

# Quantum SWITCH for Communication Enhancement

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- The quantum SWITCH is an operation that allows multiple quantum channels to act in a superposition of different orders, under the control of a **control qubit/order qubit**.
- **Classical order (fixed order):** Order of two channels  $\mathcal{A}$  and  $\mathcal{B}$  is fixed by  $\mathcal{A} \circ \mathcal{B}(\rho)$  or  $\mathcal{B} \circ \mathcal{A}(\rho)$ .
- **Quantum SWITCH (indefinite order):** Let control qubit  $|0\rangle$  controls  $\mathcal{A} \circ \mathcal{B}$  and  $|1\rangle$  controls  $\mathcal{B} \circ \mathcal{A}$ . If the control qubit is prepared as  $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ , then the entire process becomes a superposition of quantum channels, a.k.a **Indefinite Causal Order**.

# Figure Illustration

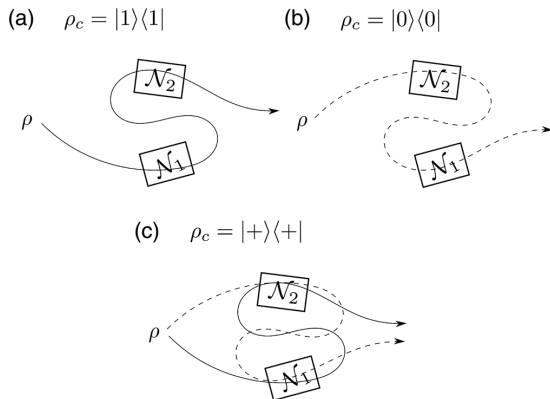


Figure 1: Superposition of Orders. Source: [Ebler et al., 2018]

# Fixed Order Channels

- Suppose we have two quantum channels (CPTP maps)  $\mathcal{E}$  and  $\mathcal{F}$ , whose action on quantum state  $\rho$  can be expressed using the Kraus representation as:

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger, \quad \mathcal{F}(\rho) = \sum_j F_j \rho F_j^\dagger \quad (1)$$

where  $\{E_i\}$  and  $\{F_j\}$  are the Kraus operators satisfying  $\sum_i E_i^\dagger E_i = \sum_j F_j^\dagger F_j = I$ .

- Assume the channels are applied sequentially, giving rise to two possible orders:

$$\mathcal{F} \circ \mathcal{E}(\rho) = \sum_{j,i} F_j E_i \rho E_i^\dagger F_j^\dagger, \quad \mathcal{E} \circ \mathcal{F}(\rho) = \sum_{i,j} E_i F_j \rho F_j^\dagger E_i^\dagger. \quad (2)$$

- The order in which the channels are applied are fixed. In  $\mathcal{F} \circ \mathcal{E}(\rho)$ ,  $\mathcal{E}$  is applied first, while in  $\mathcal{E} \circ \mathcal{F}(\rho)$  the opposite.

# Indefinite Order Channels

- The quantum SWITCH is a higher-order quantum channel constructed from  $\mathcal{E}, \mathcal{F}$  and an ancilla control qubit  $|\omega\rangle$ , defined as:

$$\mathcal{S}(\mathcal{E}, \mathcal{F}, |\omega\rangle)(\rho) = \sum_{i,j} K_{ij}(\rho \otimes \omega) K_{ij}^\dagger \quad (3)$$

where  $\omega = |\omega\rangle\langle\omega|$  and  $\{K_{ij}\}$  are the Kraus operators:

$$K_{ij} = E_i F_j \otimes |0\rangle\langle 0| + F_j E_i \otimes |1\rangle\langle 1| \quad (4)$$

- The order in which the channels  $\mathcal{E}$  and  $\mathcal{F}$  act is determined by the state of the control qubit. If  $|\omega\rangle = |0\rangle$ ,  $\mathcal{F}$  is applied first, while if  $|\omega\rangle = |1\rangle$  the opposite.
- If the control qubit is initially in a superposition state  $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ , then the quantum SWITCH creates a **superposition of orders**.
- The **Sender encodes** information into  $\rho$  and transmits it. The **Receiver** determines how to **decode** by the state of the control qubit. The control qubit is accessible only to the receiver.

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# Holevo's Theorem: Setting

Suppose Alice transmits messages, in the form of density matrices, to Bob through the following procedure:

- Alice samples  $X \in \Sigma \subseteq \{0, 1\}^n$ , where  $X = x$  with probability  $p(x)$ .
- Alice sends  $\sigma_X \in \mathbb{C}^{d \times d}$ .
- Bob picks POVM's  $\{E_y\}_{y \in \Gamma}$ , where  $\Gamma \subseteq \{0, 1\}^n$ .
- Bob measures  $\sigma_X$ , and receives output " $Y \in \Gamma$ ", where  $Y = y$  given  $X = x$  with probability  $\text{Tr}(E_y \sigma_x)$ .
- Bob tries to infer  $X$  from  $Y$ . Note:  $X, Y$  are two **classical** probability distributions.

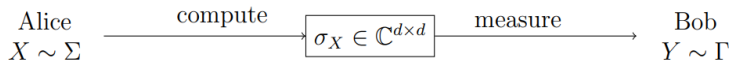


Figure 2: Communication Scheme. Source: [O'Donnell and Wright, 2015]

- Bob sees the mixed state:

$$\begin{cases} \sigma_{x_1} \text{ with prob. } p(x_1), \\ \sigma_{x_2} \text{ with prob. } p(x_2), \\ \vdots \end{cases} \equiv \sum_{x \in \Sigma} p(x) \sigma_x =: \rho_B. \quad (5)$$

- Alice sees:

$$\begin{cases} |x_1\rangle \text{ with prob. } p(x_1), \\ |x_2\rangle \text{ with prob. } p(x_2), \\ \vdots \end{cases} \equiv \sum_{x \in \Sigma} p(x) |x\rangle \langle x| =: \rho_A. \quad (6)$$

- State of the joint mixed system is:

$$\rho := \sum_{x \in \Sigma} p(x) |x\rangle \langle x| \otimes \sigma_x. \quad (7)$$

- We wish to answer the question "how much does seeing one random variable tell me about the other". Solution is given by the mutual information.
- The **classical mutual information**  $I(X; Y)$  between two random variables  $X$  and  $Y$  is

$$I(X; Y) = H(X) + H(Y) - H(X, Y), \quad (8)$$

where  $H(\cdot)$  is the Shannon entropy. Mutual information represents the amount of information one learn about  $X$  from knowing  $Y$ , and vice-versa since it is symmetric in  $X$  and  $Y$ .

- The **accessible information** is defined as:

$$I_{\text{acc}}(\sigma, p) = \max_{\substack{\text{over all} \\ \text{POVMs} \\ \{E_y\}_{y \in \Gamma}}} I(X; Y). \quad (9)$$

This represents the best Bob can do given Alice's choice of the  $\sigma_x$ 's and the distribution  $p$ .

- If  $\rho$  is the joint state of two quantum systems  $A$  and  $B$ , then the **quantum mutual information** is:

$$I(\rho_A; \rho_B) = H(\rho_A) + H(\rho_B) - H(\rho). \quad (10)$$

where  $H(\cdot)$  is the von Neumann entropy. Note: If  $\rho = \rho_A \otimes \rho_B$ , then  $I(\rho_A; \rho_B) = 0$  since  $H(\rho_A \otimes \rho_B) = H(\rho_A) + H(\rho_B)$ .

- The **Holevo information** is defined as:

$$\chi(\sigma, p) := I(\rho_A; \rho_B). \quad (11)$$

## Theorem 1 (Holevo's Theorem/Holevo's Bound)

The accessible information is upper-bounded by the Holevo information:

$$I_{\text{acc}}(\sigma, p) \leq \chi(\sigma, p). \quad (12)$$

Equivalent form of  $\chi(\sigma, p)$ :

$$\chi(\sigma, p) = \chi(\eta) := S\left(\sum_i p_i \rho_i\right) - \sum_i p_i S(\rho_i) \quad (13)$$

where  $p_i = p(x_i)$ ,  $\rho_i = \sigma_{x_i}$ ,  $\eta = \{(p_i, \rho_i)\}$  and  $S(\cdot)$  denotes the von Neumann entropy.  $\chi(\eta)$  is called the **Holevo information** or **Holevo  $\chi$  quantity**.

