

ODE Notes (I)

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1 Introduction

This is my study notes on Ordinary Differential Equations (ODEs). This notes studies common methods for solving first, second and higher-order ODEs, without delving into detailed theoretical proofs. The content is mainly based on [Prof. Chaoyu Quan's course MAT2002 Ordinary Differential Equations at CUHK\(SZ\)](#), [Prof. Jeffrey R. Chasnov's notes](#), and the textbook [\[BDM21\]](#).

2 First-Order Differential Equations

In this section we study how to solve first-order ODEs (only involving first-order derivatives). We will start from the simplest linear case (Section 2.1), then turn to more general cases.

2.1 Linear Equations

We will first give the formulation of the first-order linear ordinary differential equations.

Definition 1 (First-Order Linear ODE). *The Initial Value Problem (IVP) of the general first-order linear ODE is given by*

$$\begin{cases} \frac{dy}{dt} = p(t)y + q(t), \\ y(t_0) = y_0, \end{cases} \quad (1)$$

for some given functions $p(t)$, $q(t)$ and constants t_0 and y_0 .

Next, we will introduce the **method of integrating factors** to solve the above ODE. Multiply (1) by a function $\mu(t)$ (a.k.a., the integrating factor), leading to

$$\mu(t) \frac{dy}{dt} - \mu(t)p(t)y(t) = \mu(t)q(t). \quad (2)$$

Suppose that

$$\mu(t) \frac{dy}{dt} - \mu(t)p(t)y(t) = \frac{d}{dt} (\mu(t)y(t)), \quad (3)$$

then (2) becomes

$$\frac{d}{dt} (\mu(t)y(t)) = \mu(t)q(t) \Rightarrow \mu(t)y(t) = \int \mu(t)q(t) dt + c, \quad c \in \mathbb{R} \quad (4)$$

If $\mu(t)$ is **non-zero**, we can obtain the general solution

$$y(t) = \frac{1}{\mu(t)} \left[\int \mu(t)q(t) dt + c \right] \quad (5)$$

The problem becomes how to find such $\mu(t)$? From (3) we have

$$\mu(t)y'(t) - \mu(t)p(t)y(t) = \mu'(t)y(t) + \mu(t)y'(t) \Rightarrow y(t) \left(\frac{d\mu}{dt} + p(t)\mu(t) \right) = 0. \quad (6)$$

The equation is satisfied if $y(t) = 0$ or $\mu'(t) + p(t)\mu(t) = 0$. The first case $y(t) = 0$ is not desirable, since if the initial condition y_0 is non-zero, we have a contradiction. Therefore, we consider the second case and obtain the equation

$$\frac{d\mu}{dt} = -p(t)\mu \quad (7)$$

as the ODE for μ . This is a **separable equation** which will be detailed in Section 2.2, and revisited in Example 6. $\mu(t) \equiv 0$ is one solution but without any interest. When $\mu(t) \neq 0$, we have

$$\frac{1}{\mu} \frac{d\mu}{dt} = -p(t) \Rightarrow \ln |\mu(t)| = - \int p(t) dt + c \quad (8)$$

Choosing the arbitrary constant c to be zero, we obtain a simplest integrating factor

$$\mu(t) = \exp\left(-\int p(t) dt\right) \quad (9)$$

Plug in (5) we obtain the final solution. The general solution $y(t)$ to the ODE $y' = p(t)y + q(t)$ is given as

$$y(t) = e^{\int p(t) dt} \left[\int e^{-\int p(t) dt} q(t) dt + c \right]. \quad (10)$$

The particular solution and the constant c can be computed with the initial condition $y(t_0) = y_0$.

Example 2. Solve the ODE

$$\begin{cases} t \frac{dy}{dt} + 2y = 4t^2, \\ y(1) = 2, \end{cases} \quad (11)$$

Suppose $t \neq 0$. Write the ODE in the form $y' = p(t)y + q(t)$ and identify p, q

$$t \frac{dy}{dt} + 2y = 4t^2 \Rightarrow \frac{dy}{dt} = -\frac{2}{t}y + 4t \Rightarrow p(t) = -\frac{2}{t}, \quad q(t) = 4t. \quad (12)$$

Compute the integrating factor

$$\mu(t) = \exp\left(-\int p(t) dt\right) = t^2 \quad (13)$$

We obtain the general solution

$$y(t) = \frac{1}{t^2} \left[\int t^2 \times 4t dt + c \right] = t^2 + \frac{c}{t^2}. \quad (14)$$

From the initial condition we have $c = 1$, thus the particular solution is

$$y(t) = t^2 + \frac{1}{t^2} \quad (15)$$

The general and particular solutions are shown in Figure 1.

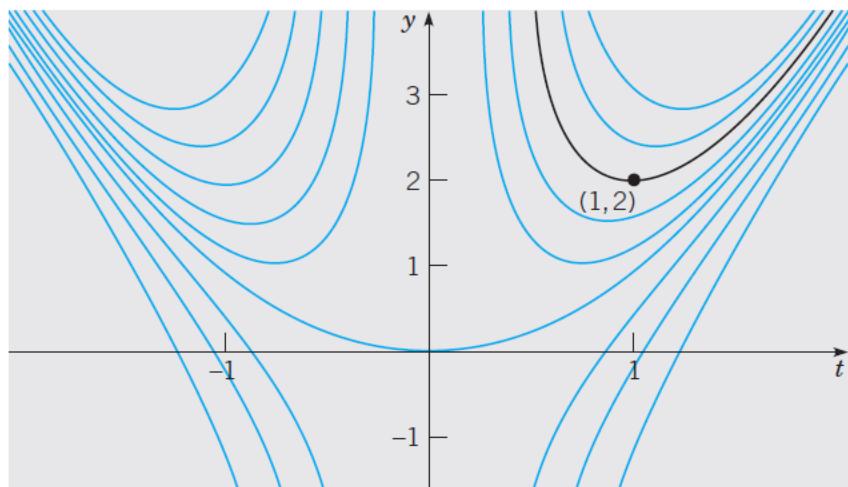


FIGURE 2.1.3 Integral curves of the differential equation $ty' + 2y = 4t^2$; the black curve passes through the point $(1, 2)$.

Figure 1

2.2 Separable Equations

Definition 3 (Separable Equation). A first order ODE $y' = f(t, y)$ is separable if it can be written in the form

$$M(t) + N(y) \frac{dy}{dt} = 0 \quad (16)$$

for some functions M, N .

Let's see an example.

Example 4. Solve the ODE

$$\begin{cases} \frac{dy}{dt} = \frac{\sin(t)}{1 - y^2}, \\ y(t_0) = y_0, \end{cases} \quad (17)$$

The key is to **separate y and t , placing them on opposite sides of the equation**. Bring y to the LHS we have

$$(1 - y^2) \frac{dy}{dt} = \sin(t) \quad (18)$$

Integrate both sides, we obtain

$$y(t) - \frac{1}{3}y(t)^3 = -\cos(t) + c, \quad c \in \mathbb{R}. \quad (19)$$

Using the initial condition to solve for c

$$\begin{cases} y(t) - \frac{1}{3}y(t)^3 = -\cos(t) + c, \\ y_0 - \frac{1}{3}y_0^3 = -\cos(t_0) + c, \end{cases} \quad (20)$$

The particular solution is given by

$$y(t) - \frac{1}{3}y(t)^3 = \cos(t_0) - \cos(t) + y_0 - \frac{1}{3}y_0^3. \quad (21)$$

Example 5. Solve the ODE

$$\frac{dy}{dt} = P(t)y \quad (22)$$

When $y \neq 0$, using the separation method we obtain

$$y = \pm e^{\bar{c}} \cdot e^{\int P(t) dt} = ce^{\int P(t) dt} \quad (23)$$

where $\bar{c} \in \mathbb{R}$ and $c = \pm e^{\bar{c}}$. Clearly $y = 0$ is also a solution to (24), so if we allow $c = 0$, then the solution $y = 0$ is also included in (23).

Example 6. Solve the ODE

$$\frac{dy}{dt} + \frac{1}{2}y = \frac{3}{2} \quad (24)$$

First write in the form

$$\frac{dy}{dt} = \frac{1}{2}(3 - y) \quad (25)$$

When $y \neq 3$, separate variables

$$\frac{1}{3-y} \frac{dy}{dt} = \frac{1}{2} \quad (26)$$

After integration and removing the absolute values we obtain

$$3 - y = \pm e^c \cdot e^{-\frac{1}{2}t} \quad (27)$$

So the final solution is

$$y = 3 + Ce^{-\frac{1}{2}t} \quad (28)$$

where $C \in \mathbb{R}$, since we included the solution $y = 3$.

2.3 Transformation Methods

There are many transformation methods, we will only discuss two of them.

2.3.1 Bernoulli Equation

Definition 7 (Bernoulli Equation). *Let n be a real number, $n \neq 0, 1$, and $p(t), q(t)$ be given functions. The Bernoulli equation is a first order non-linear ODE of the form*

$$\frac{dy}{dt} + p(t)y = q(t)y^n. \quad (29)$$

Multiply (29) with y^{-n}

$$y^{-n} \frac{dy}{dt} + p(t)y^{1-n} = q(t) \quad (30)$$

Since $\frac{d}{dt}(y^{1-n}) = (1-n)y^{-n}\frac{dy}{dt}$, (30) simplifies to

$$\frac{d}{dt}y^{1-n} + (1-n)p(t)y^{1-n} = (1-n)q(t) \quad (31)$$

Then, consider a **new variable** $v(t) = y^{1-n}(t)$, (31) becomes

$$\frac{dv}{dt} + P(t)v = Q(t), \quad P(t) = (1-n)p(t), \quad Q(t) = (1-n)q(t) \quad (32)$$

which is a first-order linear ODE for v . Let $\mu(t)$ be the integrating factor for (32), then the general solution is

$$v(t) = \frac{1}{\mu(t)} \left[\int Q(t)\mu(t) dt + c \right] \Rightarrow y(t) = \left(\frac{1}{\mu(t)} \left[\int Q(t)\mu(t) dt + c \right] \right)^{\frac{1}{1-n}} \quad (33)$$

2.3.2 Homogeneous First-Order Equation

Definition 8 (Homogeneous First-Order Equation). *A first order ODE $\frac{dy}{dt} = f(t, y)$ is called homogeneous if the function f only depends on the ratio $\frac{y}{t}$. That is, we can express*

$$f(t, y) = F\left(\frac{y}{t}\right) \quad \text{for some function } F. \quad (34)$$

We will still use a **transformation** method. Define a **new variable** $v = y/t \iff y = vt$. Then, the RHS of the ODE becomes just $F(v)$. For the LHS, by the product rule we have

$$y(t) = tv(t) \Rightarrow \frac{dy}{dt} = t\frac{dv}{dt} + v(t) \Rightarrow t\frac{dv}{dt} + v(t) = F(v). \quad (35)$$

Note that the initial condition $y(t_0) = y_0$ also transforms:

$$y(t_0) = y_0 \Rightarrow t_0v(t_0) = y_0, \quad (36)$$

and it is important to see that if $y_0 \neq 0$ then we cannot choose $t_0 = 0$, otherwise we get a contradiction. The transformed ODE in the variable v is now

$$\frac{dv}{dt} = \frac{F(v) - v}{t} \Rightarrow \frac{1}{F(v) - v} \frac{dv}{dt} = \frac{1}{t} \quad (37)$$

which is a **separable equation**.

Example 9. Solve the ODE

$$\frac{dy}{dt} = \frac{y - 4t}{t - y} = f(t, y) \quad (38)$$

Dividing numerator and denominator by t leads to

$$f(t, y) = \frac{y - 4t}{t - y} = \frac{y/t - 4}{1 - y/t} = F(y/t), \text{ where } F(s) = \frac{s - 4}{1 - s}. \quad (39)$$

Using a transformation $y = tv$ we find that v satisfies

$$\frac{1}{F(v) - v} \frac{dv}{dt} = \frac{1}{t} \Rightarrow \frac{1 - v}{(v - 2)(v + 2)} \frac{dv}{dt} = \frac{1}{t}. \quad (40)$$

Using partial fractions the coefficient can be simplified to

$$\frac{1 - v}{(v - 2)(v + 2)} = -\frac{1}{4(v - 2)} - \frac{3}{4(v + 2)}. \quad (41)$$

Then, integrating gives the general solution

$$-\frac{1}{4} \ln |v - 2| - \frac{3}{4} \ln |v + 2| = \ln |t| + c \Rightarrow -\frac{1}{4} \ln |y(t)/t - 2| - \frac{3}{4} \ln |y(t)/t + 2| = \ln |t| + c. \quad (42)$$

This gives

$$|y(t)/t - 2|^{-1/4} |y(t)/t + 2|^{-3/4} = e^c |t|, \quad c \in \mathbb{R}. \quad (43)$$

Which can be rewritten as

$$|t| |y(t)/t - 2|^{1/4} |y(t)/t + 2|^{3/4} = k, \quad k \geq 0. \quad (44)$$

2.4 Exact Equations

2.4.1 General Method

An ODE of the following form is not separable:

$$M(t, y) + N(t, y) \frac{dy}{dt} = 0 \quad (45)$$

where M, N are some functions. If the LHS of this equation can be written as $\frac{d\Psi(t,y(t))}{dt}$ for some function $\Psi(t, y)$, then integrating gives the general (implicit) solution

$$\Psi(t, y(t)) = c, \quad c \in \mathbb{R} \quad (46)$$

The requirement for

$$\frac{d\Psi(t, y(t))}{dt} = M(t, y) + N(t, y) \frac{dy}{dt} \quad (47)$$

implies

$$\frac{\partial\Psi}{\partial y}(t, y) = N(t, y), \quad \frac{\partial\Psi}{\partial t}(t, y) = M(t, y), \quad (48)$$

since $\frac{d\Psi(t,y(t))}{dt} = \frac{\partial\Psi}{\partial t}(t, y) + \frac{\partial\Psi}{\partial y}(t, y) \frac{dy}{dt}$. This is summarized in the following definition.

