This supplementary material contains the simplification of the Dome test under standardization conditions. We refer the interested readers to the paper - Xiang et.al. 2016: screening tests for lasso problems - for detailed description of the Dome test.

The lasso model is defined as the following optimization problem

$$\widehat{\boldsymbol{\beta}}(\lambda) = \underset{\boldsymbol{\beta}\in\mathbb{R}^p}{\operatorname{argmin}} \frac{1}{2n}\|\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\|^2 + \lambda\|\boldsymbol{\beta}\|_1, \tag{1}$$

where $\mathbf{y}$ is the $n \times 1$ response vector, $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_p)$ is the $n \times p$ feature matrix, $\boldsymbol{\beta} \in \mathbb{R}^p$ is the coefficient vector, and $\lambda \geq 0$ is a regularization parameter. $\|\cdot\|$ and $\|\cdot\|_1$ respectively denote the Euclidean (ℓ_2) norm and ℓ_1 norm.

The dual formulation of the lasso problem (1):

$$\widehat{\boldsymbol{\theta}}(\lambda) = \underset{\boldsymbol{\theta}\in\mathbb{R}^n}{\operatorname{argmax}} \frac{1}{2n}\|\mathbf{y}\|^2 - \frac{n\lambda^2}{2}\|\boldsymbol{\theta} - \frac{\mathbf{y}}{n\lambda}\|^2 \tag{2}$$

$$\text{subject to } |\mathbf{x}_j^T\boldsymbol{\theta}| \leq 1, \quad \forall j = 1, \cdots, p, \tag{3}$$

where $\widehat{\boldsymbol{\theta}}(\lambda)$ is the dual optimal solution of Problem (1) under the constraints (3).

First, since $\frac{\mathbf{y}}{n\lambda_{max}}$ (where $\lambda_{max} = |\frac{1}{n}\mathbf{x}_*^T\mathbf{y}|$) is a feasible solution to the dual problem of the lasso, the *default spherical bound* (based on Eq. 22 of the paper Xiang et. al. 2016) is, $\|\boldsymbol{\theta} - \frac{\mathbf{y}}{n\lambda}\| \leq \frac{\|\mathbf{y}\|}{n}(\frac{1}{\lambda} - \frac{1}{\lambda_{max}})$. We then have the following dome region (for non-negative lasso problem) based on notations in Section 4.3,

$$D(\mathbf{q}, r; \mathbf{n}, c) = \left\{\boldsymbol{\theta} : \|\boldsymbol{\theta} - \frac{\mathbf{y}}{n\lambda}\| \leq \frac{\|\mathbf{y}\|}{n}(\frac{1}{\lambda} - \frac{1}{\lambda_{max}}), \frac{1}{\sqrt{n}}\mathbf{x}_*^T\boldsymbol{\theta} \leq \frac{1}{\sqrt{n}}\right\}$$

with center $\mathbf{q} = \frac{\mathbf{y}}{n\lambda}$ and radius $r = \frac{\|\mathbf{y}\|}{n}(\frac{1}{\lambda} - \frac{1}{\lambda_{max}})$. $\mathbf{n} = \frac{1}{\sqrt{n}}\mathbf{x}_*$ is the unit norm to the half-space $\{\mathbf{x}_*^T\boldsymbol{\theta} \leq 1\}$, and $c = \frac{1}{\sqrt{n}}$.

Eqs. (24) - (26) in Section 4.3 become the following,

$$\psi_d = (\mathbf{n}^T\mathbf{q} - c)/r = \frac{1}{\sqrt{n}}(\frac{\lambda_{max}}{\lambda} - 1)/r$$

$$\mathbf{q}_d = \mathbf{q} - \psi_d r\mathbf{n} = \frac{\mathbf{y}}{n\lambda} - \frac{1}{n}(\frac{\lambda_{max}}{\lambda} - 1)\mathbf{x}_*$$

$$r_d = r\sqrt{1 - \psi_d^2} = \frac{\lambda_{max} - \lambda}{n\lambda}\sqrt{\frac{\|\mathbf{y}\|^2}{\lambda_{max}^2} - n}$$

