
6.	$\cos at$	$\frac{s}{s^2 + a^2}$, $s > 0$	15.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$, $c > 0$
7.	$\sinh at$	$\frac{a}{s^2 - a^2}$, $s > \lvert a\rvert$	16.	$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$
8.	$\cosh at$	$\frac{s}{s^2 - a^2}$, $s > \lvert a\rvert$	17.	$\delta(t-c)$*	e^{-cs}
9.	$e^{at}\sin bt$	$\frac{b}{(s-a)^2 + b^2}$, $s > a$			

*: Unit step function, to be introduced in Section 6.4.
*: Dirac delta function, to be introduced in Section 6.5.

6.2.2 Properties

Theorem 102. *If* $\mathcal{L}\{f_1(t)\}(s)$ *and* $\mathcal{L}\{f_2(t)\}(s)$ *are defined for* $s > a$, *and* c_1, c_2 *are any numbers, then*

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\}. \tag{701}$$

Proof.

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = \int_0^\infty e^{-st}[c_1 f_1(t) + c_2 f_2(t)]dt = c_1 \int_0^\infty e^{-st} f_1(t)dt + c_2 \int_0^\infty e^{-st} f_2(t)dt. \tag{702}$$

□

Example 103. *Find the Laplace transform of $f(t) = 5e^{-2t} - 3\sin 4t$, $t \geq 0$. We have*

$$\mathcal{L}\{f(t)\} = 5\mathcal{L}\{e^{-2t}\} - 3\mathcal{L}\{\sin 4t\} \tag{703}$$

$$= \frac{5}{s+2} - \frac{12}{s^2+16}, \quad s > 0. \tag{704}$$

We will introduce another theorem that will be frequently used.

Theorem 104. *Suppose that f is continuous and f' is piecewise continuous on any interval $0 \leq t \leq A$. Suppose further that there exist constants K, a, and M such that $|f(t)| \leq Ke^{at}$ for $t \geq M$. Then $\mathcal{L}\{f'(t)\}$ exists for $s > a$. Moreover,*

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0). \tag{705}$$

Proof.

$$\int_0^\infty e^{-st} f'(t)dt = \lim_{A\to\infty} \int_0^A e^{-st} f'(t)dt. \tag{706}$$

If f' has points of discontinuity (denoted by $t_1, t_2, \ldots, t_k$) in the interval $0 \leq t \leq A$, Then the above integral can be rewritten as

$$\int_0^A e^{-st} f'(t)dt = \int_0^{t_1} e^{-st} f'(t)dt + \int_{t_1}^{t_2} e^{-st} f'(t)dt + \cdots + \int_{t_k}^A e^{-st} f'(t)dt. \tag{707}$$

Using integration by parts, one has

$$\begin{aligned} \int_0^A e^{-st} f'(t)dt &= e^{-st} f(t)\Big|_0^{t_1} + e^{-st} f(t)\Big|_{t_1}^{t_2} + \cdots + e^{-st} f(t)\Big|_{t_k}^{A} \\ &\quad + s\left[\int_0^{t_1} e^{-st} f(t)dt + \int_{t_1}^{t_2} e^{-st} f(t)dt + \cdots + \int_{t_k}^A e^{-st} f(t)dt\right]. \end{aligned} \tag{708}$$

Since f is continuous, we obtain

$$\int_0^A e^{-st} f'(t)dt = e^{-sA} f(A) - f(0) + s\int_0^A e^{-st} f(t)dt. \tag{709}$$

Let $A \to \infty$, $\lim_{A\to\infty} \int_0^A e^{-st} f(t)dt = \mathcal{L}\{f(t)\}$. Further, for $A \geq M$, we have $|f(A)| \leq Ke^{aA}$; consequently, $|e^{-sA} f(A)| \leq Ke^{-(s-a)A}$. Hence $e^{-sA} f(A) \to 0$ as $A \to \infty$ whenever $s > a$. Thus the right side of (709) has the limit $s\mathcal{L}\{f(t)\} - f(0)$. Consequently, the left side of (709) also has a limit $\mathcal{L}\{f'(t)\}$. Therefore, for $s > a$, we conclude that

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0), \ s > a. \tag{710}$$

□

A corollary of this theorem is given by the following result.

Theorem 105. *Suppose that* $f, f', \dots, f^{(n-1)}$ *are continuous and* $f^{(n)}$ *is piecewise continuous on any interval* $0 \leq t \leq A$. *Suppose further that there exist constants* K, a *and* M *such that* $|f^{(i)}(t)| \leq Ke^{at}$, $i = 0, 1, \dots, n-1$ *for* $t \geq M$. *Then* $\mathcal{L}\{f^{(n)}(t)\}$ *exists for* $s > a$, *and*

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0). \tag{711}$$

where $F(s) = \mathcal{L}\{f(t)\}$.

We will conclude this section with one last property.

Theorem 106. *If* $F(s) = \mathcal{L}\{f(t)\}$ *exists for* $s > a$, *and if* c *is a constant, then*

$$\mathcal{L}\{e^{ct} f(t)\} = F(s-c), \quad s > a + c. \tag{712}$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, *then*

$$e^{ct} f(t) = \mathcal{L}^{-1}\{F(s-c)\}. \tag{713}$$

In other words, multiplication of $f(t)$ *by* e^{ct} *results in a translation of the transform* $F(s)$ *a distance* c *in the positive* s *direction, and conversely.*

Proof. We evaluate $\mathcal{L}\{e^{ct} f(t)\}$:

$$\mathcal{L}\{e^{ct} f(t)\} = \int_0^\infty e^{-st} e^{ct} f(t) dt = \int_0^\infty e^{-(s-c)t} f(t) dt = F(s-c), \tag{714}$$

which is (712). The restriction $s > a + c$ follows from $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a$. (713) is obtained by taking the inverse Laplace transform of (712). $\square$

Example 107. *Let* $f(t) = e^{ct} \sin at$, $t \geq 0$. *Since*

$$\mathcal{L}\{\sin at\} = F(s) = \int_0^\infty e^{-st} \sin at dt = \frac{a}{s^2 + a^2}, \quad s > 0, \tag{715}$$

we have

$$\mathcal{L}\{e^{ct} \sin at\} = \int_0^\infty e^{-(s-c)t} \sin at dt = F(s-c) = \frac{a}{(s-c)^2 + a^2}, \quad s > c, \tag{716}$$

and

$$\mathcal{L}^{-1}\left\{\frac{a}{(s-c)^2 + a^2}\right\} = e^{ct} \sin at, \quad s > c. \tag{717}$$

Example 108. *Find the inverse transform of*

$$G(s) = \frac{1}{s^2 - 4s + 5}. \tag{718}$$

By completing the square in the denominator, we can write

$$G(s) = \frac{1}{(s-2)^2 + 1} = F(s-2), \tag{719}$$

where $F(s) = (s^2 + 1)^{-1}$. *Since* $\mathcal{L}^{-1}\{F(s)\} = \sin t$, *it follows that*

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = e^{2t} \sin t. \tag{720}$$

6.3 Solving Initial Value Problems

The method of using the Laplace transform to solve an initial value problem is summarized as follows.

1. Use the Laplace transform to transform an initial value problem for an unknown function f in the t-domain into an algebraic problem for F in the s-domain.

2. Solve this algebraic problem to find F.

3. Recover the desired function f from its transform F by taking the inverse Laplace transform.

6.3.1 Second-Order Linear ODE

Consider the following second-order linear equation with constant coefficients

$$ay'' + by' + cy = f(t). \tag{721}$$

Assuming that the solution $y = \phi(t)$ satisfies the conditions of Theorem 105 for $n = 2$, take the transform of (721) and thereby obtain

$$a[s^2Y(s) - sy(0) - y'(0)] + b[sY(s) - y(0)] + cY(s) = F(s), \tag{722}$$

where $F(s)$ is the transform of $f(t)$. By solving (722) for $Y(s)$, we have

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}. \tag{723}$$

The problem is then solved, provided that we can find the function $y = \phi(t)$ whose Laplace transform is $Y(s)$, i.e., $\phi(t) = \mathcal{L}^{-1}\{Y(s)\}$.

Example 109. *Find the solution to the IVP*

$$y'' - y' - 2y = 0, \quad y(0) = 1, y'(0) = 0. \tag{724}$$

We assume that the IVP has a solution $y = \phi(t)$, which with its first two derivatives satisfies the conditions of Theorem 105. Then, taking the Laplace transform of the differential equation, we have

$$s^2Y(s) - sy(0) - y'(0) - (sY(s) - y(0)) - 2Y(s) = 0, \tag{725}$$

Substituting for $y(0)$ and $y'(0)$ from the initial conditions and solving for $Y(s)$, we obtain

$$Y(s) = \frac{s-1}{s^2 - s - 2} = \frac{1}{3}\frac{1}{s-2} + \frac{2}{3}\frac{1}{s+1}. \tag{726}$$

Take the inverse Laplace transform we have

$$y(t) = \mathcal{L}^{-1}(\frac{1}{3}\frac{1}{s-2}) + \mathcal{L}^{-1}(\frac{2}{3}\frac{1}{s+1}) = \frac{1}{3}\mathcal{L}^{-1}(\frac{1}{s-2}) + \frac{2}{3}\mathcal{L}^{-1}(\frac{1}{s+1}) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}. \tag{727}$$

Example 110. *Find the solution to the IVP*

$$y'' + y = \sin 2t, \quad y(0) = 2, y'(0) = 1. \tag{728}$$

We assume that the IVP has a solution $y = \phi(t)$, which with its first two derivatives satisfies the conditions of Theorem 105. Then, taking the Laplace transform of the differential equation, we have

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = 2/(s^2 + 4), \tag{729}$$

Substituting for $y(0)$ and $y'(0)$ from the initial conditions and solving for $Y(s)$, we obtain

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2 + 1)(s^2 + 4)}. \tag{730}$$

Using partial fractions, we can write $Y(s)$ in the form

$$Y(s) = \frac{as + b}{s^2 + 1} + \frac{cs + d}{s^2 + 4} = \frac{(as + b)(s^2 + 4) + (cs + d)(s^2 + 1)}{(s^2 + 1)(s^2 + 4)}. \tag{731}$$

By expanding the numerator on the right side of (731) and equating it to the numerator in (730), we find that

$$2s^3 + s^2 + 8s + 6 = (a + c)s^3 + (b + d)s^2 + (4a + c)s + (4b + d) \tag{732}$$

for all s. Then, comparing coefficients of like powers of s, we have

$$a + c = 2, \quad b + d = 1, \quad 4a + c = 8, \quad 4b + d = 6. \tag{733}$$

Consequently, $a = 2, c = 0, b = 5/3$, and $d = -2/3$, from which it follows that

$$Y(s) = \frac{2s}{s^2 + 1} + \frac{5/3}{s^2 + 1} - \frac{2/3}{s^2 + 4}. \tag{734}$$

The solution of the given initial value problem is

$$y = \phi(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t. \tag{735}$$

6.3.2 Higher Order Linear ODE

The Laplace transform method can be used with linear differential equations of higher order than second-order, as long as the coefficients in the equation are constants. Below we show how we can solve a fourth-order equation.

Example 111. *Find the solution to the IVP*

$$y^{(4)} - y = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y'''(0) = 0. \tag{736}$$

We assume that the solution $y = \phi(t)$ satisfies the conditions of Theorem 105 for $n = 4$. The Laplace transform of the differential equation is

$$s^4 Y(s) - s^3 y(0) - s^2 y'(0) - sy''(0) - y'''(0) - Y(s) = 0. \tag{737}$$

Then, using the initial conditions and solving for $Y(s)$, we have

$$Y(s) = \frac{s^2}{s^4 - 1}. \tag{738}$$

A partial fraction expansion of $Y(s)$ is

$$Y(s) = \frac{as + b}{s^2 - 1} + \frac{cs + d}{s^2 + 1}. \tag{739}$$

And it follows that

$$(as + b)(s^2 + 1) + (cs + d)(s^2 - 1) = s^2 \tag{740}$$

for all s. By setting $s = 1$ and $s = -1$, respectively, in (740), we obtain the pair of equations

$$2(a + b) = 1, \quad 2(-a + b) = 1, \tag{741}$$

and therefore $a = 0$ and $b = 1/2$. If we set $s = 0$ in (740), then $b - d = 0$, so $d = 1/2$. Finally, equating the coefficients of the cubic terms on each side of (740), we find that $a + c = 0$, so $c = 0$. Thus

$$Y(s) = \frac{1/2}{s^2 - 1} + \frac{1/2}{s^2 + 1}, \tag{742}$$

the solution of the initial value problem is

$$y = \phi(t) = \frac{\sinh t + \sin t}{2}. \tag{743}$$

6.3.3 System of Linear ODE

The Laplace transform method can be used for the system of linear differential equations, as long as the coefficients in the equation are constants.

Example 112. *Find the solution to the IVP*

$$x' = x + 2y, \quad y' = 2x + y, \quad x(0) = 0, \quad y(0) = 2. \tag{744}$$

The Laplace transform of the differential equation is

$$sX(s) - x(0) = X(s) + 2Y(s), \quad sY(s) - y(0) = 2X(s) + Y(s) \tag{745}$$

Thus

$$sX(s) = X(s) + 2Y(s), \quad sY(s) - 2 = 2X(s) + Y(s). \tag{746}$$

Solving for $X(s), Y(s)$, we have

$$X(s) = 2\frac{2}{(s-1)^2 - 4} = 2\frac{2}{(s-1)^2 - 2^2} = F_1(s-1). \tag{747}$$

where $F_1(s) = \frac{4}{s^2-2^2}$, $\mathcal{L}^{-1}(2\frac{2}{s^2-2^2}) = 2\sinh 2t$, and

$$Y(s) = \frac{2(s-1)}{(s-1)^2-4} = F_2(s-1). \tag{748}$$

where $F_2(s) = 2\frac{s}{s^2-2^2}$, $\mathcal{L}^{-1}(2\frac{s}{s^2-2^2}) = 2\cosh 2t$. Thus,

$$\begin{cases} x(t) = \mathcal{L}^{-1}(X(s)) = 2e^t\sinh(2t) = e^{3t} - e^{-t}, \\ y(t) = \mathcal{L}^{-1}(Y(s)) = 2e^t\cosh(2t) = e^{3t} + e^t. \end{cases} \tag{749}$$

6.4 Step Function and Shifting Theorem

Definition 113 (Unit Step Function)**.** *The unit step function or Heaviside function is defined by*

$$u_c(t) = \begin{cases} 0, & t < c, \\ 1, & t \geq c, \end{cases} \qquad c \geq 0. \tag{750}$$

Graph of $u_c(t)$ is shown in Figure 3.