Definition 10 (Exact Equation). A first order ODE $M(t, y) + N(t, y) \frac{dy}{dt} = 0$ is an exact equation if there exists a function $\Psi(t, y)$ such that

$$\frac{\partial\Psi}{\partial y}(t, y) = N(t, y), \quad \frac{\partial\Psi}{\partial t}(t, y) = M(t, y). \quad (49)$$

The general solution $y(t)$ to the ODE is given implicitly as $\Psi(t, y(t)) = c$, $c \in \mathbb{R}$.

Thus, the question becomes:

1. How to determine an ODE of the form $M(t, y) + N(t, y) \frac{dy}{dt} = 0$ is exact?
2. If it is an exact equation, how to find the function $\Psi(t, y)$?

Solution is given by the following theorem.

Theorem 11. Let $M(t, y)$ and $N(t, y)$ be continuous functions of t and y in some simply connected domain, and have continuous first-order partial derivatives. Then the equation

$$M(t, y) + N(t, y) \frac{dy}{dt} = 0 \quad (50)$$

is an exact differential equation if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t} \quad (51)$$

We will prove the theorem to gain a better understanding of it.

Proof. “ \Leftarrow ”. Given that (50) is an exact differential equation, (49) holds, and by taking partial derivatives on t, y we obtain

$$\frac{\partial M}{\partial y} = \frac{\partial^2 \Psi}{\partial t \partial y}, \quad \frac{\partial N}{\partial t} = \frac{\partial^2 \Psi}{\partial y \partial t} \quad (52)$$

From the continuity of $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial t}$, we know that $\frac{\partial^2 \Psi}{\partial t \partial y}$ and $\frac{\partial^2 \Psi}{\partial y \partial t}$ are continuous. Therefore, we can obtain

$$\frac{\partial^2 \Psi}{\partial t \partial y} = \frac{\partial^2 \Psi}{\partial y \partial t} \quad (53)$$

That is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t} \quad (54)$$

Thus the necessity is proven.

“ \Rightarrow ”. We want to show that if (50) satisfies (51), then we can find function $\Psi(t, y)$ satisfying (49). Integrating both sides of $\frac{\partial \Psi}{\partial t} = M(t, y)$ with respect to t , we obtain

$$\int M(t, y) dt + \varphi(y) = \Psi(t, y) \quad (55)$$

Here $\varphi(y)$ is an arbitrary differentiable function of y . We choose a suitable $\varphi(y)$ such that $\Psi(t, y)$ also satisfies $\frac{\partial \Psi}{\partial y} = N(t, y)$, that is, taking the partial derivative with respect to y on both sides of (55), we get

$$\frac{\partial \Psi}{\partial y} = \frac{\partial}{\partial y} \int M(t, y) dt + \frac{d\varphi(y)}{dy} = N(t, y). \quad (56)$$

Therefore

$$\frac{d\varphi(y)}{dy} = N(t, y) - \frac{\partial}{\partial y} \int M(t, y) dt. \quad (57)$$

Note that $\varphi(y)$ is an arbitrary differentiable function of y , so the RHS of (57) must be independent of t , which means the partial derivative of the RHS of (57) with respect to t should be zero. In fact,

$$\frac{\partial}{\partial t} \left[N(t, y) - \frac{\partial}{\partial y} \int M(t, y) dt \right] = \frac{\partial N}{\partial t} - \frac{\partial}{\partial t} \left[\frac{\partial}{\partial y} \int M(t, y) dt \right]. \quad (58)$$

Since $M(t, y)$ and $N(t, y)$ are continuous functions of t, y in some simply connected domain, and have continuous first-order partial derivatives, the order of differentiation with respect to t and y in (58) can be interchanged. Using (51), we get

$$\frac{\partial}{\partial t} \left[N(t, y) - \frac{\partial}{\partial y} \int M(t, y) dt \right] = \frac{\partial N}{\partial t} - \frac{\partial}{\partial y} \left[\frac{\partial}{\partial t} \int M(t, y) dt \right] \quad (59)$$

$$= \frac{\partial N}{\partial t} - \frac{\partial M}{\partial y} = 0. \quad (60)$$

Thus the RHS of (57) is a function of y only. Integrating both sides, we obtain

$$\varphi(y) = \int \left[N(t, y) - \frac{\partial}{\partial y} \int M(t, y) dt \right] dy. \quad (61)$$

Substituting (61) into (55), we can find

$$\Psi(t, y) = \int M(t, y) dt + \int \left[N(t, y) - \frac{\partial}{\partial y} \int M(t, y) dt \right] dy. \quad (62)$$

In this way, we have proved that if (50) satisfies condition (51), then a $\Psi(t, y)$ that satisfies (49) must exist, and its specific expression is (62), thus proving sufficiency. Combining both directions finishes the proof. \square

Example 12. Solve the ODE

$$3t^2 + 6ty^2 + (6t^2y + 4y^3)\frac{dy}{dt} = 0 \quad (63)$$

Here, $M = 3t^2 + 6ty^2$, $N = 6t^2y + 4y^3$, easy to verify that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$, so the equation is exact. Find Ψ such that it satisfies

$$\frac{\partial \Psi}{\partial t} = M = 3t^2 + 6ty^2, \quad (64)$$

$$\frac{\partial \Psi}{\partial y} = N = 6t^2y + 4y^3 \quad (65)$$

Integrating (64) with respect to t , we get

$$\Psi = t^3 + 3t^2y^2 + \varphi(y). \quad (66)$$

Taking the partial derivative of (66) with respect to y , and using (65), we get

$$\frac{\partial \Psi}{\partial y} = 6t^2y + \frac{d\varphi(y)}{dy} = 6t^2y + 4y^3 \quad (67)$$

Thus

$$\frac{d\varphi(y)}{dy} = 4y^3 \quad (68)$$

Solving this, we get

$$\varphi(y) = y^4. \quad (69)$$

Substituting $\varphi(y)$ into (66), we get

$$\Psi = t^3 + 3t^2y^2 + y^4. \quad (70)$$

Therefore, the general solution of the equation is

$$t^3 + 3t^2y^2 + y^4 = c, \quad (71)$$

where c is an arbitrary constant. Alternatively, we can directly apply (62):

$$\int M(t, y) dt + \int \left[N(t, y) - \frac{\partial}{\partial y} \int M(t, y) dt \right] dy \quad (72)$$

$$= t^3 + 3t^2y^2 + \int (6t^2y + 4y^3 - 6t^2y) dy \quad (73)$$

$$= t^3 + 3t^2y^2 + y^4 = c, \quad (74)$$

where c is arbitrary constant.

2.4.2 Exact Equations with Integrating Factor

How to solve a non-exact ODE? Similar to the way we treated the first-order linear ODEs, consider multiplying with a integrating factor μ and hope things are better. We obtain after multiplying a new ODE

$$\mu M(t, y) + \mu N(t, y) \frac{dy}{dt} = 0 \quad (75)$$

If (75) is an exact equation, then by previous theorem, the following relation must be satisfied:

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial t}(\mu N) \quad (76)$$

Let's first investigate two cases.

Case 1. μ is just a function of t , i.e., $\mu = \mu(t)$. Then (76) simplifies to

$$N(t, y) \frac{d\mu}{dt} + \mu(t) N_t(t, y) = \mu(t) M_y(t, y). \quad (77)$$

If $N(t, y) \neq 0$, then we obtain an ODE for μ :

$$\frac{d\mu}{dt} = \mu(t) \left(\frac{M_y - N_t}{N} \right) (t, y) =: \mu(t) K(t, y). \quad (78)$$

Further suppose the factor $K(t, y)$ **depends only on t** , then (78) is a first-order **linear** ODE in $\mu(t)$ which can be solved by the method of integrating factors.

Case 2. μ is just a function of y , i.e., $\mu = \mu(y)$. Then (76) simplifies to

$$M(t, y) \frac{d\mu}{dy} + \mu(y) M_y(t, y) = \mu(y) N_t(t, y). \quad (79)$$

If $M(t, y) \neq 0$, then we obtain an ODE for μ :

$$\frac{d\mu}{dy} = \mu(y) \left(\frac{N_t - M_y}{M} \right) (t, y) =: \mu(y) H(t, y). \quad (80)$$

Further suppose the factor $H(t, y)$ **depends only on y** , then (80) is a first order **linear** ODE in $\mu(y)$ (where the independent variable is now y), and again can be solved by the method of integrating factors.

After obtaining μ , plug in (75) to obtain an exact equation.

Example 13. Solve the ODE

$$3ty + y^2 + (t^2 + ty) \frac{dy}{dt} = 0 \quad (81)$$

Clearly the ODE is not exact. Compute $K = \frac{M_y - N_t}{N} = \frac{t+y}{t^2+ty} = \frac{1}{t}$ and $H = \frac{N_t - M_y}{M} = \frac{-t-y}{3ty+y^2}$. We see that K is only a function of t but H is not just a function of y . So we expect the integrating factor μ to be a function of t only, which solves the ODE

$$\frac{d\mu}{dt} = \frac{\mu(t)}{t} \Rightarrow \mu(t) = ct, \quad c \in \mathbb{R} \quad (82)$$

Multiplying this integrating factor (take $c = 1$) with the ODE yields

$$t(3ty + y^2) + t(t^2 + ty)\frac{dy}{dt} = 0, \quad (83)$$

which is now an exact equation with function $\Psi(t, y)$ given as

$$\Psi(t, y) = t^3y + \frac{1}{2}t^2y^2. \quad (84)$$

So the general (implicit) solution to the ODE is

$$t^3y(t) + \frac{1}{2}t^2y^2(t) = c, \quad c \in \mathbb{R}. \quad (85)$$

Summary on methods in Section 2:

Type	Method	Explicit/Implicit solution
$y' = p(t)y + q(t)$	Integrating factor	$y(t) = \mu(t)^{-1}(\int \mu(t)q(t)dt + c)$
$M(t) + N(y)y' = 0$	Separable equation	$m(t) + n(y(t)) = c^*$
$y' + p(t)y = q(t)y^n$	$v := y^{1-n}$	$y(t) = (\mu^{-1}(\int Q(t)\mu(t)dt + c))^{1/(1-n)}$
$y' = F(y/t)$	$v = y/t$	$1/(F(v) - v)\frac{dv}{dt} = \frac{1}{t}$
$M(t, y) + N(t, y)y' = 0$	Exact equation	$\Psi(t, y(t)) = c$

*: $m(t) = \int M(t)dt, n(y(t)) = \int N(y)dy.$

2.5 Existence and Uniqueness Theorems

For completeness, we will state the existence and uniqueness theorems for IVP of first-order ODEs. The existence and uniqueness for first-order linear ODEs is characterized by the following theorem.

Theorem 14 (Existence and Uniqueness for First-Order Linear ODE). *Suppose functions p and q are continuous on $(\alpha, \beta) \subset \mathbb{R}$ (where α, β are some real numbers). Then, for any $t_0 \in (\alpha, \beta)$, $y_0 \in \mathbb{R}$, there exists a unique function $y(t)$ satisfying*

$$\begin{cases} \frac{dy}{dt} = p(t)y + q(t), & \forall t \in (\alpha, \beta), \\ y(t_0) = y_0, \end{cases} \quad (86)$$

And the solution is defined throughout the interval (α, β) .

The above theorem states that the unique solution to the IVP exists throughout any interval (α, β) containing $t = t_0$ if the functions p and q are continuous in (α, β) . In other words, **the solution globally exists in the interval (α, β) in which p and q are continuous**.

The existence and uniqueness for first-order non-linear ODEs is characterized by the following theorem.

Theorem 15 (Existence and Uniqueness for First-Order Non-Linear ODE). *Consider the IVP*

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0. \quad (87)$$

Let R be a closed rectangle

$$R = \{(t, y) \mid |t - t_0| \leq a, |y - y_0| \leq b\} \quad (a > 0, b > 0). \quad (88)$$

Assume that both $f(t, y)$ and $\frac{\partial f}{\partial y}$ are continuous on R . Then the IVP has a unique solution $y = y(t)$ defined on the interval $(t_0 - h, t_0 + h)$, where $h = \min(\frac{b}{M}, a)$ and $M = \max_{(t,y) \in R} |f(t, y)|$.

Under the assumption of the theorem, the solution only exists in a small interval $(t_0 - h, t_0 + h) \subset [t_0 - a, t_0 + a]$ since $h = \min(\frac{b}{M}, a)$ depends on the size of the region R . And h also depends on the values of the function $f(t, y)$ in the region R ($M = \max_{(t,y) \in R} |f(t, y)|$). **The solution only locally exists in the interval $[t_0 - a, t_0 + a]$.**

3 Second-Order Linear Differential Equations

In this section we study second-order **linear** ODEs. Section 3.1 introduces general theory of homogeneous equations, Section 3.2 and 3.3 study how to solve them, and Section 3.4 deals with non-homogeneous equations.

3.1 General Theory of Homogeneous Equations

We will first present the existence and uniqueness theorem for the second-order linear equations.

Theorem 16 (Existence and Uniqueness for Second-Order Linear ODE). *Consider the IVP*

$$y'' + p(t)y' + q(t)y = r(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_1. \quad (89)$$

Suppose $I = (\alpha, \beta) \subset \mathbb{R}$ is any open interval such that $t_0 \in I$, and the functions p, q, r are continuous in I . Then, there is exactly one solution $y(t)$ to the IVP for $t \in I$. The solution $y(t)$ is defined throughout the interval where p, q, r are continuous.

Now we introduce the following classification.

