# Quantum `SWITCH` for Communication Enhancement

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August 7, 2025

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# Intuition

- The quantum `SWITCH` is an operation that allows multiple quantum channels to act in a superposition of different orders, under the control of a **control qubit/order qubit**.
- **Classical order (fixed order):** Order of two channels $\mathcal{A}$ and $\mathcal{B}$ is fixed by $\mathcal{A} \circ \mathcal{B}(\rho)$ or $\mathcal{B} \circ \mathcal{A}(\rho)$.
- **Quantum `SWITCH` (indefinite order):** Let control qubit $|0\rangle$ controls $\mathcal{A} \circ \mathcal{B}$ and $|1\rangle$ controls $\mathcal{B} \circ \mathcal{A}$. If the control qubit is prepared as $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, then the entire process becomes a superposition of quantum channels, a.k.a **Indefinite Causal Order**.

# Figure Illustration

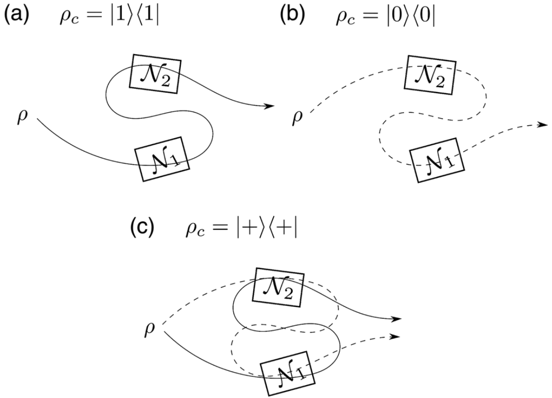


Figure 1: Superposition of Orders. Source: [Ebler et al., 2018]

# Fixed Order Channels

- Suppose we have two quantum channels (CPTP maps) $\mathcal{E}$ and $\mathcal{F}$, whose action on quantum sate $\rho$ can be expressed using the Kraus representation as:

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger, \quad \mathcal{F}(\rho) = \sum_j F_j \rho F_j^\dagger \tag{1}$$

  where $\{E_i\}$ and $\{F_j\}$ are the Kraus operators satisfying $\sum_i E_i^\dagger E_i = \sum_j F_j^\dagger F_j = I$.

- Assume the channels are applied sequentially, giving rise to two possible orders:

$$\mathcal{F} \circ \mathcal{E}(\rho) = \sum_{j,i} F_j E_i \rho E_i^\dagger F_j^\dagger, \quad \mathcal{E} \circ \mathcal{F}(\rho) = \sum_{i,j} E_i F_j \rho F_j^\dagger E_i^\dagger. \tag{2}$$

- The order in which the channels are applied are fixed. In $\mathcal{F} \circ \mathcal{E}(\rho)$, $\mathcal{E}$ is applied first, while in $\mathcal{E} \circ \mathcal{F}(\rho)$ the opposite.

# Indefinite Order Channels

- The quantum SWITCH is a higher-order quantum channel constructed from $\mathcal{E}, \mathcal{F}$ and an ancilla control qubit $|\omega\rangle$, defined as:

$$\mathcal{S}(\mathcal{E}, \mathcal{F}, |\omega\rangle)(\rho) = \sum_{i,j} K_{ij}(\rho \otimes \omega) K_{ij}^{\dagger} \tag{3}$$

where $\omega = |\omega\rangle\langle\omega|$ and $\{K_{ij}\}$ are the Kraus operators:

$$K_{ij} = E_i F_j \otimes |0\rangle\langle 0| + F_j E_i \otimes |1\rangle\langle 1| \tag{4}$$

- The order in which the channels $\mathcal{E}$ and $\mathcal{F}$ act is determined by the state of the control qubit. If $|\omega\rangle = |0\rangle$, $\mathcal{F}$ is applied first, while if $|\omega\rangle = |1\rangle$ the opposite.
- If the control qubit is initially in a superposition state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, then the quantum SWITCH creates a **superposition of orders**.
- The **Sender encodes** information into $\rho$ and transmits it. The **Receiver** determines how to **decode** by the state of the control qubit. The control qubit is accessible only to the receiver.

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# Holevo's Theorem: Setting

Suppose Alice transmits messages, in the form of density matrices, to Bob through the following procedure:

- Alice samples $X \in \Sigma \subseteq \{0,1\}^n$, where $X = x$ with probability $p(x)$.
- Alice sends $\sigma_X \in \mathbb{C}^{d\times d}$.
- Bob picks POVM's $\{E_y\}_{y\in\Gamma}$, where $\Gamma \subseteq \{0,1\}^n$.
- Bob measures $\sigma_X$, and receives output "$Y \in \Gamma$", where $Y = y$ given $X = x$ with probability $\mathrm{Tr}(E_y\sigma_x)$.
- Bob tries to infer $X$ from $Y$. Note: $X, Y$ are two **classical** probability distributions.

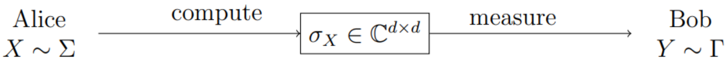


Figure 2: Communication Scheme. Source: [O'Donnell and Wright, 2015]

# Analysis

- Bob sees the mixed state:

$$\left\{\begin{array}{l} \sigma_{x_1} \text{ with prob. } p(x_1), \\ \sigma_{x_2} \text{ with prob. } p(x_2), \\ \vdots \end{array}\right. \equiv \sum_{x\in\Sigma} p(x)\sigma_x =: \rho_B. \tag{5}$$

- Alice sees:

$$\left\{\begin{array}{l} |x_1\rangle \text{ with prob. } p(x_1), \\ |x_2\rangle \text{ with prob. } p(x_2), \\ \vdots \end{array}\right. \equiv \sum_{x\in\Sigma} p(x)|x\rangle\langle x| =: \rho_A. \tag{6}$$

- State of the joint mixed system is:

$$\rho := \sum_{x\in\Sigma} p(x)|x\rangle\langle x| \otimes \sigma_x. \tag{7}$$

# Classical Information Theory

- We wish to answer the question "how much does seeing one random variable tell me about the other". Solution is given by the mutual information.
- The **classical mutual information** $I(X;Y)$ between two random variables $X$ and $Y$ is

$$I(X;Y) = H(X) + H(Y) - H(X,Y), \tag{8}$$

where $H(\cdot)$ is the Shannon entropy. Mutual information represents the amount of information one learn about $X$ from knowing $Y$, and vice-versa since it is symmetric in $X$ and $Y$.

# Quantum Information Theory

- The **accessible information** is defined as:

$$I_{\mathrm{acc}}(\sigma, p) = \max_{\substack{\text{over all} \\ \text{POVMs} \\ \{E_y\}_{y \in \Gamma}}} I(X;Y). \tag{9}$$

  This represents the best Bob can do given Alice's choice of the $\sigma_x$'s and the distribution $p$.

- If $\rho$ is the joint state of two quantum systems $A$ and $B$, then the **quantum mutual information** is:

$$I(\rho_A;\rho_B) = H(\rho_A) + H(\rho_B) - H(\rho). \tag{10}$$

  where $H(\cdot)$ is the von Neumann entropy. Note: If $\rho = \rho_A \otimes \rho_B$, then $I(\rho_A;\rho_B) = 0$ since $H(\rho_A \otimes \rho_B) = H(\rho_A) + H(\rho_B)$.

- The **Holevo information** is defined as:

$$\chi(\sigma, p) := I(\rho_A;\rho_B). \tag{11}$$

# Statement

**Theorem 1 (Holevo's Theorem/Holevo's Bound)**

The accessible information is upper-bounded by the Holevo information:

$$I_{\mathrm{acc}}(\sigma, p) \leq \chi(\sigma, p). \tag{12}$$

Equivalent form of $\chi(\sigma, p)$:

$$\chi(\sigma, p) = \chi(\eta) := S\left(\sum_i p_i \rho_i\right) - \sum_i p_i S(\rho_i) \tag{13}$$

where $p_i = p(x_i), \rho_i = \sigma_{x_i}, \eta = \{(p_i, \rho_i)\}$ and $S(\cdot)$ denotes the von Neumann entropy. $\chi(\eta)$ is called the **Holevo information** or **Holevo $\chi$ quantity**.