The Laplace transform of u_c for $c \geq 0$ is easily determined:

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t) dt = \int_c^\infty e^{-st} dt = \frac{e^{-cs}}{s}, \qquad s > 0. \tag{751}$$

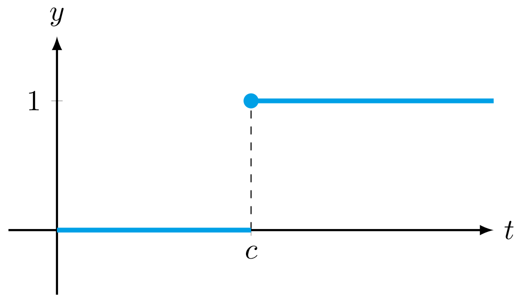
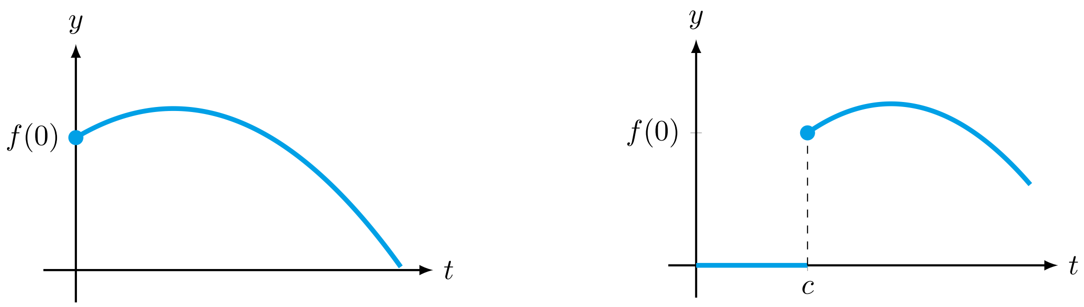


Figure 3: Graph of $y = u_c(t)$.

For a given function f defined for $t \geq 0$, we often want to consider the related function g defined by

$$y = g(t) = \begin{cases} 0, & t < c, \\ f(t-c), & t \geq c, \end{cases} \quad = u_c(t) f(t-c) \tag{752}$$

which represents a translation of f a distance c in the positive t direction; see Figure 4.



(a) $y = f(t)$ (b) $y = u_c(t)f(t-c)$

Figure 4: A translation of the given function.

Theorem 114. *If $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a$, and if c is a positive constant, then*

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s), \quad s > a. \tag{753}$$

Conversely, if $f(t) = \mathcal{L}^{-1}\{F(s)\}$, then

$$u_c(t)f(t-c) = \mathcal{L}^{-1}\{e^{-cs}F(s)\}. \tag{754}$$

Proof. Compute the transform of $u_c(t)f(t-c)$:

$$\mathcal{L}\{u_c(t)f(t-c)\} = \int_0^\infty e^{-st}u_c(t)f(t-c)dt = \int_c^\infty e^{-st}f(t-c)dt. \tag{755}$$

Introducing a new integration variable $\xi = t - c$, we have

$$\mathcal{L}\{u_c(t)f(t-c)\} = \int_0^\infty e^{-s(\xi+c)}f(\xi)d\xi = e^{-cs}\int_0^\infty e^{-s\xi}f(\xi)d\xi = e^{-cs}F(s). \tag{756}$$

Thus (753) is established; (754) follows by taking the inverse Laplace transform of both sides of (753). □

Example 115. *If the function f is defined by*

$$f(t) = \begin{cases} \sin t, & 0 \le t < \pi/4, \\ \sin t + \cos(t - \pi/4), & t \ge \pi/4, \end{cases} \tag{757}$$

find $\mathcal{L}\{f(t)\}$.

Note that $f(t) = \sin t + g(t)$, where

$$g(t) = \begin{cases} 0, & 0 \le t < \pi/4, \\ \cos(t - \pi/4), & t \ge \pi/4, \end{cases} \tag{758}$$

Thus

$$g(t) = u_{\pi/4}(t)\cos(t - \pi/4) \tag{759}$$

and

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} + \mathcal{L}\{u_{\pi/4}(t)\cos(t-\pi/4)\} = \mathcal{L}\{\sin t\} + e^{-\pi s/4}\mathcal{L}\{\cos t\}. \tag{760}$$

Introducing the transforms of $\sin t$ and $\cos t$, we obtain

$$\mathcal{L}\{f(t)\} = \frac{1}{s^2+1} + e^{-\pi s/4}\frac{s}{s^2+1} = \frac{1 + se^{-\pi s/4}}{s^2+1}. \tag{761}$$

Example 116. *Find the inverse transform of*

$$F(s) = \frac{1 - e^{-2s}}{s^2}. \tag{762}$$

From the linearity of the inverse transform, we have

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\} = t - u_2(t)(t-2). \tag{763}$$

The function f may also be written as

$$f(t) = \begin{cases} t, & 0 \le t < 2, \\ 2, & t \ge 2. \end{cases} \tag{764}$$

Example 117. *Find the solution of the differential equation*

$$2y'' + y' + 2y = g(t), \quad y(0) = 0, \quad y'(0) = 0. \tag{765}$$

where

$$g(t) = u_5(t) - u_{20}(t) = \begin{cases} 1, & 5 \le t < 20, \\ 0, & 0 \le t < 5 \quad \textit{and} \quad t \ge 20. \end{cases} \tag{766}$$

The Laplace transform of the differential equation is

$$2s^2Y(s) - 2sy(0) - 2y'(0) + sY(s) - y(0) + 2Y(s) = \mathcal{L}\{u_5(t)\} - \mathcal{L}\{u_{20}(t)\} = (e^{-5s} - e^{-20s})/s. \tag{767}$$

Using the initial values and solving for $Y(s)$, we obtain

$$Y(s) = \frac{e^{-5s} - e^{-20s}}{s(2s^2+s+2)}. \tag{768}$$

To find $y = \phi(t)$, it is convenient to write $Y(s)$ as

$$Y(s) = (e^{-5s} - e^{-20s})H(s), \quad H(s) = \frac{1}{s(2s^2+s+2)}. \tag{769}$$

Then, if $h(t) = \mathcal{L}^{-1}\{H(s)\}$, we have

$$y = \phi(t) = u_5(t)h(t-5) - u_{20}(t)h(t-20). \tag{770}$$

Finally, to determine $h(t)$, we use the partial fraction expansion of $H(s)$:

$$H(s) = \frac{a}{s} + \frac{bs+c}{2s^2+s+2}. \tag{771}$$

Upon determining the coefficients, we find that $a = 1/2$, $b = -1$, and $c = -1/2$. Thus

$$H(s) = \frac{1/2}{s} - \frac{s+1/2}{2s^2+s+2} = \frac{1/2}{s} - \frac{1}{2}\frac{(s+1/4)+1/4}{(s+\frac{1}{4})^2 + \frac{15}{16}} \tag{772}$$

$$= \frac{1/2}{s} - \frac{1}{2}\left[\frac{s+1/4}{(s+\frac{1}{4})^2 + (\frac{\sqrt{15}}{4})^2} + \frac{1}{\sqrt{15}}\frac{\sqrt{15}/4}{(s+\frac{1}{4})^2 + (\frac{\sqrt{15}}{4})^2}\right]. \tag{773}$$

By Theorem 106

$$\mathcal{L}^{-1}\left\{\frac{s+1/4}{(s+\frac{1}{4})^2 + (\frac{\sqrt{15}}{4})^2}\right\} = e^{-t/4}\cos\frac{\sqrt{15}}{4}t, \tag{774}$$

and

$$\mathcal{L}^{-1}\left\{\frac{\sqrt{15}/4}{(s+\frac{1}{4})^2+(\frac{\sqrt{15}}{4})^2}\right\} = e^{-t/4}\sin\frac{\sqrt{15}}{4}t, \tag{775}$$

Thus, we obtain

$$h(t) = \frac{1}{2} - \frac{1}{2}\left[e^{-t/4}\cos(\sqrt{15}t/4) + (\sqrt{15}/15)e^{-t/4}\sin(\sqrt{15}t/4)\right]. \tag{776}$$

6.5 Unit Impulses and Dirac Delta Function

In some applications it is necessary to deal with phenomena of an impulsive nature, for example, forces that act over very short time intervals. Such problems often lead to differential equations of the form

$$ay'' + by' + cy = g(t), \tag{777}$$

where $g(t)$ is nonzero during a short interval $t_0 - \tau < t < t_0 + \tau$ for some $\tau > 0$, and is otherwise zero. The integral $I(\tau)$, defined by

$$I(\tau) = \int_{-\infty}^{\infty} g(t)dt, \tag{778}$$

is a measure of the strength of the forcing function. In a mechanical system, where $g(t)$ is a force which is nonzero only in the interval $(t_0 - \tau, t_0 + \tau)$, $I(\tau)$ is the total impulse of the force $g(t)$ over the time interval $(t_0 - \tau, t_0 + \tau)$.

In particular, let's suppose that t_0 is zero and that $g(t)$ is given by

$$g(t) = d_\tau(t) = \begin{cases} \frac{1}{2\tau}, & -\tau < t < \tau, \\ 0, & t \leq -\tau, \text{ or } t \geq \tau, \end{cases} \tag{779}$$

where $\tau > 0$ is a small constant. In this case, $I(\tau) = 1$ independent of the value of τ, as long as $\tau \neq 0$. Graph of $g(t)$ is shown in Figure 5.

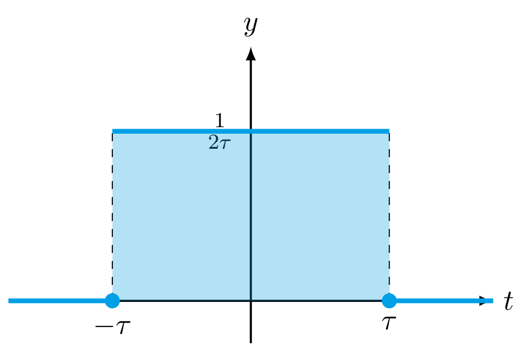


Figure 5: Graph of $y = d_\tau(t)$.

In the limit as $\tau \to 0$, as shown in Figure 6, we get an idealized unit impulse function δ, which imposes an impulse of magnitude one at $t = 0$ but is zero for all t other than zero. This function, which is not an ordinary function studied in elementary calculus, is called **Dirac delta function**.

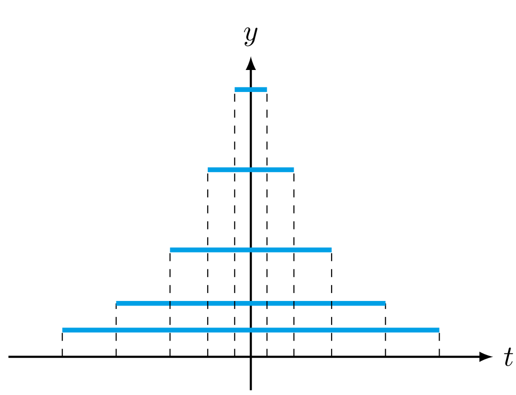


Figure 6: Graph of $y = d_\tau(t)$ as $\tau \to 0^+$.

The delta function has the following properties:

$$\delta(t) = 0,\ t \neq 0, \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)dt = 1. \tag{780}$$

A unit impulse at an arbitrary point $t = t_0$ is given by $\delta(t - t_0)$, which satisfies

$$\delta(t - t_0) = 0,\ t \neq t_0, \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t - t_0)dt = 1. \tag{781}$$

For any integrable function $f(t)$ over $(-\infty, +\infty)$,

$$\lim_{\tau \to 0^+} \int_{-\infty}^{\infty} d_\tau(t - t_0)f(t)dt = \int_{-\infty}^{\infty} \delta(t - t_0)f(t)dt. \tag{782}$$

Theorem 118. *Suppose that $f(t)$ is integrable on $(-\infty, +\infty)$ and continuous near t_0. Then*

$$\int_{-\infty}^{\infty} \delta(t - t_0)f(t)dt = f(t_0). \tag{783}$$

Proof.

$$\int_{-\infty}^{\infty} \delta(t - t_0)f(t)dt = \lim_{\tau \to 0^+} \int_{-\infty}^{\infty} d_\tau(t - t_0)f(t)dt. \tag{784}$$

Using the definition of $d_\tau(t)$ and the mean value theorem for integrals, we find that

$$\int_{-\infty}^{\infty} d_\tau(t - t_0)f(t)dt = \frac{1}{2\tau} \int_{t_0-\tau}^{t_0+\tau} f(t)dt = \frac{1}{2\tau} \cdot 2\tau \cdot f(t^*) = f(t^*), \tag{785}$$

where $t_0 - \tau < t^* < t_0 + \tau$. Hence $t^* \to t_0$ as $\tau \to 0^+$, and it follows from (784) that

$$\int_{-\infty}^{\infty} \delta(t - t_0)f(t)dt = f(t_0). \tag{786}$$

□

Laplace transform of $\delta(t - t_0)$ for $t_0 > 0$ is defined as:

$$\begin{aligned}
\mathcal{L}\{\delta(t - t_0)\} &= \lim_{\tau \to 0+} \mathcal{L}\{d_\tau(t - t_0)\} && (787)\\
&= \lim_{\tau \to 0+} \int_0^{\infty} e^{-st} d_\tau(t - t_0)dt && (788)\\
&= \lim_{\tau \to 0+} \int_{t_0-\tau}^{t_0+\tau} e^{-st} d_\tau(t - t_0)dt && (789)\\
&= \lim_{\tau \to 0+} \frac{1}{2\tau} \int_{t_0-\tau}^{t_0+\tau} e^{-st}dt && (790)\\
&= \lim_{\tau \to 0+} \frac{1}{2s\tau} e^{-st_0}(e^{s\tau} - e^{-s\tau}) = e^{-st_0} && (791)
\end{aligned}$$

Example 119. *Find the solution of the IVP*

$$2y'' + y' + 2y = \delta(t - 5), \quad y(0) = 0, \quad y'(0) = 0. \tag{792}$$

To solve the problem, we take the Laplace transform of the differential equation and use the initial conditions, obtaining

$$(2s^2 + s + 2)Y(s) = e^{-5s}. \tag{793}$$

Thus

$$Y(s) = \frac{e^{-5s}}{2s^2 + s + 2} = \frac{e^{-5s}}{2}\frac{1}{(s + \frac{1}{4})^2 + \frac{15}{16}}. \tag{794}$$

By Theorem 106

$$\mathcal{L}^{-1}\left\{\frac{1}{(s + \frac{1}{4})^2 + \frac{15}{16}}\right\} = \frac{4}{\sqrt{15}}e^{-t/4}\sin\frac{\sqrt{15}}{4}t. \tag{795}$$

Hence, by Theorem 114, we have

$$y = \mathcal{L}^{-1}\{Y(s)\} = \frac{2}{\sqrt{15}}u_5(t)e^{-(t-5)/4}\sin\frac{\sqrt{15}}{4}(t-5), \tag{796}$$

which is the formal solution of the given problem. It is also possible to write y in the form

$$y = \begin{cases} 0, & t < 5, \\ \frac{2}{\sqrt{15}}e^{-(t-5)/4}\sin\frac{\sqrt{15}}{4}(t-5), & t \geq 5. \end{cases} \tag{797}$$

6.6 The Convolution Theorem

We first present the definition of convolution operation.