Definition 17 (Homogeneous). *A second order linear ODE*

$$p(t)y'' + q(t)y' + r(t)y = s(t), \quad p(t) \neq 0, \quad (90)$$

is called homogeneous if $s(t) \equiv 0$. Otherwise, if $s(t) \neq 0$, the ODE is called non-homogeneous.

For second-order homogeneous linear equations we have the following **principle of superposition**.

Theorem 18 (Principle of Superposition). *If y_1 and y_2 are two solutions of the ODE*

$$p(t)y'' + q(t)y' + r(t)y = 0. \quad (91)$$

Then for any constants $c_1, c_2 \in \mathbb{R}$, the function $c_1y_1(t) + c_2y_2(t)$ is also a solution to the ODE.

Clearly the principle of superposition **holds for homogeneous linear equations of any order**, which can be easily verified due to the linear structure.

Let's return to the second-order case. In other words, from two solutions we can construct infinite solutions to the homogeneous linear ODE. We can define a family of solutions

$$S = \{y = c_1y_1 + c_2y_2 \mid c_1, c_2 \in \mathbb{R}\} \quad (92)$$

to the ODE. The next question is: Given two solutions $y_1(t)$ and $y_2(t)$, can **any** solution to the ODE be expressed as a linear combination of $y_1(t)$ and $y_2(t)$?

Definition 19 (Wronskian). Given $y_1(t)$ and $y_2(t)$,

$$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y'_1(t) & y'_2(t) \end{vmatrix} \quad (93)$$

is called the Wronskian for y_1 and y_2 .

Indeed, we have the following theorem.

Theorem 20. Suppose that I is an open interval in which $p(t)$ and $q(t)$ are continuous. Let $y_1(t)$ and $y_2(t)$ be two solutions to the ODE

$$y'' + p(t)y' + q(t)y = 0 \quad (94)$$

for $t \in I$. Then, any solution $y(t)$ to the ODE can be expressed as

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \quad (95)$$

for constants c_1 and $c_2 \iff \exists t_0 \in I$ such that the Wronskian $W(y_1, y_2)[t_0] \neq 0$.

The theorem says that if $y_1(t)$ and $y_2(t)$ are two solutions to the above ODE and $W(y_1, y_2)[t_0] \neq 0$, then the general solution to the above ODE is given by the (95). In this case, we say that (y_1, y_2) form a **fundamental set of solutions** (FSS) to the ODE.

Example 21. $y_1(t) = \exp(-2t)$ and $y_2(t) = \exp(-3t)$ are solutions to the ODE

$$y'' + 5y' + 6y = 0. \quad (7)$$

$$W[y_1, y_2](t) = \begin{vmatrix} \exp(-2t) & \exp(-3t) \\ -2\exp(-2t) & -3\exp(-3t) \end{vmatrix} = -\exp(-5t) \neq 0 \quad (96)$$

Any solution to the ODE $y'' + 5y' + 6y = 0$ can be written as the linear combination of $y_1(t) = \exp(-2t)$ and $y_2(t) = \exp(-3t)$, they form a FSS to the ODE.

Example 22. For the ODE

$$2t^2y'' + 3ty' - y = 0, \quad t > 0, \quad (97)$$

the function $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ are solutions. Let us compute the Wronskian

$$W(y_1, y_2)[t] = -\frac{3}{2}t^{-3/2}, \quad (98)$$

which is non-zero for $t > 0$. Therefore we can deduce that (y_1, y_2) form a FSS for the ODE, and a general solution y to the ODE can be expressed as

$$y(t) = c_1 t^{1/2} + c_2 t^{-1}, \quad (99)$$

for some constants c_1, c_2 .

Next we will examine further the properties of the Wronskian of two solutions to the second-order linear homogeneous ODE. We will show an explicit formula for the Wronskian even if the two solutions are unknown.

Theorem 23 (Abel's Identity). *Let I be an open interval in which p and q are continuous. Suppose y_1 and y_2 are two non-zero solutions to the ODE*

$$y'' + p(t)y' + q(t)y = 0. \quad (100)$$

Then, the Wronskian is given as

$$W(y_1, y_2)[t] = c \exp \left(- \int p(t) dt \right), \quad (101)$$

where the constant c depends on y_1 and y_2 , but not on t . Consequently, $W(y_1, y_2)[t] = 0$ if and only if $c = 0$. In particular, if $W(y_1, y_2)[t_0] \neq 0$ for some $t_0 \in I$, then it holds that $W(y_1, y_2)[t] \neq 0$ for all $t \in I$. And, if $W(y_1, y_2)[t_0] = 0$ for some $t_0 \in I$, then it holds that $W(y_1, y_2)[t] = 0$ for all $t \in I$.

Proof. The idea is to derive an ODE for the Wronskian W . Going back to the ODE, as y_1 is a solution we have

$$y_1'' + p(t)y_1' + q(t)y_1 = 0 \Rightarrow y_2y_1'' + y_2p(t)y_1' + y_2q(t)y_1 = 0. \quad (102)$$

Similarly, as y_2 is a solution,

$$y_1y_2'' + y_1p(t)y_2' + y_1q(t)y_2 = 0. \quad (103)$$

Subtracting one from another gives

$$(y_1y_2'' - y_2y_1'') + p(t)(y_1y_2' - y_2y_1') = 0. \quad (104)$$

Noting that

$$W(y_1, y_2)[t] = y_1(t)y_2'(t) - y_2(t)y_1'(t) \Rightarrow W'(y_1, y_2)[t] = y_1(t)y_2''(t) - y_2(t)y_1''(t). \quad (105)$$

From (104) we have

$$W' + p(t)W = 0, \quad (106)$$

which is a linear first-order equation. By integrating factors, we find the general solution

$$W(y_1, y_2)[t] = c \exp \left(- \int p(t) dt \right), \quad (107)$$

for some constant $c \in \mathbb{R}$. As a constant of integration, c does not depend on t . \square

An implication of the theorem is that (y_1, y_2) form a FSS to $y'' + p(t)y' + q(t)y = 0$ if and only if $W(y_1, y_2)[t] \neq 0, \forall t \in I$.

Does a FSS always exist? This is answered in the next theorem.

Theorem 24. Let I be an open interval of \mathbb{R} , p and q are continuous functions in I . For any $t_0 \in I$, let $y_1(t)$ be the (unique) solution to the IVP

$$y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = 1, \quad y'(t_0) = 0, \quad (108)$$

and $y_2(t)$ be the (unique) solution to the IVP

$$y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = 0, \quad y'(t_0) = 1. \quad (109)$$

Then, (y_1, y_2) forms a FSS to the ODE.

Proof. Note that the existence of y_1 and y_2 to the corresponding IVPs is guaranteed by the Existence and Uniqueness Theorem. We only need to show that the Wronskian $W(y_1, y_2)[t_0]$ is non-zero. Computing gives

$$W(y_1, y_2)[t_0] = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1. \quad (110)$$

□

Indeed, the FSS are not unique. There are many different choices for $y_1(t_0), y'_1(t_0), y_2(t_0), y'_2(t_0)$ such that the corresponding solutions y_1, y_2 satisfy $W(y_1, y_2)[t_0] \neq 0$.

The FSS is closely related to the concept of linear (in)dependence in linear algebra.

Definition 25 (Linear Dependence). Consider 2 functions $x_1(t), x_2(t)$ defined on an interval $I \subset \mathbb{R}$. We say that $x_1(t), x_2(t)$ are linearly dependent if there are non-zero constants α_1, α_2 , such that

$$\alpha_1 x_1(t) + \alpha_2 x_2(t) = 0 \quad \forall t \in I. \quad (111)$$

Let $x_1(t), x_2(t)$ be defined on an interval $I \subset \mathbb{R}$. If $x_1(t), x_2(t)$ are not linearly dependent, then they are linearly independent.

Theorem 26. If $y_1(t)$ and $y_2(t)$ are two solutions to the ODE $y'' + p(t)y' + q(t)y = 0$, $t \in I$, where p, q are given continuous functions in I (some open interval). Then $y_1(t)$ and $y_2(t)$ are linearly independent $\iff W[y_1, y_2](t) \neq 0$, $\forall t \in I$ ((y_1, y_2) forms a FSS).

Proof. “ \Leftarrow ”.

$$\begin{pmatrix} y_1(t) & y_2(t) \\ y'_1(t) & y'_2(t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (112)$$

has only the zero solution, since the determinant $W[y_1, y_2](t) \neq 0$. Thus, $c_1 y_1(t) + c_2 y_2(t) = 0$ implies $c_1 = c_2 = 0$, meaning that $y_1(t)$ and $y_2(t)$ are linearly independent.

“ \Rightarrow ”. If $W[y_1, y_2](t_0) = 0$ for some $t_0 \in I$. Then the linear system

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (113)$$

has non-zero solution $(c_1^*, c_2^*) \neq (0, 0)$. Define $\phi(t) = c_1^*y_1(t) + c_2^*y_2(t)$, $t \in I$, then $\phi(t)$ is the solution to the ODE $y'' + p(t)y' + q(t)y = 0$ with initial conditions $y(t_0) = 0, y'(t_0) = 0$. But $y(t) = 0, t \in I$ is also the solution to the ODE with initial conditions $y(t_0) = 0, y'(t_0) = 0$. By the existence and uniqueness theorem, $\phi(t) = c_1^*y_1(t) + c_2^*y_2(t) = 0$. This implies that $y_1(t)$ and $y_2(t)$ are linearly dependent, which is a contradiction.

□

Clearly the proof also works for linear homogeneous ODEs of higher order. Another remark is that **a n -th order linear homogeneous ODE has at most n linear independent solutions**.

Based on the above results, the strategy to solve

$$y'' + p(t)y' + q(t)y = 0, \quad t \in I, \quad (114)$$

can be summarized as follows:

1. **Find two solutions y_1, y_2 satisfying the ODE.**
2. Find $t_* \in I$ such that the Wronskian $W(y_1, y_2)[t_*]$ is non-zero. Then, the general solution to the ODE is

$$y(t) = c_1y_1(t) + c_2y_2(t) \quad (115)$$

for some constants c_1, c_2 .

3. If initial conditions are prescribed at some $t_0 \in I$, compute c_1 and c_2 to determine the particular solution.

Step 1 is highly nontrivial, and is the basis to all methods in Section 3 (and Section 4).

3.2 Homogeneous Equations with Constant Coefficients

3.2.1 General Method

Although the FSS for the second-order linear homogeneous ODE $y'' + p(t)y' + q(t)y = 0$ always exist, but unfortunately, **there is no method to find the FSS explicitly**. However, when p, q are **constants**, we can find the FSS for $y'' + py' + qy = 0$ explicitly.

We will study the solutions to the ODE

$$ay'' + by' + cy = 0 \quad (116)$$

for fixed real constants $a, b, c \in \mathbb{R}$ with $a \neq 0$. Consider substituting a **trial function** $y(t) = \exp(rt)$ for some constant r into (116), which yields

$$(ar^2 + br + c)\exp(rt) = 0. \quad (117)$$

Since $\exp(rt) > 0$, we have

$$ar^2 + br + c = 0. \quad (118)$$

(118) is known as the **characteristic equation** for the ODE (116). If we can find the roots of the characteristic equation, then we know that $\exp(rt)$, where r is a root, is a solution to (116). By the quadratic formula we obtain

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \quad (119)$$

Three possibilities:

1. Two distinct real roots r_1, r_2 if $b^2 > 4ac$.
2. Two complex roots (complex conjugate pairs) r_1, \bar{r}_1 if $b^2 < 4ac$.
3. A repeated real root r if $b^2 = 4ac$.

Discriminant $\Delta := b^2 - 4ac$.

Case 1. Two distinct real roots $\Delta > 0$. In the case $b^2 > 4ac$, we obtain two real roots

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \quad (120)$$

This gives us two solutions

$$y_1(t) = \exp(r_1 t), \quad y_2(t) = \exp(r_2 t). \quad (121)$$

Check the Wronskian:

$$W(y_1, y_2)[t] = y_1(t)y'_2(t) - y_2(t)y'_1(t) \quad (122)$$

$$= r_2 \exp((r_1 + r_2)t) - r_1 \exp((r_1 + r_2)t) \quad (123)$$

$$= (r_2 - r_1) \exp((r_1 + r_2)t). \quad (124)$$

Clearly $W(y_1, y_2)[t] \neq 0$ for all $t \in \mathbb{R}$. Thus, $y_1(t) = \exp(r_1 t)$, $y_2(t) = \exp(r_2 t)$ is the FSS of the ODE (116). From previous theorems, any solution $y(t)$ to the ODE is of the form

$$y(t) = c_1 \exp(r_1 t) + c_2 \exp(r_2 t) \quad (125)$$

for some constants c_1 and c_2 .

