Entry, Exit, and firm Dynamics in Long Run Equilibrium

Author(s): Hugo A. Hopenhayn

ENTRY, EXIT, AND FIRM DYNAMICS IN LONG RUN EQUILIBRIUM[1]

By Hugo A. Hopenhayn[2]

This paper develops and analyzes a dynamic stochastic model for a competitive industry which endogenously determines processes for entry and exit and for individual firms' output and employment. The concept of stationary equilibrium is introduced, extending long run industry equilibrium theory to account for entry, exit, and heterogeneity in the size and growth rate of firms. Conditions under which there will be entry and exit in the stationary equilibrium are given. Cross-sectional properties—across size and age cohorts—are analyzed and compared to the data. Implications for the equilibrium distributions of profits and the value of firms are analyzed. The effect of changes in the parameters describing the technological and market conditions of the industry on the equilibrium size distribution and turnover rates are also analyzed.

KEYWORDS: Entry and exit; firm dynamics; selection; survival analysis.

1. INTRODUCTION

This paper studies an equilibrium model of entry and exit and firm dynamics. Recent empirical work and research have emphasized the predominant role that firm specific sources of uncertainty have in accounting for firm size dynamics. As a consequence of this idiosyncratic uncertainty substantial amounts of resources are reallocated across firms, from contracting and exiting ones to new and expanding ones. The main objective of the paper is to contribute to our understanding of some economic determinants of this process and to build up a tractable structure for its analysis. This is achieved by developing the steady state analysis of the industry equilibrium model.

The quantitative significance of this reallocation is reflected in the high turnover rates of jobs and firms. Approximately a third of the stock of jobs and over forty percent of the firms in manufacturing disappear over five year periods and are replaced by new ones.[3] In connection to this process of job/firm creation and destruction, two important facts arise from the recent empirical studies: (i) firm specific uncertainty dominates firm level dynamics and (ii) entry and exit rates are highly correlated across industries and most of their variation is accounted for by these industry effects. That is, there are high and low

---

[1] The original name was "A Dynamic Stochastic Model of Entry and Exit to an Industry."

[3] For evidence on job creation and destruction for U.S. manufactures, see Davis and Haltiwanger (1991), Dunne, Roberts, and Samuelson (1989b). Leonard (1987) studies the relationship between growth and size for Wisconsin firms. Evans (1987) and Hall (1987) provide evidence on growth properties of firms also by age. For evidence on entry and exit, see Dunne, Roberts, and Samuelson (1989a, 1989b).

turnover industries. Our modeling strategy is influenced by these two observations.

In our model, firms are faced with individual productivity shocks and this is the only source of uncertainty. On the basis of their shocks, they decide optimally when to exit the industry. As firms exit the industry, new ones come in. Entry requires an investment that is nonrecoverable and becomes sunk thereafter. In the steady state the entry and exit rates are equal and so is job creation and destruction. The steady state also implies stationary distributions for firm size, profits, and value. Though these distributions and all aggregates remain constant through time, this is by the offsetting effect of firm level entry, exit, growth, and contraction. For given aggregate industry demand and input supply functions, the characteristics of the process for firm shocks, the cost of entry, and production technology determine the stationary equilibrium distributions and the entry and exit rate. The comparative statics analysis of the stationary equilibrium is developed in the paper.

The idea that the dynamics of firm size can be explained by stochastic models of firm evolution with purely idiosyncratic shocks has been in the literature for a long time. Hart and Prais (1956), Simon and Bonini (1958) and Adelman (1958) had remarkable success in fitting statistical models to the data by specifying particular processes for firm size. Competitive equilibrium theory of industry evolution was first developed by Lucas and Prescott (1971). The equilibrium in their model implies a stochastic process for price, aggregate output, and investment but no firm level heterogeneity and no entry and exit. Lucas (1978) analyzed size distribution of firms in a more specialized model. His model, however, does not focus explicitly on entry and exit. Dynamic models of entry and exit were first developed by Brock (1972) and Vernon Smith (1974). In these models firms have identical size and in the limit there is no entry and exit.

The equilibrium models described above have no firm specific stochastic elements that can give rise to the observed firm dynamics. Jovanovic (1982) introduced the first model of this kind. In his model firms are subject to productivity shocks drawn from a distribution with unknown mean but known variance. The mean is specific to a firm and realizations are independent across firms. The equilibrium leads to selection through exit and entry. His model is equivalent to one where productivity shocks follow a particular (nonstationary) process. Pakes and Ericson (1990) discuss the implications of more general versions of this learning model and compare them to those of a model of their own, which is based on the idea that firms' production is affected by investments with uncertain outcomes (see also Ericson and Pakes (1989)).