According to **Theorem 3**, the screening test for a nondegenerate dome $D(\mathbf{q}, r; \mathbf{n}, c)$ at jth feature $\mathbf{x}_j$:

$$T_{D(\mathbf{q},r;\mathbf{n},c)}(\mathbf{x}_j) = \begin{cases} 1; \text{ if } V_l(\frac{1}{\sqrt{n}}\mathbf{x}_*^T\mathbf{x}_j, \|\mathbf{x}_j\|) < \frac{\mathbf{y}^T}{n\lambda}\mathbf{x}_j < V_u(\frac{1}{\sqrt{n}}\mathbf{x}_*^T\mathbf{x}_j, \|\mathbf{x}_j\|); \\ 0; \text{ otherwise;} \end{cases} \tag{4}$$

The Dome test rejects $\mathbf{x}_j$ if $T_{D(\mathbf{q},r;\mathbf{n},c)}(\mathbf{x}_j) = 1$.

For the lasso problem, $V_u(t_1, t_2) = 1 - M_1(t_1, t_2)$ and $V_l(t_1, t_2) = -V_u(-t_1, t_2) = -1 + M_1(-t_1, t_2)$. $M_1(t_1, t_2)$ is given by the following definition with $|\psi_d| \leq 1$.

$$M_1(t_1, t_2) = \begin{cases} rt_2, & \text{if } t_1 < -\psi_d t_2; \\ -\psi_d r t_1 + r\sqrt{t_2^2 - t_1^2}\sqrt{1 - \psi_d^2}, & \text{if } t_1 \geq -\psi_d t_2. \end{cases}$$

Let $t_1 = \frac{1}{\sqrt{n}}\mathbf{x}_*^T\mathbf{x}_j, t_2 = \|\mathbf{x}_j\| = \sqrt{n}$ (due to standardization). With some algebra, $t_1 < -\psi_d t_2$ implies $\mathbf{x}_*^T\mathbf{x}_j < -\frac{n\lambda_{max}}{\|\mathbf{y}\|}$. $M_1(t_1, t_2), V_l(t_1, t_2), V_u(t_1, t_2)$ can be rewritten as,

$$M_1(\frac{1}{\sqrt{n}}\mathbf{x}_*^T\mathbf{x}_j, \|\mathbf{x}_j\|) = \begin{cases} \frac{\|\mathbf{y}\|}{\sqrt{n}}(\frac{1}{\lambda} - \frac{1}{\lambda_{max}}), & \text{if } \mathbf{x}_*^T\mathbf{x}_j < -\frac{n\lambda_{max}}{\|\mathbf{y}\|}; \\ -\frac{\lambda_{max}-\lambda}{n\lambda}\mathbf{x}_*^T\mathbf{x}_j + \frac{\lambda_{max}-\lambda}{n\lambda}\sqrt{n - \frac{1}{n}(\mathbf{x}_*^T\mathbf{x}_j)^2}\sqrt{\frac{\|\mathbf{y}\|^2}{\lambda_{max}^2} - n}, & \text{if } \mathbf{x}_*^T\mathbf{x}_j \geq -\frac{n\lambda_{max}}{\|\mathbf{y}\|}; \end{cases}$$

$$V_u(\frac{1}{\sqrt{n}}\mathbf{x}_*^T\mathbf{x}_j, \|\mathbf{x}_j\|) = \begin{cases} 1 - \frac{\|\mathbf{y}\|}{\sqrt{n}}(\frac{1}{\lambda} - \frac{1}{\lambda_{max}}), & \text{if } \mathbf{x}_*^T\mathbf{x}_j < -\frac{n\lambda_{max}}{\|\mathbf{y}\|}; \\ 1 + \frac{\lambda_{max}-\lambda}{n\lambda}\mathbf{x}_*^T\mathbf{x}_j - \frac{\lambda_{max}-\lambda}{n\lambda}\sqrt{n - \frac{1}{n}(\mathbf{x}_*^T\mathbf{x}_j)^2}\sqrt{\frac{\|\mathbf{y}\|^2}{\lambda_{max}^2} - n}, & \text{if } \mathbf{x}_*^T\mathbf{x}_j \geq -\frac{n\lambda_{max}}{\|\mathbf{y}\|}; \end{cases}$$