Definition 120. *If $f(t)$ and $g(t)$ are defined on $[0, \infty)$, then the convolution $f * g$ is the function defined by*

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau. \tag{798}$$

Convolution satisfies the following properties.

Property 121.

$$f * g = g * f \qquad \textit{Commutative Law} \tag{799}$$
$$f * (g_1 + g_2) = f * g_1 + f * g_2 \qquad \textit{Distributive Law} \tag{800}$$
$$(f * g) * h = f * (g * h) \qquad \textit{Associative Law} \tag{801}$$
$$f * 0 = 0 * f = 0 \qquad \textit{Zero Property} \tag{802}$$

Laplace transform of convolution is given by the following theorem.

Theorem 122 (Convolution Theorem)**.** *If $F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$ both exist for $s > a$, then*

$$\mathcal{L}\{(f * g)(t)\} = F(s)G(s), \quad s > a. \tag{803}$$

As a result, if $\mathcal{L}^{-1}\{F(s)\} = f(t)$ and $\mathcal{L}^{-1}\{G(s)\} = g(t)$, then

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t). \tag{804}$$

Proof.

$$F(s)G(s) = \int_0^{+\infty} e^{-s\xi} f(\xi)d\xi \int_0^{+\infty} e^{-s\tau} g(\tau)d\tau \tag{805}$$
$$= \int_0^{+\infty} g(\tau) \left[\int_0^{+\infty} e^{-s(\xi+\tau)} f(\xi)d\xi\right] d\tau. \tag{806}$$

Let $\xi + \tau = t$, then we have

$$F(s)G(s) = \int_0^{+\infty} g(\tau) \left[\int_\tau^{+\infty} e^{-st} f(t-\tau)dt\right] d\tau \tag{807}$$
$$= \int_0^{+\infty} e^{-st} \left[\int_0^t f(t-\tau)g(\tau)d\tau\right] dt \tag{808}$$
$$= \int_0^{+\infty} e^{-st}(f * g)(t)dt \tag{809}$$
$$= \mathcal{L}\{(f * g)(t)\}. \tag{810}$$

□

Similar result holds for Fourier transform; see Section 4.2 of [GW18] for details.

Example 123. *Find the solution of the IVP*

$$y'' + 4y = g(t), \quad y(0) = 3, \quad y'(0) = -1. \tag{811}$$

where the forcing function g is given.

By taking the Laplace transform of the differential equation and using the initial conditions, we obtain

$$s^2 Y(s) - 3s + 1 + 4Y(s) = G(s) \tag{812}$$

Thus,

$$Y(s) = \frac{3s-1}{s^2+4} + \frac{G(s)}{s^2+4}. \tag{813}$$

Observe that the first and second terms on the right side of (813) contain the dependence of $Y(s)$ on the initial conditions and forcing function, respectively. It is convenient to write $Y(s)$ in the form

$$Y(s) = 3\frac{s}{s^2+4} - \frac{1}{2}\frac{2}{s^2+4} + \frac{1}{2}\frac{2}{s^2+4}G(s). \tag{814}$$

Then, using Theorem 122, we obtain

$$y = 3\cos 2t - \frac{1}{2}\sin 2t + \frac{1}{2}\int_0^t \sin 2(t-\tau) g(\tau) d\tau. \tag{815}$$

If a specific forcing function g is given, then the integral in (815) can be evaluated (by numerical computations, if necessary).

6.7 Definition for Inverse Laplace Transform

Question: If two functions have the same Laplace transform, are the two functions the same? The answer is given by the following theorem.

Theorem 124. *If $f(t)$ and $g(t)$ are continuous on $[0, +\infty)$ of exponential order and $\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\}$, then $f = g$.*

The proof is based on Weierstrass Approximation Theorem of continuous functions by polynomials.

Theorem 125 (Weierstrass Approximation Theorem)**.** *If $f(t)$ is a continuous function on a closed interval $[a, b]$, then for every $\varepsilon > 0$ there exists a polynomial $q_\varepsilon(t)$ such that*

$$\max_{t\in[a,b]} |f(t) - q_\varepsilon(t)| < \varepsilon. \tag{816}$$

Next, we will prove Theorem 124, with the additional assumption that f and g share an exponential order s_0.

Proof. Let $u = f - g$, then $\mathcal{L}\{u\}(s) = \mathcal{L}\{f - g\}(s) = 0$ for all $s > s_0$, and u is of exponential order s_0. There exists two positive constants k, T such that

$$|u(t)| < ke^{s_0 t}, \quad t > T. \tag{817}$$

Evaluate $\mathcal{L}\{u\}(s)$ at $\hat{s} = s_1 + n + 1$, where s_1 is any real number such that $s_1 > s_0$ and n is any non-negative integer, one has

$$\mathcal{L}\{u\}(\hat{s}) = \int_0^\infty e^{(-s_1-n-1)t}u(t)dt = \int_0^\infty e^{-s_1 t}e^{-(n+1)t}u(t)dt \tag{818}$$

Let $e^{-t} = y$, then $dy = -e^{-t}dt$, then

$$\mathcal{L}\{u\}(\hat{s}) = \int_0^1 y^n y^{s_1} u(-\ln(y))dy \tag{819}$$

Now introduce $v(y) = y^{s_1}u(-\ln(y)) = e^{-s_1 t}u(t)$,

$$\lim_{y\to 0+} |v(y)| = \lim_{t\to\infty} e^{-s_1 t}|u(t)| < \lim_{t\to\infty} ke^{-(s_1-s_0)t} = 0. \tag{820}$$

Thus, the function v does not diverge at $y = 0$. Since $\mathcal{L}\{u\}(\hat{s}) = 0$ for all $\hat{s} > s_0$,

$$\int_0^1 y^n v(y)dy = 0, \quad n = 0, 1, \cdots \tag{821}$$

Thus, for **any** polynomial $p(y)$, one has

$$\int_0^1 p(y)v(y)dy = 0. \tag{822}$$

Now,

$$\int_0^1 v^2(y)dy = \int_0^1 (v(y) - p(y))v(y)dy + \int_0^1 p(y)v(y)dy \tag{823}$$
$$= \int_0^1 (v(y) - p(y))v(y)dy \tag{824}$$
$$\leq \int_0^1 |p(y) - v(y)||v(y)|dy \tag{825}$$
$$\leq \max_{y\in[0,1]} |v(y)| \int_0^1 |p(y) - v(y)|dy \tag{826}$$

this is true for **any** polynomial $p(y)$. For any $\varepsilon > 0$, there exists a polynomial $p_\varepsilon(y)$, s.t.

$$\max_{y\in[0,1]} |v(y) - p_\varepsilon(y)| \leq \varepsilon \tag{827}$$

Thus,

$$\int_0^1 v^2(y)dy \leq \max_{y\in[0,1]} |v(y)| \int_0^1 |p_\varepsilon(y) - v(y)|dy \tag{828}$$
$$\leq \max_{y\in[0,1]} |v(y)|\varepsilon \tag{829}$$

Therefore,

$$\int_0^1 v^2(y)dy = 0. \tag{830}$$

$v(y)$ is continuous, thus, $v(y) \equiv 0$, and $u(t) \equiv 0$, $f(t) = g(t)$. □

If $f(t), g(t)$ are piecewise continuous on the interval $0 \leq t \leq A$ for any positive A, and $\mathcal{L}\{f\} = \mathcal{L}\{g\}$, then $f(t)$ and $g(t)$ are equal except at a set of discrete points where jump discontinuity happens, these two functions can be considered as the same (indeed, they are almost equal everywhere). The proof can be found in Section 69 of [Chu72].

7 Nonlinear Differential Equations and Stability

In all of our previous study, we have mainly focused on linear equations to compute explicit solutions. For nonlinear equations, we have been restricted to certain techniques such as separable equations and exact equations (in the first-order nonlinear ODE case). For a general nonlinear equation

$$y^{(n)} = F(t, y, y', \ldots, y^{(n-1)}), \tag{831}$$

or nonlinear systems of differential equations

$$\frac{d\mathbf{y}(t)}{dt} = \mathbf{P}(t, \mathbf{y}(t)). \tag{832}$$

we can only say something about existence and uniqueness of solutions in a small time interval if F or $\mathbf{P}$ are continuous with continuous derivatives. Although unfortunately, explicit formulas for solutions are usually unavailable, we can use geometric methods to deduce more information about the solutions. This will be the focus of this section.

7.1 Geometric Approach for A Single First-Order ODE

Given a first-order nonlinear ODE

$$y'(t) = f(t, y), \tag{833}$$

for continuous functions f and $\frac{\partial f}{\partial y}$, we know that there is exactly one solution to the IVP when initial conditions are given. We can plot the graph of t vs y using the equation. Furthermore, at each point (t, y) we can draw a line segment with slope $f(t, y)$, this gives a **direction field** for the ODE. Putting an arrow at the end of each line segment gives a **vector field** in the $t - y$ plane. See Figure 7 for example.

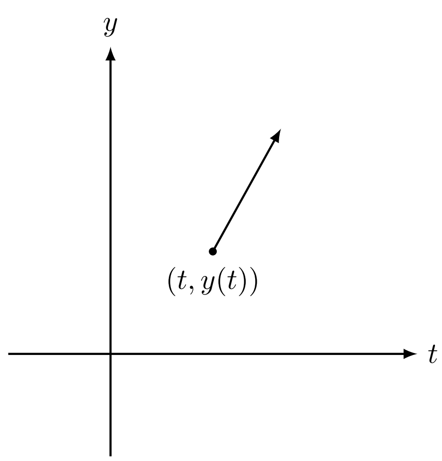
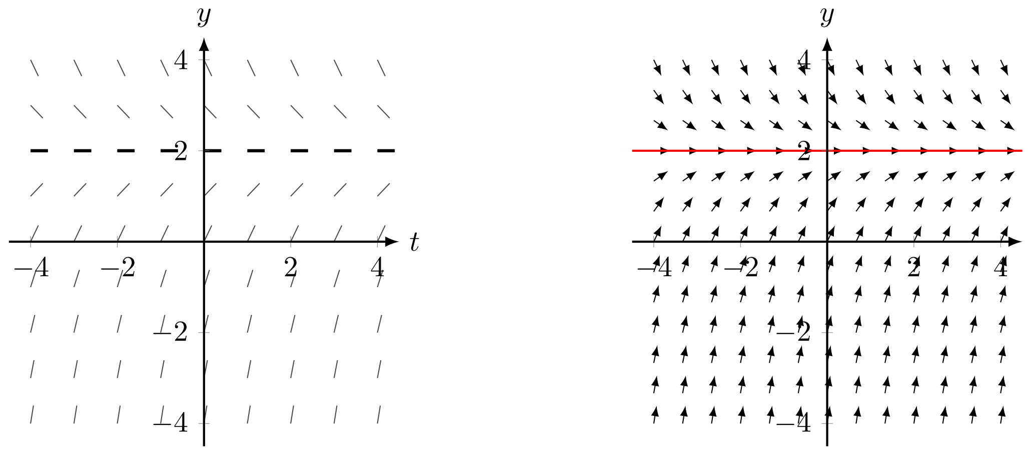


Figure 7: Vector representation at point $(t, y(t))$.

Example 126. *Consider the first-order ODE*

$$y' = f(y) = 2 - y. \tag{834}$$

As f does not depend on t, the slopes of the line segments at a fixed y-coordinate are all the same. We plot the direction field and vector field in Figure 8, with arrows indicating the change of y. Notice that the line segments have zero slope whenever the points lie on the line $\{y = 2\}$.



(a) Direction field

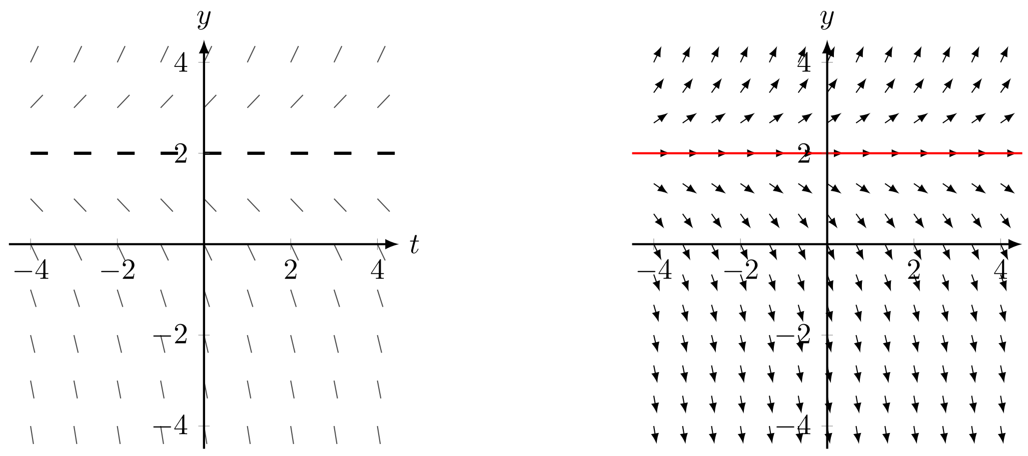
(b) Vector field

Figure 8: Geometric representation of $y' = 2 - y$ in (t, y) plane.