We will prove (13).

# Proof of (13)

Write out all the states:

$$\rho_A = \sum_i p_i |i\rangle\langle i|, \quad \rho_B = \sum_i p_i \rho_i, \quad \rho_{AB} = \sum_i p_i |i\rangle\langle i| \otimes \rho_i \quad (14)$$

Compute von Neumann entropy:

$$H(\rho_A) = H(\{p_i\}), \quad H(\rho_B) = S\left(\sum_i p_i \rho_i\right) \quad (15)$$

where  $H(\{p_i\})$  is the Shannon entropy of  $\{p_i\}$ . Moreover we have:

$$H(\rho_{AB}) = H(\{p_i\}) + \sum_i p_i S(\rho_i) \quad (16)$$

Plug  $H(\rho_A)$ ,  $H(\rho_B)$  and  $H(\rho_{AB})$  into the definition of  $I(\rho_A; \rho_B)$  will get the result.

# Proof of (13) (cont'd)

Finally we will show that (16) holds.  $\rho_{AB}$  can be written as a block diagonal matrix:

$$\rho_{AB} = \begin{pmatrix} p_1\rho_1 & 0 & 0 & \cdots \\ 0 & p_2\rho_2 & 0 & \cdots \\ 0 & 0 & p_3\rho_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (17)$$

Each  $\rho_i$  has its own eigenvalue  $r_{ik}$  (where  $k$  is the index of the eigenstate of  $\rho_i$ ). The eigenvalues of  $\rho_{AB}$  are all  $p_i r_{ik}$ . So we have:

$$H(\rho_{AB}) = - \sum_{i,k} p_i r_{ik} \log(p_i r_{ik}) \quad (18)$$

$$= - \sum_{i,k} p_i r_{ik} (\log p_i + \log r_{ik}) \quad (19)$$

$$= - \sum_i p_i \log p_i \left( \sum_k r_{ik} \right) - \sum_i p_i \sum_k r_{ik} \log r_{ik} \quad (20)$$

$$= - \sum_i p_i \log p_i + \sum_i p_i S(\rho_i) \quad (21)$$

$$= H(\{p_i\}) + \sum_i p_i S(\rho_i) \quad (22)$$

where  $\sum_k r_{ik} = 1$  (each density matrix's trace is 1) and  $-\sum_k r_{ik} \log r_{ik} = S(\rho_i)$ . □

# HSW Theorem

The classical channel capacity  $C$  of a quantum channel is given by the HSW theorem:

## Theorem 2 (Holevo-Schumacher-Westmoreland Theorem)

Suppose quantum channel  $\mathcal{E}$  has the Kraus representation  $\mathcal{E}(\sigma) = \sum_j E_j \sigma E_j^\dagger$ .  $\rho := \sum_i p_i \rho_i$ . Then the classical channel capacity of  $\mathcal{E}$  is given by the following regularized expression:

$$C(\mathcal{E}) = \lim_{n \rightarrow \infty} \frac{1}{n} \chi^* (\mathcal{E}^{\otimes n}) \quad (23)$$

where  $\chi^*(\mathcal{E})$  is the **maximum Holevo information**:

$$\chi^*(\mathcal{E}) = \max_{\{p_i, \rho_i\}} \left[ S(\mathcal{E}(\rho)) - \sum_i p_i S(\mathcal{E}(\rho_i)) \right] \quad (24)$$

$$= \max_{\{p_i, \rho_i\}} \left[ S \left( \sum_i p_i \sum_j E_j \rho_i E_j^\dagger \right) - \sum_i p_i S \left( \sum_j E_j \rho_i E_j^\dagger \right) \right] \quad (25)$$

$\rho_i$  is the input quantum state,  $p_i$  is the probability distribution of the quantum state.



- Unlike the classical case,  $C(\mathcal{E})$  is not given by a single-letter formula involving a single copy of the  $\mathcal{E}$ . Instead it is given by a **regularization of the maximum Holevo information**, which involves many copies of  $\mathcal{E}$ . In general, it is unclear how to compute such a formula, and a simpler expression for  $C(\mathcal{E})$  is not yet known.
- $\mathcal{E}^{\otimes k} : \mathcal{B}(\mathcal{H}_A^{\otimes k}) \rightarrow \mathcal{B}(\mathcal{H}_B^{\otimes k})$  means composing  $k$  independent copies of  $\mathcal{E}$  in parallel. For any input density operator  $\rho^{(k)} \in \mathcal{B}(\mathcal{H}_A^{\otimes k})$ , the output is:

$$\mathcal{E}^{\otimes k}(\rho^{(k)}) = (\mathcal{E} \otimes \mathcal{E} \otimes \cdots \otimes \mathcal{E})(\rho^{(k)}) \quad (26)$$

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# Motivation for Quantum SWITCH

- In this section, we will focus on the communication of **classical** information.
- Any non-constant quantum channel  $\mathcal{N}$  has positive Holevo information:
  - There exist at least two pure states  $|\phi\rangle$  and  $|\psi\rangle$ , such that  $\mathcal{N}(|\phi\rangle\langle\phi|) \neq \mathcal{N}(|\psi\rangle\langle\psi|)$ .
  - Set  $p(\phi) = p(\psi) = \frac{1}{2}$ , then the joint state becomes

$$\sigma = \frac{1}{2}\mathcal{N}(|\phi\rangle\langle\phi|) + \frac{1}{2}\mathcal{N}(|\psi\rangle\langle\psi|) \quad (27)$$

- So the Holevo information is

$$\chi(\mathcal{N}) = S(\sigma) - \left(\frac{1}{2}S(\mathcal{N}(|\phi\rangle\langle\phi|)) + \frac{1}{2}S(\mathcal{N}(|\psi\rangle\langle\psi|))\right) > 0 \quad (28)$$

due to the concavity of  $S(\cdot)$ .

- The Holevo information of a constant channel, e.g. **completely depolarizing channel**, is zero, which makes it impossible to perform classical communication. However, this can be achieved with quantum SWITCH.

# Kraus Operator for Depolarizing Channel

- A completely depolarizing channel  $\mathcal{N}^D$  on a  $d$ -dimensional quantum system can be represented by  $d^2$  orthogonal ( $\text{Tr}[U_i^\dagger U_j] = d \cdot \delta_{ij}$ ) unitary operators  $U_i$ , such that its action on a state  $\rho$  is

$$\mathcal{N}^D(\rho) = \frac{1}{d^2} \sum_{i=1}^{d^2} U_i \rho U_i^\dagger = \text{Tr}[\rho] \frac{I}{d}. \quad (29)$$

Example for  $d = 2$ :

$$U_1 = \sqrt{2}|0\rangle\langle 0|, U_2 = \sqrt{2}|0\rangle\langle 1|, U_3 = \sqrt{2}|1\rangle\langle 0|, U_4 = \sqrt{2}|1\rangle\langle 1|.$$

- Thus, according to (4), the overall quantum channel resulting from the quantum SWITCH of **two completely depolarizing channels** has the Kraus operator:

$$W_{ij} = \frac{1}{d^2} (U_i U_j \otimes |0\rangle\langle 0|_c + U_j U_i \otimes |1\rangle\langle 1|_c) \quad (30)$$

