Capacity Measures and Additivity

Siqi Yao

SCHOOL OF DATA SCIENCE

August 15, 2025

Table of Contents

Table of Contents

Classical Channel Capacity

- The basic mathematical model for a communication system is shown in Figure 1, where:
  - X/Y are the channel input/output symbol taken in an alphabet $\mathcal{X}/\mathcal{Y}$.
  - $p(y \mid x) = p_{Y|X}(y \mid x)$ is the conditional probability distribution.
- Model X, Y as random variables. Since $p_{Y|X}(y \mid x)$ is a fixed property of the channel, the choice of the marginal distribution $p_X(x)$ completely determines the joint distribution $p_{X,Y}(x, y)$.
- Thus, we can define the **classical channel capacity** as

$$C = \sup_{p_X(x)} I(X;Y) \tag{1}$$

where $I(X;Y) = \sum_{x\in\mathcal{X}} \sum_{y\in\mathcal{Y}} P_{X,Y}(x,y) \log \frac{P_{X,Y}(x,y)}{P_X(x)P_Y(y)}$ is the **mutual information**.

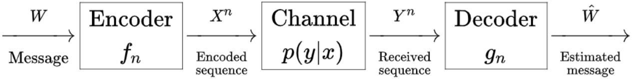

Figure 1: Communication Model. Source: [Wikipedia contributors, 2025]

Additivity of Classical Channel Capacity

- Classical channel capacity is **additive** over independent **classical** channels: using two independent channels in parallel provides the same capacity as the sum of each.
- Let p_1, p_2 be two independent channels. Define the product (parallel) channel $p_1 \times p_2$ as:

$$(p_1 \times p_2)\left((y_1, y_2) \mid (x_1, x_2)\right) = p_1(y_1 \mid x_1) \cdot p_2(y_2 \mid x_2). \tag{2}$$

Then the additivity states that:

$$C(p_1 \times p_2) = C(p_1) + C(p_2). \tag{3}$$

Classical Channel Capacity of Quantum Channels

- However, classical channel capacity of **quantum** channels (classical input and output, transmission through quantum channels) does not satisfy additivity.
- The classical capacity of a quantum channel $\mathcal{N}$ is given by the HSW theorem:

$$C(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n}\chi^*(\mathcal{N}^{\otimes n}) \tag{4}$$

where $\chi^*(\mathcal{N})$ is the maximum Holevo information of $\mathcal{N}$:

$$\chi^*(\mathcal{N}) = \max_{\{p_x,\rho_x\}} \left[S\left(\sum_x p_x \mathcal{N}(\rho_x)\right) - \sum_x p_x S(\mathcal{N}(\rho_x))\right] \tag{5}$$

- Since $\chi^*(\mathcal{N})$ does not satisfy additivity, same is for $C(\mathcal{N})$.

Table of Contents

Super-additivity of Holevo Information

- [Shor, 2004] showed that the following four conjectures are either all true or all false:
  - Additivity of the minimum entropy output of a quantum channel;
  - Additivity of the maximum Holevo information of a quantum channel;
  - Additivity of the entanglement of formation;
  - Strong super-additivity of the entanglement of formation.
- [Hastings, 2009] showed that all of these conjectures are false.
- In fact, $\chi^*(\mathcal{N})$ satisfies **super-additivity**, i.e., for n independent quantum channels $\{\Phi_i\}_{i=1}^n$, when combined in parallel as a new channel $\mathcal{N} := \Phi_1 \otimes \cdots \otimes \Phi_n$, we have the inequality

$$\chi^*(\mathcal{N}) \geq \sum_{i=1}^{n} \chi^*(\Phi_i) \tag{6}$$

which can be strict.

Coherent Information and Quantum Capacity

- For a quantum channel Φ, the **coherent information** is defined as:

$$\mathcal{Q}^{(1)}(\Phi) := \max_{\rho} I_c(\rho, \Phi) = \max_{\rho}[S(\Phi(\rho)) - S(\Phi^c(\rho))] \tag{7}$$

- The **quantum capacity** of Φ is given by the regularized formula:

$$\mathcal{Q}(\Phi) = \lim_{n\to\infty} \frac{\mathcal{Q}^{(1)}(\Phi^{\otimes n})}{n} = \sup_{n\in\mathbb{N}} \frac{1}{n}\mathcal{Q}^{(1)}\left(\Phi^{\otimes n}\right) \tag{8}$$

  where the second equality is due to the **Fekete's lemma**.
- The coherent information is also **super-additive**, i.e., for n independent quantum channels $\{\Phi_i\}_{i=1}^n$, and $\mathcal{N} := \Phi_1 \otimes \cdots \otimes \Phi_n$, we have the inequality

$$\mathcal{Q}^{(1)}(\mathcal{N}) \geq \sum_{i=1}^{n} \mathcal{Q}^{(1)}(\Phi_i) \tag{9}$$

  which can be strict.

Weak Additivity

- The super-additivity can be divided into two categories based on the violation of **weak and strong additivity**. Take coherent information as example.
- If

$$\mathcal{Q}^{(1)}(\Phi^{\otimes n}) = n\,\mathcal{Q}^{(1)}(\Phi) \quad \forall n \in \mathbb{N}, \tag{10}$$

  i.e., in the setting of **multiple independent and parallel uses of Φ**, then the coherent information of Φ satisfies **weak additivity**.
- Once weak additivity holds, the quantum capacity simplifies to

$$\mathcal{Q}(\Phi) = \mathcal{Q}^{(1)}(\Phi) \tag{11}$$

- Violation of weak additivity $\Rightarrow$ weak superadditivity: There exists channels such that

$$\mathcal{Q}^{(1)}(\Phi^{\otimes n}) > n\,\mathcal{Q}^{(1)}(\Phi). \tag{12}$$

Strong Additivity

- For a pair of channel Φ_1, Φ_2, if we fix Φ_1 and the equality

$$\mathcal{Q}^{(1)}(\Phi_1 \otimes \Phi_2) = \mathcal{Q}^{(1)}(\Phi_1) + \mathcal{Q}^{(1)}(\Phi_2). \tag{13}$$

  holds for all Φ_2, then the coherent information of Φ_1 satisfies **strong additivity**.