$$V_l(\frac{1}{\sqrt{n}}\mathbf{x}_*^T\mathbf{x}_j, \|\mathbf{x}_j\|) = \begin{cases} -1 + \frac{\|\mathbf{y}\|}{\sqrt{n}}(\frac{1}{\lambda} - \frac{1}{\lambda_{max}}), & \text{if } \mathbf{x}_*^T\mathbf{x}_j > \frac{n\lambda_{max}}{\|\mathbf{y}\|}; \\ -1 + \frac{\lambda_{max}-\lambda}{n\lambda}\mathbf{x}_*^T\mathbf{x}_j + \frac{\lambda_{max}-\lambda}{n\lambda}\sqrt{n - \frac{1}{n}(\mathbf{x}_*^T\mathbf{x}_j)^2}\sqrt{\frac{\|\mathbf{y}\|^2}{\lambda_{max}^2} - n}, & \text{if } \mathbf{x}_*^T\mathbf{x}_j \leq \frac{n\lambda_{max}}{\|\mathbf{y}\|}; \end{cases}$$

Plugging these terms into (4) with some algebra yields to the Dome test in three cases:

- If $\mathbf{x}_*^T\mathbf{x}_j < -\frac{n\lambda_{max}}{\|\mathbf{y}\|}$, the Dome test (4) rejects $\mathbf{x}_j$ if,

$$-n\lambda + (\lambda_{max} - \lambda)\mathbf{x}_*^T\mathbf{x}_j + (\lambda_{max} - \lambda)\sqrt{n - \frac{1}{n}(\mathbf{x}_*^T\mathbf{x}_j)^2}\sqrt{\frac{\|\mathbf{y}\|^2}{\lambda_{max}^2} - n} < \mathbf{y}^T\mathbf{x}_j < n\lambda - \sqrt{n}\|\mathbf{y}\|\frac{\lambda_{max} - \lambda}{\lambda_{max}}$$

- If $\mathbf{x}_*^T\mathbf{x}_j > \frac{n\lambda_{max}}{\|\mathbf{y}\|}$, the Dome test (4) rejects $\mathbf{x}_j$ if,

$$-n\lambda + \sqrt{n}\|\mathbf{y}\|\frac{\lambda_{max}-\lambda}{\lambda_{max}} < \mathbf{y}^T\mathbf{x}_j < n\lambda + (\lambda_{max}-\lambda)\mathbf{x}_*^T\mathbf{x}_j - (\lambda_{max}-\lambda)\sqrt{n-\frac{1}{n}(\mathbf{x}_*^T\mathbf{x}_j)^2}\sqrt{\frac{\|\mathbf{y}\|^2}{\lambda_{max}^2}-n}$$

- If $-\frac{n\lambda_{max}}{\|\mathbf{y}\|} \leq \mathbf{x}_*^T\mathbf{x}_j \leq \frac{n\lambda_{max}}{\|\mathbf{y}\|}$, the Dome test (4) rejects $\mathbf{x}_j$ if,

$$\begin{aligned}
&-n\lambda + (\lambda_{max}-\lambda)\mathbf{x}_*^T\mathbf{x}_j + (\lambda_{max}-\lambda)\sqrt{n-\frac{1}{n}(\mathbf{x}_*^T\mathbf{x}_j)^2}\sqrt{\frac{\|\mathbf{y}\|^2}{\lambda_{max}^2}-n} \\
&< \mathbf{y}^T\mathbf{x}_j \\
&< n\lambda + (\lambda_{max}-\lambda)\mathbf{x}_*^T\mathbf{x}_j - (\lambda_{max}-\lambda)\sqrt{n-\frac{1}{n}(\mathbf{x}_*^T\mathbf{x}_j)^2}\sqrt{\frac{\|\mathbf{y}\|^2}{\lambda_{max}^2}-n}
\end{aligned}$$