We will prove (13).

# Proof of (13)

Write out all the states:

$$\rho_A = \sum_i p_i |i\rangle\langle i|, \quad \rho_B = \sum_i p_i \rho_i, \quad \rho_{AB} = \sum_i p_i |i\rangle\langle i| \otimes \rho_i \tag{14}$$

Compute von Neumann entropy:

$$H(\rho_A) = H(\{p_i\}), \quad H(\rho_B) = S\left(\sum_i p_i \rho_i\right) \tag{15}$$

where $H(\{p_i\})$ is the Shannon entropy of $\{p_i\}$. Moreover we have:

$$H(\rho_{AB}) = H(\{p_i\}) + \sum_i p_i S(\rho_i) \tag{16}$$

Plug $H(\rho_A), H(\rho_B)$ and $H(\rho_{AB})$ into the definition of $I(\rho_A;\rho_B)$ will get the result.

# Proof of (13) (cont'd)

Finally we will show that (16) holds. $\rho_{AB}$ can be written as a block diagonal matrix:

$$\rho_{AB} = \begin{pmatrix} p_1\rho_1 & 0 & 0 & \cdots \\ 0 & p_2\rho_2 & 0 & \cdots \\ 0 & 0 & p_3\rho_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{17}$$

Each $\rho_i$ has its own eigenvalue $r_{ik}$ (where $k$ is the index of the eigenstate of $\rho_i$). The eigenvalues of $\rho_{AB}$ are all $p_i r_{ik}$. So we have:

$$\begin{aligned} H(\rho_{AB}) &= -\sum_{i,k} p_i r_{ik} \log(p_i r_{ik}) && (18) \\ &= -\sum_{i,k} p_i r_{ik} (\log p_i + \log r_{ik}) && (19) \\ &= -\sum_i p_i \log p_i \left( \sum_k r_{ik} \right) - \sum_i p_i \sum_k r_{ik} \log r_{ik} && (20) \\ &= -\sum_i p_i \log p_i + \sum_i p_i S(\rho_i) && (21) \\ &= H(\{p_i\}) + \sum_i p_i S(\rho_i) && (22) \end{aligned}$$

where $\sum_k r_{ik} = 1$ (each density matrix's trace is 1) and $-\sum_k r_{ik} \log r_{ik} = S(\rho_i)$. □

# HSW Theorem

The classical channel capacity $C$ of a quantum channel is given by the HSW theorem:

## Theorem 2 (Holevo-Schumacher-Westmoreland Theorem)

Suppose quantum channel $\mathcal{E}$ has the Kraus representation $\mathcal{E}(\sigma) = \sum_j E_j \sigma E_j^\dagger$. $\rho := \sum_i p_i \rho_i$. Then the classical channel capacity of $\mathcal{E}$ is given by the following regularized expression:

$$C(\mathcal{E}) = \lim_{n\to\infty} \frac{1}{n} \chi^* \left(\mathcal{E}^{\otimes n}\right) \tag{23}$$

where $\chi^*(\mathcal{E})$ is the **maximum Holevo information**:

$$\begin{aligned}\chi^*(\mathcal{E}) &= \max_{\{p_i,\rho_i\}} \left[ S\left(\mathcal{E}(\rho)\right) - \sum_i p_i S(\mathcal{E}(\rho_i)) \right] && (24) \\ &= \max_{\{p_i,\rho_i\}} \left[ S\left( \sum_i p_i \sum_j E_j \rho_i E_j^\dagger \right) - \sum_i p_i S\left( \sum_j E_j \rho_i E_j^\dagger \right) \right] && (25)\end{aligned}$$

$\rho_i$ is the input quantum state, $p_i$ is the probability distribution of the quantum state.

- Unlike the classical case, $C(\mathcal{E})$ is not given by a single-letter formula involving a single copy of the $\mathcal{E}$. Instead it is given by a **regularization of the maximum Holevo information**, which involves many copies of $\mathcal{E}$. In general, it is unclear how to compute such a formula, and a simpler expression for $C(\mathcal{E})$ is not yet known.
- $\mathcal{E}^{\otimes k} : \mathcal{B}(\mathcal{H}_A^{\otimes k}) \to \mathcal{B}(\mathcal{H}_B^{\otimes k})$ means composing $k$ independent copies of $\mathcal{E}$ in parallel. For any input density operator $\rho^{(k)} \in \mathcal{B}(\mathcal{H}_A^{\otimes k})$, the output is:

$$\mathcal{E}^{\otimes k}(\rho^{(k)}) = (\mathcal{E} \otimes \mathcal{E} \otimes \cdots \otimes \mathcal{E})(\rho^{(k)}) \tag{26}$$

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# Motivation for Quantum SWITCH

- In this section, we will focus on the communication of **classical** information.
- Any non-constant quantum channel $\mathcal{N}$ has positive Holevo information:
  - There exist at least two pure states $|\phi\rangle$ and $|\psi\rangle$, such that $\mathcal{N}(|\phi\rangle\langle\phi|) \neq \mathcal{N}(|\psi\rangle\langle\psi|)$.
  - Set $p(\phi) = p(\psi) = \frac{1}{2}$, then the joint state becomes

$$\sigma = \frac{1}{2}\mathcal{N}(|\phi\rangle\langle\phi|) + \frac{1}{2}\mathcal{N}(|\psi\rangle\langle\psi|) \tag{27}$$

  - So the Holevo information is

$$\chi(\mathcal{N}) = S(\sigma) - (\frac{1}{2}S(\mathcal{N}(|\phi\rangle\langle\phi|)) + \frac{1}{2}S(\mathcal{N}(|\psi\rangle\langle\psi|))) > 0 \tag{28}$$

    due to the concavity of $S(\cdot)$.
- The Holevo information of a constant channel, e.g. **completely depolarizing channel**, is zero, which makes it impossible to perform classical communication. However, this can be achieved with quantum SWITCH.

# Kraus Operator for Depolarizing Channel

- A completely depolarizing channel $\mathcal{N}^D$ on a $d$-dimensional quantum system can be represented by $d^2$ orthogonal ($\mathrm{Tr}[U_i^\dagger U_j] = d \cdot \delta_{ij}]$) unitary operators $U_i$, such that its action on a state $\rho$ is

$$\mathcal{N}^D(\rho) = \frac{1}{d^2} \sum_{i=1}^{d^2} U_i \rho U_i^\dagger = \mathrm{Tr}[\rho] \frac{I}{d}. \tag{29}$$

  Example for $d = 2$:
  $U_1 = \sqrt{2}|0\rangle\langle 0|, U_2 = \sqrt{2}|0\rangle\langle 1|, U_3 = \sqrt{2}|1\rangle\langle 0|, U_4 = \sqrt{2}|1\rangle\langle 1|.$

- Thus, according to (4), the overall quantum channel resulting from the quantum `SWITCH` of **two completely depolarizing channels** has the Kraus operator:

$$W_{ij} = \frac{1}{d^2} \left( U_i U_j \otimes |0\rangle\langle 0|_c + U_j U_i \otimes |1\rangle\langle 1|_c \right) \tag{30}$$