Example 27. Solve the ODE

$$y'' + 9y' + 20y = 0 \quad (126)$$

Consider a trial function $y(t) = \exp(rt)$. The characteristic equation is

$$r^2 + 9r + 20 = (r + 4)(r + 5) = 0. \quad (127)$$

We have two real roots $r_1 = -4$ and $r_2 = -5$. Hence, the general solution is

$$y(t) = c_1 \exp(-4t) + c_2 \exp(-5t), \quad c_1, c_2 \in \mathbb{R}. \quad (128)$$

Case 2. Complex roots $\Delta < 0$. We now consider the case $\Delta = b^2 - 4ac < 0$. Then, the roots to the characteristic equation $ar^2 + br + c = 0$ is a complex-conjugate pair:

$$r_1 = \lambda + i\mu, \quad \lambda = \frac{-b}{2a}, \quad \mu = \frac{\sqrt{4ac - b^2}}{2a}, \quad i := \sqrt{-1}, \quad r_2 = \bar{r}_1 = \lambda - i\mu. \quad (129)$$

We obtain two functions

$$y_1(t) = \exp(r_1 t) = \exp((\lambda + i\mu)t), \quad y_2(t) = \exp(r_2 t) = \exp((\lambda - i\mu)t). \quad (130)$$

Using Euler's formula, we arrive at

$$y_1(t) = \exp(\lambda t) (\cos(\mu t) + i \sin(\mu t)), \quad y_2(t) = \exp(\lambda t) (\cos(\mu t) - i \sin(\mu t)) \quad (131)$$

Let's check that y_1 and y_2 are linearly independent. Suppose there are constants $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$\alpha_1 y_1(t) + \alpha_2 y_2(t) = 0 \quad \forall t \in I \Rightarrow e^{\lambda t} ((\alpha_1 + \alpha_2) \cos(\mu t) + i(\alpha_1 - \alpha_2) \sin(\mu t)) = 0. \quad (132)$$

The exponential is non-zero for all $t \in \mathbb{R}$, so to make the above expression zero, we need

$$\alpha_1 + \alpha_2 = 0, \quad \alpha_1 - \alpha_2 = 0 \quad \Rightarrow \quad \alpha_1 = \alpha_2 = 0. \quad (133)$$

So y_1 and y_2 are linearly independent. We can also calculate the Wronskian $W(y_1, y_2)[t] = -2i\mu e^{2\lambda t} \neq 0$, since $\mu \neq 0$ (otherwise we will not have $b^2 - 4ac < 0$). Thus, any solution $y(t)$ to the ODE is of the form

$$y(t) = e^{\lambda t} ((c_1 + c_2) \cos(\mu t) + i(c_1 - c_2) \sin(\mu t)) \quad (134)$$

or

$$y(t) = e^{\lambda t} (d_1 \cos(\mu t) + d_2 i \sin(\mu t)) \quad (135)$$

for some constants d_1 and d_2 . However, the solution is expressed as a complex-valued function. Since the coefficients of the ODE are real numbers, it would be better for us to obtain a real-valued function as a solution.

Theorem 28. Given an ODE

$$y'' + p(t)y' + q(t)y = 0 \quad (136)$$

with p and q are continuous real-valued functions. If $y(t) = u(t) + iv(t)$ is a complex-valued solution to the ODE, where u and v are real-valued functions, then its real part $u(t)$ and its imaginary part $v(t)$ are both solutions to the ODE.

Proof. Substituting the complex-valued solution into the ODE gives

$$0 = u''(t) + iv''(t) + p(t)u'(t) + ip(t)v'(t) + q(t)u(t) + iq(t)v(t) \quad (137)$$

$$= (u''(t) + p(t)u'(t) + q(t)u(t)) + i(v''(t) + p(t)v'(t) + q(t)v(t)). \quad (138)$$

A complex number is zero if and only if its real and imaginary parts are both zero, thus we have

$$u'' + p(t)u' + q(t)u = 0, \quad v'' + p(t)v' + q(t)v = 0. \quad (139)$$

□

Clearly the proof also works for linear homogeneous ODEs of higher order, and this result will be utilized again in Section 4.1.

From (135) we get the real-valued functions

$$u(t) = e^{\lambda t} \cos(\mu t), \quad v(t) = e^{\lambda t} \sin(\mu t) \quad (140)$$

Clearly u and v are linearly independent, and the Wronskian can be computed as $W(u, v)[t] = \mu e^{2\lambda t} \neq 0$, since $\mu \neq 0$. Thus, any solution y to the ODE $ay'' + by' + cy = 0$ with $b^2 - 4ac < 0$ can be expressed as

$$y(t) = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t) \quad (141)$$

which is a real-valued function.

Case 3. One repeated root $\Delta = 0$. The last case is when $b^2 - 4ac = 0$ and we have a repeated root to the characteristic equation:

$$r_1 = r_2 = -\frac{b}{2a} \quad (142)$$

The problem is apparent: both roots give the same function

$$y_1(t) = y_2(t) = \exp\left(-\frac{b}{2a}t\right). \quad (143)$$

We will use the Wronskian to find a solution that is linearly independent to $y_1(t)$. By Theorem 23, if $y_1 = \exp(-\frac{b}{2a}t)$ and y_2 are two solutions to the ODE $ay'' + by' + cy = 0$, then the Wronskian is

$$W(y_1, y_2)[t] = d \exp\left(-\int \frac{b}{a} dt\right) = d \exp\left(-\frac{b}{a}t\right) \quad (144)$$

for some constant $d \in \mathbb{R}$. On the other hand we have

$$W(y_1, y_2)[t] = y_1(t)y'_2(t) - y'_1(t)y_2(t) = e^{-\frac{b}{2a}t}y'_2(t) + \frac{b}{2a}e^{-\frac{b}{2a}t}y_2(t). \quad (145)$$

Choose $d = 1$, we have

$$e^{-\frac{b}{2a}t}y'_2(t) + \frac{b}{2a}e^{-\frac{b}{2a}t}y_2(t) = e^{-\frac{b}{a}t} \Rightarrow y'_2(t) + \frac{b}{2a}y_2(t) = e^{-\frac{b}{2a}t} \quad (146)$$

which is a first-order linear ODE for y_2 , the solution is

$$y_2(t) = te^{-\frac{b}{2a}t} \quad (147)$$

where we have neglected any constants of integration. Now check the linear independence for $y_1 = e^{-\frac{b}{2a}t}$ and $y_2 = te^{-\frac{b}{2a}t}$. Suppose α_1 and α_2 are two constants such that

$$\alpha_1 y_1(t) + \alpha_2 y_2(t) = 0 \quad \forall t \in I \Rightarrow e^{-\frac{b}{2a}t}(\alpha_1 + t\alpha_2) = 0. \quad (148)$$

Since the exponential is never zero, for $\alpha_1 + t\alpha_2$ to be zero for all $t \in I$, we must have $\alpha_1 = \alpha_2 = 0$. We can also compute the Wronskian $W(y_1, y_2)[t] = e^{-\frac{b}{a}t} \neq 0$. Thus any solution y to the ODE $ay'' + by' + cy = 0$ with $b^2 - 4ac = 0$ can be expressed as

$$y(t) = c_1 e^{-\frac{b}{2a}t} + c_2 t e^{-\frac{b}{2a}t} \quad (149)$$

for constants $c_1, c_2 \in \mathbb{R}$.

Summary of Section 3.2.1:

For the second order linear ODE

$$ay'' + by' + cy = 0 \quad (150)$$

with constants a, b, c . Let r_1 and r_2 be the roots to the characteristic equation

$$ar^2 + br + c = 0 \quad (151)$$

- If $b^2 > 4ac$, then r_1 and r_2 are real numbers, and the general solution is given as

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \quad (152)$$

- If $b^2 < 4ac$, then r_1 and r_2 are complex numbers such that $r_1 = \lambda + i\mu$ and $r_2 = \overline{r_1} = \lambda - i\mu$ for real numbers λ, μ . Then, the general solution is given as

$$y(t) = e^{\lambda t} (c_1 \cos(\mu t) + c_2 \sin(\mu t)) \quad (153)$$

- If $b^2 = 4ac$, then $r_1 = r_2 = r$. Then the general solution is given as

$$y(t) = c_1 e^{-\frac{b}{2a}t} + c_2 t e^{-\frac{b}{2a}t} \quad (154)$$

3.2.2 Euler Equations

Euler equations (a.k.a Cauchy-Euler equations) are the differential equations of the form

$$x^2 \frac{d^2 y}{dx^2} + Ax \frac{dy}{dx} + By = 0, \quad x > 0 \quad (155)$$

where A and B are constants. This is a second-order homogeneous linear ODE with **non-constant** coefficients, but we will convert it into an ODE with **constant** coefficients. Introducing a new independent variable

$$t = \ln x, \quad \text{or} \quad x = e^t, \quad (156)$$

and let

$$Y(t) = y(e^t) = y(x). \quad (157)$$

Taking the derivative we have

$$\frac{dy(x)}{dx} = \frac{dY(t)}{dx} = \frac{dY}{dt} \cdot \frac{dt}{dx} = Y'(t) \frac{1}{x}. \quad (158)$$

Then,

$$x \frac{dy(x)}{dx} = Y'(t). \quad (159)$$

Taking derivative again we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(Y'(t) \frac{1}{x} \right) \quad (160)$$

$$= \frac{1}{x} \frac{d}{dx} Y'(t) + Y'(t) \left(-\frac{1}{x^2} \right) \quad (161)$$

$$= \frac{1}{x} \frac{d}{dt} Y'(t) \frac{dt}{dx} - \frac{1}{x^2} Y'(t) \quad (162)$$

$$= \frac{1}{x^2} (Y''(t) - Y'(t)). \quad (163)$$

Then,

$$x^2 \frac{d^2y}{dx^2} = Y''(t) - Y'(t). \quad (164)$$

Substituting $x \frac{dy}{dx}$ and $x^2 \frac{d^2y}{dx^2}$ into the Euler equation we get

$$Y''(t) + (A - 1)Y'(t) + BY(t) = 0. \quad (165)$$

This is a constant coefficient linear equation, the general solution $Y(t)$ can be obtained. Then, the general solution of the Euler equation is

$$y(x) = Y(\ln x). \quad (166)$$

An alternative method for solving Euler equations is using **trial solution** $y = x^r$ (r is the power to be determined), then $y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$, then

$$r(r-1)x^r + Arx^r + Bx^r = 0. \quad (167)$$

Thus,

$$r^2 + (A - 1)r + B = 0. \quad (168)$$

- Case 1: Two distinct real roots r_1, r_2 . the general solution is

$$y(t) = c_1 x^{r_1} + c_2 x^{r_2}. \quad (169)$$

- Case 2: One repeated real root $r_1 = r_2$. The general solution is

$$y(x) = c_1 x^{r_1} + c_2 \ln(x) x^{r_1}. \quad (170)$$

- Case 3: two distinct complex roots: $\lambda \pm i\mu$, $\lambda, \mu \in \mathbb{R}$. The general solution is

$$y(x) = c_1 x^\lambda \cos(\mu \ln(x)) + c_2 x^\lambda \sin(\mu \ln(x)). \quad (171)$$

The Euler equation has a **regular singular point** at $x = 0$, which is related to the series solution of ODEs. Details can be found in Section 5.2 of [Prof. Jeffrey R. Chasnov's notes](#).

3.3 Homogeneous Equations with Non-Constant Coefficients

The **reduction of order** method can be applied to a second-order homogeneous ODE with **non-constant** coefficient. Although the general method for finding a FSS for $y'' + p(t)y' + q(t)y = 0$ is not available, but if we can find one nonzero-solution $y_1(t)$ of the ODE, then we can use the reduction of order method to find $y_2(t)$ so that (y_1, y_2) forms a FSS.

Consider the ODE

$$y'' + p(t)y' + q(t)y = 0. \quad (172)$$

Suppose $y_1(t)$ is a non-zero solution to the ODE. To find a second solution, consider the function

$$y(t) = v(t)y_1(t). \quad (173)$$

Then, the product rule gives

$$y'(t) = v'(t)y_1(t) + v(t)y'_1(t), \quad (174)$$

$$y''(t) = v''(t)y_1(t) + 2v'(t)y'_1(t) + v(t)y''_1(t). \quad (175)$$

If y is a solution to the ODE, we have

$$0 = y'' + p(t)y' + q(t)y \quad (176)$$

$$= v''y_1 + 2v'y'_1 + vy''_1 + p(t)(v'y_1 + vy'_1) + q(t)vy_1 \quad (177)$$

$$= y_1v'' + (2y'_1 + p(t)y_1)v'. \quad (178)$$

This gives us a second-order ODE for v that only involves v'' and v' . Define a new function $z = v'$, leading to

$$y_1(t)z' + (2y'_1(t) + p(t)y_1(t))z = 0. \quad (179)$$

Here we treat y_1 and y'_1 as given functions. Note that this is a first-order linear ODE

$$\frac{dz}{dt} + \frac{2y'_1 + py_1}{y_1}z = 0, \quad (180)$$

since $y_1 \neq 0$. Solving this gives

$$v'(t) = z(t) = \exp\left(-\int \frac{2y'_1 + py_1}{y_1} dt\right) \quad (181)$$

$$= \exp\left(-\int p(t)dt - 2\ln(y_1(t))\right) \quad (182)$$

$$= \frac{1}{y_1^2(t)} \exp\left(-\int p(t)dt\right). \quad (183)$$

Integrating once more leads to

$$v(t) = \int (y_1(t))^{-2} e^{-\int p(t)dt} dt \quad (184)$$

and the second solution to the ODE is given as

$$y_2(t) = y_1(t) \int (y_1(t))^{-2} e^{-\int p(t)dt} dt. \quad (185)$$

Example 29. Given that $y_1(t) = t^{-1}$ is a solution of

$$2t^2y'' + 3ty' - y = 0, \quad t > 0, \quad (186)$$

find a FSS.

The ODE can be written as $y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0$, thus $p(t) = \frac{3}{2t}$. Plug in (185) we obtain

$$y_2 = \frac{1}{t} \int t^2 e^{-\frac{3}{2} \ln(t) + c_1} \quad (187)$$

$$= e^{c_1} \frac{1}{t} \int t^{\frac{1}{2}} \quad (188)$$

$$= e^{c_1} \frac{1}{t} \left(\frac{2}{3} t^{\frac{3}{2}} + c_2 \right) \quad (189)$$

Take $e^{c_1} = \frac{3}{2}$ and $c_2 = 0$ we obtain $y_2 = t^{\frac{1}{2}}$. We can verify the Wronskian $W(y_1, y_2)[t] = \frac{3}{2}t^{-\frac{3}{2}} \neq 0$ for $t > 0$. Consequently, y_1, y_2 form a FSS for the ODE.

Note that this method can be used to find a second solution to the ODE if you **already have one solution**. The difficulty actually lies in finding a first solution to the ODE.