Example 127. *Similarly, for the first-order ODE*

$$y' = y - 2, \tag{835}$$

we have the direction field and vector field as Figure 9. Also notice that the line segments have zero slope whenever the points lie on the line $\{y = 2\}$.



(a) Direction field

(b) Vector field

Figure 9: Geometric representation of $y' = y - 2$ in (t, y) plane.

If we put a particle inside this vector field at initial position (t_0, y_0), the particle will move along with the vector field, this traces out a **trajectory** $\{(t, y(t)) : t \in I\}$ for the solution to the ODE.

In the first example $y' = 2 - y$, all trajectories will go to the line $\{y = 2\}$ as $t \to \infty$, while for the second example $y' = y - 2$, if trajectories start with initial position $y_0 = 2$, then the trajectories will stay on the line $\{y = 2\}$ as $t \to \infty$. However, if $y_0 > 2$, then the trajectories will move up and away from $\{y = 2\}$, and correspondingly in $y_0 < 2$, the trajectories will move down and away from $\{y = 2\}$.

In the above examples, the line $\{y = 2\}$ is what we will call **equilibrium/stationary** solutions to the ODE, as the values of y do not change as time progresses.

Definition 128 (Critical Point and Stationary Solution)**.** *Given a continuous function $f(t, y)$, suppose $y_* \in \mathbb{R}$ is a point such that*

$$f(t, y_*) = 0 \quad \forall t \in I, \tag{836}$$

then y_* is a critical point of f. We call the constant function

$$\phi(t) = y_* \quad \forall t \in I \tag{837}$$

a stationary solution to the ODE $y' = f(t, y)$.

Besides the direction field and vector field, another useful graphic for the nonlinear **autonomous** ODE

$$y' = f(y) \tag{838}$$

is the graph y vs $f(y)$.

Example 129. *The Logistic equation is given as follows: for positive constants r and K,*

$$y' = f(y) = ry\left(1 - \frac{y}{K}\right) \tag{839}$$

for population dynamics. It is easy to see that $y = 0$ and $y = K$ are stationary solutions, and if y is not equal to 0 or K, then

$$y(t) = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}} \quad \text{for } y(0) = y_0. \tag{840}$$

Hence, we can deduce that for nonnegative initial values y_0,

$$y(t) \to K \text{ as } t \to \infty \text{ if } y_0 > 0, \quad y(t) = 0 \text{ for all } t > 0 \text{ if } y_0 = 0. \tag{841}$$

We can plot the vector field for the case $r = 1$ and $K = 4$ in Figure 10, and observe that the line segments with y-coordinate equal to 0 or 4 ($y = 0$ or $y = 4$) have zero slope.

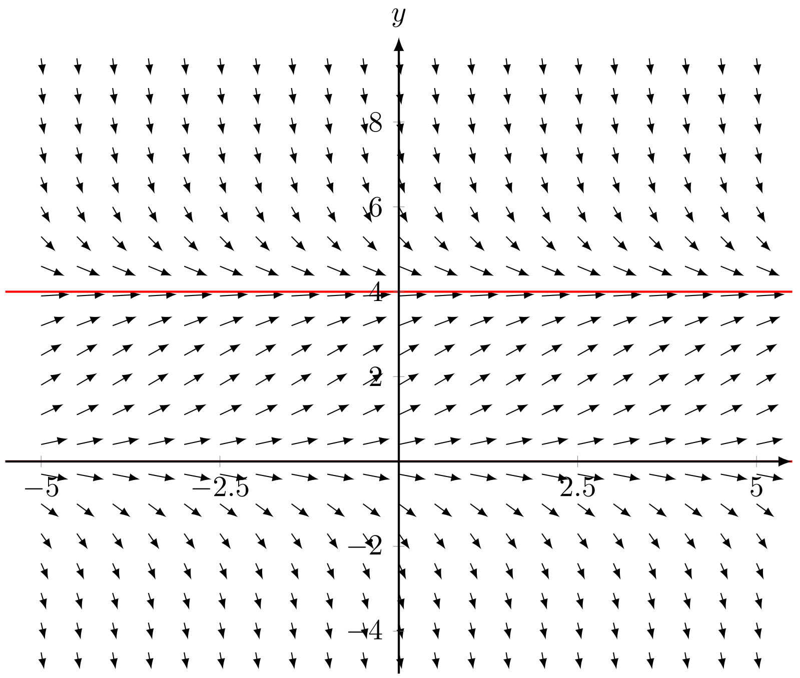


Figure 10: Vector field for the Logistic equation in (t, y) plane with $r = 1$ and $K = 4$.

We now plot the graph y vs $f(y)$ in Figure 11, which is a parabola that intersects the horizontal axis at two points $y = 0$ and $y = K$, corresponding to two stationary solutions.

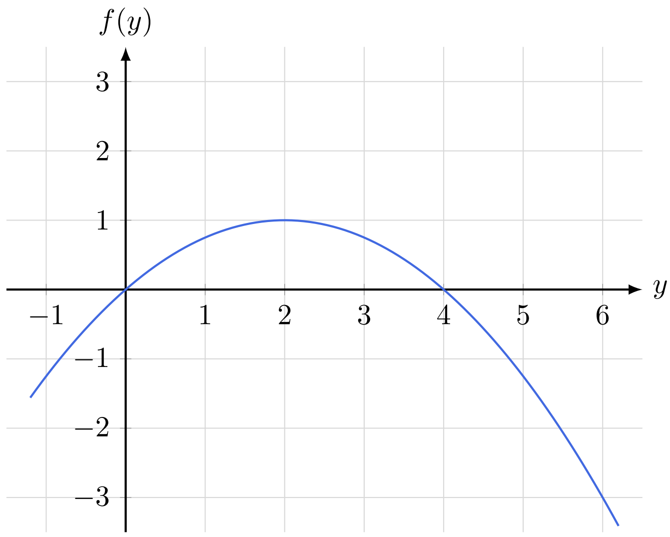


Figure 11: The plot y vs $f(y)$ for the Logistic equation.

From this plot we can also deduce some behaviour of the solution to the ODE. Suppose we start with an initial condition x_0 in between 0 and K, then $f(x_0)$ is positive, so the solution y will increase in value, until it reaches $y = K$ where the derivative y' is zero. Similarly, if we start with an initial condition $x_1 > K$, then $f(x_1)$ is negative. Hence, the solution y will decrease in value, until it reaches $y = K$. Similarly, if we start with an initial value $x_2 < 0$, then $f(x_2)$ is negative and the solution y will decrease, moving away from the stationary solution $y = 0$. This can be summarized in Figure 12, where we include arrows to demonstrate the behaviour of the solution.

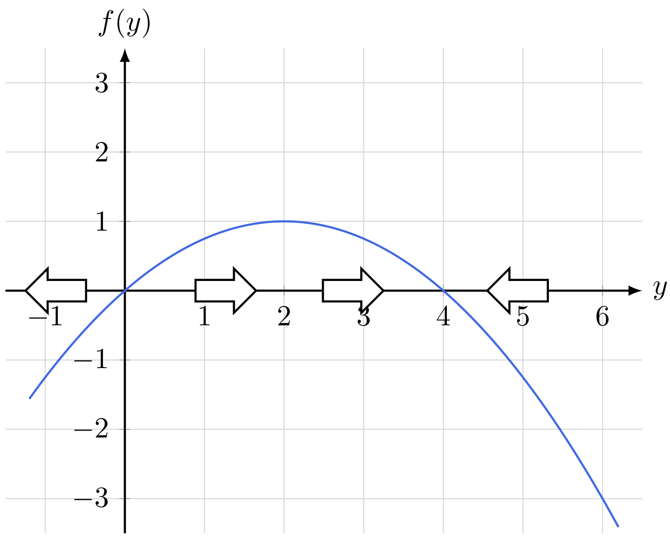


Figure 12: The plot y vs $f(y)$ for the Logistic equation with indicating arrows.

Example 130. *We study a modification of the Logistic equation, called Logistic equation with threshold. Let $r > 0, 0 < T < K$ be positive constants, and consider the equation*

$$y' = f(y) = -r\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right)y. \tag{842}$$

First we identify the critical points, which are $y_1 = 0, y_2 = T$ and $y_3 = K$. Next, plotting the graph y vs $f(y)$ (see Figure 13 for $r = 1, T = 4$ and $K = 8$) leads to a cubic graph.

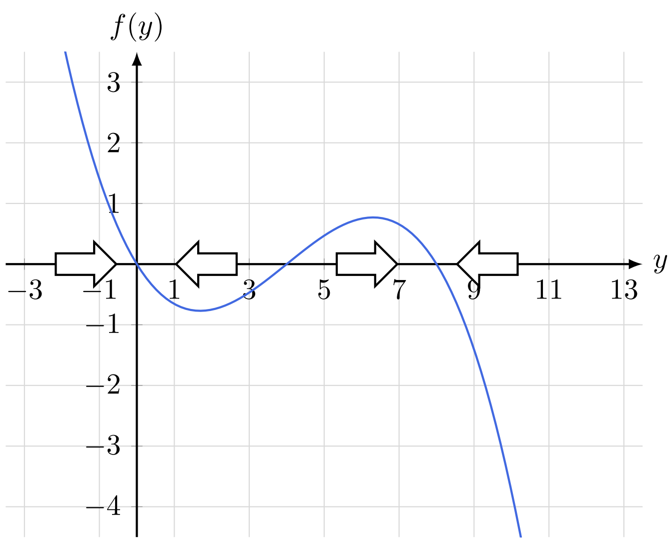


Figure 13: The plot y vs $f(y)$ for the modified Logistic equation.

We have the following observation for the ODE (842):

- if initial condition $y(0) = y_0 \in (0, T)$, then $f(y_0)$ is negative and the solution y should decrease;
- if initial condition $y(0) = y_0 \in (T, K)$, then $f(y_0)$ is positive and the solution y should increase;

- if initial condition $y(0) = y_0 > K$, then $f(y_0)$ is negative and the solution y should decrease.

From this we deduce that

$$y(t) \to 0 \text{ if } 0 < y_0 < T, \quad y(t) \to T \text{ if } y_0 = T, \quad y(t) \to K \text{ if } y_0 > T. \tag{843}$$

Furthermore, the vector field plot in Figure 14 also supports our observations on the solution behavior.

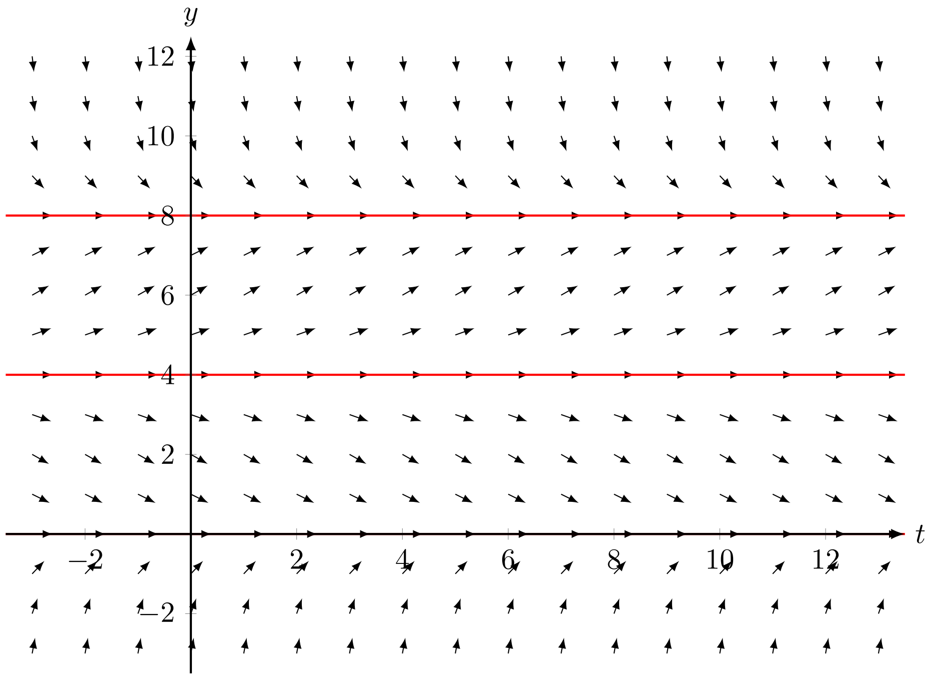


Figure 14: Vector field for the modified Logistic equation with threshold in (t, y) plane.

In the above examples we saw that there are instances where if we start "close" to a stationary solution, we either converge to the stationary solution, or move away to another stationary solution, or the solution $y(t)$ goes to $\pm\infty$ as $t \to \infty$. This is formally defined as follows.

Definition 131 (Asymptotic Stability)**.** *Given an autonomous first-order ODE $y' = f(y)$, and a stationary solution y_*, we say that y_* is* ***asymptotically*** *stable if there is an $\delta_0 > 0$ (depending only on y_*) such that for any solution $\phi(t)$ to the IVP*

$$y' = f(y) \text{ for } t \in I, \quad y(t_0) = y_0 \text{ with } t_0 \in I, \tag{844}$$

the following property is satisfied:

$$|y_0 - y_*| < \delta_0 \implies \phi(t) \to y_* \text{ as } t \to \infty, \tag{845}$$

which means if we start close to y_*, we will move towards y_* as time progresses.

The unstable stationary solution / point is defined as follows.

Definition 132. *For the stationary point y_*, if we start close to y_*, the trajectory moves far away from y_* as time progresses, then we call this stationary point unstable. Denote $\phi(t)$ the solution to the IVP*

$$y' = f(y) \text{ for } t \in I, \quad y(t_0) = y_0 \text{ with } t_0 \in I, \tag{846}$$

Then for an unstable stationary solution y_*, except for $y_0 = y_*$ (which implies that $\phi(t) = y_*$ for all $t \geq t_0$), any other initial condition would lead to $|\phi(t) - y_*| \not\to 0$ ($t \to +\infty$), so the solution $\phi(t)$ will never reach y_* for an unstable stationary solution $y = y_*$.