# Quantum SWITCH for Depolarizing Channel

- Set control state to $\rho_c := |\psi_c\rangle\langle\psi_c|$, where $|\psi_c\rangle := \sqrt{p}|0\rangle + \sqrt{1-p}|1\rangle$.
- Set the input state to $\rho$, then the receiver will get the output state

$$\mathcal{S}(\mathcal{N}^D, \mathcal{N}^D, \rho_c)(\rho) = \frac{1}{d^4}\sum_{i,j}\Big(p|0\rangle\langle 0|_c \otimes U_i U_j \rho U_j^\dagger U_i^\dagger + (1-p)|1\rangle\langle 1|_c \otimes U_j U_i \rho U_i^\dagger U_j^\dagger \tag{31}$$

$$+ \sqrt{p(1-p)}|0\rangle\langle 1|_c \otimes U_i U_j \rho U_i^\dagger U_j^\dagger + \sqrt{p(1-p)}|1\rangle\langle 0|_c \otimes U_j U_i \rho U_j^\dagger U_i^\dagger\Big)$$

$$= p|0\rangle\langle 0|_c \otimes \frac{I}{d} + (1-p)|1\rangle\langle 1|_c \otimes \frac{I}{d} \tag{32}$$

$$+ \sqrt{p(1-p)}\frac{|0\rangle\langle 1|_c}{d^2} \otimes \sum_j \mathrm{Tr}[U_j\rho]\frac{U_j^\dagger}{d}$$

$$+ \sqrt{p(1-p)}\frac{|1\rangle\langle 0|_c}{d^2} \otimes \sum_j \mathrm{Tr}[\rho U_j^\dagger]\frac{U_j}{d}.$$

$$= (p|0\rangle\langle 0|_c + (1-p)|1\rangle\langle 1|_c) \otimes \frac{I}{d} + \sqrt{p(1-p)}(|0\rangle\langle 1|_c + |1\rangle\langle 0|_c) \otimes \frac{\rho}{d^2}. \tag{33}$$

# Brief Explanation

- (31) results from (3).
- $\sum_j \mathrm{Tr}[U_j \rho] \frac{U_j^\dagger}{d}$ results from viewing $U_j \rho$ as a single state and applying (29).
- (33) follows from the fact that $\{U_j\}$ forms an orthonormal basis for the set of $d \times d$ matrices, i.e., $\rho = \sum_{j=1}^{d^2} \mathrm{Tr}[U_j \rho] \ \frac{U_j^\dagger}{d}$. This is related to the **Hilbert-Schmidt operator and Hilbert-Schmidt space**.
- The quantum SWITCH of two depolarizing channels depends on state $\rho$. Thus, **we can communicate classical information at a nonzero rate**.

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# Extending to > 2 Channels

- When $N = 2$, completely depolarizing channels are unable to transmit quantum data, even when applying quantum SWITCH.
- We consider $N(\geq 3)$ completely depolarizing channels combined in a superposition of $N$ causal orders related to each other by **cyclic permutations**.
- The intermediate nodes in Figure 3 (purple) are identity operations

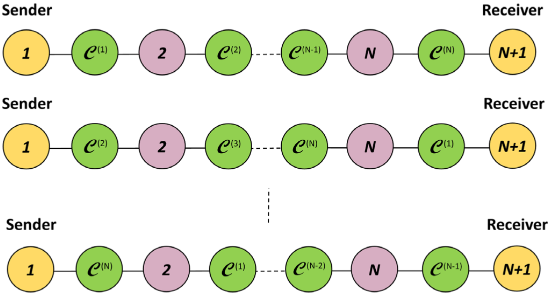


Figure 3: Cyclic Permutations. Source: [Chiribella et al., 2021a]

# Settings

- Take $N = 3$ for simplicity, construct the quantum `SWITCH` $\mathcal{S}(\mathcal{E}, \mathcal{F}, \mathcal{G}, |\omega\rangle)(\rho)$ where:
  - $\mathcal{E}, \mathcal{F}, \mathcal{G}$ are three completely depolarizing channels.
  - $|\omega\rangle$ is the control **qutrit**.
- Set:
  - $\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger$
  - $\mathcal{F}(\rho) = \sum_j F_j \rho F_j^\dagger$
  - $\mathcal{G}(\rho) = \sum_k G_k \rho G_k^\dagger$
- Let the **orthonormal** basis of the control space be:
  - $|\pi_0\rangle$ denotes the order $\mathcal{E} \to \mathcal{F} \to \mathcal{G}$
  - $|\pi_1\rangle$ denotes the order $\mathcal{F} \to \mathcal{G} \to \mathcal{E}$
  - $|\pi_2\rangle$ denotes the order $\mathcal{G} \to \mathcal{E} \to \mathcal{F}$
- Let the control state be the equal superposition $|\omega\rangle = \frac{1}{\sqrt{3}} (|\pi_0\rangle + |\pi_1\rangle + |\pi_2\rangle)$ and let $\mathbf{S}$ denote the set of permutations. The density matrix becomes:

$$\omega = |\omega\rangle\langle\omega| = \frac{1}{3} \sum_{\pi, \pi' \in \mathbf{S}} |\pi\rangle\langle\pi'| \tag{34}$$

# Kraus Operators

- $G_k F_j E_i := K_{ijk}^{(\pi_0)}, E_i G_k F_j := K_{ijk}^{(\pi_1)}, F_j E_i G_k := K_{ijk}^{(\pi_2)}$. Extending from (4) we have:

$$K_{ijk} = K_{ijk}^{(\pi_0)} \otimes |\pi_0\rangle\langle\pi_0| + K_{ijk}^{(\pi_1)} \otimes |\pi_1\rangle\langle\pi_1| + K_{ijk}^{(\pi_2)} \otimes |\pi_2\rangle\langle\pi_2| \tag{35}$$

$$= \sum_{\pi} K_{ijk}^{(\pi)} \otimes |\pi\rangle\langle\pi| \tag{36}$$

- So the channel becomes:

$$\mathcal{S} = \sum_{i,j,k} K_{ijk}(\rho \otimes \frac{1}{3}\sum_{\pi,\pi'} |\pi\rangle\langle\pi'|)K_{ijk}^{\dagger} \tag{37}$$

$$= \sum_{i,j,k}\sum_{\pi} K_{ijk}^{(\pi)} \otimes |\pi\rangle\langle\pi|(\rho \otimes \frac{1}{3}\sum_{\pi,\pi'} |\pi\rangle\langle\pi'|) \sum_{\pi'} K_{ijk}^{(\pi')\dagger} \otimes |\pi'\rangle\langle\pi'| \tag{38}$$

$$= \sum_{i,j,k}\sum_{\pi,\pi'} K_{ijk}^{(\pi)} \rho K_{ijk}^{(\pi')\dagger} \otimes \frac{1}{3}|\pi\rangle\langle\pi'| \tag{39}$$

$$= \sum_{\pi,\pi'} \mathcal{C}_{\pi\pi'}(\rho) \otimes \omega_{\pi,\pi'}\, |\pi\rangle\langle\pi'| := \mathcal{C}_{\text{eff}}(\rho) \tag{40}$$

  where $\mathcal{C}_{\pi\pi'}(\rho) := \sum_{i,j,k} K_{ijk}^{(\pi)} \rho K_{ijk}^{(\pi')\dagger}, \omega_{\pi,\pi'} = \frac{1}{3}$.
- Notation for general $N$ channel setting: $C_{\pi\pi'}(\rho) = \sum_{s_1,\ldots,s_N} K_{(s_1,\ldots,s_N)}^{(\pi)}\, \rho\, K_{(s_1,\ldots,s_N)}^{(\pi')\dagger}$, where $s_i$'s are indexes of Kraus operators corresponding to each channel.

# Channel Construction

- Recall: $\omega = |\omega\rangle\langle\omega|$ where $|\omega\rangle = \frac{1}{\sqrt{N}} \sum_\pi |\pi\rangle$.
- [Chiribella et al., 2021b] showed that:

$$\mathcal{C}_{\pi\pi}(\rho) = \frac{I}{d} \quad \text{and} \quad \mathcal{C}_{\pi\pi'}(\rho) = \frac{\rho}{d^2} \quad \forall\, \pi \neq \pi' \tag{41}$$

- Plug into (40) yields:

$$\mathcal{C}_{\text{eff}}(\rho) = \sum_\pi \frac{I}{d} \otimes \frac{1}{N} |\pi\rangle\langle\pi| + \sum_{\pi\neq\pi'} \frac{\rho}{d^2} \otimes \frac{1}{N} |\pi\rangle\langle\pi'| \tag{42}$$

$$= \frac{I}{d} \otimes \frac{I}{N} + \frac{\rho}{Nd^2} \otimes \sum_{\pi\neq\pi'} |\pi\rangle\langle\pi'| \tag{43}$$

$$= \frac{I}{d} \otimes \frac{I}{N} + \frac{\rho}{Nd^2} \otimes \left(N|\omega\rangle\langle\omega| - I\right), \tag{44}$$

where (44) comes from the relations $N|\omega\rangle\langle\omega| = \sum_{\pi,\pi'} |\pi\rangle\langle\pi'|$ and $I = \sum_\pi |\pi\rangle\langle\pi|$.