- Hence, the quantum capacity satisfies:

$$\mathcal{Q}(\Phi_1 \otimes \Phi_2) = \mathcal{Q}(\Phi_1) + \mathcal{Q}(\Phi_2). \tag{14}$$

- Violation of strong additivity $\Rightarrow$ strong superadditivity: There exists a pair of channels such that

$$\mathcal{Q}^{(1)}(\Phi_1 \otimes \Phi_2) > \mathcal{Q}^{(1)}(\Phi_1) + \mathcal{Q}^{(1)}(\Phi_2). \tag{15}$$

  This implies that **using different channels in parallel** can transmit more information than using them individually.

Private Information and Private Capacity

- The private capacity $\mathcal{P}(\Phi)$ of a quantum channel Φ is defined as the largest rate at which classical information can be faithfully sent through the channel, such that the environment gains no meaningful knowledge about the information.
- The channel's private information is defined as:

$$\mathcal{P}^{(1)}(\Phi) = \max_{\{p_x, \rho_x\}} \left[I_c(\sum_x p_x \rho_x, \Phi) - \sum_x p_x I_c(\rho_x, \Phi) \right] \tag{16}$$

which satisfies **supper-additivity**.
- The channel's private capacity is given by the regularized expression:

$$\mathcal{P}(\Phi) = \lim_{n \to \infty} \frac{\mathcal{P}^{(1)}(\Phi^{\otimes n})}{n} = \sup_{n \in \mathbb{N}} \frac{1}{n} \mathcal{P}^{(1)} \left(\Phi^{\otimes n} \right) \tag{17}$$

Equivalent Expression of Private Information

- Denote the quantum channel and its complementary respectively as $\mathcal{B}, \mathcal{C}$. $\bar{\rho}_a := \sum_x p_x \rho_x$. We have:

$$\sigma_B = \mathcal{B}(\bar{\rho}_a), \quad \sigma_C = \mathcal{C}(\bar{\rho}_a), \quad \sigma_B^x = \mathcal{B}(\rho_x), \quad \sigma_C^x = \mathcal{C}(\rho_x) \tag{18}$$

- We can derive:

$$H(\sigma_{XB}) = H(p_x) + \sum_x p_x H(\sigma_B^x), \quad H(\sigma_X) = H(p_x) \tag{19}$$

- Thus,

$$I(X;B)_\sigma = H(\sigma_X) + H(\sigma_B) - H(\sigma_{XB}) = H(\sigma_B) - \sum_x p_x H(\sigma_B^x) \tag{20}$$

$$= H(\mathcal{B}(\bar{\rho}_a)) - \sum_x p_x H(\mathcal{B}(\rho_x)) \tag{21}$$

- Similarly:

$$I(X;C)_\sigma = H(\mathcal{C}(\bar{\rho}_a)) - \sum_x p_x H(\mathcal{C}(\rho_x)) \tag{22}$$

Equivalent Expression of Private Information (cont'd)

- So we have:

$$I(X;B)_\sigma - I(X;C)_\sigma = \underbrace{[H(\mathcal{B}(\bar{\rho}_a)) - H(\mathcal{C}(\bar{\rho}_a))]}_{=I_c(\bar{\rho}_a,\mathcal{B})} - \sum_x p_x \underbrace{[H(\mathcal{B}(\rho_x)) - H(\mathcal{C}(\rho_x))]}_{=I_c(\rho_x,\mathcal{B})} \quad (23)$$

$$= I_c(\bar{\rho}_a,\mathcal{B}) - \sum_x p_x I_c(\rho_x,\mathcal{B}) \quad (24)$$

- At last we obtain an equivalent expression for (16):

$$\mathcal{P}^{(1)}(\Phi) = \max_{\{p_x,\rho_x\}} [I(X;\Phi)_\sigma - I(X;\Phi^c)_\sigma] \quad (25)$$

- Full derivation details can be found in the " Quantum Information Theory Preliminaries " section of my notes **Quantum `SWITCH` for Communication Enhancement**.

References I

Hastings, M. B. (2009).
Superadditivity of communication capacity using entangled inputs.
Nature Physics, 5(4):255–257.

Leditzky, F., Leung, D., Siddhu, V., Smith, G., and Smolin, J. A. (2023).
The platypus of the quantum channel zoo.
IEEE Transactions on Information Theory, 69(6):3825–3849.

Shor, P. W. (2004).
Equivalence of additivity questions in quantum information theory.
Communications in Mathematical Physics, 246(3):453–472.

Siddhu, V. (2021).
Entropic singularities give rise to quantum transmission.
Nature Communications, 12(1):5750.

References II

Singh, S. and Datta, N. (2022).
Coherent information of a quantum channel or its complement is generically positive.
Quantum, 6:775.

Wikipedia contributors (2025).
Channel capacity — Wikipedia, The Free Encyclopedia.
https://en.wikipedia.org/wiki/Channel_capacity.

Wu, Z., Zhao, Q., and Ma, Z. (2025).
Super-additivity of quantum capacity in simple channels.
arXiv preprint arXiv:2505.24661.

Thank you!
Any questions?