# Quantum SWITCH for Depolarizing Channel

- Set control state to  $\rho_c := |\psi_c\rangle\langle\psi_c|$ , where  $|\psi_c\rangle := \sqrt{p}|0\rangle + \sqrt{1-p}|1\rangle$ .
- Set the input state to  $\rho$ , then the receiver will get the output state

$$\mathcal{S}(\mathcal{N}^D, \mathcal{N}^D, \rho_c)(\rho) = \frac{1}{d^4} \sum_{i,j} \left( p|0\rangle\langle 0|_c \otimes U_i U_j \rho U_j^\dagger U_i^\dagger + (1-p)|1\rangle\langle 1|_c \otimes U_j U_i \rho U_i^\dagger U_j^\dagger \right. \\ \left. + \sqrt{p(1-p)}|0\rangle\langle 1|_c \otimes U_i U_j \rho U_i^\dagger U_j^\dagger + \sqrt{p(1-p)}|1\rangle\langle 0|_c \otimes U_j U_i \rho U_j^\dagger U_i^\dagger \right) \quad (31)$$

$$= p|0\rangle\langle 0|_c \otimes \frac{I}{d} + (1-p)|1\rangle\langle 1|_c \otimes \frac{I}{d} \quad (32)$$

$$+ \sqrt{p(1-p)} \frac{|0\rangle\langle 1|_c}{d^2} \otimes \sum_j \text{Tr}[U_j \rho] \frac{U_j^\dagger}{d} \\ + \sqrt{p(1-p)} \frac{|1\rangle\langle 0|_c}{d^2} \otimes \sum_j \text{Tr}[\rho U_j^\dagger] \frac{U_j}{d}. \\ = (p|0\rangle\langle 0|_c + (1-p)|1\rangle\langle 1|_c) \otimes \frac{I}{d} + \sqrt{p(1-p)}(|0\rangle\langle 1|_c + |1\rangle\langle 0|_c) \otimes \frac{\rho}{d^2}. \quad (33)$$

- (31) results from (3).
- $\sum_j \text{Tr}[U_j \rho] \frac{U_j^\dagger}{d}$  results from viewing  $U_j \rho$  as a single state and applying (29).
- (33) follows from the fact that  $\{U_j\}$  forms an orthonormal basis for the set of  $d \times d$  matrices, i.e.,  $\rho = \sum_{j=1}^{d^2} \text{Tr}[U_j \rho] \frac{U_j^\dagger}{d}$ . This is related to the **Hilbert-Schmidt operator and Hilbert-Schmidt space**.
- The quantum SWITCH of two depolarizing channels depends on state  $\rho$ . Thus, **we can communicate classical information at a nonzero rate**.

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# Extending to $> 2$ Channels

- When  $N = 2$ , completely depolarizing channels are unable to transmit quantum data, even when applying quantum SWITCH.
- We consider  $N(\geq 3)$  completely depolarizing channels combined in a superposition of  $N$  causal orders related to each other by **cyclic permutations**.
- The intermediate nodes in Figure 3 (purple) are identity operations

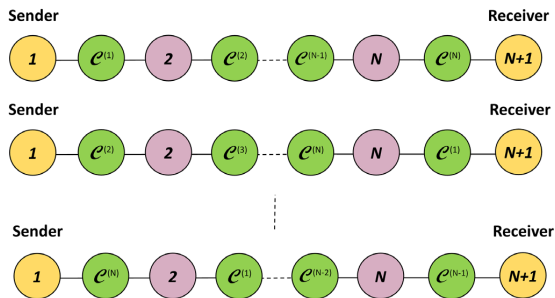


Figure 3: Cyclic Permutations. Source: [Chiribella et al., 2021a]



# Settings

- Take  $N = 3$  for simplicity, construct the quantum SWITCH  $\mathcal{S}(\mathcal{E}, \mathcal{F}, \mathcal{G}, |\omega\rangle)(\rho)$  where:
  - $\mathcal{E}, \mathcal{F}, \mathcal{G}$  are three completely depolarizing channels.
  - $|\omega\rangle$  is the control **qutrit**.
- Set:
  - $\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger$
  - $\mathcal{F}(\rho) = \sum_j F_j \rho F_j^\dagger$
  - $\mathcal{G}(\rho) = \sum_k G_k \rho G_k^\dagger$
- Let the **orthonormal** basis of the control space be:
  - $|\pi_0\rangle$  denotes the order  $\mathcal{E} \rightarrow \mathcal{F} \rightarrow \mathcal{G}$
  - $|\pi_1\rangle$  denotes the order  $\mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{E}$
  - $|\pi_2\rangle$  denotes the order  $\mathcal{G} \rightarrow \mathcal{E} \rightarrow \mathcal{F}$
- Let the control state be the equal superposition  $|\omega\rangle = \frac{1}{\sqrt{3}} (|\pi_0\rangle + |\pi_1\rangle + |\pi_2\rangle)$  and let  $\mathbf{S}$  denote the set of permutations. The density matrix becomes:

$$\omega = |\omega\rangle\langle\omega| = \frac{1}{3} \sum_{\pi, \pi' \in \mathbf{S}} |\pi\rangle\langle\pi'| \quad (34)$$

- $G_k F_j E_i := K_{ijk}^{(\pi_0)}$ ,  $E_i G_k F_j := K_{ijk}^{(\pi_1)}$ ,  $F_j E_i G_k := K_{ijk}^{(\pi_2)}$ . Extending from (4) we have:

$$K_{ijk} = K_{ijk}^{(\pi_0)} \otimes |\pi_0\rangle\langle\pi_0| + K_{ijk}^{(\pi_1)} \otimes |\pi_1\rangle\langle\pi_1| + K_{ijk}^{(\pi_2)} \otimes |\pi_2\rangle\langle\pi_2| \quad (35)$$

$$= \sum_{\pi} K_{ijk}^{(\pi)} \otimes |\pi\rangle\langle\pi| \quad (36)$$

- So the channel becomes:

$$\mathcal{S} = \sum_{i,j,k} K_{ijk} (\rho \otimes \frac{1}{3} \sum_{\pi,\pi'} |\pi\rangle\langle\pi'|) K_{ijk}^\dagger \quad (37)$$

$$= \sum_{i,j,k} \sum_{\pi} K_{ijk}^{(\pi)} \otimes |\pi\rangle\langle\pi| (\rho \otimes \frac{1}{3} \sum_{\pi,\pi'} |\pi\rangle\langle\pi'|) \sum_{\pi'} K_{ijk}^{(\pi')\dagger} \otimes |\pi'\rangle\langle\pi'| \quad (38)$$

$$= \sum_{i,j,k} \sum_{\pi,\pi'} K_{ijk}^{(\pi)} \rho K_{ijk}^{(\pi')\dagger} \otimes \frac{1}{3} |\pi\rangle\langle\pi'| \quad (39)$$

$$= \sum_{\pi,\pi'} \mathcal{C}_{\pi\pi'}(\rho) \otimes \omega_{\pi,\pi'} |\pi\rangle\langle\pi'| := \mathcal{C}_{\text{eff}}(\rho) \quad (40)$$

where  $\mathcal{C}_{\pi\pi'}(\rho) := \sum_{i,j,k} K_{ijk}^{(\pi)} \rho K_{ijk}^{(\pi')\dagger}$ ,  $\omega_{\pi,\pi'} = \frac{1}{3}$ .

- Notation for general  $N$  channel setting:

$\mathcal{C}_{\pi\pi'}(\rho) = \sum_{s_1,\dots,s_N} K_{(s_1,\dots,s_N)}^{(\pi)} \rho K_{(s_1,\dots,s_N)}^{(\pi')\dagger}$ , where  $s_i$ 's are indexes of Kraus operators corresponding to each channel.

# Channel Construction

- Recall:  $\omega = |\omega\rangle\langle\omega|$  where  $|\omega\rangle = \frac{1}{\sqrt{N}} \sum_{\pi} |\pi\rangle$ .
- [Chiribella et al., 2021b] showed that:

$$\mathcal{C}_{\pi\pi}(\rho) = \frac{I}{d} \quad \text{and} \quad \mathcal{C}_{\pi\pi'}(\rho) = \frac{\rho}{d^2} \quad \forall \pi \neq \pi' \quad (41)$$

- Plug into (40) yields:

$$\mathcal{C}_{\text{eff}}(\rho) = \sum_{\pi} \frac{I}{d} \otimes \frac{1}{N} |\pi\rangle\langle\pi| + \sum_{\pi \neq \pi'} \frac{\rho}{d^2} \otimes \frac{1}{N} |\pi\rangle\langle\pi'| \quad (42)$$

$$= \frac{I}{d} \otimes \frac{I}{N} + \frac{\rho}{Nd^2} \otimes \sum_{\pi \neq \pi'} |\pi\rangle\langle\pi'| \quad (43)$$

$$= \frac{I}{d} \otimes \frac{I}{N} + \frac{\rho}{Nd^2} \otimes (N|\omega\rangle\langle\omega| - I), \quad (44)$$

where (44) comes from the relations  $N|\omega\rangle\langle\omega| = \sum_{\pi, \pi'} |\pi\rangle\langle\pi'|$  and  $I = \sum_{\pi} |\pi\rangle\langle\pi|$ .