# Channel as a Mixture of Channels

- Rearranging (44) we have:

$$\mathcal{C}_{\text{eff}}(\rho) = \mathcal{E}_0(\rho) \otimes (1-p)\rho_0 + \mathcal{E}_1(\rho) \otimes p\rho_1, \tag{45}$$

where $\rho_0 := |\omega\rangle\langle\omega|$ and $\rho_1 := \frac{I-|\omega\rangle\langle\omega|}{N-1}, p := \frac{(N-1)(d^2-1)}{Nd^2}$. $\mathcal{E}_0, \mathcal{E}_0$ are quantum channels defined by

$$\mathcal{E}_0(\rho) := \frac{N-1}{N-1+d^2}\rho + \frac{d^2}{N-1+d^2}\frac{I}{d} \tag{46}$$

and

$$\mathcal{E}_1(\rho) := \frac{d^2}{d^2-1}\frac{I}{d} - \frac{1}{d^2-1}\rho \tag{47}$$

- $\mathcal{C}_{\text{eff}}$ is a mixture of two channels $\mathcal{E}_0$ and $\mathcal{E}_1$. By measuring $\rho_0$ and $\rho_1$, it is possible to determine the occurrence of the channels $\mathcal{E}_0$ and $\mathcal{E}_1$.

# Transmitting Quantum Data

- $\mathcal{E}_1$ is unable to transmit quantum data, see [Chiribella et al., 2021a].
- $\mathcal{E}_0$ is a depolarizing channel, with probability of depolarization equal to $\frac{d^2}{N+d^2-1}$.
- $\lim_{N\to\infty} \frac{d^2}{N+d^2-1} = 0 \Rightarrow \lim_{N\to\infty} \mathcal{E}_0(\rho) = \rho$. Therefore, as long as we wait until the control system is measured to be $\rho_0$, we achieve **nearly perfect quantum information transmission**.

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# Key Quantity

- Next, we discuss quantum `SWITCH` constructed by **general channels and permutations**. Let $n$ denote the total number of channels, $m := |\mathbf{S}|$ denote the number of permutations in $\mathbf{S}$. $n$ may not equal $m$.
- [Wu et al., 2025a] introduced the key quantity

$$\mathcal{P}_n = 1 - \frac{1}{m^2} \min_{\rho} \sum_{\pi, \pi' \in \mathbf{S}} \mathrm{Tr}(C_{\pi\pi'}(\rho)) \tag{48}$$

- $\mathcal{P}_n$ can be viewed as the maximum probability (as $\rho$ varies over all input states) of obtaining the measurement outcome $F_2$ associated with the POVM $\{F_1 = |\omega\rangle\langle\omega| = \frac{1}{m}\sum_{i,j\in\mathbf{S}} |i\rangle\langle j|,\ F_2 = I - |\omega\rangle\langle\omega|\}$. Proof in next page.

# Proof

Probability of obtaining outcome $F_1$ can be computed as:

$$\begin{aligned}
\Pr[F_1] &= \mathrm{Tr}\left[(I \otimes F_1)\, \mathcal{C}_{\text{eff}}(\rho)\right]. && (49)\\
&= \mathrm{Tr}\left[(I \otimes F_1) \sum_{\pi,\pi'} \mathcal{C}_{\pi\pi'}(\rho) \otimes \frac{1}{m} |\pi\rangle\langle\pi'|\right] && (50)\\
&= \frac{1}{m} \sum_{\pi,\pi'} \mathrm{Tr}\left[\mathcal{C}_{\pi\pi'}(\rho) \otimes (F_1 |\pi\rangle\langle\pi'|)\right] && (51)\\
&= \frac{1}{m} \sum_{\pi,\pi'} \mathrm{Tr}(\mathcal{C}_{\pi\pi'}(\rho)) \cdot \mathrm{Tr}(F_1 |\pi\rangle\langle\pi'|) && (52)\\
&= \frac{1}{m^2} \sum_{\pi,\pi'} \mathrm{Tr}(\mathcal{C}_{\pi\pi'}(\rho)) && (53)
\end{aligned}$$

So probability of obtaining outcome $F_2$ is:

$$\Pr[F_2] = 1 - \Pr[F_1] = 1 - \frac{1}{m^2} \sum_{\pi,\pi'} \mathrm{Tr}(\mathcal{C}_{\pi\pi'}(\rho)) \tag{54}$$

So we have:

$$\mathcal{P}_n = \max_{\rho} \Pr[F_2] = 1 - \min_{\rho} \frac{1}{m^2} \sum_{\pi,\pi'} \mathrm{Tr}(\mathcal{C}_{\pi\pi'}(\rho)) \tag{55}$$

# Proposition on S-invariance and Commutativity

## Proposition 1

The quantity $\mathcal{P}_n = 0$ if and only if the Kraus operators $\{C_i\}$ are **S**-invariant, i.e.,

$$K^{(\pi)}_{(s_1,\ldots,s_n)} = K^{(\pi')}_{(s_1,\ldots,s_n)} \tag{56}$$

for all indices $s_1, \ldots, s_n$ and for all $\pi, \pi' \in \mathbf{S}$. That is, the Kraus operators are **independent of the causal order**. In particular, for $n = m = 2$, **S**-invariance is equivalent to commutativity: $E_i F_j = F_j E_i \quad \forall\, i, j$ for two sets of Kraus operators $\{E_i\}, \{F_j\}$.

# Proof of Proposition 1

First we will show a simple fact that $\mathrm{Tr}(C_{\pi\pi}(\rho)) = 1 \quad \forall \pi \in \mathbf{S}$.
Recall:

$$C_{\pi\pi}(\rho) = \sum_{s_1,\ldots,s_n} K^{(\pi)}_{(s_1,\ldots,s_n)} \, \rho \, K^{(\pi)\dagger}_{(s_1,\ldots,s_n)} \tag{57}$$

Here, $K^{(\pi)}_{(s_1,\ldots,s_n)}$ is the composite Kraus operator under the permutation $\pi$:

$$K^{(\pi)}_{(s_1,\ldots,s_n)} := A^{(\pi)}_{s_1,\ldots,s_n} = A_{\pi(1),s_1} A_{\pi(2),s_2} \cdots A_{\pi(n),s_n}, \tag{58}$$

$$K^{(\pi)\dagger}_{(s_1,\ldots,s_n)} := A^{(\pi)\dagger}_{s_1,\ldots,s_n} = A^{\dagger}_{\pi(n),s_n} A^{\dagger}_{\pi(n-1),s_{n-1}} \cdots A^{\dagger}_{\pi(1),s_1} \tag{59}$$

where each $A_{i,s_i}$ is the $s_i$-th Kraus operator of the $i$-th quantum channel.
Since each channel is a valid CPTP map, the composite channel must also be a CPTP map.
So we have:

$$\mathrm{Tr}(C_{\pi\pi}(\rho)) = 1 \quad \forall \pi \in \mathbf{S} \tag{60}$$

Let $\rho$ be an arbitrary input state, and let $\pi, \pi' \in \mathbf{S}$ be arbitrary permutations in $\mathbf{S}$. We then have

$$\begin{aligned}
\mathrm{Tr}(C_{\pi\pi'}(\rho)) &= \sum_{s_1,\ldots,s_n} \mathrm{Tr}\left(K^{(\pi)}_{(s_1,\ldots,s_n)}\, \rho\, K^{(\pi')\dagger}_{(s_1,\ldots,s_n)}\right) && (61)\\
&= \sum_{s_1,\ldots,s_n} \mathrm{Tr}\left(K^{(\pi)}_{(s_1,\ldots,s_n)}\sqrt{\rho}\left(K^{(\pi')}_{(s_1,\ldots,s_n)}\sqrt{\rho}\right)^{\dagger}\right) && (62)\\
&\leq \frac{1}{2}\left[\mathrm{Tr}(C_{\pi\pi}(\rho)) + \mathrm{Tr}(C_{\pi'\pi'}(\rho))\right] = 1 && (63)
\end{aligned}$$

where the final equality follows from (60), and the first inequality follows from combining the Cauchy-Schwarz and AM-GM inequalities, which for arbitrary operators $A$ and $B$ yields

$$\begin{aligned}
\mathrm{Tr}(AB^\dagger) &\leq \sqrt{\mathrm{Tr}(AA^\dagger)\mathrm{Tr}(BB^\dagger)} && (64)\\
&\leq \frac{1}{2}\left(\mathrm{Tr}(AA^\dagger) + \mathrm{Tr}(BB^\dagger)\right). && (65)
\end{aligned}$$