Recall Figure 12, for the Logistic equation, the stationary solution $y_* = 0$ is unstable, and $y_* = K$ is asymptotically stable for any initial condition $y_0 > 0$. And Figure 13 shows that for the Logistic equation with threshold, $y_* = 0$ and $y_* = K$ are asymptotically stable, while $y_* = T$ is unstable.

7.2 Geometric Approach for First-Order Linear System

We now turn to first-order linear systems of the form

$$\frac{d\mathbf{y}(t)}{dt} = \mathbf{A}\mathbf{y}(t), \tag{847}$$

where $\mathbf{A} \in \mathbb{R}^{2\times 2}$ is a constant matrix with real coefficients. For the upcoming analysis, we will assume that:

$$\mathbf{A} \text{ is non-singular} \iff \det \mathbf{A} \neq 0, \text{ and 0 is not an eigenvalue of } \mathbf{A}. \tag{848}$$

Then, the only possible solution to

$$\frac{d\mathbf{y}}{dt} = \mathbf{A}\mathbf{y} = \mathbf{0} \tag{849}$$

is the zero vector $\mathbf{y} = \mathbf{0}$, i.e., $\mathbf{0}$ is the **unique critical point**.

Compare to first-order equations, the solution to $\mathbf{y}' = \mathbf{A}\mathbf{y}(t)$ is a vector $\mathbf{y} = (y_1, y_2)$. Without going to a three dimensional plot $(t, y_1(t), y_2(t))$, we can still obtain information on the behavior of the solution $\mathbf{y}(t)$.

Definition 133. *We call the (y_1, y_2) plane as the* ***phase plane****. A solution $\mathbf{y}(t) = (y_1(t), y_2(t))$ for $t \in I$ traces out a curve in the phase plane, which we call a* ***trajectory****. As it is impossible to draw all trajectories, for a representative set of trajectories (meaning that they indicate all possible behavior of different trajectories) we call a* ***phase portrait****.*

The phase portrait will yield crucial information about the stability of the critical points, which are determined by the eigenvalues of the matrix $\mathbf{A}$. For a 2×2 matrix, we have the following three possibilities for eigenvalues:

A Real, distinct eigenvalues $r_1 \neq r_2$,

B Real, repeated eigenvalues $r_1 = r_2$.

C Complex conjugate pairs of eigenvalues $r_1 = \lambda + i\mu, r_2 = \overline{r_1}$,

We are mainly interested in the tendency of the trajectories when $t \to \infty$, but we will also look at the tendency of the trajectories when $t \to -\infty$ in order to get full insights of the trajectories and phase portrait. Also, for the ODE system

$$\frac{d\mathbf{y}(t)}{dt} = \mathbf{f}(\mathbf{y}), \tag{850}$$

where $\mathbf{y}(t) = (y_1(t), y_2(t))^T, \mathbf{f} = (f_1(y_1, y_2), f_2(y_1, y_2))^T$, for each point (y_1, y_2) in the phase plane, we can draw a line segment with arrow in the direction of vector $(f_1(y_1, y_2), f_2(y_1, y_2))^T$, then we can get the direction field of the ODE system.

7.2.1 Case A (1): Real Distinct Eigenvalues with Same Sign

Recall that if $r_1 \neq r_2$, then the eigenvectors $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$ corresponding to r_1 and r_2 are linearly independent, and the general solution to $\mathbf{y}' = \mathbf{A}\mathbf{y}(t)$ is

$$\mathbf{y}(t) = c_1\boldsymbol{\xi}_1 e^{r_1 t} + c_2\boldsymbol{\xi}_2 e^{r_2 t}. \tag{851}$$

Subcase 1. $r_1 < r_2 < 0$. If both r_1 and r_2 are negative, then as $t \to \infty$, we have that $\mathbf{y}(t) \to \mathbf{0}$, i.e., all solutions tend to the critical point. We now illustrate how this happens in the phase portrait.

First, if $c_2 = 0$, which means the initial condition $\mathbf{y}(0) = \mathbf{y}_0$ is a constant multiple of $\boldsymbol{\xi}_1$, then $\mathbf{y}(t) = c_1\boldsymbol{\xi}_1 e^{r_1 t}$ and thus the solution always stays on the line spanned by the vector $\boldsymbol{\xi}_1$. Similarly, if $c_1 = 0$, meaning that $\mathbf{y}_0$ is a constant multiple of $\boldsymbol{\xi}_2$, then $\mathbf{y}(t)$ always stays on the line spanned by $\boldsymbol{\xi}_2$.

This is illustrated in Figure 15 for the matrix $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ -1 & -0.25 \end{pmatrix}$ with eigenvalues $r_1 = -1, r_2 = -0.25$ and eigenvectors $\boldsymbol{\xi}_1 = (3,4)^T$ and $\boldsymbol{\xi}_2 = (0,1)^T$. The red lines indicate the lines spanned by the eigenvectors $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$, which need not be perpendicular.

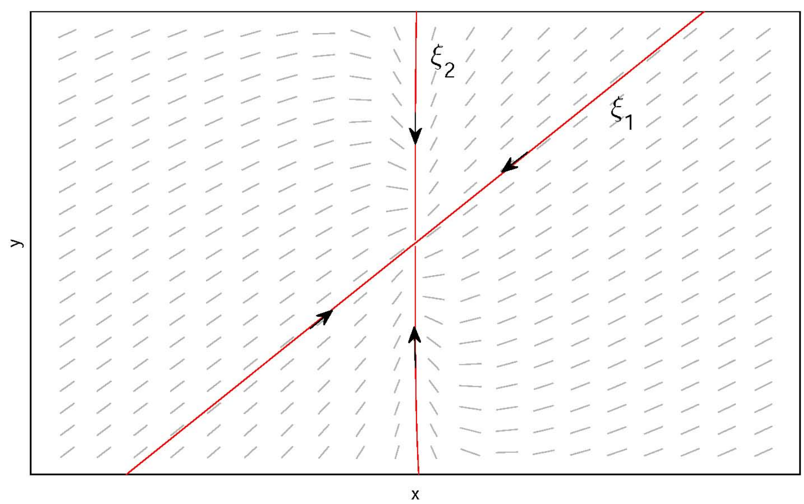


Figure 15: Phase portrait for the system with matrix $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ -1 & -0.25 \end{pmatrix}$.

What about $c_1 \neq 0, c_2 \neq 0$ (the initial condition $\mathbf{y}_0$ does not lie on the lines spanned by $\boldsymbol{\xi}_1$ or $\boldsymbol{\xi}_2$)? Rewrite the expression for the general solution into

$$\mathbf{y}(t) = e^{r_2 t}(c_1\boldsymbol{\xi}_1 e^{(r_1 - r_2)t} + c_2\boldsymbol{\xi}_2) \text{ if } r_1 < r_2 < 0. \tag{852}$$

Since $r_1 < r_2$, then as $t \to \infty$, the term $c_1\boldsymbol{\xi}_1 e^{(r_1-r_2)t}$ is negligible. Therefore, the trajectories tend towards the line spanned by $\boldsymbol{\xi}_2$. Indeed, the trajectories tend to tangent with $\boldsymbol{\xi}_2$ at the critical point as $t \to \infty$.

We can also write the expression as

$$\mathbf{y}(t) = e^{r_1 t}(c_1\boldsymbol{\xi}_1 + c_2\boldsymbol{\xi}_2 e^{(r_2 - r_1)t}) \text{ if } r_1 < r_2 < 0. \tag{853}$$

Also note that as $t \to -\infty$ (running backwards in time), the term $c_2\boldsymbol{\xi}_2 e^{(r_2-r_1)t}$ is negligible, hence, the trajectories would have nearly the same slope as $\boldsymbol{\xi}_1$ as $t \to -\infty$.

For the above cases, the eigenvalues are negative, we have $\lim_{t\to\infty} \mathbf{y}(t) = \mathbf{0}$, we call the critical point $\mathbf{0}$ a **nodal sink**, since all trajectories point towards $\mathbf{0}$. For $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} -1 & 0 \\ -1 & -0.25 \end{pmatrix}\mathbf{y}$, this leads to Figure 16.

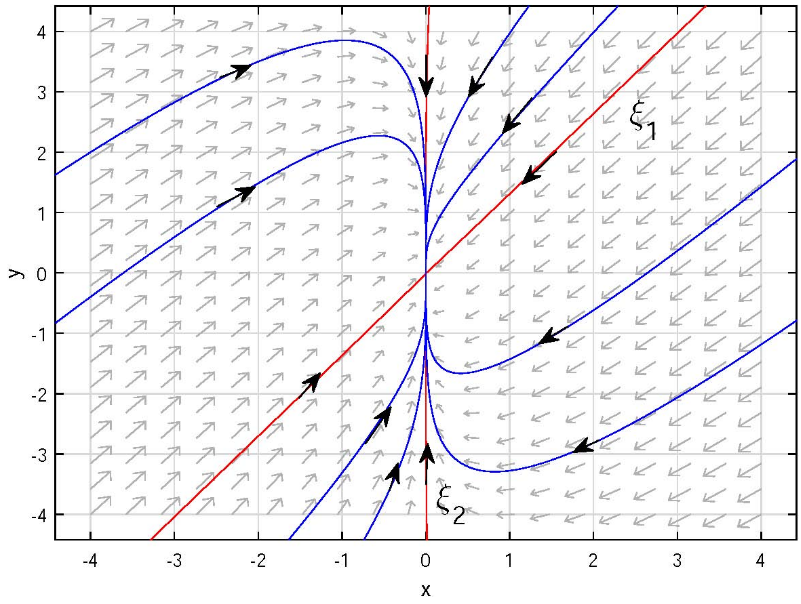


Figure 16: Phase portrait for the system with matrix $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ -1 & -0.25 \end{pmatrix}$.

Example 134. *For*

$$\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} -7/2 & 5/2 \\ 5/2 & -7/2 \end{pmatrix} \mathbf{y}(t) =: \mathbf{A}\mathbf{y}(t) \tag{854}$$

determine the critical points and their stability. Draw the phase portrait.

Since $\det \mathbf{A} = 6 \neq 0$, the matrix $\mathbf{A}$ is invertible and $\mathbf{0}$ is the only critical point. The eigenvalues $r_1 = -6$ and $r_2 = -1$ are real distinct. Computing the eigenvectors yields $\boldsymbol{\xi}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\boldsymbol{\xi}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then the general solution is

$$\mathbf{y}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-6t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} = e^{-t} \left(c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-5t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \tag{855}$$

For large $t > 0$, $\mathbf{y}(t) \to 0$ with trajectories parallel to $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, this yields the phase portrait in Figure 17. For large $t < 0$, the dominating term is $c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-5t}$, so the trajectories are parallel to $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

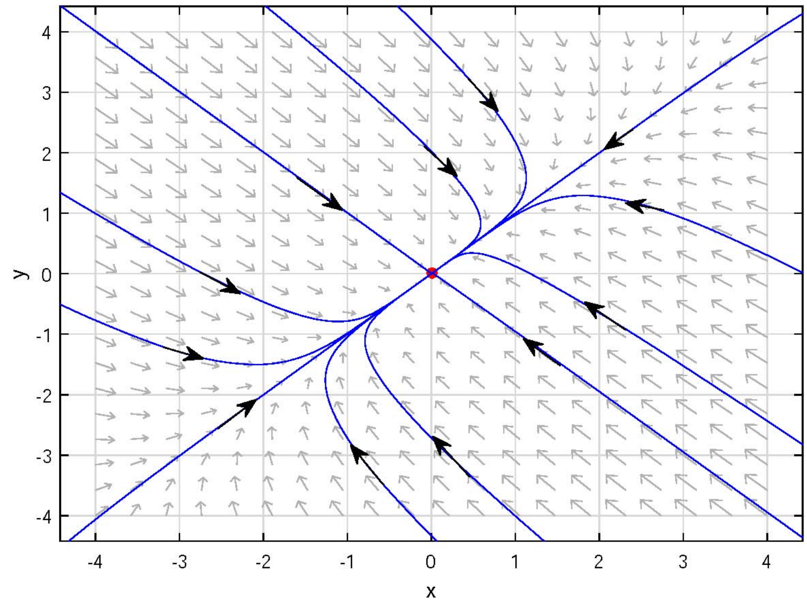


Figure 17: Phase portrait for $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} -7/2 & 5/2 \\ 5/2 & -7/2 \end{pmatrix} \mathbf{y}(t)$.

Subcase 2. $0 < r_2 < r_1$**.** In this case, we get the similar phase portrait for $\mathbf{y}(t) = c_1\boldsymbol{\xi}_1 e^{r_1 t} + c_2\boldsymbol{\xi}_2 e^{r_2 t}$, but the direction of motion is reversed.

(a) If $c_1 = 0$, and $c_2 \neq 0$, then $\mathbf{y}(t) = c_2\boldsymbol{\xi}_2 e^{r_2 t} \to \mathbf{0}$ as $t \to -\infty$.

(b) If $c_2 = 0$, and $c_1 \neq 0$, then $\mathbf{y}(t) = c_1\boldsymbol{\xi}_1 e^{r_1 t} \to \mathbf{0}$ as $t \to -\infty$.

(c) If $c_1 \neq 0, c_2 \neq 0$, rewriting

$$\mathbf{y}(t) = e^{r_1 t}(c_1\boldsymbol{\xi}_1 + c_2\boldsymbol{\xi}_2 e^{(r_2 - r_1)t}) \tag{856}$$

for $t \to \infty$, with $r_2 - r_1 < 0$, the term $c_2\boldsymbol{\xi}_2 e^{(r_2-r_1)t}$ is negligible. Hence, the trajectories tend to parallel to the line spanned by $\boldsymbol{\xi}_1$ as $t \to \infty$.

(d) If $c_1 \neq 0, c_2 \neq 0$, rewriting

$$\mathbf{y}(t) = e^{r_2 t}(c_1\boldsymbol{\xi}_1 e^{(r_1 - r_2)t} + c_2\boldsymbol{\xi}_2), \tag{857}$$

for $t \to -\infty$, with $r_1 - r_2 > 0$, the term $c_1\boldsymbol{\xi}_1 e^{(r_1-r_2)t}$ is negligible. Hence, the trajectories tend to tangent with the line spanned by $\boldsymbol{\xi}_2$ as $t \to -\infty$.