# Channel as a Mixture of Channels

- Rearranging (44) we have:

$$\mathcal{C}_{\text{eff}}(\rho) = \mathcal{E}_0(\rho) \otimes (1-p)\rho_0 + \mathcal{E}_1(\rho) \otimes p\rho_1, \quad (45)$$

where  $\rho_0 := |\omega\rangle\langle\omega|$  and  $\rho_1 := \frac{I-|\omega\rangle\langle\omega|}{N-1}$ ,  $p := \frac{(N-1)(d^2-1)}{Nd^2}$ .  $\mathcal{E}_0, \mathcal{E}_1$  are quantum channels defined by

$$\mathcal{E}_0(\rho) := \frac{N-1}{N-1+d^2}\rho + \frac{d^2}{N-1+d^2}\frac{I}{d} \quad (46)$$

and

$$\mathcal{E}_1(\rho) := \frac{d^2}{d^2-1}\frac{I}{d} - \frac{1}{d^2-1}\rho \quad (47)$$

- $\mathcal{C}_{\text{eff}}$  is a mixture of two channels  $\mathcal{E}_0$  and  $\mathcal{E}_1$ . By measuring  $\rho_0$  and  $\rho_1$ , it is possible to determine the occurrence of the channels  $\mathcal{E}_0$  and  $\mathcal{E}_1$ .

- $\mathcal{E}_1$  is unable to transmit quantum data, see [Chiribella et al., 2021a].
- $\mathcal{E}_0$  is a depolarizing channel, with probability of depolarization equal to  $\frac{d^2}{N+d^2-1}$ .
- $\lim_{N \rightarrow \infty} \frac{d^2}{N+d^2-1} = 0 \Rightarrow \lim_{N \rightarrow \infty} \mathcal{E}_0(\rho) = \rho$ . Therefore, as long as we wait until the control system is measured to be  $\rho_0$ , we achieve **nearly perfect quantum information transmission**.

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- Next, we discuss quantum SWITCH constructed by **general channels and permutations**. Let  $n$  denote the total number of channels,  $m := |\mathbf{S}|$  denote the number of permutations in  $\mathbf{S}$ .  $n$  may not equal  $m$ .
- [Wu et al., 2025a] introduced the key quantity

$$\mathcal{P}_n = 1 - \frac{1}{m^2} \min_{\rho} \sum_{\pi, \pi' \in \mathbf{S}} \text{Tr}(C_{\pi\pi'}(\rho)) \quad (48)$$

- $\mathcal{P}_n$  can be viewed as the maximum probability (as  $\rho$  varies over all input states) of obtaining the measurement outcome  $F_2$  associated with the POVM  $\{F_1 = |\omega\rangle\langle\omega| = \frac{1}{m} \sum_{i,j \in \mathbf{S}} |i\rangle\langle j|, F_2 = I - |\omega\rangle\langle\omega|\}$ . Proof in next page.



Probability of obtaining outcome  $F_1$  can be computed as:

$$\Pr[F_1] = \text{Tr}[(I \otimes F_1) \mathcal{C}_{\text{eff}}(\rho)]. \quad (49)$$

$$= \text{Tr} \left[ (I \otimes F_1) \sum_{\pi, \pi'} \mathcal{C}_{\pi\pi'}(\rho) \otimes \frac{1}{m} |\pi\rangle\langle\pi'| \right] \quad (50)$$

$$= \frac{1}{m} \sum_{\pi, \pi'} \text{Tr} [\mathcal{C}_{\pi\pi'}(\rho) \otimes (F_1 |\pi\rangle\langle\pi'|)] \quad (51)$$

$$= \frac{1}{m} \sum_{\pi, \pi'} \text{Tr}(\mathcal{C}_{\pi\pi'}(\rho)) \cdot \text{Tr}(F_1 |\pi\rangle\langle\pi'|) \quad (52)$$

$$= \frac{1}{m^2} \sum_{\pi, \pi'} \text{Tr}(\mathcal{C}_{\pi\pi'}(\rho)) \quad (53)$$

So probability of obtaining outcome  $F_2$  is:

$$\Pr[F_2] = 1 - \Pr[F_1] = 1 - \frac{1}{m^2} \sum_{\pi, \pi'} \text{Tr}(\mathcal{C}_{\pi\pi'}(\rho)) \quad (54)$$

So we have:

$$\mathcal{P}_n = \max_{\rho} \Pr[F_2] = 1 - \min_{\rho} \frac{1}{m^2} \sum_{\pi, \pi'} \text{Tr}(\mathcal{C}_{\pi\pi'}(\rho)) \quad (55)$$

## Proposition 1

The quantity  $\mathcal{P}_n = 0$  if and only if the Kraus operators  $\{C_i\}$  are **S**-invariant, i.e.,

$$K_{(s_1, \dots, s_n)}^{(\pi)} = K_{(s_1, \dots, s_n)}^{(\pi')} \quad (56)$$

for all indices  $s_1, \dots, s_n$  and for all  $\pi, \pi' \in \mathbf{S}$ . That is, the Kraus operators are **independent of the causal order**. In particular, for  $n = m = 2$ , **S**-invariance is equivalent to commutativity:

$E_i F_j = F_j E_i \quad \forall i, j$  for two sets of Kraus operators  $\{E_i\}, \{F_j\}$ .

# Proof of Proposition 1

First we will show a simple fact that  $\text{Tr}(C_{\pi\pi}(\rho)) = 1 \quad \forall \pi \in \mathbf{S}$ .

Recall:

$$C_{\pi\pi}(\rho) = \sum_{s_1, \dots, s_n} K_{(s_1, \dots, s_n)}^{(\pi)} \rho K_{(s_1, \dots, s_n)}^{(\pi)\dagger} \quad (57)$$

Here,  $K_{(s_1, \dots, s_n)}^{(\pi)}$  is the composite Kraus operator under the permutation  $\pi$ :

$$K_{(s_1, \dots, s_n)}^{(\pi)} := A_{s_1, \dots, s_n}^{(\pi)} = A_{\pi(1), s_1} A_{\pi(2), s_2} \cdots A_{\pi(n), s_n}, \quad (58)$$

$$K_{(s_1, \dots, s_n)}^{(\pi)\dagger} := A_{s_1, \dots, s_n}^{(\pi)\dagger} = A_{\pi(n), s_n}^\dagger A_{\pi(n-1), s_{n-1}}^\dagger \cdots A_{\pi(1), s_1}^\dagger \quad (59)$$

where each  $A_{i, s_i}$  is the  $s_i$ -th Kraus operator of the  $i$ -th quantum channel.

Since each channel is a valid CPTP map, the composite channel must also be a CPTP map.