# Explanations on the Cauchy-Schwarz Inequality

To apply Cauchy-Schwarz inequality, we define the **Hilbert-Schmidt inner product** as follows (this is also related to the Hilbert-Schmidt space):

$$\langle A, B \rangle := \mathrm{Tr}(AB^\dagger) \tag{66}$$

The norm induced by this inner product is:

$$\|A\|_{\mathrm{HS}} := \sqrt{\mathrm{Tr}(AA^\dagger)} \tag{67}$$

Applying the Cauchy-Schwarz inequality, we obtain:

$$|\mathrm{Tr}(AB^\dagger)| \leq \sqrt{\mathrm{Tr}(AA^\dagger)} \cdot \sqrt{\mathrm{Tr}(BB^\dagger)} \tag{68}$$

By setting $A_{(s_1,\dots,s_n)} = K^{(\pi)}_{(s_1,\dots,s_n)} \sqrt{\rho}, B_{(s_1,\dots,s_n)} = K^{(\pi')}_{(s_1,\dots,s_n)} \sqrt{\rho}$, and using Cauchy-Schwarz inequality for each pair of $A_{(s_1,\dots,s_n)}, B_{(s_1,\dots,s_n)}$, we get the inequality last page.

# Conclusion and Remarks

Since the equality holds if and only if $A_{(s_1,\ldots,s_n)} = B_{(s_1,\ldots,s_n)} \quad \forall s_1,\ldots,s_n$, it follows that

$$\operatorname{Tr}(C_{\pi\pi'}(\rho)) = 1, \tag{69}$$

if and only if

$$K^{(\pi)}_{(s_1,\ldots,s_n)}\sqrt{\rho} = K^{(\pi')}_{(s_1,\ldots,s_n)}\sqrt{\rho} \quad \forall s_1,\ldots,s_n. \tag{70}$$

From the definition of $\mathcal{P}_n$, it follows that $\mathcal{P}_n = 0$ if and only if (70) holds for all states $\rho$, for all permutations $\pi, \pi' \in \mathbf{S}$, and for all indices $s_1,\ldots,s_n$. By considering pure states, we see that $\mathcal{P}_n = 0$ if and only if

$$K^{(\pi)}_{(s_1,\ldots,s_n)} = K^{(\pi')}_{(s_1,\ldots,s_n)} \tag{71}$$

for all $s_1,\ldots,s_n$, as desired. In particular, when $n = m = 2$, the permutation set $\mathbf{S}$ only contains two permutations $(1,2)$ and $(2,1)$, thus S-invariance implies

$$C^1_{s_1}C^2_{s_2} = C^2_{s_2}C^1_{s_1} \quad \text{for all } s_1, s_2, \tag{72}$$

i.e., the Kraus operators of the channels $\mathcal{C}^1$ and $\mathcal{C}^2$ pairwise commute.

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# Notations

- Denote the quantum `SWITCH` channel $\mathcal{C}_{\text{eff}}(\rho)$ as $\mathcal{S}^n(\rho)$, where $n$ is the number of channels.
- In the following, we will consider the case $\mathcal{C}^i = \mathcal{N}$ for $i = 1, \ldots, n$, where $\mathcal{N}$ is a fixed quantum channel, and let $\mathcal{N}^n$ denote the $n$-fold composition $\mathcal{N} \circ \cdots \circ \mathcal{N}$.

# Tracing Out Control Recovers $\mathcal{N}^n$

- $\mathcal{N}^n$ can be obtained from $\mathcal{S}^n$ by tracing out the control system:

$$\mathcal{N}^n(\rho) = \mathcal{N} \circ \cdots \circ \mathcal{N}(\rho) = \sum_{s_1,\ldots,s_n} C_{s_n} \cdots C_{s_1} \rho C_{s_1}^\dagger \cdots C_{s_n}^\dagger \tag{73}$$

Take partial trace:

$$\begin{aligned}
\mathrm{Tr}_C\left[\mathcal{S}^n(\rho)\right] &= \sum_{\pi,\pi' \in \mathbf{S}} C_{\pi\pi'}(\rho) \cdot \mathrm{Tr}\left(\omega_{\pi,\pi'} |\pi\rangle\langle\pi'|\right) &(74)\\
&= \sum_{\pi,\pi' \in \mathbf{S}} C_{\pi\pi'}(\rho) \cdot \frac{1}{m} \delta_{\pi\pi'} &(75)\\
&= \frac{1}{m} \sum_{\pi \in \mathbf{S}} C_{\pi\pi}(\rho) &(76)
\end{aligned}$$

Clearly $C_{\pi\pi}(\rho) = \mathcal{N}^n(\rho)$ (since $C^i = \mathcal{N} \quad \forall i$), thus:

$$\mathrm{Tr}_C\left[\mathcal{S}^n(\rho)\right] = \mathcal{N}^n(\rho). \tag{77}$$

# Quantifying Communication Enhancement

- For all capacity measure $f$ satisfying the data-processing inequality $f(\mathcal{A} \circ \mathcal{B}) \leq f(\mathcal{B})$, for arbitrary quantum channels $\mathcal{A}$, $\mathcal{B}$, we obtain the **bottleneck inequality**:

$$f(\mathcal{N}^n) \leq f(\mathcal{S}^n). \tag{78}$$

  since taking partial trace is also a CPTP map.

- Capacity measures such as classical capacity, quantum capacity, Holevo information and coherent information all satisfy (78). Thus, we define the associated **causal gain** by

$$\delta_f = f(\mathcal{S}^n) - f(\mathcal{N}^n). \tag{79}$$

  $\delta_f$ is a direct measure of the communication enhancement of the channel $\mathcal{N}$ which is achieved by inputting $n$-copies of the channel into the quantum `SWITCH`.

# Sufficient Condition for $\delta_f = 0$

- $\mathcal{P}_n = 0$ if and only if $K^{(\pi)}_{(s_1,\dots,s_n)} = K^{(\pi')}_{(s_1,\dots,s_n)}$ for all $s_1, \dots, s_n$. Hence, for a given subset of permutations $\mathbf{S}$, $\mathcal{P}_n = 0$ implies that $C_{\pi\pi'}(\rho) = C_{\pi\pi}(\rho) = C_{\pi'\pi'}(\rho) = C(\rho)$ for arbitrary permutations $\pi, \pi' \in \mathbf{S}$.
- Therefore $\mathcal{S}^n(\rho) = \mathcal{N}^n(\rho) \otimes \omega$, where $\omega$ is the control state and is independent of the input state $\rho$. Proof:

$$\mathcal{S}^n(\rho) = \sum_{\pi,\pi' \in \mathbf{S}} C_{\pi\pi'}(\rho) \otimes \omega_{\pi,\pi'} |\pi\rangle\langle\pi'| \tag{80}$$

$$= \sum_{\pi,\pi' \in \mathbf{S}} \mathcal{N}^n(\rho) \otimes \omega_{\pi,\pi'} |\pi\rangle\langle\pi'| \tag{81}$$

$$= \mathcal{N}^n(\rho) \otimes \left( \sum_{\pi,\pi' \in \mathbf{S}} \omega_{\pi,\pi'} |\pi\rangle\langle\pi'| \right) \tag{82}$$

$$= \mathcal{N}^n(\rho) \otimes \omega \tag{83}$$

- Since $\omega$ is a constant, we have $f(\mathcal{S}^n) = f(\mathcal{N}^n)$, so that $\delta_f = 0$.