In all above cases, the trajectories will move away from the critical point $\mathbf{0}$ when $t \to \infty$. In this case we call $\mathbf{0}$ a **nodal source**. See Figure 18 for phase portrait of $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} 1 & 0 \\ 1 & 0.25 \end{pmatrix} \mathbf{y}(t)$, where eigenvalues are positive and distinct ($r_1 = 1, r_2 = 0.25$). The red lines indicate the lines spanned by the eigenvectors $\boldsymbol{\xi}_1 = (3,4)^T$ and $\boldsymbol{\xi}_2 = (0,1)^T$.

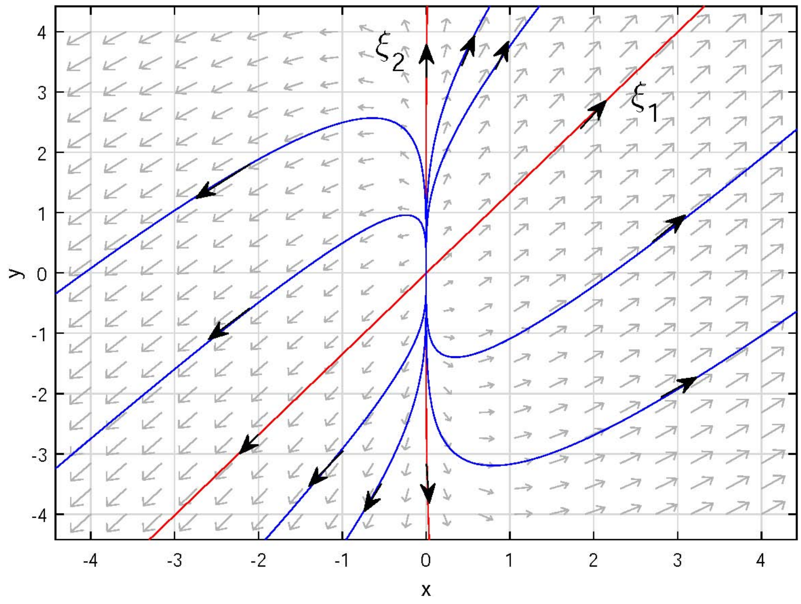


Figure 18: Phase portrait for $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} 1 & 0 \\ 1 & 0.25 \end{pmatrix} \mathbf{y}(t)$.

Summary: When $r_1 \neq r_2$ and r_1, r_2 have the same sign, the critical point is called the **node**. If $r_1 < 0, r_2 < 0$, the critical point is the **nodal sink**. When $r_1 > 0, r_2 > 0$, the critical point is the **nodal source**.

7.2.2 Case A (2): Real Distinct Eigenvalues with Opposite Sign

Without loss of generality, suppose $r_2 < 0 < r_1$. Then, from the expression for the general solution $\mathbf{y}(t) = c_1 \boldsymbol{\xi}_1 e^{r_1 t} + c_2 \boldsymbol{\xi}_2 e^{r_2 t}$, we observe the following:

(a) If $c_1 = 0$, and $c_2 \neq 0$, then $\mathbf{y}(t) = c_2 \boldsymbol{\xi}_2 e^{r_2 t} \to \mathbf{0}$ as $t \to \infty$.

(b) If $c_2 = 0$, and $c_1 \neq 0$, then $\mathbf{y}(t) = c_1 \boldsymbol{\xi}_1 e^{r_1 t} \to \mathbf{0}$ as $t \to -\infty$.

(c) If $c_1 \neq 0, c_2 \neq 0$, rewriting

$$\mathbf{y}(t) = e^{r_1 t}(c_1 \boldsymbol{\xi}_1 + c_2 \boldsymbol{\xi}_2 e^{(r_2 - r_1)t}) \tag{858}$$

for $t \to \infty$, with $r_2 - r_1 < 0$, the term $c_2 \boldsymbol{\xi}_2 e^{(r_2 - r_1)t}$ is negligible. Hence, the trajectories approach the line spanned by $\boldsymbol{\xi}_1$ as $t \to \infty$.

(d) If $c_1 \neq 0, c_2 \neq 0$, rewriting

$$\mathbf{y}(t) = e^{r_2 t}(c_1 \boldsymbol{\xi}_1 e^{(r_1 - r_2)t} + c_2 \boldsymbol{\xi}_2), \tag{859}$$

for $t \to -\infty$, with $r_1 - r_2 > 0$, the term $c_1 \boldsymbol{\xi}_1 e^{(r_1 - r_2)t}$ is negligible. Hence, the trajectories tend to parallel to the line spanned by $\boldsymbol{\xi}_2$ as $t \to -\infty$.

These observations yield the phase portrait in Figure 19 for the matrix $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -2 & -2 \end{pmatrix}$ with eigenvalues $r_1 = 2, r_2 = -1$ and eigenvectors $\boldsymbol{\xi}_1 = (-2, 1)^T$ (lower red line), $\boldsymbol{\xi}_2 = (-1, 2)^T$ (higher red line). For the case $r_1 < 0 < r_2$ we obtain the same portrait with arrows reversed. In this case where two eigenvalues have different sign, the critical point is called **saddle point**.

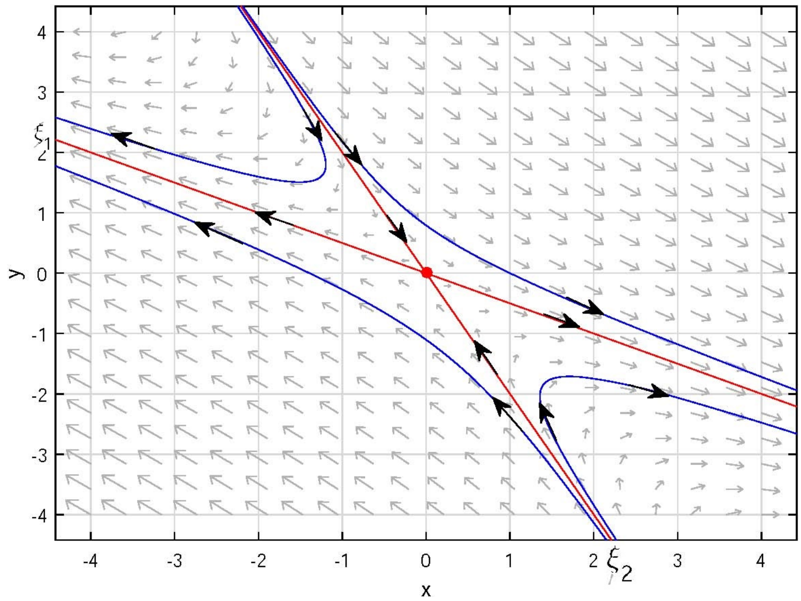


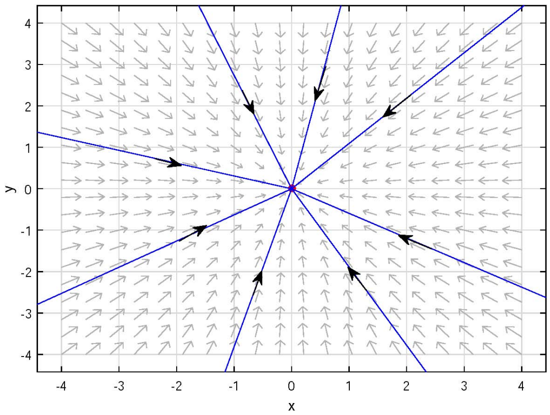
Figure 19: Phase portrait for $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} 3 & 2 \\ -2 & -2 \end{pmatrix} \mathbf{y}(t)$.

7.2.3 Case B (1): Equal Eigenvalues, Two Linearly Independent Eigenvectors

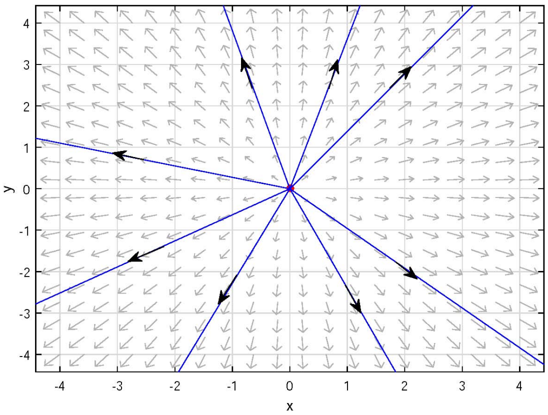
Assume we have a repeated eigenvalue $r_1 = r_2 = r$, with two linearly independent eigenvectors $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$. Then the expression for the general solution is

$$\mathbf{y}(t) = (c_1\boldsymbol{\xi}_1 + c_2\boldsymbol{\xi}_2)e^{rt}. \tag{860}$$

If $r < 0$ then $\mathbf{y} \to \mathbf{0}$ as $t \to \infty$ independent of the sign of c_1 and c_2 since $c_1\boldsymbol{\xi}_1 + c_2\boldsymbol{\xi}_2$ represent a line spanned by $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$. This means that every trajectory is a **straight line** through the critical point $\mathbf{0}$, see Figure 20 (a). Similarly, if $r > 0$, then $\mathbf{y}(t) \to \mathbf{0}$ as $t \to -\infty$ independent of the sign of c_1 and c_2, see Figure 20 (b). The trajectories are also straight lines through the critical point. In these cases, we call the critical point a **proper node** or **star point**.



(a) Phase portrait for $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{y}(t)$.



(b) Phase portrait for $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{y}(t)$.

Figure 20: (a) Stable proper node & nodel sink. (b) Unstable proper node & nodel source.

7.2.4 Case B (2): Equal Eigenvalues, One Linearly Independent Eigenvector

Let $r_1 = r_2 = r$ be the repeated eigenvalue, and $\boldsymbol{\xi}$ the associated eigenvector. Then, the general solution is

$$\mathbf{y}(t) = c_1\boldsymbol{\xi}e^{rt} + c_2(\boldsymbol{\xi}t + \boldsymbol{\eta})e^{rt} = e^{rt}((c_1\boldsymbol{\xi} + c_2\boldsymbol{\eta}) + tc_2\boldsymbol{\xi}) =: e^{rt}\mathbf{z}(t), \tag{861}$$

where we recall that $\boldsymbol{\eta}$ is a **generalized eigenvector** to the eigenvalue r, i.e.,

$$(\mathbf{A} - r\mathbf{I})\boldsymbol{\xi} = \mathbf{0}, \qquad (\mathbf{A} - r\mathbf{I})\boldsymbol{\eta} = \boldsymbol{\xi}. \tag{862}$$

First consider $r < 0$. To sketch the trajectories, note for fixed c_1 and c_2, the vector function $\mathbf{z}(t) = (c_1\boldsymbol{\xi} + c_2\boldsymbol{\eta}) + c_2\boldsymbol{\xi}t$ is a straight line through the point $c_1\boldsymbol{\xi} + c_2\boldsymbol{\eta}$ (initial position corresponding to $t = 0$) in the direction of $c_2\boldsymbol{\xi}$ (parallel to $\boldsymbol{\xi}$), which is the direction of **increasing** t. (Note that the direction of increasing t is different for $c_2 > 0$ and for $c_2 < 0$). Writing the solution $\mathbf{y}(t)$ as $\mathbf{y}(t) = e^{rt}\mathbf{z}(t)$ allows us to interpret that $\mathbf{z}(t)$ determines the **direction** of the trajectory and e^{rt} determine its magnitude.

If $c_2 = 0$, the trajectory is just the line spanned by $\boldsymbol{\xi}$. However, if $c_2 \neq 0$, then the dominating term is $c_2t\boldsymbol{\xi}e^{rt}$ for large (positive/negative) values of t, and we expect the trajectories to be tangent to the line spanned by $\boldsymbol{\xi}$ as $t \to \infty$ and parallel to the line spanned by $\boldsymbol{\xi}$ as $t \to -\infty$ since $r < 0$.

Now we draw the lines $c_1\boldsymbol{\xi} + c_2\boldsymbol{\eta} + tc_2\boldsymbol{\xi}$ passing through the point $c_1\boldsymbol{\xi} + c_2\boldsymbol{\eta}$ in the direction of $c_2\boldsymbol{\xi}$, which is the direction of **increasing** t. As t increases, the direction of the trajectories follow the direction of increasing t.

However, due to $r < 0$, the magnitude is shrinking exponentially, and the trajectories tend to origin as $t \to \infty$. Since the trajectories are supposed to travel along the direction of increasing t, we expect that the trajectories (except the line spanned by $\boldsymbol{\xi}$) will do a **sharp turn** and tangent to the line spanned by $\boldsymbol{\xi}$ at the origin. The orientation of the trajectories depends on the relative positions of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, two possible situations are shown in Figure 21. In these cases where the geometric multiplicity of the repeated eigenvalue is equal to one, we call the critical point an **improper node** or a **degenerate node**.