So we have:

$$\text{Tr}(C_{\pi\pi}(\rho)) = 1 \quad \forall \pi \in \mathbf{S} \quad (60)$$

# Proof of Proposition 1 (cont'd)

Let  $\rho$  be an arbitrary input state, and let  $\pi, \pi' \in \mathbf{S}$  be arbitrary permutations in  $\mathbf{S}$ . We then have

$$\mathrm{Tr}(C_{\pi\pi'}(\rho)) = \sum_{s_1, \dots, s_n} \mathrm{Tr} \left( K_{(s_1, \dots, s_n)}^{(\pi)} \rho K_{(s_1, \dots, s_n)}^{(\pi')\dagger} \right) \quad (61)$$

$$= \sum_{s_1, \dots, s_n} \mathrm{Tr} \left( K_{(s_1, \dots, s_n)}^{(\pi)} \sqrt{\rho} \left( K_{(s_1, \dots, s_n)}^{(\pi')} \sqrt{\rho} \right)^\dagger \right) \quad (62)$$

$$\leq \frac{1}{2} [\mathrm{Tr}(C_{\pi\pi}(\rho)) + \mathrm{Tr}(C_{\pi'\pi'}(\rho))] = 1 \quad (63)$$

where the final equality follows from (60), and the first inequality follows from combining the Cauchy-Schwarz and AM-GM inequalities, which for arbitrary operators  $A$  and  $B$  yields

$$\mathrm{Tr}(AB^\dagger) \leq \sqrt{\mathrm{Tr}(AA^\dagger)\mathrm{Tr}(BB^\dagger)} \quad (64)$$

$$\leq \frac{1}{2} \left( \mathrm{Tr}(AA^\dagger) + \mathrm{Tr}(BB^\dagger) \right). \quad (65)$$

# Explanations on the Cauchy-Schwarz Inequality

To apply Cauchy-Schwarz inequality, we define the **Hilbert-Schmidt inner product** as follows (this is also related to the Hilbert-Schmidt space):

$$\langle A, B \rangle := \text{Tr}(AB^\dagger) \quad (66)$$

The norm induced by this inner product is:

$$\|A\|_{\text{HS}} := \sqrt{\text{Tr}(AA^\dagger)} \quad (67)$$

Applying the Cauchy-Schwarz inequality, we obtain:

$$|\text{Tr}(AB^\dagger)| \leq \sqrt{\text{Tr}(AA^\dagger)} \cdot \sqrt{\text{Tr}(BB^\dagger)} \quad (68)$$

By setting  $A_{(s_1, \dots, s_n)} = K_{(s_1, \dots, s_n)}^{(\pi)} \sqrt{\rho}$ ,  $B_{(s_1, \dots, s_n)} = K_{(s_1, \dots, s_n)}^{(\pi')} \sqrt{\rho}$ , and using Cauchy-Schwarz inequality for each pair of  $A_{(s_1, \dots, s_n)}$ ,  $B_{(s_1, \dots, s_n)}$ , we get the inequality last page.

# Conclusion and Remarks

Since the equality holds if and only if  $A_{(s_1, \dots, s_n)} = B_{(s_1, \dots, s_n)} \quad \forall s_1, \dots, s_n$ , it follows that

$$\text{Tr}(C_{\pi\pi'}(\rho)) = 1, \quad (69)$$

if and only if

$$K_{(s_1, \dots, s_n)}^{(\pi)} \sqrt{\rho} = K_{(s_1, \dots, s_n)}^{(\pi')} \sqrt{\rho} \quad \forall s_1, \dots, s_n. \quad (70)$$

From the definition of  $\mathcal{P}_n$ , it follows that  $\mathcal{P}_n = 0$  if and only if (70) holds for all states  $\rho$ , for all permutations  $\pi, \pi' \in \mathbf{S}$ , and for all indices  $s_1, \dots, s_n$ . By considering pure states, we see that  $\mathcal{P}_n = 0$  if and only if

$$K_{(s_1, \dots, s_n)}^{(\pi)} = K_{(s_1, \dots, s_n)}^{(\pi')} \quad (71)$$

for all  $s_1, \dots, s_n$ , as desired. In particular, when  $n = m = 2$ , the permutation set  $\mathbf{S}$  only contains two permutations  $(1, 2)$  and  $(2, 1)$ , thus S-invariance implies

$$C_{s_1}^1 C_{s_2}^2 = C_{s_2}^2 C_{s_1}^1 \quad \text{for all } s_1, s_2, \quad (72)$$

i.e., the Kraus operators of the channels  $\mathcal{C}^1$  and  $\mathcal{C}^2$  pairwise commute.

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- Denote the quantum SWITCH channel  $\mathcal{C}_{\text{eff}}(\rho)$  as  $\mathcal{S}^n(\rho)$ , where  $n$  is the number of channels.
- In the following, we will consider the case  $\mathcal{C}^i = \mathcal{N}$  for  $i = 1, \dots, n$ , where  $\mathcal{N}$  is a fixed quantum channel, and let  $\mathcal{N}^n$  denote the  $n$ -fold composition  $\mathcal{N} \circ \dots \circ \mathcal{N}$ .



# Tracing Out Control Recovers $\mathcal{N}^n$

- $\mathcal{N}^n$  can be obtained from  $\mathcal{S}^n$  by tracing out the control system:

$$\mathcal{N}^n(\rho) = \mathcal{N} \circ \dots \circ \mathcal{N}(\rho) = \sum_{s_1, \dots, s_n} C_{s_n} \dots C_{s_1} \rho C_{s_1}^\dagger \dots C_{s_n}^\dagger \quad (73)$$

Take partial trace:

$$\text{Tr}_C [\mathcal{S}^n(\rho)] = \sum_{\pi, \pi' \in \mathbf{S}} C_{\pi\pi'}(\rho) \cdot \text{Tr} (\omega_{\pi, \pi'} |\pi\rangle\langle\pi'|) \quad (74)$$

$$= \sum_{\pi, \pi' \in \mathbf{S}} C_{\pi\pi'}(\rho) \cdot \frac{1}{m} \delta_{\pi\pi'} \quad (75)$$

$$= \frac{1}{m} \sum_{\pi \in \mathbf{S}} C_{\pi\pi}(\rho) \quad (76)$$

Clearly  $C_{\pi\pi}(\rho) = \mathcal{N}^n(\rho)$  (since  $C^i = \mathcal{N} \quad \forall i$ ), thus:

$$\text{Tr}_C [\mathcal{S}^n(\rho)] = \mathcal{N}^n(\rho). \quad (77)$$

# Quantifying Communication Enhancement

- For all capacity measure  $f$  satisfying the data-processing inequality  $f(\mathcal{A} \circ \mathcal{B}) \leq f(\mathcal{B})$ , for arbitrary quantum channels  $\mathcal{A}, \mathcal{B}$ , we obtain the **bottleneck inequality**:

$$f(\mathcal{N}^n) \leq f(\mathcal{S}^n). \quad (78)$$

since taking partial trace is also a CPTP map.

- The classical, quantum, and private capacity, and the Holevo information and coherent information all satisfy (78). Thus, we define the associated **causal gain** by

$$\delta_f = f(\mathcal{S}^n) - f(\mathcal{N}^n). \quad (79)$$

$\delta_f$  is a direct measure of the communication enhancement of the channel  $\mathcal{N}$  which is achieved by inputting  $n$ -copies of the channel into the quantum SWITCH.

# Sufficient Condition for $\delta_f = 0$

- $\mathcal{P}_n = 0$  if and only if  $K_{(s_1, \dots, s_n)}^{(\pi)} = K_{(s_1, \dots, s_n)}^{(\pi')}$  for all  $s_1, \dots, s_n$ .  
Hence, for a given subset of permutations  $\mathbf{S}$ ,  $\mathcal{P}_n = 0$  implies that  $C_{\pi\pi'}(\rho) = C_{\pi\pi}(\rho) = C_{\pi'\pi'}(\rho) = C(\rho)$  for arbitrary permutations  $\pi, \pi' \in \mathbf{S}$ .
- Therefore  $\mathcal{S}^n(\rho) = \mathcal{N}^n(\rho) \otimes \omega$ , where  $\omega$  is the control state and is independent of the input state  $\rho$ . Proof:

$$\mathcal{S}^n(\rho) = \sum_{\pi, \pi' \in \mathbf{S}} C_{\pi\pi'}(\rho) \otimes \omega_{\pi, \pi'} |\pi\rangle \langle \pi'| \quad (80)$$

$$= \sum_{\pi, \pi' \in \mathbf{S}} \mathcal{N}^n(\rho) \otimes \omega_{\pi, \pi'} |\pi\rangle \langle \pi'| \quad (81)$$

$$= \mathcal{N}^n(\rho) \otimes \left( \sum_{\pi, \pi' \in \mathbf{S}} \omega_{\pi, \pi'} |\pi\rangle \langle \pi'| \right) \quad (82)$$

$$= \mathcal{N}^n(\rho) \otimes \omega \quad (83)$$

- Since  $\omega$  is a constant, we have  $f(\mathcal{S}^n) = f(\mathcal{N}^n)$ , so that  $\delta_f = 0$ .