These two models are rich in their implications for firm level dynamics, which has been the focus of their analysis. But from the aggregate point of view, they are complex dynamical systems, so their general analysis or even numerical computation turns out to be a complicated task. The main objective of this paper is to provide a simpler framework to address some questions relating to the process of job and firm reallocation. The concept of a *stationary equilibrium*

developed here, which corresponds to the steady state analysis of a dynamical system, provides this more tractable structure. As this concept extends to other models of firm dynamics, our research is complementary to existing work.[4]

Steady state analysis has been used in economics to study the long run properties of dynamic models. We use it here to understand how changes in the structural characteristics of an industry (as given by parameters of the model) affect turnover, growth of firms, and the distributions of size, profits, and value of firms. In particular we analyze the effects of changes in the costs of entry, fixed costs, demand, and some characteristics of the process for firms' shocks. A stationary equilibrium need not have positive entry and exit. As an example, Jovanovic's model is one which exhibits no entry and exit in the limit. In the paper we discuss conditions under which positive entry and exit will occur.

The stationary equilibrium implies a size distribution of firms by age cohorts. Empirical evidence indicates that this size distribution is stochastically increasing in the age of firm cohorts. In our model this distribution is derived from the exogenous process for firms' shocks, production decisions, and the selection that results from the endogenous exit decision. We provide general conditions on the exogenous stochastic process under which such empirical regularity holds.

The paper is organized as follows. Section 2 describes the model. Section 3 defines an *industry equilibrium* and proves existence and uniqueness. Section 4 develops similar results for a stationary equilibrium. Section 5 provides the analysis of the model. It is divided in three parts. The first deals with life cycle properties of firms: size distribution and age. The second one develops the comparative statics. The third one provides some results concerning profits and the value of firms. Section 6 provides the final remarks.

2. THE MODEL

The industry is composed by a continuum of firms which produce a homogeneous product. Firms behave competitively, taking prices in the output and input markets as given. Aggregate demand is given by the inverse demand function $D(Q)$ and the input price by $W(N)$, where N is total industry demand for the input. We consider here the case of a single input—e.g. labor—but all results extend to the case of multiple inputs under standard homotheticity assumptions. We make the following assumption:

ASSUMPTION A.1: (a) *D is continuous, strictly decreasing, and* $\lim_{x \to \infty} D(x) = 0$. (b) *$W$ is continuous, nondecreasing, and strictly bounded above zero.*

---

[4] Lippman and Rummelt (1982) also develop steady state analysis and the comparative statics in a model with firm specific shocks. However, in their model the firm specific shock does not change through time, so all uncertainty is resolved after entry. Hence, no reallocation takes place in the steady state. Still, this model provides some interesting insights about the relationship between size distribution and the (ex-post) excess returns of firms.

The output of an individual firm is $q = f(\varphi, n)$, where $\varphi \in S \equiv [0,1]$ is a productivity shock which follows a Markov process independent across firms with conditional distribution $F(\varphi'|\varphi)$. In addition, a fixed cost c_f must be paid every period by incumbent firms. Note that a positive fixed cost is necessary for exit to take place and is equivalent to the existence of a fixed outside opportunity cost for some resources (e.g. managerial ability) used by the firm. For given output and input prices p and w, let $\pi(\varphi, p, w)$, $q(\varphi, p, w)$, and $n(\varphi, p, w)$ denote, respectively, the profit, output supply, and input demand functions. Note that $\pi(\varphi, 0, w) = -c_f$. Firms discount profits with a constant factor $0 < \beta < 1$. We assume the following:

ASSUMPTION A.2: (a) q and n are single valued, strictly increasing in φ, and continuous; (b) π is continuous and strictly increasing in φ; (c) $\lim_{Q \to 0} \pi(0, D(Q), w) > 0$.

ASSUMPTION A.3: (a) F is continuous in φ and φ'; (b) F is strictly decreasing in φ.

ASSUMPTION A.4: (Recurrence): For any $\varepsilon > 0$ there exists an integer n such that $F^n(\varepsilon|\varphi) > 0$, where $F^n(\cdot|\varphi)$ gives the distribution of φ_{t+n} given $\varphi_t = \varphi$.