3.4 Non-Homogeneous Equations

We now turn our attention to ODE of the form

$$y'' + p(t)y' + q(t)y = r(t), \quad (190)$$

for given functions p , q , and r that are continuous in an interval I . The corresponding homogeneous equation is

$$y'' + p(t)y' + q(t)y = 0. \quad (191)$$

We have the following observation. Let Z_1 and Z_2 be solutions to the non-homogeneous problem (190). Then, the difference $Z := Z_1 - Z_2$ satisfies

$$Z'' + p(t)Z' + q(t)Z = r - r = 0. \quad (192)$$

That is, the difference Z satisfies the homogeneous equation (191). If (y_1, y_2) are a FSS to (191), then we can write $Z = Z_1 - Z_2$ as

$$Z_1(t) - Z_2(t) = c_1y_1(t) + c_2y_2(t), \quad (193)$$

for some constants c_1, c_2 . We have actually derived a general expression for the solution to the non-homogeneous equation (190). Let $Y(t)$ denote a solution to (190), then any solution y to (190) can be expressed as

$$y(t) = Y(t) + c_1y_1(t) + c_2y_2(t), \quad (194)$$

where (y_1, y_2) is a FSS to the homogeneous problem (191).

Definition 30 (Complementary Solution, Particular Solution). *For a solution expression*

$$y(t) = c_1y_1(t) + c_2y_2(t) + Y(t) \quad (195)$$

to the ODE

$$y'' + p(t)y' + q(t)y = r(t), \quad (196)$$

we call the function

$$y_c(t) := c_1y_1(t) + c_2y_2(t) \quad (197)$$

the complementary solution, which is a solution to the homogeneous equation, and the function $Y(t)$ the particular solution, which is a solution to the non-homogeneous equation.

This gives us the way of solving non-homogeneous second-order linear ODEs:

1. Obtain a FSS (y_1, y_2) to the homogeneous problem (191).
2. Find a solution $Y(t)$ to the non-homogeneous problem (190).
3. The general solution to (191) is then given as

$$y(t) = Y(t) + c_1y_1(t) + c_2y_2(t). \quad (198)$$

So the key questions become:

- How do we find y_1 and y_2 ?
- How do we find $Y(t)$?

Our discussion in Section 3.4.1 and 3.4.2 will focus on these two questions.

3.4.1 Method of Undetermined Coefficients

The general method for finding the second-order linear ODE with non-constant coefficient $a(t)y'' + b(t)y' + c(t)y = r(t)$ is still missing. We will first look at the special cases **when a, b, c are real constants and $r(t)$ is in some particular form**. In other words, we will show how to obtain a solution Y to the ODE

$$ay'' + by' + cy = r(t) \quad (199)$$

for some specific forms of $r(t)$.

The method for this case is the **method of undetermined coefficients**, which makes a guess on what the particular solution $Y(t)$ could look like. There are only certain classes of functions for $r(t)$ which $Y(t)$ could be obtained explicitly. We will consider $r(t)$ to be a mixture of **polynomials, exponential, sine and cosine**.

Example 31. Solve

$$y'' - 3y' - 4y = 3e^{2t}. \quad (200)$$

In the standard form we have

$$r(t) = 3e^{2t}. \quad (201)$$

Since the derivative of exponential function is also exponential, a **possible choice for the particular solution Y would involve exponential**. Solving the homogeneous problem $y'' - 3y' - 4y = 0$, the complementary solution is obtained as

$$y_c(t) = c_1 e^{4t} + c_2 e^{-t}. \quad (202)$$

Returning to the non-homogeneous problem, **assume $Y(t)$ is of the form**

$$Y(t) = Ae^{qt} \quad (203)$$

for some coefficients A and q that are not determined yet. Plugging into the non-homogeneous equations gives

$$Y'' - 3Y' - 4Y = Aq^2 e^{qt} - 3Aqe^{qt} - 4Ae^{qt} = A(q^2 - 3q - 4)e^{qt} = 3e^{2t}. \quad (204)$$

Therefore, it makes sense to choose

$$q = 2, \quad A(q^2 - 3q - 4) = 3 \quad \Rightarrow \quad A = -\frac{1}{2} \quad \Rightarrow \quad Y(t) = -\frac{1}{2}e^{2t}. \quad (205)$$

Hence, the general solution y to the ODE $y'' - 3y' - 4y = 3e^{2t}$ can be expressed as

$$y(t) = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{2}e^{2t}. \quad (206)$$

Example 32. Solve

$$y'' - 3y' - 4y = 2e^{-t}. \quad (207)$$

Since $r(t)$ is an exponential, try $Y(t) = Ae^{-t}$ and determine the value of A . However,

$$Y'' - 3Y' - 4Y = A(1 + 3 - 4)e^{-t} = 0. \quad (208)$$

So no choice of A would satisfy the non-homogeneous ODE. Actually, a FSS to the homogeneous ODE $y'' - 3y' - 4y = 0$ is $y_1 = e^{4t}$ and $y_2 = e^{-t}$. That is, the guess function $Y(t) = Ae^{-t}$ actually is a solution to the homogeneous problem, and consequently, it cannot be a solution to the non-homogeneous problem! In this case, where the assumed form of the particular solution Y is a duplicate of one of the solutions to the homogeneous problem, we can consider a new guess for Y which looks like

$$Y(t) = Ate^{-t}, \quad (209)$$

for undetermined constant A , which is similar to the FSS $(e^{-\frac{b}{2a}t}, te^{-\frac{b}{2a}t})$ for the ODE $ay'' + by' + cy = 0$ when $b^2 = 4ac$. Trying this new guess yields

$$Y'' - 3Y' - 4Y = -5Ae^{-t} = 2e^{-t}. \quad (210)$$

This means that we should take

$$A = -\frac{1}{5} \Rightarrow Y(t) = -\frac{2}{5}te^{-t}. \quad (211)$$

Thus a general solution y to the ODE $y'' - 3y' - 4y = 2e^{-t}$ is

$$y(t) = c_1e^{4t} + c_2e^{-t} - \frac{2}{5}te^{-t}. \quad (212)$$

Example 33. *Solve*

$$y'' - 3y' - 4y = t^2 + t + 1. \quad (213)$$

We know the complementary solution is $y_c = c_1e^{4t} + c_2e^{-t}$. Since $r(t)$ is a polynomial of degree 2, a possible guess is that the particular solution Y is also a polynomial of the same degree, that is $Y(t) = At^2 + Bt + C$ for some undetermined coefficients A, B, C . Then, plugging into the equation gives

$$Y'' - 3Y' - 4Y = 2A - 3(2At + B) - 4(At^2 + Bt + C) \quad (214)$$

$$= -4At^2 - (4B + 6A)t + (2A - 3B - 4C) = t^2 + t + 1. \quad (215)$$

Comparing coefficients immediately gives

$$A = -\frac{1}{4}, \quad B = \frac{1}{8}, \quad C = -\frac{15}{32}, \quad (216)$$

so the general solution y to the ODE $y'' - 3y' - 4y = t^2 + t + 1$ can be expressed as

$$y(t) = c_1e^{4t} + c_2e^{-t} - \frac{1}{4}t^2 + \frac{1}{8}t - \frac{15}{32}. \quad (217)$$

What if $r(t)$ involves the multiplication of exponential function and polynomials? The method is summarized as follows.

Case 1. $r(t) = P_n(t)e^{\alpha t}$. A possible guess is

$$Y(t) = t^s Q_n(t) e^{\alpha t}, \quad (218)$$

where $Q_n(t) = A_0 + A_1 t + \dots + A_n t^n$ is a polynomial with undetermined coefficients A_0, \dots, A_n , and $s \in \{0, 1, 2\}$ is an exponent determined by the following criterion:

$$s = \begin{cases} 0 & \text{if } \alpha \neq r_1, \alpha \neq r_2, \\ 1 & \text{if } \alpha = r_1 \neq r_2, \\ 2 & \text{if } r_1 = r_2 = \alpha. \end{cases} \quad (219)$$

where r_1 and r_2 are the roots to the characteristic equation

$$ar^2 + br + c = 0. \quad (220)$$

In fact, s is the **multiplicity** of α as a root of the characteristic equation. The guess (218) includes Example 31 to Example 33!

The problem of determining a particular solution to the ODE

$$ay'' + by' + cy = P_n(t)e^{\alpha t} \quad (221)$$

can be done by a substitution. Let

$$Y(t) = e^{\alpha t} u(t), \quad (222)$$

and by substituting this into the ODE we obtain

$$e^{\alpha t} [a[u'' + 2\alpha u' + \alpha^2 u] + b[u' + \alpha u] + cu] = e^{\alpha t} P_n(t) \quad (223)$$

$$\Rightarrow au'' + (2\alpha a + b)u' + (a\alpha^2 + ba + c)u = P_n(t). \quad (224)$$

To equal polynomial degree on both sides, it is reasonable to take

$$u(t) = \begin{cases} A_n t^n + \dots + A_0 & \text{if } a\alpha^2 + ba + c \neq 0, \\ t(A_n t^n + \dots + A_0) & \text{if } a\alpha^2 + ba + c = 0, 2a\alpha + b \neq 0, \\ t^2(A_n t^n + \dots + A_0) & \text{if } a\alpha^2 + ba + c = 0, 2a\alpha + b = 0. \end{cases} \quad (225)$$

$$= t^s (A_n t^n + \dots + A_0), \quad s = \begin{cases} 0 & \text{if } \alpha \neq r_1, \alpha \neq r_2, \\ 1 & \text{if } \alpha = r_1 \neq r_2, \\ 2 & \text{if } r_1 = r_2 = \alpha. \end{cases} \quad (226)$$

Example 34. Find a particular solution of

$$y'' - 3y' - 4y = te^{-t}. \quad (227)$$

e^{-t} is a solution to the homogeneous problem, and the non-homogeneous term is $r(t) = te^{-t}$. In this case we have $r_2 = \alpha = -1$ and $r_1 = 4$. Taking $s = 1$ we try a particular solution Y of the form

$$Y(t) = t(A_1 t + A_0)e^{-t} = (A_1 t^2 + A_0 t)e^{-t}. \quad (228)$$

Take derivatives

$$Y'(t) = (-A_1 t^2 + (2A_1 - A_0)t + A_0)e^{-t}, \quad Y''(t) = (A_1 t^2 + (A_0 - 4A_1)t + 2A_1 - 2A_0)e^{-t}. \quad (229)$$

Substituting these into the equation, we get $(-10A_1 t + 2A_1 - 5A_0)e^{-t} = te^{-t}$. Thus, $-10A_1 = 1$, $2A_1 - 5A_0 = 0$. Therefore, $A_1 = -\frac{1}{10}$, $A_0 = -\frac{1}{25}$. The particular solution is

$$Y(t) = t \left(-\frac{1}{10}t - \frac{1}{25} \right) e^{-t}. \quad (230)$$

What if $r(t)$ involves the multiplication of exponential function and polynomial as well as sine(cosine) function?

Example 35. Solve

$$y'' - 3y' - 4y = 2\sin(t) \quad (231)$$

The complementary solution is $y_c = c_1 e^{4t} + c_2 e^{-t}$. Since the non-homogeneous term $r(t) = 2\sin(t)$, a possible solution would involve sine and cosine, so consider

$$Y(t) = a \sin(\alpha t) + b \cos(\beta t) \quad (232)$$

for undetermined coefficients a, b, α, β . Plugging into the non-homogeneous equations gives

$$\begin{aligned} Y'' - 3Y' - 4Y &= -a\alpha^2 \sin(\alpha t) - b\beta^2 \cos(\beta t) - 3(a\alpha \cos(\alpha t) - b\beta \sin(\beta t)) \\ &\quad - 4(a \sin(\alpha t) + b \cos(\beta t)) \end{aligned} \quad (233)$$

$$\begin{aligned} &= \sin(\alpha t)[-a\alpha^2 - 4a] + \cos(\beta t)[-b\beta^2 - 4b] \\ &\quad + \cos(\alpha t)[-3a\alpha] + \sin(\beta t)[3b\beta] \end{aligned} \quad (234)$$

$$= 2\sin(t) \quad (235)$$

Since the RHS only involves $\sin(t)$, we can set

$$\alpha = 1, \quad \beta = 1. \quad (236)$$

This simplifies the above calculation to

$$\sin(t)[-5a + 3b] + \cos(t)[-5b - 3a] = 2\sin(t). \quad (237)$$

Since there is no term involving the cosine on the RHS, we must have

$$-5a + 3b = 2, \quad -5b - 3a = 0 \quad \Rightarrow \quad a = -\frac{5}{17}, \quad b = \frac{3}{17}. \quad (238)$$

Therefore, the general solution y to the ODE can be expressed as

$$y(t) = c_1 e^{4t} + c_2 e^{-t} - \frac{5}{17} \sin(t) + \frac{3}{17} \cos(t). \quad (239)$$

Case 2. $r(t) = e^{\alpha t} P_n(t) \cos(\beta t)$ or $e^{\alpha t} P_n(t) \sin(\beta t)$. Using Euler's formula: $\cos(\beta t) = \frac{1}{2} (e^{\beta it} + e^{-\beta it})$, $\sin(\beta t) = \frac{1}{2i} (e^{\beta it} - e^{-\beta it})$, the ODE becomes

$$ay'' + by' + cy = \frac{1}{2} P_n(t) (e^{(\alpha+\beta i)t} + e^{(\alpha-\beta i)t}) \quad (240)$$

$$ay'' + by' + cy = \frac{1}{2i} P_n(t) (e^{(\alpha+\beta i)t} - e^{(\alpha-\beta i)t}). \quad (241)$$