# Main Conjecture

We have shown that  $\mathcal{P}_n > 0$  is a necessary condition for positive causal gain, so it is natural to propose the following conjecture:

## Conjecture

For all channels outside **a set of measure-zero**, the condition  $\mathcal{P}_n > 0$  is necessary and sufficient for  $\delta_f > 0$ .

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The **Pauli channel** describes a probabilistic mixture of Pauli operations applied to a single qubit (density matrix)  $\rho$ :

$$\mathcal{N}(\rho) = \sum_{i=0}^3 p_i \sigma_i \rho \sigma_i^\dagger, \quad \text{with} \quad \sum_i p_i = 1 \quad (84)$$

where:

- $\sigma_0 = I$ ,  $\sigma_1 = X$ ,  $\sigma_2 = Y$ ,  $\sigma_3 = Z$  are the four Pauli operators,
- $p_i \geq 0$  is the probability of applying operation  $\sigma_i$ .
- Completely depolarizing channel is a special case of Pauli channel:  
 $\mathcal{N}(\rho) = \frac{1}{4} (I\rho I + X\rho X + Y\rho Y + Z\rho Z) = \frac{I}{2}.$

$$\begin{aligned} \sigma_0 = I &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \sigma_1 = X &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \sigma_2 = Y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, & \sigma_3 = Z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

- In this section, we verify the conjecture in the space of all Pauli channels  $\mathcal{N}(\rho) = \sum_{i=0}^3 p_i \sigma_i \rho \sigma_i^\dagger$ , and the permutation set  $\mathbf{S}$  contains the identity permutation  $(1, \dots, n)$  and its reversal  $(n, \dots, 1)$ .
- In such case, the quantum SWITCH places  $n$  copies of a Pauli channel  $\mathcal{N}$  in a superposition of the forward and backward orders  $\mathcal{N}_n \circ \dots \circ \mathcal{N}_1$  and  $\mathcal{N}_1 \circ \dots \circ \mathcal{N}_n$  (where  $\mathcal{N}_i = \mathcal{N}$  for all  $i$ ).
- The forward and backward orders are indistinguishable when used individually. We show that when such orders are placed in a superposition via quantum SWITCH, an enhancement of classical capacity and coherent information occurs almost surely

- Fix the control state to be  $\omega = |+\rangle\langle +|$  with  $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ , then apply the projective measurement  $\{F_1 = |\omega\rangle\langle\omega|, F_2 = I - |\omega\rangle\langle\omega|\}$ .
- The probability of obtaining  $F_2$  is independent of the initial state  $\rho$  (proof on next few pages), i.e., for all states  $\rho$ ,

$$\text{Tr}((I \otimes F_2)\mathcal{S}^n(\rho)) = \mathcal{P}_n. \quad (85)$$



Recall, probability of obtaining outcome  $F_2$  is:

$$\Pr[F_2] = 1 - \frac{1}{m^2} \sum_{\pi, \pi'} \text{Tr}(C_{\pi\pi'}(\rho)) \quad (86)$$

For Pauli channel  $\mathcal{N}(\rho) = \sum_{i=0}^3 p_i \sigma_i \rho \sigma_i^\dagger$ , its Kraus representation is:

$$\mathcal{N}(\rho) = \sum_i E_i \rho E_i^\dagger, \quad E_i = \sqrt{p_i} \sigma_i \quad (87)$$

For a quantum SWITCH of  $n$  channels, the composite Kraus operator is given by:

$$K_{(s_1, \dots, s_n)}^{(\pi)} = \sqrt{p_{s_1} \cdots p_{s_n}} \cdot \sigma_{s_{\pi(1)}} \cdots \sigma_{s_{\pi(n)}}, \quad (88)$$

where  $\pi$  is the permutation (either forward or reverse).

Take trace:

$$\text{Tr}(C_{\pi\pi'}(\rho)) = \sum_{s_1, \dots, s_n} \text{Tr} \left( \sigma_{s_{\pi(1)}} \cdots \sigma_{s_{\pi(n)}} \rho \sigma_{s_{\pi'(n)}}^\dagger \cdots \sigma_{s_{\pi'(1)}}^\dagger \right) \cdot p_{s_1} \cdots p_{s_n} \quad (89)$$

$$(90)$$

# Proof (cont'd)

If  $\pi = \pi'$ :

$$\text{Tr}(C_{\pi\pi}(\rho)) = \sum_{s_1, \dots, s_n} \text{Tr} \left( \sigma_{s_{\pi(1)}} \cdots \sigma_{s_{\pi(n)}} \rho \sigma_{s_{\pi(n)}}^\dagger \cdots \sigma_{s_{\pi(1)}}^\dagger \right) \cdot p_{s_1} \cdots p_{s_n} \quad (91)$$

$$= \sum_{s_1, \dots, s_n} \text{Tr} \left( \rho \sigma_{s_{\pi(n)}}^\dagger \cdots \sigma_{s_{\pi(1)}}^\dagger \sigma_{s_{\pi(1)}} \cdots \sigma_{s_{\pi(n)}} \right) \cdot p_{s_1} \cdots p_{s_n} \quad (92)$$

$$= \sum_{s_1, \dots, s_n} \text{Tr}(\rho) \cdot p_{s_1} \cdots p_{s_n} \quad (93)$$

$$= \sum_{s_1, \dots, s_n} p_{s_1} \cdots p_{s_n} \quad (94)$$

$$= \left( \sum_{s_1} p_{s_1} \right) \cdots \left( \sum_{s_n} p_{s_n} \right) = 1 \cdots 1 = 1. \quad (95)$$

# Proof (cont'd)

If  $\pi \neq \pi'$ , take  $n = 2$  as example,  $\pi = (1, 2), \pi' = (2, 1)$ :

$$\text{Tr}(C_{\pi\pi'}(\rho)) = \sum_{s_1, s_2} \text{Tr}(\sigma_{s_1} \sigma_{s_2} \rho \sigma_{s_1}^\dagger \sigma_{s_2}^\dagger) \cdot p_{s_1} p_{s_2} \quad (96)$$

$$= \sum_{s_1, s_2} \text{Tr}(\sigma_{s_1} \sigma_{s_2} \rho \sigma_{s_1} \sigma_{s_2}) \cdot p_{s_1} p_{s_2} \quad (97)$$

$$= \sum_{s_1, s_2} \text{Tr}(\rho \sigma_{s_1} \sigma_{s_2} \sigma_{s_1} \sigma_{s_2}) \cdot p_{s_1} p_{s_2} \quad (98)$$

Case 1,  $s_1 = s_2$ :

$$\sigma_{s_1} \sigma_{s_1} \sigma_{s_1} \sigma_{s_1} = I \quad (99)$$

Case 2,  $s_1 \neq s_2$ :

- If  $i \neq j$  and  $i, j \neq 0$ , use the relationship  $\sigma_i \sigma_j = -\sigma_j \sigma_i$ :

$$\sigma_{s_1} \sigma_{s_2} \sigma_{s_1} \sigma_{s_2} = -\sigma_{s_1}^2 \sigma_{s_2}^2 = -I \quad (100)$$

- If  $s_1 = 0$  or  $s_2 = 0$  (i.e.,  $\sigma_0 = I$ ), the product is  $I$ .

In all cases, the outcome is independent of  $\rho$ .

# Structural Characterization of $\mathcal{P}_n = 0$

[Wu et al., 2025b] proved the following proposition that characterizes the condition for  $\mathcal{P}_n = 0$ :

## Proposition 2

The quantity  $\mathcal{P}_n$  is zero if and only if:

- ①  $n$  is even and the Kraus operators of  $\mathcal{N}$  are commutative.
- ②  $n$  is odd and the number of Kraus operators of  $\mathcal{N}$  is not more than 2.