# Main Conjecture

We have shown that $\mathcal{P}_n > 0$ is a necessary condition for positive causal gain, so it is natural to propose the following conjecture:

**Conjecture**

For all channels outside **a set of measure-zero**, the condition $\mathcal{P}_n > 0$ is necessary and sufficient for $\delta_f > 0$.

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# Pauli Channel

The **Pauli channel** describes a probabilistic mixture of Pauli operations applied to a single qubit (density matrix) $\rho$:

$$\mathcal{N}(\rho) = \sum_{i=0}^{3} p_i \sigma_i \rho \sigma_i^\dagger, \quad \text{with } \sum_i p_i = 1 \tag{84}$$

where:

- $\sigma_0 = I$, $\sigma_1 = X$, $\sigma_2 = Y$, $\sigma_3 = Z$ are the four Pauli operators,
- $p_i \geq 0$ is the probability of applying operation $\sigma_i$.
- Completely depolarizing channel is a special case of Pauli channel: $\mathcal{N}(\rho) = \frac{1}{4}\left(I\rho I + X\rho X + Y\rho Y + Z\rho Z\right) = \frac{I}{2}$.

$$\sigma_0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_2 = Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

# Setting

- In this section, we verify the conjecture in the space of all Pauli channels $\mathcal{N}(\rho) = \sum_{i=0}^{3} p_i \sigma_i \rho \sigma_i^\dagger$, and the permutation set $\mathbf{S}$ contains the identity permutation $(1, \ldots, n)$ and its reversal $(n, \ldots, 1)$.
- In such case, the quantum `SWITCH` places $n$ copies of a Pauli channel $\mathcal{N}$ in a superposition of the forward and backward orders $\mathcal{N}_n \circ \cdots \circ \mathcal{N}_1$ and $\mathcal{N}_1 \circ \cdots \circ \mathcal{N}_n$ (where $\mathcal{N}_i = \mathcal{N}$ for all $i$).
- The forward and backward orders are indistinguishable when used individually. We show that when such orders are placed in a superposition via quantum `SWITCH`, an enhancement of classical capacity and coherent information occurs almost surely

# Operational Meaning of $\mathcal{P}_n$

- Fix the control state to be $\omega = |+\rangle\langle+|$ with $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, then apply the projective measurement $\{F_1 = |\omega\rangle\langle\omega|, F_2 = I - |\omega\rangle\langle\omega|\}$.
- The probability of obtaining $F_2$ is independent of the initial state $\rho$ (proof on next few pages), i.e., for all states $\rho$,

$$\mathrm{Tr}((I \otimes F_2)\mathcal{S}^n(\rho)) = \mathcal{P}_n. \tag{85}$$

# Proof

Recall, probability of obtaining outcome $F_2$ is:

$$\Pr[F_2] = 1 - \frac{1}{m^2} \sum_{\pi,\pi'} \mathrm{Tr}(\mathcal{C}_{\pi\pi'}(\rho)) \tag{86}$$

For Pauli channel $\mathcal{N}(\rho) = \sum_{i=0}^{3} p_i \sigma_i \rho \sigma_i^\dagger$, its Kraus representation is:

$$\mathcal{N}(\rho) = \sum_i E_i \rho E_i^\dagger, \quad E_i = \sqrt{p_i}\sigma_i \tag{87}$$

For a quantum `SWITCH` of $n$ channels, the composite Kraus operator is given by:

$$K^{(\pi)}_{(s_1,\ldots,s_n)} = \sqrt{p_{s_1} \cdots p_{s_n}} \cdot \sigma_{s_{\pi(1)}} \cdots \sigma_{s_{\pi(n)}}, \tag{88}$$

where $\pi$ is the permutation (either forward or reverse).
Take trace:

$$\mathrm{Tr}(C_{\pi\pi'}(\rho)) = \sum_{s_1,\ldots,s_n} \mathrm{Tr}\left(\sigma_{s_{\pi(1)}} \cdots \sigma_{s_{\pi(n)}} \rho \sigma^\dagger_{s_{\pi'(n)}} \cdots \sigma^\dagger_{s_{\pi'(1)}}\right) \cdot p_{s_1} \cdots p_{s_n} \tag{89}$$

(90)

# Proof (cont'd)

If $\pi = \pi'$:

$$\begin{aligned}
\mathrm{Tr}(C_{\pi\pi}(\rho)) &= \sum_{s_1,\ldots,s_n} \mathrm{Tr}\left(\sigma_{s_{\pi(1)}} \cdots \sigma_{s_{\pi(n)}} \rho \sigma^\dagger_{s_{\pi(n)}} \cdots \sigma^\dagger_{s_{\pi(1)}}\right) \cdot p_{s_1} \cdots p_{s_n} && (91)\\
&= \sum_{s_1,\ldots,s_n} \mathrm{Tr}\left(\rho \sigma^\dagger_{s_{\pi(n)}} \cdots \sigma^\dagger_{s_{\pi(1)}} \sigma_{s_{\pi(1)}} \cdots \sigma_{s_{\pi(n)}}\right) \cdot p_{s_1} \cdots p_{s_n} && (92)\\
&= \sum_{s_1,\ldots,s_n} \mathrm{Tr}\left(\rho\right) \cdot p_{s_1} \cdots p_{s_n} && (93)\\
&= \sum_{s_1,\ldots,s_n} p_{s_1} \cdots p_{s_n} && (94)\\
&= \left(\sum_{s_1} p_{s_1}\right) \cdots \left(\sum_{s_n} p_{s_n}\right) = 1 \cdots 1 = 1. && (95)
\end{aligned}$$

# Proof (cont'd)

If $\pi \neq \pi'$, take $n = 2$ as example, $\pi = (1,2), \pi' = (2,1)$:

$$\operatorname{Tr}(C_{\pi\pi'}(\rho)) = \sum_{s_1,s_2} \operatorname{Tr}(\sigma_{s_1}\sigma_{s_2}\rho\sigma_{s_1}^\dagger\sigma_{s_2}^\dagger)\cdot p_{s_1}p_{s_2} \tag{96}$$

$$= \sum_{s_1,s_2} \operatorname{Tr}(\sigma_{s_1}\sigma_{s_2}\rho\sigma_{s_1}\sigma_{s_2})\cdot p_{s_1}p_{s_2} \tag{97}$$

$$= \sum_{s_1,s_2} \operatorname{Tr}(\rho\sigma_{s_1}\sigma_{s_2}\sigma_{s_1}\sigma_{s_2})\cdot p_{s_1}p_{s_2} \tag{98}$$

Case 1, $s_1 = s_2$:

$$\sigma_{s_1}\sigma_{s_1}\sigma_{s_1}\sigma_{s_1} = I \tag{99}$$

Case 2, $s_1 \neq s_2$:

- If $i \neq j$ and $i, j \neq 0$, use the relationship $\sigma_i\sigma_j = -\sigma_j\sigma_i$:

$$\sigma_{s_1}\sigma_{s_2}\sigma_{s_1}\sigma_{s_2} = -\sigma_{s_1}^2\sigma_{s_2}^2 = -I \tag{100}$$

- If $s_1 = 0$ or $s_2 = 0$ (i.e., $\sigma_0 = I$), the product is $I$.

In all cases, the outcome is independent of $\rho$.

# Structural Characterization of $\mathcal{P}_n = 0$

[Wu et al., 2025b] proved the following proposition that characterizes the condition for $\mathcal{P}_n = 0$:

**Proposition 2**

The quantity $\mathcal{P}_n$ is zero if and only if:

1. $n$ is even and the Kraus operators of $\mathcal{N}$ are commutative.
2. $n$ is odd and the number of Kraus operators of $\mathcal{N}$ is not more than 2.