A possible guess for the above two ODEs is

$$Y(t) = t^s (Q_n(t) \cos(\beta t) + R_n(t) \sin(\beta t)) e^{\alpha t}, \quad (242)$$

where $Q_n(t) = A_0 + A_1 t + \dots + A_n t^n$, $R_n(t) = B_0 + B_1 t + \dots + B_n t^n$ are polynomials with undetermined coefficients $A_0, \dots, A_n, B_0, \dots, B_n$, and $s \in \{0, 1\}$ is an exponent determined by

$$s = \begin{cases} 0 & \text{if } \alpha + i\beta \text{ is not a root of the characteristic equation,} \\ 1 & \text{if } \alpha + i\beta \text{ is a root of the characteristic equation.} \end{cases} \quad (243)$$

To see the reason behind this, let us consider the case $r(t) = e^{\alpha t} P_n(t) \sin(\beta t)$ since the two cases are similar. We consider

$$Y(t) = e^{\alpha t} (Q(t) \cos(\beta t) + R(t) \sin(\beta t)), \quad (244)$$

for some functions Q and R , and upon differentiating:

$$\begin{aligned} Y'(t) &= \alpha e^{\alpha t} (Q(t) \cos(\beta t) + R(t) \sin(\beta t)) + e^{\alpha t} \beta (-Q(t) \sin(\beta t) + R(t) \cos(\beta t)) \\ &\quad + e^{\alpha t} (Q'(t) \cos(\beta t) + R'(t) \sin(\beta t)), \end{aligned} \quad (245)$$

$$\begin{aligned} Y''(t) &= \alpha^2 e^{\alpha t} (Q(t) \cos(\beta t) + R(t) \sin(\beta t)) + 2e^{\alpha t} \alpha \beta (-Q(t) \sin(\beta t) + R(t) \cos(\beta t)) \\ &\quad + 2\alpha e^{\alpha t} (Q'(t) \cos(\beta t) + R'(t) \sin(\beta t)) + \beta^2 e^{\alpha t} (-Q(t) \cos(\beta t) - R(t) \sin(\beta t)) \\ &\quad + 2\beta e^{\alpha t} (-Q'(t) \sin(\beta t) + R'(t) \cos(\beta t)) + e^{\alpha t} (Q''(t) \cos(\beta t) + R''(t) \sin(\beta t)). \end{aligned} \quad (246)$$

Plugging the above expression into the ODE yields

$$e^{\alpha t} P_n(t) \sin(\beta t) = aY'' + bY' + cY \quad (247)$$

$$\begin{aligned} &= e^{\alpha t} \cos(\beta t) [(a\alpha^2 - a\beta^2 + b\alpha + c)Q + (2a\alpha + b)(\beta R + Q') + 2a\beta R' + aQ''] \\ &\quad + e^{\alpha t} \sin(\beta t) [(a\alpha^2 - a\beta^2 + b\alpha + c)R + (2a\alpha + b)(-\beta Q + R') - 2a\beta Q' + aR'']. \end{aligned} \quad (248)$$

Equating coefficients means that

$$(a\alpha^2 - a\beta^2 + b\alpha + c)Q + (2a\alpha + b)(\beta R + Q') + 2a\beta R' + aQ'' = 0, \quad (249)$$

$$(a\alpha^2 - a\beta^2 + b\alpha + c)R + (2a\alpha + b)(-\beta Q + R') - 2a\beta Q' + aR'' = P_n. \quad (250)$$

Observe that, $\alpha + i\beta$ is a root of the characteristic equation if and only if

$$a(\alpha + i\beta)^2 + b(\alpha + i\beta) + c = [a\alpha^2 - a\beta^2 + b\alpha + c] + i(2a\alpha + b)\beta = 0. \quad (251)$$

Using the fact that a complex number is zero if and only if the real and imaginary parts are zero, we have

$$\alpha + i\beta \text{ is a root} \iff a(\alpha^2 - \beta^2) + b\alpha + c = 0, \quad (2a\alpha + b)\beta = 0. \quad (252)$$

As the RHS of (250) are polynomials, we may take Q and R to be polynomials. The question is the degree.

- Case 1: $\alpha + i\beta$ is not a root of the characteristic equation. Then, $(a\alpha^2 - a\beta^2 + b\alpha + c)$ and $(2a\alpha + b)\beta$ are not all zeros. We can take Q and R to have the **same degree** as the polynomial P_n , i.e.,

$$Q(t) = A_n t^n + \cdots + A_0, \quad R(t) = B_n t^n + \cdots + B_0$$

- Case 2: $\alpha + i\beta$ is a root of the characteristic equation, then (249), (250) simplifies to

$$(2a\alpha + b)Q' + 2a\beta R' + aQ'' = 0, \quad (253)$$

$$(2a\alpha + b)R' - 2a\beta Q' + aR'' = P_n. \quad (254)$$

and from the second equation, we see that the degree of the LHS would be the degree of R' or Q' (which ever is higher), thus we take

$$Q(t) = t(A_n t^n + \cdots + A_1 t + A_0), \quad R(t) = t(B_n t^n + \cdots + B_1 t + B_0), \quad (255)$$

in order to match the degree with the RHS.

Example 36. Find a particular solution of

$$y'' - 3y' - 4y = -8e^t \cos 2t. \quad (256)$$

We guess our particular solution $Y(t)$ is the product of e^t and a linear combination of $\cos 2t$ and $\sin 2t$, i.e.

$$Y(t) = Ae^t \cos 2t + Be^t \sin 2t \quad (257)$$

It follows that

$$Y'(t) = [A \cos 2t - 2A \sin 2t]e^t + [B \sin 2t + 2B \cos 2t]e^t \quad (258)$$

$$= (A + 2B)e^t \cos 2t + (-2A + B)e^t \sin 2t \quad (259)$$

and

$$Y''(t) = [(A + 2B)\cos 2t - 2(A + 2B)\sin 2t]e^t + [(-2A + B)\sin 2t + 2(-2A + B)\cos 2t]e^t \quad (260)$$

$$= (-3A + 4B)e^t \cos 2t + (-4A - 3B)e^t \sin 2t \quad (261)$$

After substituting for y , y' and y'' in (256) we obtain:

$$\begin{aligned} & e^t \cos 2t[(-3A + 4B) - 3(A + 2B) - 4A] \\ & + e^t \sin 2t[(-4A - 3B) - 3(-2A + B) - 4B] = -8e^t \cos 2t \end{aligned} \quad (262)$$

Hence:

$$\begin{cases} -10A - 2B = -8, \\ 2A - 10B = 0, \end{cases} \Rightarrow \begin{cases} A = \frac{10}{13}, \\ B = \frac{2}{13}, \end{cases} \quad (263)$$

Hence our particular solution is:

$$Y(t) = \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin 2t. \quad (264)$$

Summary of Section 3.4.1:

For

$$ay'' + by' + cy = r(t) \quad (265)$$

the trial function $Y(t)$ vs. $r(t)$ is listed as follows:

$r(t)$	$Y(t)$	The value for s
$P_n(t)e^{\alpha t}$	$Q_n(t)t^s e^{\alpha t}$	$s = \begin{cases} 0, & \alpha \text{ is not a root.} \\ 1, & \alpha = r_1 \neq r_2 \\ 2, & \alpha = r_1 = r_2 \end{cases}$
		r_1, r_2 are roots of $ar^2 + br + c = 0$
$\begin{cases} P_n e^{\alpha t} \sin \beta t \\ P_n e^{\alpha t} \cos \beta t \end{cases}$	$\begin{cases} [Q_n(t) \cos \beta t \\ + R_n(t) \sin \beta t] t^s e^{\alpha t} \end{cases}$	$s = \begin{cases} 0, & \text{if } \alpha + i\beta \text{ is not a root of } ar^2 + br + c = 0. \\ 1, & \text{if } \alpha + i\beta \text{ is a root of } ar^2 + br + c = 0. \end{cases}$

We will conclude this section with another theorem.

Theorem 37. Suppose Y_1 is a solution to

$$ay'' + by' + cy = g_1(t), \quad (266)$$

and Y_2 is a solution to

$$ay'' + by' + cy = g_2(t). \quad (267)$$

Then the sum $Y_1 + Y_2$ is a solution to

$$ay'' + by' + cy = g_1(t) + g_2(t). \quad (268)$$

Proof. Since Y_1 is a solution to $ay'' + by' + cy = g_1(t)$. and Y_2 is a solution to $ay'' + by' + cy = g_2(t)$, we have

$$aY_1'' + bY_1' + cY_1 = g_1(t) \quad (269)$$

$$aY_2'' + bY_2' + cY_2 = g_2(t) \quad (270)$$

Sum the two equations, we have

$$[aY_1'' + bY_1' + cY_1] + [aY_2'' + bY_2' + cY_2] \quad (271)$$

$$= a[Y_1'' + Y_2''] + b[Y_1' + Y_2'] + c[Y_1 + Y_2] \quad (272)$$

$$= a[Y_1 + Y_2]'' + b[Y_1 + Y_2]' + c[Y_1 + Y_2] \quad (273)$$

$$= g_1(t) + g_2(t) = g(t). \quad (274)$$

□

Clearly the result also holds when a, b, c are not constants.

Example 38. Find a particular solution of

$$y'' - 3y' - 4y = 3e^{2t} + 2e^{-t} + 2\sin t - 8e^t \cos 2t. \quad (275)$$

Combining previous results, we have

$$Y(t) = -\frac{1}{2}e^{2t} - \frac{2}{5}te^{-t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t + \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin 2t. \quad (276)$$

3.4.2 Variation of Parameters

The method of undetermined coefficients is straightforward, but requires that the non-homogeneous term $r(t)$ to be in a special form. We need a more general method that in principle can be applied to any equation. One such method is the **variation of parameters**.

Consider a general 2nd-order linear ODE

$$y'' + p(t)y' + q(t)y = r(t), \quad (277)$$

and suppose (y_1, y_2) forms a FSS to the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0. \quad (278)$$

How to find a particular solution to the non-homogeneous equation (277)? Consider for some functions $u_1(t), u_2(t)$ such that the new function

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) \quad (279)$$

solves (277). We now determine what equations u_1 and u_2 have to satisfy. Differentiating (279) yields

$$y' = u'_1y_1 + u_1y'_1 + u'_2y_2 + u_2y'_2. \quad (280)$$

In order to simplify the computation, **impose a condition**

$$u'_1y_1 + u'_2y_2 = 0. \quad (281)$$

Then the derivative becomes

$$y' = u_1y'_1 + u_2y'_2. \quad (282)$$

Differentiating again leads to

$$y'' = u'_1y'_1 + u_1y''_1 + u'_2y'_2 + u_2y''_2 \quad (283)$$

Substitute into the non-homogeneous ODE gives

$$y'' + p(t)y' + q(t)y = u_1(y''_1 + p(t)y'_1 + q(t)y_1) + u_2(y''_2 + p(t)y'_2 + q(t)y_2) \quad (284)$$

$$+ u'_1y'_1 + u'_2y'_2 \quad (285)$$

$$= u'_1y'_1 + u'_2y'_2 = r(t). \quad (286)$$

Thus, we obtain two conditions for u_1 and u_2 :

$$u'_1y_1 + u'_2y_2 = 0, \quad u'_1y'_1 + u'_2y'_2 = r(t), \quad (287)$$

which can be summarized as

$$\begin{pmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{pmatrix} \begin{pmatrix} u'_1 \\ u'_2 \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix} \quad (288)$$

Since the determinant is the Wronskian $W(y_1, y_2)[t]$ which is non-zero since (y_1, y_2) is a FSS, (u'_1, u'_2) can be solved. Using Cramer's rule, we have

$$u'_1(t) = -\frac{y_2 r}{W(y_1, y_2)}(t), \quad u'_2(t) = \frac{y_1 r}{W(y_1, y_2)}(t). \quad (289)$$

Integrating gives

$$u_1(t) = -\int \frac{y_2 r}{W(y_1, y_2)}(t) dt + d_1, \quad u_2(t) = \int \frac{y_1 r}{W(y_1, y_2)}(t) dt + d_2, \quad (290)$$

for constants $d_1, d_2 \in \mathbb{R}$, and the general solution to the non-homogeneous equation is

$$y(t) = (c_1 + d_1)y_1(t) + (c_2 + d_2)y_2(t) - y_1 \int \frac{y_2 r}{W(y_1, y_2)}(t) dt + y_2 \int \frac{y_1 r}{W(y_1, y_2)}(t) dt. \quad (291)$$

We can simply take $d_1 = d_2 = 0$, so the final solution becomes

$$y(t) = c_1 y_1(t) + c_2 y_2(t) - y_1 \int \frac{y_2 r}{W(y_1, y_2)}(t) dt + y_2 \int \frac{y_1 r}{W(y_1, y_2)}(t) dt. \quad (292)$$

for constants $c_1, c_2 \in \mathbb{R}$.

This method is able to treat rather general second-order ODEs (since $p(t)$ and $q(t)$ need not be constants). However, **it is not easy to find a FSS** (if $p(t)$ and $q(t)$ are not constants). Another difficulty lies in the evaluation of the integrals:

$$-\int \frac{y_2 r}{W(y_1, y_2)}(t) dt, \quad \int \frac{y_1 r}{W(y_1, y_2)}(t) dt \quad (293)$$

which may not be possible if r, y_1, y_2 are complicated functions.