# Degradable Channels

- For general  $n$ ,  $\mathcal{P}_n = 0$  only if the **Choi rank** of  $\mathcal{N} \leq 2$ , and such Pauli channels are shown to be either **degradable** or **antidegradable**.
- A quantum channel has a Stinespring representation:

$$\mathcal{N}(\rho) = \text{Tr}_E U_{\mathcal{N}}(\rho) \quad (101)$$

where  $E$  is the environment,  $\mathcal{N} : \mathcal{H}_A \rightarrow \mathcal{H}_B$ ,  $U_{\mathcal{N}} : \mathcal{H}_A \rightarrow \mathcal{H}_B \otimes \mathcal{H}_E$ .

- The complementary channel  $\mathcal{N}^c : \mathcal{H}_A \rightarrow \mathcal{H}_E$  is defined by

$$\mathcal{N}^c(\rho) = \text{Tr}_B U_{\mathcal{N}}(\rho). \quad (102)$$

- A channel  $\mathcal{N}$  is degradable when there exists a CPTP map  $\mathcal{T} : \mathcal{H}_B \rightarrow \mathcal{H}_E$  such that:

$$\mathcal{N}^c = \mathcal{T} \circ \mathcal{N}. \quad (103)$$

- Anti-degradable channel: a channel whose complement is degradable, i.e. there exists a CPTP map  $\mathcal{S} : \mathcal{H}_E \rightarrow \mathcal{H}_B$  such that  $\mathcal{N} = \mathcal{S} \circ \mathcal{N}^c$ .

## $\mathcal{P}_n = 0$ Lies on a Measure-Zero Set

- The set of Pauli channels is parametrized by the three-dimensional simplex of probability vectors in  $\mathbb{R}^4$ , as shown in Figure 4
- [Wu et al., 2025b] also showed that the subset satisfying  $\mathcal{P}_n = 0$  lies in the edges of this simplex, which is a **measure-zero set**.
- Thus,  $\mathcal{P}_n > 0$  almost surely.

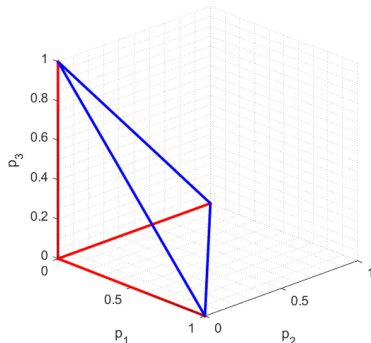


Figure 4: Simplex Representing Pauli Channels. Source: [Wu et al., 2025a]

# Explicit Decomposition of $\mathcal{S}^n$ via $\mathcal{P}_n$

- Now, we establish the connection between  $\mathcal{P}_n$  and  $\delta_f$  when  $f$  is either the classical capacity  $C$  or the coherent information  $I_c$
- [Wu et al., 2025b] proved the following expression for  $\mathcal{S}^n$ :

$$\mathcal{S}^n(\rho) = (1 - \mathcal{P}_n)\Phi_+(\rho) \otimes \omega + \mathcal{P}_n\Phi_-(\rho) \otimes (I - \omega), \quad (104)$$

where  $\omega = |+\rangle\langle+|$  and  $\Phi_{\pm}$  are two Pauli channels.

- Take partial trace we have:

$$\text{Tr}_C[\mathcal{S}^n(\rho)] = (1 - \mathcal{P}_n)\Phi_+(\rho) \cdot \text{Tr}(\omega) + \mathcal{P}_n\Phi_-(\rho) \cdot \text{Tr}(1 - \omega) \quad (105)$$

$$= (1 - \mathcal{P}_n)\Phi_+(\rho) + \mathcal{P}_n\Phi_-(\rho) = \mathcal{N}^n(\rho) \quad (106)$$

where we used the fact that  $\text{Tr}(\omega) = \text{Tr}(1 - \omega) = 1$ . Thus,

$$\mathcal{N}^n = (1 - \mathcal{P}_n)\Phi_+ + \mathcal{P}_n\Phi_- \quad (107)$$

[Wu et al., 2025b] proved that the classical capacity  $C$  of  $\mathcal{S}^n$  takes a similar form:

## Theorem 3

Let  $\Phi_{\pm}$  be as in (104). Then

$$C(\mathcal{S}^n) = (1 - \mathcal{P}_n)C(\Phi_+) + \mathcal{P}_nC(\Phi_-). \quad (108)$$



# Deriving Classical Causal Gain for Qubit Unital Channels

- We now use Theorem 3 to derive an explicit expression for the **classical causal gain**  $\delta_C$ .
- Qubit unital channel are channels that satisfy the following conditions:
  - **Action on Qubit:** The input and output quantum states are density matrices in a two-dimensional Hilbert space (i.e.,  $2 \times 2$  matrices).
  - **Unital Property:** The channel maps the identity matrix  $I$  to itself, i.e.,

$$\mathcal{M}(I) = I.$$

For a single qubit,  $I$  is the  $2 \times 2$  identity matrix.

- **Examples:** Pauli channels, completely depolarizing channel, unital channel.

# Classical Capacity of Qubit Unital Channels

- Classical capacity of qubit unital Channels can be computed via the formula

$$C(\mathcal{M}) = \chi(\mathcal{M}) = 1 - H^{\min}(\mathcal{M}) \quad (109)$$

where  $\chi(\mathcal{M})$  is the Holevo information,  $H^{\min}(\mathcal{M}) = \min_{\rho} H(\mathcal{M}(\rho))$  is the minimum output von Neumann entropy of the channel. More general result can be found in [Müller-Hermes, 2021b].

- For qubit unital Channels,  $\mathcal{M}$ 's eigenvalues  $\alpha_i$  (satisfying  $\mathcal{M}(A) = \alpha_i A$ ) determine the minimum output entropy:

$$H^{\min}(\mathcal{M}) = h(\alpha), \quad \alpha = \max_i |\alpha_i|, \quad (110)$$

where  $h(x)$  is the binary entropy function

$$H_b\left(\frac{1+x}{2}\right) = -\frac{1+x}{2} \log \frac{1+x}{2} - \frac{1-x}{2} \log \frac{1-x}{2}, \text{ with domain } [-1, 1].$$

# Range of Eigenvalues of Pauli Channels

- Pauli channel  $\mathcal{N}(\rho) = \sum_{i=0}^3 p_i \sigma_i \rho \sigma_i^\dagger$  is Pauli diagonal:

$$\mathcal{N}(\sigma_j) = \sum_{i=0}^3 p_i \sigma_i \sigma_j \sigma_i = \lambda_j \sigma_j, \quad \text{for } j = 0, 1, 2, 3 \quad (111)$$

where  $\lambda_j$  is the  $j$ th eigenvalue.

- Apply the following relationship:

$$\sigma_i \sigma_j \sigma_i = \begin{cases} \sigma_j, & \text{if } \sigma_i \text{ commutes with } \sigma_j, \\ -\sigma_j, & \text{if } \sigma_i \text{ anti-commutes with } \sigma_j. \end{cases} \quad (112)$$

- Define  $\chi_{ij} = \pm 1$  to represent  $\sigma_i \sigma_j \sigma_i = \chi_{ij} \sigma_j$ , we have:

$$\mathcal{N}(\sigma_j) = \left( \sum_{i=0}^3 p_i \chi_{ij} \right) \sigma_j \quad (113)$$

# Range of Eigenvalues of Pauli Channels (cont'd)

- Finally we have the eigenvalues:

$$\lambda_j = \sum_{i=0}^3 p_i \cdot \chi_{ij}, \quad \text{where } \chi_{ij} = \begin{cases} 1 & \text{if } [\sigma_i, \sigma_j] = 0 \\ -1 & \text{if } \{\sigma_i, \sigma_j\} = 0 \end{cases} \quad (114)$$

- For  $\sigma_0 = I : \mathcal{N}(I) = I \Rightarrow \lambda_0 = 1$ .
- For  $\sigma_1 = X$ :

$$\sigma_i X \sigma_i = \begin{cases} X & i = 0, 1 \\ -X & i = 2, 3 \end{cases} \Rightarrow \lambda_1 = p_0 + p_1 - p_2 - p_3 \quad (115)$$

- For  $\sigma_2 = Y$ :

$$\sigma_i Y \sigma_i = \begin{cases} Y & i = 0, 2 \\ -Y & i = 1, 3 \end{cases} \Rightarrow \lambda_2 = p_0 + p_2 - p_1 - p_3 \quad (116)$$

- For  $\sigma_3 = Z$ :

$$\sigma_i Z \sigma_i = \begin{cases} Z & i = 0, 3 \\ -Z & i = 1, 2 \end{cases} \Rightarrow \lambda_3 = p_0 + p_3 - p_1 - p_2 \quad (117)$$

# Range of Eigenvalues of Pauli Channels (cont'd)

- Take  $\lambda_1$  for example. Since  $\sum_{i=0}^3 p_i = 1$ , we have  $\lambda_1 = 2(p_0 + p_1) - 1$ . Thus:

$$-1 \leq \lambda_1 \leq 1 \quad (118)$$

where  $\lambda_1 = 1$  iff  $p_0 + p_1 = 1$ ,  $\lambda_1 = -1$  iff  $p_2 + p_3 = 1$ .