# Degradable Channels

- For general $n$, $\mathcal{P}_n = 0$ only if the **Choi rank** of $\mathcal{N} \leq 2$, and such Pauli channels are shown to be either **degradable or antidegradable**.
- A quantum channel has a Stinespring representation:

$$\mathcal{N}(\rho) = \mathrm{Tr}_E\, U_{\mathcal{N}}(\rho) \tag{101}$$

  where $E$ is the environment, $\mathcal{N} : \mathcal{H}_A \to \mathcal{H}_B, U_{\mathcal{N}} : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$.
- The complementary channel $\mathcal{N}^c : \mathcal{H}_A \to \mathcal{H}_E$ is defined by

$$\mathcal{N}^c(\rho) = \mathrm{Tr}_B\, U_{\mathcal{N}}(\rho). \tag{102}$$

- A channel $\mathcal{N}$ is degradable when there exists a CPTP map $\mathcal{T} : \mathcal{H}_B \to \mathcal{H}_E$ such that:

$$\mathcal{N}^c = \mathcal{T} \circ \mathcal{N}. \tag{103}$$

- Anti-degradable channel: a channel whose complement is degradable, i.e. there exists a CPTP map $\mathcal{S} : \mathcal{H}_E \to \mathcal{H}_B$ such that $\mathcal{N} = \mathcal{S} \circ \mathcal{N}^c$.

# $\mathcal{P}_n = 0$ Lies on a Measure-Zero Set

- The set of Pauli channels is parametrized by the three-dimensional simplex of probability vectors in $\mathbb{R}^4$, as shown in Figure 4
- [Wu et al., 2025b] also showed that the subset satisfying $\mathcal{P}_n = 0$ lies in the edges of this simplex, which is a **measure-zero set**.
- Thus, $\mathcal{P}_n > 0$ almost surely.

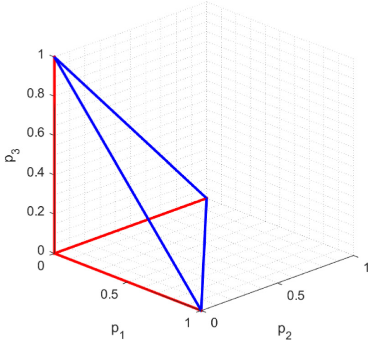


Figure 4: Simplex Representing Pauli Channels. Source: [Wu et al., 2025a]

# Explicit Decomposition of $\mathcal{S}^n$ via $\mathcal{P}_n$

- Now, we establish the connection between $\mathcal{P}_n$ and $\delta_f$ when $f$ is either the classical capacity $C$ or the coherent information $I_c$
- [Wu et al., 2025b] proved the following expression for $\mathcal{S}^n$:

$$\mathcal{S}^n(\rho) = (1 - \mathcal{P}_n)\Phi_+(\rho) \otimes \omega + \mathcal{P}_n \Phi_-(\rho) \otimes (I - \omega), \tag{104}$$

where $\omega = |+\rangle\langle+|$ and $\Phi_\pm$ are two Pauli channels.

- Take partial trace we have:

$$\mathrm{Tr}_C[\mathcal{S}^n(\rho)] = (1 - \mathcal{P}_n)\Phi_+(\rho) \cdot \mathrm{Tr}(\omega) + \mathcal{P}_n \Phi_-(\rho) \cdot \mathrm{Tr}(1 - \omega) \tag{105}$$

$$= (1 - \mathcal{P}_n)\Phi_+(\rho) + \mathcal{P}_n \Phi_-(\rho) = \mathcal{N}^n(\rho) \tag{106}$$

where we used the fact that $\mathrm{Tr}(\omega) = \mathrm{Tr}(1 - \omega) = 1$. Thus,

$$\mathcal{N}^n = (1 - \mathcal{P}_n)\Phi_+ + \mathcal{P}_n \Phi_- \tag{107}$$

# Classical Capacity of $\mathcal{S}^n$

[Wu et al., 2025b] proved that the classical capacity $C$ of $\mathcal{S}^n$ takes a similar form:

**Theorem 3**

Let $\Phi_\pm$ be as in (104). Then

$$C(\mathcal{S}^n) = (1 - \mathcal{P}_n)C(\Phi_+) + \mathcal{P}_n C(\Phi_-). \tag{108}$$

# Deriving Classical Causal Gain for Qubit Unital Channels

- We now use Theorem 3 to derive an explicit expression for the **classical causal gain** $\delta_C$.
- Qubit unital channel are channels that satisfy the following conditions:
  - **Action on Qubit**: The input and output quantum states are density matrices in a two-dimensional Hilbert space (i.e., $2 \times 2$ matrices).
  - **Unital Property**: The channel maps the identity matrix $I$ to itself, i.e.,

    $$\mathcal{M}(I) = I.$$

    For a single qubit, $I$ is the $2 \times 2$ identity matrix.
  - **Examples**: Pauli channels, completely depolarizing channel, unit channel.

# Classical Capacity of Qubit Unital Channels

- Classical capacity of qubit unital Channels can be computed via the formula

$$C(\mathcal{M}) = \chi(\mathcal{M}) = 1 - H^{\min}(\mathcal{M}) \tag{109}$$

where $\chi(\mathcal{M})$ is the Holevo information, $H^{\min}(M) = \min_\rho H(\mathcal{M}(\rho))$ is the minimum output von Neumann entropy of the channel. More general result can be found in [Müller-Hermes, 2021b].

- For qubit unital Channels, $\mathcal{M}$'s eigenvalues $\alpha_i$ (satisfying $\mathcal{M}(A) = \alpha_i A$) determine the minimum output entropy:

$$H^{\min}(\mathcal{M}) = h(\alpha), \quad \alpha = \max_i |\alpha_i|, \tag{110}$$

where $h(x)$ is the binary entropy function $H_b\left(\frac{1+x}{2}\right) = -\frac{1+x}{2}\log\frac{1+x}{2} - \frac{1-x}{2}\log\frac{1-x}{2}$, with domain $[-1, 1]$.

# Range of Eigenvalues of Pauli Channels

- Pauli channel $\mathcal{N}(\rho) = \sum_{i=0}^{3} p_i \sigma_i \rho \sigma_i^\dagger$ is Pauli diagonal:

$$\mathcal{N}(\sigma_j) = \sum_{i=0}^{3} p_i \sigma_i \sigma_j \sigma_i = \lambda_j \sigma_j, \quad \text{for } j = 0, 1, 2, 3 \tag{111}$$

  where $\lambda_j$ is the $j$th eigenvalue.

- Apply the following relationship:

$$\sigma_i \sigma_j \sigma_i = \begin{cases} \sigma_j, & \text{if } \sigma_i \text{ commutes with } \sigma_j, \\ -\sigma_j, & \text{if } \sigma_i \text{ anti-commutes with } \sigma_j. \end{cases} \tag{112}$$

- Define $\chi_{ij} = \pm 1$ to represent $\sigma_i \sigma_j \sigma_i = \chi_{ij} \sigma_j$, we have:

$$\mathcal{N}(\sigma_j) = \left( \sum_{i=0}^{3} p_i \chi_{ij} \right) \sigma_j \tag{113}$$

# Range of Eigenvalues of Pauli Channels (cont'd)

- Finally we have the eigenvalues:

$$\lambda_j = \sum_{i=0}^{3} p_i \cdot \chi_{ij}, \quad \text{where } \chi_{ij} = \begin{cases} 1 & \text{if } [\sigma_i, \sigma_j] = 0 \\ -1 & \text{if } \{\sigma_i, \sigma_j\} = 0 \end{cases} \tag{114}$$

- For $\sigma_0 = I : \mathcal{N}(I) = I \Rightarrow \lambda_0 = 1$.
- For $\sigma_1 = X$:

$$\sigma_i X \sigma_i = \begin{cases} X & i = 0, 1 \\ -X & i = 2, 3 \end{cases} \Rightarrow \lambda_1 = p_0 + p_1 - p_2 - p_3 \tag{115}$$

- For $\sigma_2 = Y$:

$$\sigma_i Y \sigma_i = \begin{cases} Y & i = 0, 2 \\ -Y & i = 1, 3 \end{cases} \Rightarrow \lambda_2 = p_0 + p_2 - p_1 - p_3 \tag{116}$$

- For $\sigma_3 = Z$:

$$\sigma_i Z \sigma_i = \begin{cases} Z & i = 0, 3 \\ -Z & i = 1, 2 \end{cases} \Rightarrow \lambda_3 = p_0 + p_3 - p_1 - p_2 \tag{117}$$

# Range of Eigenvalues of Pauli Channels (cont'd)

- Take $\lambda_1$ for example. Since $\sum_{i=0}^{3} p_i = 1$, we have $\lambda_1 = 2(p_0 + p_1) - 1$. Thus:

$$-1 \leq \lambda_1 \leq 1 \tag{118}$$

where $\lambda_1 = 1$ iff $p_0 + p_1 = 1$, $\lambda_1 = -1$ iff $p_2 + p_3 = 1$.