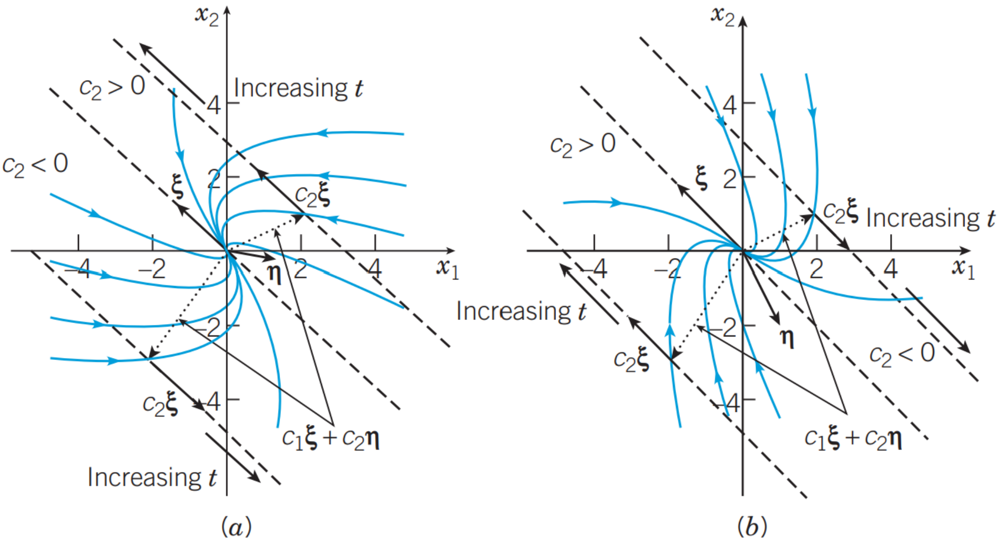
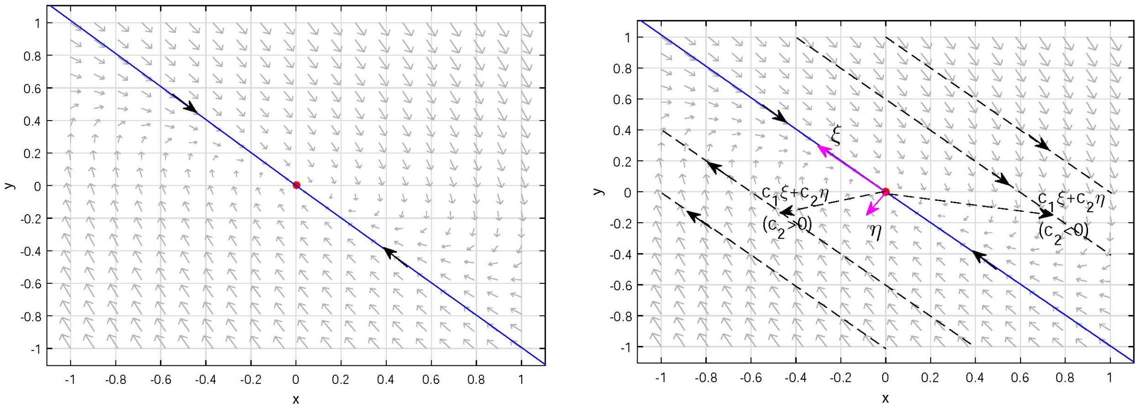


Figure 21: Asymptotically stable improper node.

Example 135. *Draw the phase portrait of* $\mathbf{y}' = \mathbf{A}\mathbf{y}(t)$*, where* $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ -4 & -7 \end{pmatrix}$ *with a repeated eigenvalue* $r = -3$*, eigenvector* $\boldsymbol{\xi} = (-1, 1)^T$*, and the generalized eigenvector* $\boldsymbol{\eta} = \left(-\frac{1}{8}, -\frac{1}{8}\right)^T$.

The first thing to draw is the line given by $c_1\boldsymbol{\xi}e^{rt} = (-c_1e^{-3t}, c_1e^{-3t})^T$. Next, we draw the lines $c_1\boldsymbol{\xi} + c_2\boldsymbol{\eta} + tc_2\boldsymbol{\xi}$ passing through the point $c_1\boldsymbol{\xi} + c_2\boldsymbol{\eta}$ in the direction of $c_2\boldsymbol{\xi}$, which is the direction of **increasing** t. This is given in Figure 22.



(a) The line given by $c_1\boldsymbol{\xi}e^{rt}$.

(b) The dotted lines given by $c_1\boldsymbol{\xi} + c_2\boldsymbol{\eta} + tc_2\boldsymbol{\xi}$.

Figure 22: Phase portrait for $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} 1 & 4 \\ -4 & -7 \end{pmatrix} \mathbf{y}(t)$.

From the expression $\mathbf{y}(t) = (c_1\boldsymbol{\xi} + c_2\boldsymbol{\eta} + c_2t\boldsymbol{\xi})e^{rt}$, the dominating term is $c_2t\boldsymbol{\xi}e^{rt}$ for large (positive/negative) values of t, so we expect the trajectories to be parallel to the line spanned by $\boldsymbol{\xi}$. As t increases, the direction of the trajectories follow the direction of increasing t. Due to $r < 0$, we expect that the trajectory to travel along the direction of increasing t but do a **sharp turn** to go back to the origin. This is reflected in Figure 23.

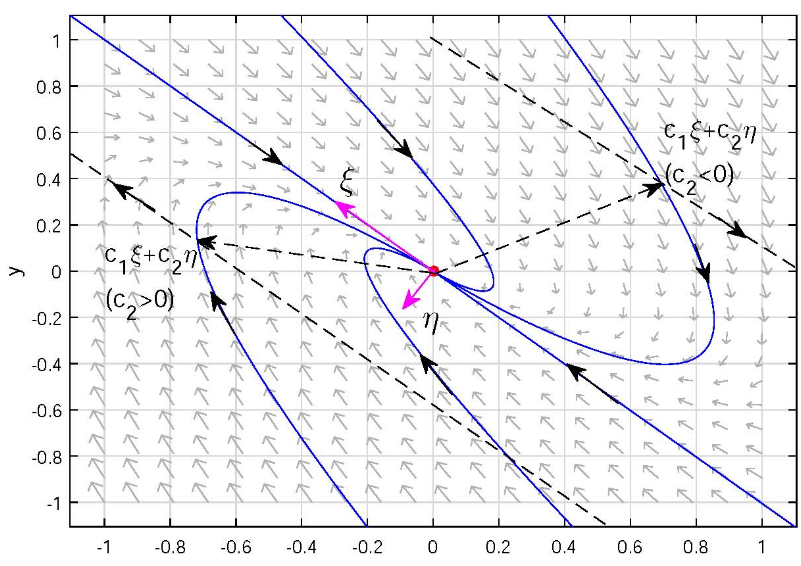


Figure 23: Phase portrait for $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} 1 & 4 \\ -4 & -7 \end{pmatrix} \mathbf{y}(t)$. Nodal sink.

Next consider $r > 0$. For this case, we have the similar phase portrait, but the direction of the trajectories are reversed, i.e., the origin is unstable and every trajectory is leaving the origin. An example with $r_1 = r_2 > 0$ is given by Figure 24, where the matrix $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$ with repeated eigenvalue $r = 1$ and eigenvector $\boldsymbol{\xi} = (-1, 2)^T$, the generalized eigenvector $\boldsymbol{\eta} = (0, -1)^T$. This gives an unstable critical point at the origin.

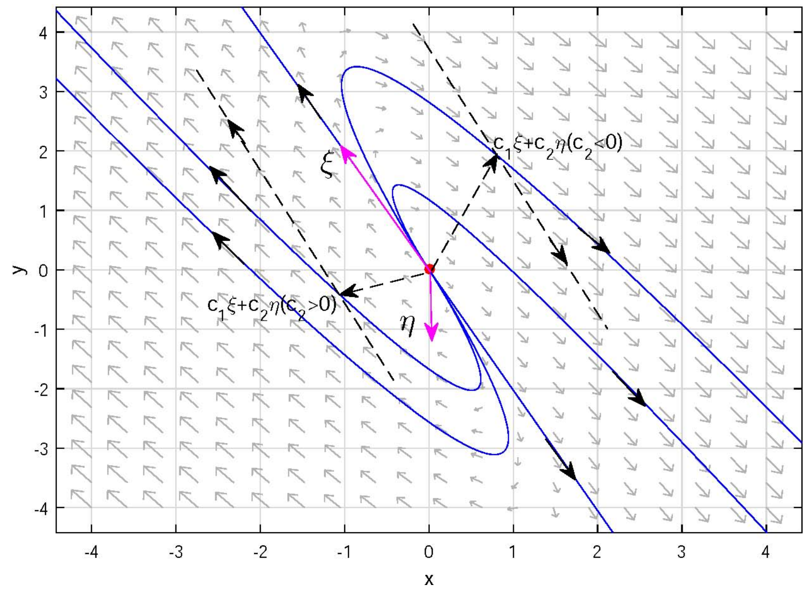


Figure 24: Phase portrait for $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix} \mathbf{y}(t)$. Nodal source.

7.2.5 Case C (1): Complex Eigenvalues, Non-Zero Real Part

Suppose $r_1 = \lambda + i\mu$, for $\lambda, \mu \in \mathbb{R}$ with $\lambda \neq 0$, so $r_2 = \lambda - i\mu$. Denote the corresponding eigenvectors to be $\boldsymbol{\xi}_1 = \mathbf{u} + i\mathbf{v}$ with $\boldsymbol{\xi}_2 = \mathbf{u} - i\mathbf{v}$, and set

$$\begin{cases} \mathbf{y}_1(t) = e^{\lambda t}(\mathbf{u}\cos(\mu t) - \mathbf{v}\sin(\mu t)) = e^{\lambda t}\mathbf{z}_1(t), \\ \mathbf{y}_2(t) = e^{\lambda t}(\mathbf{u}\sin(\mu t) + \mathbf{v}\cos(\mu t)) = e^{\lambda t}\mathbf{z}_2(t), \end{cases} \tag{863}$$

we have the general solution

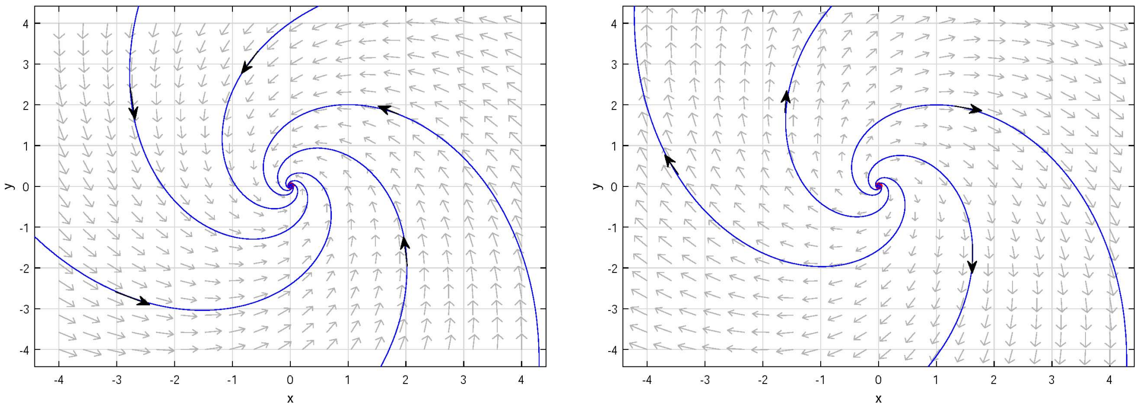
$$\mathbf{y}(t) = e^{\lambda t}(c_1\mathbf{z}_1(t) + c_2\mathbf{z}_2(t)), \tag{864}$$

for $c_1, c_2 \in \mathbb{R}$. Note that $\mathbf{z}_1$ and $\mathbf{z}_2$ are functions of cosine and sine, therefore are periodic bounded functions in t. We expect that

(1) if $\lambda < 0$, then $\mathbf{y}(t) \to \mathbf{0}$ as $t \to \infty$;

(2) if $\lambda > 0$, then $\mathbf{y}(t) \to \mathbf{0}$ as $t \to -\infty$.

So, for $\lambda < 0$, we expect the trajectories tend to $\mathbf{0}$ like a spiral. For example, see Figure 25 (a) for $\mathbf{A} = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix}$ with $r_{1(2)} = -1 \pm \sqrt{2}i$. For $\lambda > 0$, we have the same phase portrait but spirals outwards, see Figure 25 (b) for $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$ with $r_{1(2)} = 1 \pm \sqrt{2}i$.



(a) Phase portrait for $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix} \mathbf{y}(t)$. (b) Phase portrait for $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} \mathbf{y}(t)$.

Figure 25: (a) Critical point is a **spiral sink**. (b) Critical point is a **spiral source**..

One way to determine whether the trajectories spiral "clockwise" or "anticlockwise" is to look at the transformation of a point by the matrix $\mathbf{A}$. For example

$$\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix} \mathbf{y}(t) \tag{865}$$

has a matrix $\mathbf{A}$ with eigenvalues $-1 \pm \sqrt{2}i$. Applying the matrix $\mathbf{A}$ to the point $\mathbf{x} = (0, 1)^T$ yields

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}. \tag{866}$$

The vector $(-1, -1)^T$ provides a direction which the trajectories will be traveling. Therefore, if a trajectory starts at $(0, 1)^T$, it will move in a direction $(-1, -1)^T$, so the trajectories spiral in the anticlockwise direction when t increases. We call the critical point **0** a **spiral point** in this case where the eigenvalues of the matrix $\mathbf{A}$ are complex conjugate pairs with non-zero real part. If $\lambda < 0$ we have a **spiral sink** and if $\lambda > 0$ we have a **spiral source**.

7.2.6 Case C (2): Purely Imaginary Eigenvalues

We now consider the case where the eigenvalues of the matrix $\mathbf{A}$ are purely imaginary, i.e., $r_1 = i\mu, r_2 = -i\mu$ for $\mu \in \mathbb{R}$. In this case the general solution is

$$\mathbf{y}(t) = c_1\mathbf{z}_1(t) + c_2\mathbf{z}_2(t) = c_1(\mathbf{u}\cos(\mu t) - \mathbf{v}\sin(\mu t)) + c_2(\mathbf{u}\sin(\mu t) + \mathbf{v}\cos(\mu t)), \tag{867}$$

where $\boldsymbol{\xi}_1 = \mathbf{u} + i\mathbf{v}$ is the eigenvector corresponding to r_1. Due to the periodic nature of $\mathbf{z}_1$ and $\mathbf{z}_2$ we expect the trajectories to encircle the critical point, but neither approach nor move away as $t \to \infty$. See Figure 26 for the phase portrait of the system with $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -5 & -2 \end{pmatrix}$ and $r_1 = i$, $r_2 = -i$.