Example 39. Solve the ODE

$$y'' - 3y' + 2y = \frac{e^{3t}}{e^t + 1} \quad (294)$$

First look at the homogeneous problem

$$y'' - 3y' + 2y = 0, \quad (295)$$

the complementary solution is given as

$$y_c(t) = c_1 e^t + c_2 e^{2t}. \quad (296)$$

We now compute u_1 and u_2 , where we use

$$y_1 = e^t, \quad y_2 = e^{2t}, \quad r = \frac{e^{3t}}{e^t + 1}, \quad W(y_1, y_2)[t] = e^{3t}. \quad (297)$$

We have

$$u'_1(t) = -\frac{e^{2t}}{e^t + 1}, \quad u'_2(t) = \frac{e^t}{e^t + 1}. \quad (298)$$

Integrating gives

$$u_1(t) = \ln(e^t + 1) - e^t, \quad u_2(t) = \ln(e^t + 1). \quad (299)$$

Hence, a particular solution is

$$Y(t) = u_1 y_1 + u_2 y_2 = e^t \ln(e^t + 1) + e^{2t} \ln(e^t + 1) - e^{2t}. \quad (300)$$

The general solution to the ODE is

$$y(t) = c_1 e^t + c_2 e^{2t} + e^t \ln(e^t + 1) + e^{2t} \ln(e^t + 1) \quad (301)$$

where c_1, c_2 are arbitrary constants.

4 Higher-Order Linear Differential Equations

4.1 General Theory

The general n -th order linear ODE is of the form

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = g(t), \quad (302)$$

and for an IVP we provide initial conditions

$$y(t_0) = x_0, y'(t_0) = x_1, \dots, y^{(n-1)}(t_0) = x_{n-1}. \quad (303)$$

We first state the existence and uniqueness theorem.

Theorem 40 (Existence and Uniqueness for n -th Order Linear ODE). *Let $I \subset \mathbb{R}$ be an open interval and suppose $g, p_0, p_1, \dots, p_{n-1}$ are continuous functions in I . For $t_0 \in I$ and $x_0, \dots, x_{n-1} \in \mathbb{R}$, there is exactly one solution to the IVP*

$$\begin{cases} y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = g(t), \\ y(t_0) = x_0, y'(t_0) = x_1, \dots, y^{(n-1)}(t_0) = x_{n-1}. \end{cases} \quad (304)$$

Linear (in)dependence is defined in similar way.

Definition 41 (Linear Dependence). *The functions $f_1(t), \dots, f_n(t)$ are linearly dependent on the interval I if there exists a set of numbers $(\alpha_1, \dots, \alpha_n) \neq (0, \dots, 0)$, such that*

$$\alpha_1 f_1(t) + \cdots + \alpha_n f_n(t) = 0, \quad (305)$$

for all $t \in I$. Otherwise, we say that the functions $f_1(t), \dots, f_n(t)$ are linearly independent.

Similar to Theorem 18, we also have the principle of superposition.

Theorem 42 (Principle of Superposition). *Let y_1, \dots, y_n be solutions to the homogeneous equation*

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = 0, \quad (306)$$

then, for any constants $c_1, \dots, c_n \in \mathbb{R}$, the function

$$\phi(t) = c_1 y_1(t) + \cdots + c_n y_n(t) \quad (307)$$

is also a solution to the above homogeneous equation.

We also have the Wronskian.

Definition 43 (Wronskian). Given functions f_1, \dots, f_n that are differentiable up to order $n - 1$, we define the Wronskian W as

$$W(f_1, \dots, f_n)[t] = \det \begin{pmatrix} f_1 & f_2 & \dots & f_n \\ f'_1 & f'_2 & \dots & f'_n \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \dots & f_n^{(n-1)} \end{pmatrix} [t]. \quad (308)$$

The natural question is: given n solutions y_1, \dots, y_n to the **homogeneous equation**

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \dots + p_1(t)y' + p_0(t)y = 0. \quad (309)$$

Can **any** solution ϕ to the homogeneous equation be expressed as a linear combination of y_1, \dots, y_n ? Similarly with the case of the second-order ODE, we have the following theorem.

Theorem 44. If p_0, \dots, p_{n-1} are continuous functions in I , and y_1, \dots, y_n are solutions to the above homogeneous equation, then every solution ϕ to the homogeneous equation can be expressed as a linear combination of y_1, \dots, y_n if and only if $W(y_1, \dots, y_n)[t_0] \neq 0$ for some $t_0 \in I$. In this case, we call (y_1, \dots, y_n) a **fundamental set of solutions (FSS)** to the homogeneous equation.

An analogous result to Abel's identity (Theorem 23):

Theorem 45. Let y_1, \dots, y_n be solutions to the homogeneous equation

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \dots + p_1(t)y' + p_0(t)y = 0, \quad (310)$$

for $t \in I$. Then,

$$W(y_1, \dots, y_n)[t] = ce^{-\int p_{n-1}(t)dt} \quad (311)$$

for a constant c not dependent on $t \in I$.

Proof. The idea is to derive an equation satisfied by the Wronskian. Recall the rule for taking derivatives on determinants: **The derivative of an $n \times n$ determinant is equal to the sum of n determinants, where the k -th determinant is obtained by differentiating the k -th row of the original determinant, and keeping the other rows unchanged.**

For example, denote $D := \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ where a, b, c, \dots, i are functions of t . Then we have

$$\frac{dD}{dt} = \begin{vmatrix} a' & b' & c' \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & c \\ d' & e' & f' \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & c \\ d & e & f \\ g' & h' & i' \end{vmatrix} \quad (312)$$

Thus, we have

$$\begin{aligned} \frac{d}{dt}W[t] &= \left| \begin{array}{cccc} y'_1 & y'_2 & \cdots & y'_n \\ y'_1 & y'_2 & \cdots & y'_n \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{array} \right| + \left| \begin{array}{cccc} y_1 & y_2 & \cdots & y_n \\ y''_1 & y''_2 & \cdots & y''_n \\ y''_1 & y''_2 & \cdots & y''_n \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{array} \right| \\ &+ \cdots + \left| \begin{array}{cccc} y_1 & y_2 & \cdots & y_n \\ y'_1 & y'_2 & \cdots & y'_n \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{array} \right| = \left| \begin{array}{cccc} y_1 & y_2 & \cdots & y_n \\ y'_1 & y'_2 & \cdots & y'_n \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{array} \right|. \quad (313) \end{aligned}$$

The first $n - 1$ determinants all have two identical rows, thus they are all zero and only the last determinant is nonzero. Using that for each $1 \leq k \leq n$,

$$y_k^{(n)} = -p_{n-1}y_k^{(n-1)} - \cdots - p_1y_k' - p_0y_k, \quad (314)$$

then applying elementary row operations we find that

$$\frac{d}{dt}W[t] = \left| \begin{array}{cccc} y_1 & y_2 & \cdots & y_n \\ y'_1 & y'_2 & \cdots & y'_n \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ -p_{n-1}y_1^{(n-1)} & -p_{n-1}y_2^{(n-1)} & \cdots & -p_{n-1}y_n^{(n-1)} \end{array} \right| = -p_{n-1}W[t]. \quad (315)$$

Thus,

$$W(y_1, \dots, y_n)[t] = ce^{-\int p_{n-1}(t)dt} \quad (316)$$

for a constant c not dependent on $t \in I$.

□

Finally, the relationship between linear (in)dependence and Wronskian.

Theorem 46. If y_1, \dots, y_n are solutions to the ODE $y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = 0$, $t \in I$, then y_1, \dots, y_n are linearly independent $\iff W[y_1, \dots, y_n](t) \neq 0$, $\forall t \in I$ ((y_1, \dots, y_n) forms a FSS).

4.2 Homogeneous Equations with Constant Coefficients

We will study, for constants $a_n \neq 0, a_{n-1}, \dots, a_0 \in \mathbb{R}$, the equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0. \quad (317)$$

Still, consider a trial function $\phi = e^{rt}$ for $r \in \mathbb{R}$. Substituting this gives the **characteristic equation**

$$a_n r^n + \dots + a_1 r + a_0 = 0. \quad (318)$$

The characteristic polynomial is

$$Z(r) = a_n r^n + \dots + a_1 r + a_0. \quad (319)$$

From the fundamental theorem of algebra, every polynomial with real coefficients of degree n has n complex roots. Hence

$$Z(r) = a_n(r - r_1)(r - r_2) \cdots (r - r_n), \quad (320)$$

where r_1, \dots, r_n are complex numbers, it is possible that some roots are repeated.

Definition 47 (Multiplicity). *Let $P_k(x)$ be a polynomial of degree k in x . A root r has multiplicity $m \in \mathbb{N}, m \geq 1$, if there is another polynomial $S_{k-m}(x)$ of degree $k - m$ such that $S_{k-m}(r) \neq 0$ and*

$$P_k(x) = S_{k-m}(x)(x - r)^m. \quad (321)$$

Case 1. Real and distinct roots. If the roots of $Z(r) = 0$ are all real and distinct, then we have the solutions

$$y_1(t) = e^{r_1 t}, \dots, y_n(t) = e^{r_n t}. \quad (322)$$

They are linearly independent solutions and form a FSS. Compute the Wronskian

$$W(e^{r_1 t}, e^{r_2 t}, \dots, e^{r_n t})(t) = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} & \dots & e^{r_n t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} & \dots & r_n e^{r_n t} \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1} e^{r_1 t} & r_2^{n-1} e^{r_2 t} & \dots & r_n^{n-1} e^{r_n t} \end{vmatrix} \quad (323)$$

$$= e^{(r_1 + \dots + r_n)t} \begin{vmatrix} 1 & 1 & \dots & 1 \\ r_1 & r_2 & \dots & r_n \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1} & r_2^{n-1} & \dots & r_n^{n-1} \end{vmatrix} \quad (324)$$

$$= e^{(r_1 + \dots + r_n)t} \prod_{1 \leq i < j \leq n} (r_j - r_i) \neq 0. \quad (325)$$

Example 48. Solve the ODE

$$y^{(4)} - 7y''' + 6y'' + 30y' - 36y = 0 \quad (326)$$

The characteristic equation is:

$$r^4 - 7r^3 + 6r^2 + 30r - 36 = 0 \quad (327)$$

which can be factorized as

$$(r - 3)(r + 2)(r^2 - 6r + 6) = 0 \quad (328)$$

Hence $r_1 = -2, r_2 = 3, r_3 = 3 - \sqrt{3}, r_4 = 3 + \sqrt{3}$. The general solution is given by:

$$y = c_1 e^{-2t} + c_2 e^{3t} + c_3 e^{(3-\sqrt{3})t} + c_4 e^{(3+\sqrt{3})t}. \quad (329)$$

Case 2. Some roots are complex. If some roots are complex, they must appear in pairs, i.e. $\lambda \pm i\mu$ (see the [Complex conjugate root theorem](#)). Thus, we could replace the complex-valued solutions $e^{(\lambda+i\mu)t}$ and $e^{(\lambda-i\mu)t}$ by the real-valued solutions $e^{\lambda t} \cos \mu t, e^{\lambda t} \sin \mu t$. (Recall [Case 2](#) of Section 3.2)

Example 49. Solve the ODE

$$y^{(4)} - y = 0 \quad (330)$$

The characteristic equation is:

$$r^4 - 1 = 0. \quad (331)$$

We have $r = 1, -1, \pm i$. Thus $\lambda = 0, \mu = 1$. Hence $\{e^t, e^{-t}, \cos t, \sin t\}$ forms a FSS. The general solution is given by:

$$y = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t. \quad (332)$$

Case 3. Some roots are repeated.

Subcase 1: If one of the real root r_1 is repeated with multiplicity s , then the corresponding linearly independent solutions corresponding to root r_1 are:

$$e^{r_1 t}, t e^{r_1 t}, t^2 e^{r_1 t}, \dots, t^{s-1} e^{r_1 t}. \quad (333)$$

Subcase 2: If the complex root $r_1 = \lambda + i\mu$ is repeated with multiplicity s , then the corresponding conjugate $\bar{r}_1 = \lambda - i\mu$ is also the root with multiplicity s . In this case, we could replace the complex-valued solutions $e^{(\lambda+i\mu)t}, \dots, t^{s-1} e^{(\lambda+i\mu)t}$ and $e^{(\lambda-i\mu)t}, \dots, t^{s-1} e^{(\lambda-i\mu)t}$ by the real valued solutions as follows:

$$e^{\lambda t} \cos \mu t, t e^{\lambda t} \cos \mu t, t^2 e^{\lambda t} \cos \mu t, \dots, t^{s-1} e^{\lambda t} \cos \mu t \quad \text{from real parts} \quad (334)$$

$$e^{\lambda t} \sin \mu t, t e^{\lambda t} \sin \mu t, t^2 e^{\lambda t} \sin \mu t, \dots, t^{s-1} e^{\lambda t} \sin \mu t \quad \text{from imaginary parts} \quad (335)$$

These are linearly independent solutions corresponding to the repeated roots $r_1 = \lambda + i\mu$ and $\bar{r}_1 = \lambda - i\mu$.

Example 50. Solve the ODE

$$y^{(4)} + 2y'' + y = 0 \quad (336)$$

The characteristic equation is:

$$r^4 + 2r^2 + 1 = (r^2 + 1)(r^2 + 1) = 0. \quad (337)$$

We have $r = i, i, -i, -i$, thus $\lambda = 0, \mu = 1$. The fundamental solution is:

$$e^{it}, te^{it}, e^{-it}, te^{-it}. \quad (338)$$

The general solution is given by:

$$y = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t. \quad (339)$$

4.3 Non-Homogeneous Equations

4.3.1 Method of Undetermined Coefficients

Consider the non-homogeneous equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(t). \quad (340)$$

If Y_1 and Y_2 are both solutions to the non-homogeneous problem, then $Y_1 - Y_2$ is a solution to the corresponding homogeneous equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0. \quad (341)$$

Given a FSS (y_1, \dots, y_n) to the homogeneous equation, a general solution to the non-homogeneous equation (340) is

$$y(t) = c_1 y_1(t) + \cdots + c_n y_n(t) + Y(t), \quad (342)$$

where $Y(t)$ is a particular solution to the non-homogeneous equation, $c_1 y_1(t) + \cdots + c_n y_n(t)$ is the complementary solution (solution to the homogeneous equation).