- Similarly, we have $-1 \leq \lambda_2 \leq 1, -1 \leq \lambda_3 \leq 1$. As a result:

$$\lambda_j \in [-1, 1] \quad \forall j \tag{119}$$

- Set $\gamma$, $\mu$ and $\nu$ to be the maximum of absolute values of eigenvalues of $\mathcal{N}^n$, $\Phi_+$ and $\Phi_-$ respectively.
- Assume the eigenvalues of $\Phi_+$ are $\{\alpha_i\}$ and those of $\Phi_-$ are $\{\beta_i\}$. Then the eigenvalues of $\mathcal{N}^n$ are given by:

$$\lambda_i = (1 - \mathcal{P}_n)\alpha_i + \mathcal{P}_n\beta_i. \tag{120}$$

- $\gamma$ satisfies:

$$\gamma = \max_i |\lambda_i| \leq (1 - \mathcal{P}_n) \max_i |\alpha_i| + \mathcal{P}_n \max_i |\beta_i| \leq 1. \tag{121}$$

# Explicit Formula for Classical Causal Gain

- The classical capacity of the effective channel is

$$\begin{aligned} C(\mathcal{S}^n) &= (1-\mathcal{P}_n)C(\Phi_+) + \mathcal{P}_n C(\Phi_-) &(122)\\ &= (1-\mathcal{P}_n)(1-h(\mu)) + \mathcal{P}_n(1-h(\nu))) &(123)\\ &= 1-(1-\mathcal{P}_n)h(\mu) - \mathcal{P}_n h(\nu) &(124)\end{aligned}$$

- The classical causal gain is then

$$\begin{aligned} \delta_C &= C(\mathcal{S}^n) - C(\mathcal{N}^n) &(125)\\ &= C(\mathcal{S}^n) - (1-h(\gamma))) &(126)\\ &= h(\gamma) - (1-\mathcal{P}_n)h(\mu) - \mathcal{P}_n h(\nu) &(127)\end{aligned}$$

- Note: The composition channel $\mathcal{N} = \mathcal{N} \circ \cdots \circ \mathcal{N}$ is still a Pauli channel, due to the (anti-)commutativity of Pauli operators.

# Sufficiency Proof of the Conjecture

- We will prove the sufficiency part of the conjecture ($\mathcal{P}_n > 0 \Rightarrow \delta_C > 0$) holds for Pauli channels except the completely depolarizing channel with forward and backward orders.
- Combining (121), and the concavity and monotonicity of $h(x)$, we have:

$$h(\gamma) \geq h\left((1-\mathcal{P}_n)\mu + \mathcal{P}_n\nu\right) \geq (1-\mathcal{P}_n)h(\mu) + \mathcal{P}_n h(\nu). \tag{128}$$

  Therefore,

$$\delta_C = h(\gamma) - (1-\mathcal{P}_n)h(\mu) - \mathcal{P}_n h(\nu) \geq 0. \tag{129}$$

- The equality holds iff $\mu = \nu$. E.g., for completely depolarizing channel, $p_0 = \cdots = p_3 = \frac{1}{4}, \lambda_1 = \cdots = \lambda_3 = 0$. Thus, $\mu = \nu = 0$.

# Important Note

- When computing $\alpha = \max_i |\alpha_i|$ for $\Phi_+$ and $\Phi_-$, we have ruled out the trivial eigenvalue $\lambda_0 = 1$.
- $I$ is an **indistinguishable** input state: the completely mixed state carries no information.
- The true contributors to information transmission are the non-trivial actions of operators like $X, Y, Z$.
- Therefore, the eigenvalues actually used for $\alpha = \max_i |\alpha_i|$ are those corresponding to $\sigma_1, \sigma_2, \sigma_3$.

# Coherent Information

- Next, we will discuss the **coherent information causal gain** $\delta_I$.
- For a channel $\mathcal{N}$, the coherent information is defined as:

$$I_c(\mathcal{N}) = \max_{\rho} I_c(\rho, \mathcal{N}) = \max_{\rho} \left[ S(\mathcal{N}(\rho)) - S(\mathcal{N}^c(\rho)) \right], \tag{130}$$

  where $\rho$ is the input state, $S(\cdot)$ is the von Neumann entropy and $\mathcal{N}^c$ is the complementary channel of $\mathcal{N}$.
- $S(\mathcal{N}(\rho))$: The entropy of the output state, quantifying the information obtained by the receiver.
- $S(\mathcal{N}^c(\rho))$: The entropy of the environment state, quantifying the amount of quantum information leaked to the environment.
- Quantum information transmission is enabled when $I_c > 0$.

# Quantum Causal Gain in Coherent Information

- For a Pauli channel $\mathcal{N}$ , the coherent information attains a maximum on the completely mixed state $\frac{I}{2}$ , which is called the **hashing bound**:

$$I_c\left(\mathcal{N}\right) = I_c\left(\frac{I}{2}, \mathcal{N}\right) = 1 - H(\vec{p}) \tag{131}$$

where $\vec{p} = (p_0, p_1, p_2, p_3)$ is the probability vector associated with the Pauli channel $N$, and $H(\cdot)$ is the Shannon entropy.

- Since $\Phi_{\pm}$ are both Pauli channels, it follows from (104) that the coherent information of $\mathcal{S}^n$ also attains a maximum on $\frac{I}{2}$:

$$I_c(\mathcal{S}^n) = 1 - (1 - \mathcal{P}_n)H(\vec{s}) - \mathcal{P}_n H(\vec{t}) \tag{132}$$

- So we have the causal gain:

$$\delta_I = H(\vec{q}) - (1 - \mathcal{P}_n)H(\vec{s}) - \mathcal{P}_n H(\vec{t}) \tag{133}$$

- Similar to the case of $\delta_c$, it is then straightforward to deduce that $\mathcal{P}_n > 0$ is a sufficient condition for $\delta_I > 0$.

# Necessary and Sufficient Condition for Causal Gain

Based on previous results, [Wu et al., 2025b] proved the following theorem:

**Theorem 4**

Let $f$ be the classical capacity or coherent information, and let $\delta_f$ be the causal gain associated with forward and backward orders. Then for all Pauli channels (except the completely depolarizing channel when $n$ is odd), the condition $\mathcal{P}_n > 0 \iff \delta_f > 0$ holds.

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# Qudit Depolarizing Channels

A qudit depolarizing channel $\mathcal{D}_p^d$ is given by:

$$\mathcal{D}_p^d(\rho) = (1-p)\rho + p\,\mathrm{Tr}(\rho)\frac{I}{d} \tag{134}$$

where $p \in (0,1)$ and $d > 1$.
[Wu et al., 2025b] proved the following result:

**Theorem 5**

Let $\delta_C$ be the classical causal gain with respect to forward and backward orders. Then for all qudit depolarizing channels (except the completely depolarizing channel), the condition $\mathcal{P}_n > 0 \iff \delta_C > 0$ holds.

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Thank you!
Any questions?