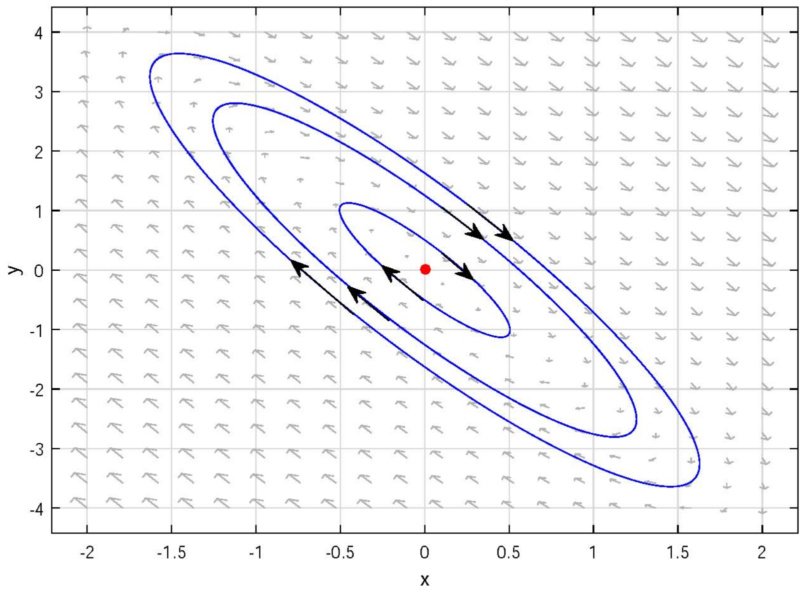


Figure 26: Phase portrait for $\frac{d\mathbf{y}(t)}{dt} = \begin{pmatrix} 2 & 1 \\ -5 & -2 \end{pmatrix} \mathbf{y}(t)$. Each trajectory is an elliptic curve.

This can also be seen from rewriting the general solution:

$$\mathbf{y}(t) = \sqrt{c_1^2 + c_2^2}\left(\mathbf{u}\cos(\mu t)\frac{c_1}{\sqrt{c_1^2+c_2^2}} + \mathbf{u}\sin(\mu t)\frac{c_2}{\sqrt{c_1^2+c_2^2}}\right)$$

$$+ \sqrt{c_1^2 + c_2^2}\left(\mathbf{v}\cos(\mu t)\frac{c_2}{\sqrt{c_1^2+c_2^2}} - \mathbf{v}\sin(\mu t)\frac{c_1}{\sqrt{c_1^2+c_2^2}}\right) \tag{868}$$

$$= \sqrt{c_1^2 + c_2^2}(\mathbf{u}\sin(\theta + \mu t) + \mathbf{v}\cos(\theta + \mu t)), \tag{869}$$

where $\theta \in [0, 2\pi]$ is a constant such that $\sin(\theta) = \frac{c_1}{\sqrt{c_1^2+c_2^2}}$ and $\cos(\theta) = \frac{c_2}{\sqrt{c_1^2+c_2^2}}$. The last line shows that the trajectory $\{\mathbf{y}(t)\}_{t\in I}$ can be seen as an ellipse centered at the origin with a fixed distance that is not changing in time.

Again, the direction of the trajectories "clockwise" or "anticlockwise" can be determined by testing one point with the matrix $\mathbf{A}$. For example starting from the point $\mathbf{x} = (0, 1)^T$ we have

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}. \tag{870}$$

Therefore, if a trajectory starts at $(0, 1)^T$, it will move in the direction $(1, -2)^T$, so the trajectories move in the clockwise direction when t increases. For this case where the eigenvalues of $\mathbf{A}$ are purely imaginary, we call the critical point a **center**.

Summary on Section 7.2. The behavior of trajectories for the system $\frac{d\mathbf{y}(t)}{dt} = \mathbf{A}\mathbf{y}(t)$ where the origin $\mathbf{0}$ is a critical point depends on the non-zero eigenvalues r_1, r_2. One of the following three situations can occur:

- All trajectories approach $\mathbf{0}$ as $t \to \infty$, then $\mathbf{0}$ is either a **nodal sink** or a **spiral sink**.
- All trajectories remains bounded (contained in a bounded set in the phase space) but do not approach $\mathbf{0}$ as $t \to \infty$. Then $\mathbf{0}$ is a **center**.
- Some trajectories (possibly all) except the trajectory $\mathbf{y}_*(t) = \mathbf{0}$ for all t, becomes unbounded as $t \to \infty$. Then $\mathbf{0}$ is either a **nodal source**, a **spiral source** or a **saddle point**.

Note that due to the **uniqueness**, through each point (y_1, y_2) of the phase plane, there is **only** one trajectory passing through that point. This implies that trajectories **do not cross each other**.

7.3 Lyapunov Stability

Definition 136 (Stability / Lyapunov Stability). *Let $\mathbf{y}_*$ be a critical point of the autonomous system*

$$\frac{d\mathbf{y}(t)}{dt} = \mathbf{f}(\mathbf{y}(t)), \tag{871}$$

i.e., $\mathbf{f}(\mathbf{y}_*) = \mathbf{0}$*. We say that $\mathbf{y}_*$ is **stable** if for any $\varepsilon > 0$, there exists a $\delta > 0$ (depending on $\mathbf{y}_*$ and ε) such that any solution $\mathbf{y} = \boldsymbol{\phi}(t)$ to $\frac{d\mathbf{y}(t)}{dt} = \mathbf{f}(\mathbf{y}(t))$ and $\mathbf{y}(t_0) = \boldsymbol{\phi}(t_0)$ satisfies*

$$\text{if } \|\boldsymbol{\phi}(t_0) - \mathbf{y}_*\| < \delta \quad \text{then} \quad \|\boldsymbol{\phi}(t) - \mathbf{y}_*\| < \varepsilon \quad \forall t \geq t_0, \tag{872}$$

where t_0 is some real number. Important remarks:

1. *$\mathbf{y}_*$ is **unstable** if it is not stable.*
2. *$\mathbf{y}_*$ is **asymptotically stable** if there exists $\delta > 0$ (depending only on $\mathbf{y}_*$) such that*

$$\text{if } \|\boldsymbol{\phi}(t_0) - \mathbf{y}_*\| < \delta \quad \text{then} \quad \boldsymbol{\phi}(t) \to \mathbf{y}_* \text{ as } t \to \infty. \tag{873}$$

***Asymptotically stable** must be **stable**, **stable** may not be **asymptotically stable**. See Figure 27.*

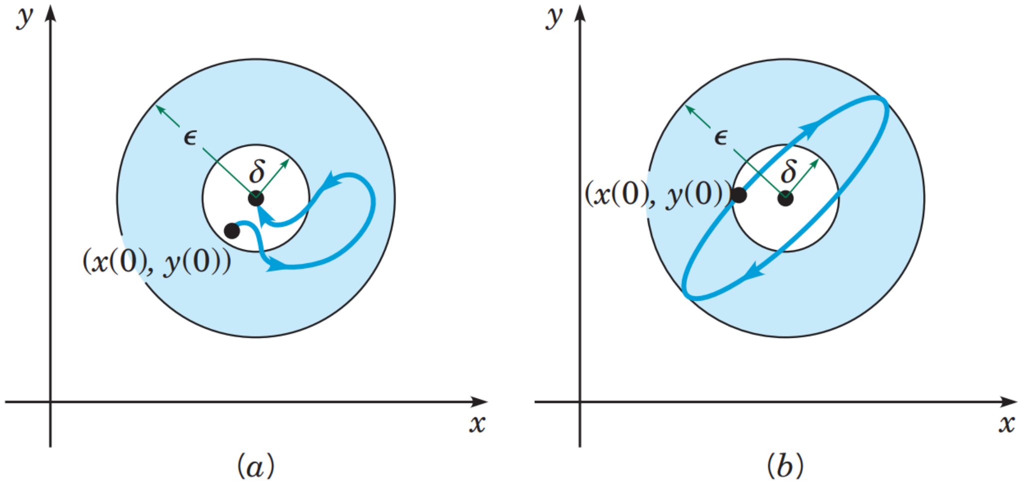


Figure 27: Graph of trajectories that exhibit (a) asymptotic stability and (b) stability.

The concepts of asymptotic stability, stability, and instability can be easily visualized in terms of an oscillating pendulum. Consider the configuration shown in Figure 28, in which an object with mass m is attached to one end of a rigid weightless rod of length L. The other end of the rod is fixed at the origin O, and the rod is free to rotate in the entire plane of the paper.

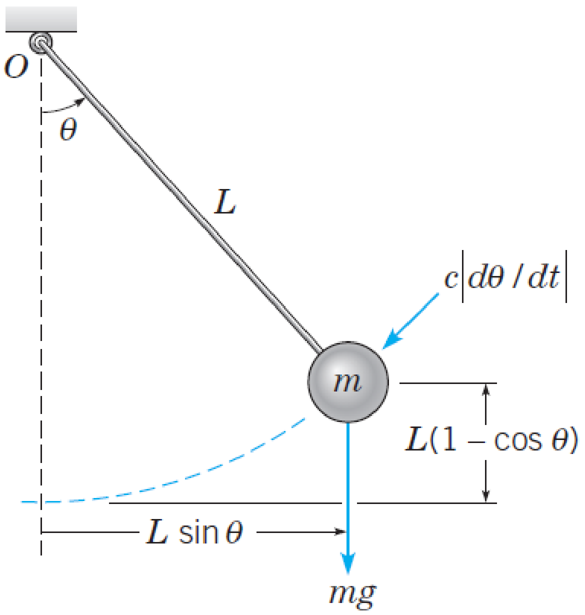


Figure 28: An oscillating pendulum.

If the object is slightly displaced from the lower equilibrium position, it will oscillate back and forth with gradually decreasing amplitude, eventually converging to the equilibrium position as the initial potential energy is dissipated by the damping force. This type of motion illustrates **asymptotic stability** and is shown in Figure 29 (a).

On the other hand, if the object is slightly displaced from the upper equilibrium position, it will fall very fast due to the gravity, and will ultimately converge to the lower equilibrium position. This type of motion illustrates **instability**. See Figure 29 (b). In practice, it is impossible to maintain the pendulum in its upward equilibrium position since a slight perturbation will cause the object to fall.

Finally, consider the ideal situation in which the damping coefficient is zero. In this case, if the object is displaced slightly from its lower equilibrium position, it will oscillate ∞ times with constant amplitude around the equilibrium position. Since there is no dissipation in the system, the object will remain near the equilibrium position but will not approach it asymptotically. This type of motion is **stable but not asymptotically stable**, as indicated in Figure 29 (c). In general, this motion is impossible to achieve in reality, because a slight air resistance or friction will eventually cause the pendulum to converge to its lower equilibrium position.

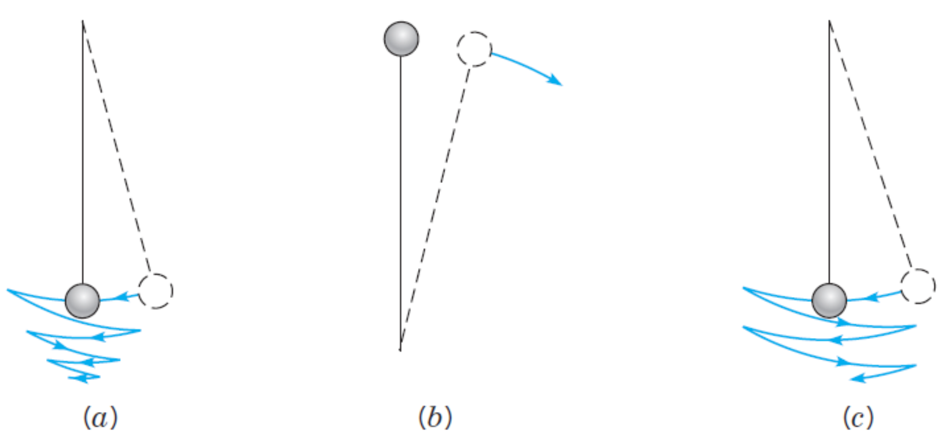


Figure 29: Qualitative motion of a pendulum. (a) With air resistance. (b) With or without air resistance. (c) Without air resistance.

We can now classify for

$$\frac{d\mathbf{y}}{dt} = \mathbf{A}\mathbf{y} \quad (\mathbf{A} \in \mathbb{R}^{2\times 2}) \tag{874}$$

the stability of the critical point $\mathbf{0}$:

Eigenvalues	Type	Stability
$r_1 > r_2 > 0$	Node	Unstable
$r_1 > 0 > r_2$	Saddle	Unstable
$r_1 < r_2 < 0$	Node	Asym.stable
$r_1 = r_2 < 0$	Proper/improper node	Asym.stable
$r_1 = r_2 > 0$	Proper/improper node	Unstable
$r_1 = \lambda + i\mu$	Spiral	Unstable ($\lambda > 0$), Asym.stable ($\lambda < 0$)
$r_1 = i\mu$	Center	Stable

7.4 Other Topics

- **More on Lyapunov Stability, Lyapunov's Second Method:** Wikipedia. Section 9.2 and 9.6 of [BDM21].
- **Limit Cycle:** Wikipedia.
- **Poincaré–Bendixson Theorem:** Lecture by Oliver Knill. Section 9.7 of [BDM21].
- **Chaos Theory:** Wikipedia. Section 9.8 of [BDM21].
  - **Lorentz System:** Wikipedia.
  - **Sharkovskii's Theorem:** Wikipedia. [LY75].
  - **Hénon Map:** Wikipedia.
  - **Duffing Equation:** Wikipedia.

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A Linear Algebra Notations

In our study of first-order systems, we will deal with the case where the entries of the matrix $\mathbf{A}$ are functions of the independent variable t, hence we can define a matrix function of t as $\mathbf{A}(t)$ where

$$\mathbf{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}(t) & a_{m2}(t) & \dots & a_{mn}(t) \end{pmatrix}. \tag{875}$$

We say that $\mathbf{A}(t)$ is **continuous** if all the entries $a_{11}(t), \dots, a_{mn}(t)$ are continuous functions of t. Similarly, we say $\mathbf{A}(t)$ is **differentiable** if all its entries are differentiable functions. Then

$$\frac{d}{dt}\mathbf{A}(t) = \begin{pmatrix} a'_{11}(t) & a'_{12}(t) & \dots & a'_{1n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a'_{m1}(t) & a'_{m2}(t) & \dots & a'_{mn}(t) \end{pmatrix}. \tag{876}$$

We can also define the (indefinite) integral of $\mathbf{A}(t)$ as

$$\int \mathbf{A}(t)dt = \left(\int a_{ij}(t)dt \right)_{1 \le i \le m, 1 \le j \le n}. \tag{877}$$

We also have the chain rule

$$\frac{d(\mathbf{A}(t)\mathbf{B}(t))}{dt} = \frac{d\mathbf{A}(t)}{dt}\mathbf{B}(t) + \mathbf{A}(t)\frac{d\mathbf{B}(t)}{dt}. \tag{878}$$