Similar to second-order equations, we now find a particular solution Y to the non-homogeneous equation (340) if $g(t)$ is a **sum/product of exponentials, cosine, sine and polynomials**. But the main difference is that the multiplicity of roots to the characteristic equation can be **greater than two**. Thus, higher powers of t need to be multiplied to get the solution to the non-homogeneous equation.

We again investigate the cases:

1. $g(t) = e^{\alpha t} P_m(t)$,
2. $g(t) = e^{\alpha t} P_m(t) \cos(\beta t)$, or $g(t) = e^{\alpha t} P_m(t) \sin(\beta t)$.

Remember the characteristic equation for the homogeneous equation is

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0. \quad (343)$$

The possible particular solutions can be used are

1. $Y(t) = t^s e^{\alpha t} Q_m(t)$, where

$$Q_m(t) = A_m t^m + \cdots + A_1 t + A_0 \quad (344)$$

for undetermined coefficients A_m, \dots, A_0 , and

$$s = \begin{cases} 0, & \text{if } \alpha \text{ is not a root of the characteristic equation.} \\ m, & \text{if } \alpha \text{ is a root of the characteristic equation with multiplicity } m \end{cases} \quad (345)$$

2. $Y(t) = t^s e^{\alpha t} [Q_m(t) \cos(\beta t) + R_m(t) \sin(\beta t)]$, where

$$Q_m = A_m t^m + \cdots + A_1 t + A_0, R_m = B_m t^m + \cdots + B_1 t + B_0 \quad (346)$$

are polynomials of degree m with undetermined coefficients $A_m, \dots, A_0, B_m, \dots, B_0$, and

$$s = \begin{cases} 0, & \text{if } \alpha + i\beta \text{ is not a root of the characteristic equation.} \\ m, & \text{if } \alpha + i\beta \text{ is a root of the characteristic equation with multiplicity } m. \end{cases} \quad (347)$$

Example 51. Solve

$$y''' - 3y'' + 3y' - y = 4e^t \quad (348)$$

For the homogeneous equation, the associated characteristic equation is

$$r^3 - 3r^2 + 3r - 1 = (r - 1)^3 = 0, \quad (349)$$

so $r_1 = r_2 = r_3 = 1$, i.e., a repeated eigenvalue of multiplicity three. So set

$$y_1 = e^t, \quad y_2 = te^t, \quad y_3 = t^2e^t, \quad (350)$$

and the complementary solution (to the homogeneous equation) is

$$y_c(t) = c_1e^t + c_2te^t + c_3t^2e^t. \quad (351)$$

Since $g(t) = 4e^t$ and $\alpha = 1$ is a root of the characteristic equation with multiplicity 3, consider $s = 3$ and a trial solution

$$Y(t) = At^3e^t. \quad (352)$$

Computing gives

$$Y''' - 3Y'' + 3Y' - Y = 6Ae^t = 4e^t \Rightarrow A = \frac{2}{3}, \quad (353)$$

so the general solution to the non-homogeneous ODE is

$$y(t) = c_1e^t + c_2te^t + c_3t^2e^t + \frac{2}{3}t^3e^t. \quad (354)$$

Example 52. Solve

$$y^{(4)} + 2y'' + y = 3 \sin t \quad (355)$$

The characteristic equation corresponding to the homogeneous equation is

$$r^4 + 2r^2 + 1 = (r^2 + 1)(r^2 + 1) = 0 \quad (356)$$

so $r_1 = r_3 = i, r_2 = r_4 = -i$, i.e., a repeated pair of complex conjugate roots (multiplicity = 2). Thus we have

$$y_1 = \cos t, \quad y_2 = \sin t, \quad y_3 = t \cos t, \quad y_4 = t \sin t, \quad (357)$$

and the complementary solution to the homogeneous equation is

$$y_c(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t. \quad (358)$$

The non-homogeneous term $g(t) = 3 \sin t$, we have $\alpha = 0, \beta = 1, \alpha + i\beta = i$ is the root with multiplicity 2. Thus, $s = 2$. Consider a trial solution

$$Y(t) = At^2 \sin t + Bt^2 \cos t. \quad (359)$$

Then,

$$Y^{(4)} + 2Y'' + Y = -8A \sin t - 8B \cos t = 3 \sin t \Rightarrow B = 0, \quad A = -\frac{3}{8}. \quad (360)$$

Hence, the general solution to the non-homogeneous equation is

$$y(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t - \frac{3}{8} t^2 \sin t. \quad (361)$$

4.3.2 Variation of Parameters

Similar to second-order equations, there is also a method to treat rather general high order equations

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \cdots + p_1(t)y' + p_0(t)y = g(t), \quad t \in I. \quad (362)$$

Suppose we have a FSS y_1, \dots, y_n to the homogeneous equation. Then, the complementary solution is

$$y_c(t) = c_1 y_1(t) + \cdots + c_n y_n(t). \quad (363)$$

Now, we consider a trial solution for the non-homogeneous equation of the form

$$Y(t) = u_1(t)y_1(t) + \cdots + u_n(t)y_n(t) \quad (364)$$

for unknown functions u_1, \dots, u_n . Differentiating gives

$$Y'(t) = u_1(t)y'_1(t) + \cdots + u_n(t)y'_n(t) + u'_1(t)y_1(t) + \cdots + u'_n(t)y_n(t). \quad (365)$$

As before we set the constraint

$$u'_1(t)y_1(t) + u'_2(t)y_2(t) + \cdots + u'_n(t)y_n(t) = 0, \quad (366)$$

so that Y' simplifies to

$$Y'(t) = u_1(t)y'_1(t) + u_2(t)y'_2(t) + \cdots + u_n(t)y'_n(t). \quad (367)$$

Computing Y'' and setting

$$u'_1(t)y'_1(t) + \cdots + u'_n(t)y'_n(t) = 0 \quad (368)$$

leads to the simplified expression for the second derivative

$$Y''(t) = u_1(t)y''_1(t) + \cdots + u_n(t)y''_n(t). \quad (369)$$

Repeat this procedure (differentiating and then setting the sum of terms involving u'_1, \dots, u'_n to zero), we have (after simplification)

$$Y^{(n-1)}(t) = u_1(t)y_1^{(n-1)}(t) + \cdots + u_n(t)y_n^{(n-1)}(t) \quad (370)$$

Thus, the final expression for $Y^{(n)}(t)$ is

$$Y^{(n)}(t) = u_1(t)y_1^{(n)}(t) + \cdots + u_n(t)y_n^{(n)}(t) + u'_1(t)y_1^{(n-1)}(t) + \cdots + u'_n(t)y_n^{(n-1)}(t) \quad (371)$$

In summary, we obtain $n - 1$ equations

$$u'_1(t)y_1^{(m)}(t) + \cdots + u'_n(t)y_n^{(m)}(t) = 0 \quad \forall 0 \leq m \leq n - 2, \quad (372)$$

as well as a simplified expression for $Y^{(m)}$:

$$Y^{(m)}(t) = u_1(t)y_1^{(m)}(t) + \cdots + u_n(t)y_n^{(m)}(t), \quad m = 0, \dots, n - 1, \quad (373)$$

$$Y^{(n)}(t) = u_1(t)y_1^{(n)}(t) + \cdots + u_n(t)y_n^{(n)}(t) + u'_1(t)y_1^{(n-1)}(t) + \cdots + u'_n(t)y_n^{(n-1)}(t). \quad (374)$$

Now, substitute $Y, Y', \dots, Y^{(n-1)}, Y^{(n)}$ into the LHS of the non-homogeneous ODE (362):

$$\text{LHS} = Y^{(n)} + p_{n-1}Y^{(n-1)} + \cdots + p_1Y' + p_0Y \quad (375)$$

Substituting:

$$\begin{aligned} \text{LHS} &= \overbrace{[u_1y_1^{(n)} + \cdots + u_ny_n^{(n)}] + [u'_1y_1^{(n-1)} + \cdots + u'_ny_n^{(n-1)}]}^{\text{from } Y^{(n)}} \\ &\quad + p_{n-1}[u_1y_1^{(n-1)} + \cdots + u_ny_n^{(n-1)}] + \cdots + p_1[u_1y'_1 + \cdots + u_ny'_n] \\ &\quad + p_0[u_1y_1 + \cdots + u_ny_n] \end{aligned} \quad (376)$$

Regroup all terms by u_1, u_2, \dots, u_n :

- Collect all terms of u_1 : $u_1y_1^{(n)} + p_{n-1}u_1y_1^{(n-1)} + \cdots + p_1u_1y'_1 + p_0u_1y_1 = u_1 \left[y_1^{(n)} + p_{n-1}y_1^{(n-1)} + \cdots + p_1y'_1 + p_0y_1 \right]$
- Collect all terms of u_2 : $u_2 \left[y_2^{(n)} + p_{n-1}y_2^{(n-1)} + \cdots + p_1y'_2 + p_0y_2 \right]$
- \dots
- Collect all terms of u_n : $u_n \left[y_n^{(n)} + p_{n-1}y_n^{(n-1)} + \cdots + p_1y'_n + p_0y_n \right]$
- The remaining terms (only u'_i): $u'_1y_1^{(n-1)} + \cdots + u'_ny_n^{(n-1)}$

Since y_1, \dots, y_n are all solutions to the homogeneous equation, all expressions in the square brackets $[\dots]$ in the previous step are equal to 0.

$$\text{LHS} = u_1 \cdot (0) + u_2 \cdot (0) + \cdots + u_n \cdot (0) + u'_1y_1^{(n-1)} + \cdots + u'_ny_n^{(n-1)} \quad (377)$$

Thus, we obtain the last equation:

$$u'_1y_1^{(n-1)} + \cdots + u'_ny_n^{(n-1)} = g(t) \quad (378)$$

Collecting all expressions involving u'_1, \dots, u'_n , we obtain

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_{n-1} & y_n \\ y'_1 & y'_2 & \cdots & y'_{n-1} & y'_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_{n-1}^{(n-2)} & y_n^{(n-2)} \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_{n-1}^{(n-1)} & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} u'_1 \\ u'_2 \\ \vdots \\ u'_{n-1} \\ u'_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(t) \end{pmatrix}. \quad (379)$$

Thus, the derivatives of the unknown functions u_1, \dots, u_n can be found by inverting the matrix of derivatives, whose determinant is the non-zero Wronskian, since (y_1, \dots, y_n) forms a FSS. Denote the matrix as $M(t)$. Use Cramer's rule, by setting

$$M_i(t) = \begin{pmatrix} y_1 & \dots & 0 & \dots & y_n \\ y'_1 & \dots & 0 & \dots & y'_n \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & \dots & 0 & \dots & y_n^{(n-2)} \\ y_1^{(n-1)} & \dots & 1 & \dots & y_n^{(n-1)} \end{pmatrix}, \quad (380)$$

i.e., replace the i th column of $M(t)$ with the vector $(0, \dots, 0, 1)^T$. Then Cramer's rule gives

$$u'_i(t) = \frac{g(t) \det M_i(t)}{\det M(t)}, \quad (381)$$

and by integrating we get an expression for $u_i(t)$. The particular solution to the non-homogeneous equation is therefore

$$Y(t) = y_1(t) \int \frac{g(t) \det M_1(t)}{\det M(t)} dt + \dots + y_n(t) \int \frac{g(t) \det M_n(t)}{\det M(t)} dt. \quad (382)$$

However, in general the evaluation of the integrals can be difficult, but we can always use Abel's identity to simplify, since

$$\det M(t) = W(y_1, \dots, y_n)[t] = ce^{-\int p_{n-1}(t)dt}. \quad (383)$$

Example 53. Solve

$$y''' + y' = \sec^2(t) \text{ for } t \in (-\pi/2, \pi/2). \quad (384)$$

The characteristic equation for the homogeneous problem is $r^3 + r = 0$ and so $r_1 = 0, r_2 = i$ and $r_3 = -i$. Hence the complementary solution is

$$y_c(t) = c_1 + c_2 \cos t + c_3 \sin t. \quad (385)$$

By variation of parameters we look for a particular solution of the form

$$Y(t) = u_1 y_1 + u_2 y_2 + u_3 y_3 = u_1(t) + u_2(t) \cos t + u_3(t) \sin t, \quad (386)$$

with

$$M(t) \begin{pmatrix} u'_1 \\ u'_2 \\ u'_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sec^2(t) \end{pmatrix}, \quad M(t) = \begin{pmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{pmatrix}. \quad (387)$$

Define

$$M_1(t) = \begin{pmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 1 & -\cos t & -\sin t \end{pmatrix}, \quad M_2(t) = \begin{pmatrix} 1 & 0 & \sin t \\ 0 & 0 & \cos t \\ 0 & 1 & -\sin t \end{pmatrix}, \quad M_3(t) = \begin{pmatrix} 1 & \cos t & 0 \\ 0 & -\sin t & 0 \\ 0 & -\cos t & 1 \end{pmatrix} \quad (388)$$

We can compute

$$\det M(t) = 1, \quad \det M_1(t) = 1, \quad \det M_2(t) = -\cos t, \quad \det M_3(t) = -\sin t, \quad (389)$$

so

$$u_1 = \int \sec^2(t) dt = \tan(t), \quad (390)$$

$$u_2 = \int -\sec^2(t) \cos(t) dt = -\ln(|\sec(t) + \tan(t)|), \quad (391)$$

$$u_3 = \int -\sec^2(t) \sin(t) dt = -\sec(t). \quad (392)$$

Hence, the particular solution is

$$Y(t) = \tan(t) - \cos(t) \ln(|\sec(t) + \tan(t)|) - \sin(t) \sec(t) \quad (393)$$

$$= -\cos(t) \ln(|\sec(t) + \tan(t)|). \quad (394)$$

References

- [BDM21] W. E. Boyce, R. C. DiPrima, and D. B. Meade. *Elementary differential equations and boundary value problems*. John Wiley & Sons, 2021.