- Similarly, we have  $-1 \leq \lambda_2 \leq 1$ ,  $-1 \leq \lambda_3 \leq 1$ . As a result:

$$\lambda_j \in [-1, 1] \quad \forall j \quad (119)$$

- Set  $\gamma$ ,  $\mu$  and  $\nu$  to be the maximum of absolute values of eigenvalues of  $\mathcal{N}^n$ ,  $\Phi_+$  and  $\Phi_-$  respectively.
- Assume the eigenvalues of  $\Phi_+$  are  $\{\alpha_i\}$  and those of  $\Phi_-$  are  $\{\beta_i\}$ . Then the eigenvalues of  $\mathcal{N}^n$  are given by:

$$\lambda_i = (1 - \mathcal{P}_n)\alpha_i + \mathcal{P}_n\beta_i. \quad (120)$$

- $\gamma$  satisfies:

$$\gamma = \max_i |\lambda_i| \leq (1 - \mathcal{P}_n) \max_i |\alpha_i| + \mathcal{P}_n \max_i |\beta_i| \leq 1. \quad (121)$$

# Explicit Formula for Classical Causal Gain

- The classical capacity of the effective channel is

$$C(\mathcal{S}^n) = (1 - \mathcal{P}_n)C(\Phi_+) + \mathcal{P}_n C(\Phi_-) \quad (122)$$

$$= (1 - \mathcal{P}_n)(1 - h(\mu)) + \mathcal{P}_n(1 - h(\nu)) \quad (123)$$

$$= 1 - (1 - \mathcal{P}_n)h(\mu) - \mathcal{P}_n h(\nu) \quad (124)$$

- The classical causal gain is then

$$\delta_C = C(\mathcal{S}^n) - C(\mathcal{N}^n) \quad (125)$$

$$= C(\mathcal{S}^n) - (1 - h(\gamma)) \quad (126)$$

$$= h(\gamma) - (1 - \mathcal{P}_n)h(\mu) - \mathcal{P}_n h(\nu) \quad (127)$$

- Note: The composition channel  $\mathcal{N} = \mathcal{N} \circ \dots \circ \mathcal{N}$  is still a Pauli channel, due to the (anti-)commutativity of Pauli operators.

# Sufficiency Proof of the Conjecture

- We will prove the sufficiency part of the conjecture  $(\mathcal{P}_n > 0 \Rightarrow \delta_C > 0)$  holds for Pauli channels except the completely depolarizing channel with forward and backward orders.
- Combining (121), and the concavity and monotonicity of  $h(x)$ , we have:

$$h(\gamma) \geq h((1 - \mathcal{P}_n)\mu + \mathcal{P}_n\nu) \geq (1 - \mathcal{P}_n)h(\mu) + \mathcal{P}_nh(\nu). \quad (128)$$

Therefore,

$$\delta_C = h(\gamma) - (1 - \mathcal{P}_n)h(\mu) - \mathcal{P}_nh(\nu) \geq 0. \quad (129)$$

- The equality holds iff  $\mu = \nu$ . E.g., for completely depolarizing channel,  $p_0 = \cdots = p_3 = \frac{1}{4}$ ,  $\lambda_1 = \cdots = \lambda_3 = 0$ . Thus,  $\mu = \nu = 0$ .

# Important Note

- When computing  $\alpha = \max_i |\alpha_i|$  for  $\Phi_+$  and  $\Phi_-$ , we have ruled out the trivial eigenvalue  $\lambda_0 = 1$ .
- $I$  is an **indistinguishable** input state: the completely mixed state carries no information.
- The true contributors to information transmission are the non-trivial actions of operators like  $X, Y, Z$ .
- Therefore, the eigenvalues actually used for  $\alpha = \max_i |\alpha_i|$  are those corresponding to  $\sigma_1, \sigma_2, \sigma_3$ .



- Next, we will discuss the **coherent information causal gain**  $\delta_I$ .
- For a channel  $\mathcal{N}$ , the coherent information is defined as:

$$I_c(\mathcal{N}) = \max_{\rho} I_c(\rho, \mathcal{N}) = \max_{\rho} [S(\mathcal{N}(\rho)) - S(\mathcal{N}^c(\rho))], \quad (130)$$

where  $\rho$  is the input state,  $S(\cdot)$  is the von Neumann entropy and  $\mathcal{N}^c$  is the complementary channel of  $\mathcal{N}$ .

- $S(\mathcal{N}(\rho))$ : The entropy of the output state, quantifying the information obtained by the receiver.
- $S(\mathcal{N}^c(\rho))$ : The entropy of the environment state, quantifying the amount of quantum information leaked to the environment.
- Quantum information transmission is enabled when  $I_c > 0$ .

# Quantum Causal Gain in Coherent Information

- For a Pauli channel  $\mathcal{N}$ , the coherent information attains a maximum on the completely mixed state  $\frac{I}{2}$ , which is called the **hashing bound**:

$$I_c(\mathcal{N}) = I_c\left(\frac{I}{2}, \mathcal{N}\right) = 1 - H(\vec{p}) \quad (131)$$

where  $\vec{p} = (p_0, p_1, p_2, p_3)$  is the probability vector associated with the Pauli channel  $\mathcal{N}$ , and  $H(\cdot)$  is the Shannon entropy.

- Since  $\Phi_{\pm}$  are both Pauli channels, it follows from (104) that the coherent information of  $\mathcal{S}^n$  also attains a maximum on  $\frac{I}{2}$ :

$$I_c(\mathcal{S}^n) = 1 - (1 - \mathcal{P}_n)H(\vec{s}) - \mathcal{P}_n H(\vec{t}) \quad (132)$$

- So we have the causal gain:

$$\delta_I = H(\vec{q}) - (1 - \mathcal{P}_n)H(\vec{s}) - \mathcal{P}_n H(\vec{t}) \quad (133)$$

- Similar to the case of  $\delta_c$ , it is then straightforward to deduce that  $\mathcal{P}_n > 0$  is a sufficient condition for  $\delta_I > 0$ .

# Necessary and Sufficient Condition for Causal Gain

Based on previous results, [Wu et al., 2025b] proved the following theorem:

## Theorem 4

Let  $f$  be the classical capacity or coherent information, and let  $\delta_f$  be the causal gain associated with forward and backward orders. Then for all Pauli channels (except the completely depolarizing channel when  $n$  is odd), the condition  $\mathcal{P}_n > 0 \iff \delta_f > 0$  holds.

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# Qudit Depolarizing Channels

A qudit depolarizing channel  $\mathcal{D}_p^d$  is given by:

$$\mathcal{D}_p^d(\rho) = (1 - p)\rho + p \operatorname{Tr}(\rho) \frac{I}{d} \quad (134)$$

where  $p \in (0, 1)$  and  $d > 1$ .

[Wu et al., 2025b] proved the following result:

## Theorem 5

Let  $\delta_C$  be the classical causal gain with respect to forward and backward orders. Then for all qudit depolarizing channels (except the completely depolarizing channel), the condition  $\mathcal{P}_n > 0 \iff \delta_C > 0$  holds.



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Thank you!  
Any questions?