Assumption A.3.b says the higher is the productivity shock in period t, the more likely are higher shocks in period $t + 1$. Together with A.2 this implies that expected discounted profits are an increasing function of a firm's current shock φ. In equilibrium firms exit the industry whenever their state falls below a reservation level x. Given this cutoff point, Assumption A.4 implies that the life span of a firm is almost surely finite, preventing the mass of firms from escaping to a *no exit* region. Also note that as a consequence of A.2, for given $p \geqslant 0$ and $w > 0$ profits are uniformly bounded.

Each period, before the new shocks are realized, incumbent firms may exit the industry and potential entrants may enter. A firm that exits secures a present value which we normalize to zero. To enter, a firm must pay an entry cost $c_e \geqslant 0$. After paying this cost its productivity shock, which is drawn from an initial distribution ν, is revealed. We make the following assumption:

ASSUMPTION A.5: ν has a continuous distribution function G.

After observing their shocks, firms make their output decisions and prices are determined competitively to equate aggregate demand and supply in the respective markets. These equilibrium prices will obviously depend on the number and shocks of firms producing in the industry that period. This information is summarized by a measure μ_t over firms' shocks; the total mass $M_t = \mu_t(S)$ is a

measure of the total size of the industry, while for any borel set $A \subset S$, $\mu_t(A)$ is the mass of firms with shocks in A. We call μ_t the *state of the industry* in period t. Aggregating output supply and input demand functions of all firms we obtain

$$(1) \qquad Q^s(\mu, p, w) = \int q(\varphi, p, w) \mu(d\varphi), \quad \text{and}$$

$$(2) \qquad N^d(\mu, p, w) = \int n(\varphi, p, w) \mu(d\varphi).$$

The only source of uncertainty in the model are the firm specific productivity shocks. Since there are a large number of firms, and since the conditional distribution function F and the probability measure ν over initial states are the same for all firms, the frequency distribution for the idiosyncratic shocks each period coincides with the probability distribution dictated by the initial distribution, the conditional distribution function, and the entry and exit rules.[5] Since there are no industry-wide shocks, this implies that aggregate output, employment, prices, and the frequency distribution for φ follow deterministic paths. That is, a competitive equilibrium, given the initial measure μ_0 over types of firms has a deterministic sequence $\{(p_t, w_t, \mu_t)\}$. Therefore, in the firm's decision problem analyzed in the following section, the sequence $\{(p_t, w_t)\}$ is deterministic; only its own productivity parameter φ_t is stochastic.

Letting μ_0 be the initial distribution of firm's shocks, the industry is thus defined by the following list of elements: $\{D(\cdot), W(\cdot), c_f, c_e, f, S, F, \nu, \mu_0\}$.

3. EQUILIBRIUM

Potential entrants and incumbent firms maximize expected discounted profits with perfect foresight on future prices. In consequence, exit decisions are a function of the sequence of future prices. Given price sequences $z = \{p_t, w_t\}$ the problem of an incumbent firm is defined recursively by

$$(3) \qquad v_t(\varphi, z) = \max \left\{ \pi(\varphi, p_t, w_t) \right\} + \beta \max \left\{ 0, \int v_{t+1}(\varphi', z) F(d\varphi' | \varphi) \right\}.$$

Hence v_t gives the value of a firm of type φ at period t after the realization of its new shock. Also note that the exit decision is made prior to observing next

---

[5] Technical problems exist when, as in our case, there is a continuum of random variables. Judd (1985), Feldman and Gilles (1985), Uhlig (1987), and Green (1989) discuss ways in which a law of large numbers can be justified. Since independence does not play any role in our model we can assume that the realizations for 'closely' located firms are correlated in the way shown by Feldman and Gilles. With this assumption the distribution of realizations across firms will coincide with the distribution of the process.

period's shock and will involve a reservation rule:

$$(4) \qquad x_t = \inf\left\{\varphi \in S : \int v_{t+1}(\varphi', z)F(d\varphi'|\varphi) \geqslant 0\right\} \quad \text{or}$$

$$x_t = 1 \quad \text{if this set is empty.}$$

A firm will exit the industry the first time its shock gets below this reservation value, i.e. the first time $\varphi_t < x_t$.

PROPOSITION 1 (Properties of v_t): *Let z be a bounded sequence of prices and wages such that $w_t \geqslant \underline{w} > 0$. Then: (i) the functions v_t are continuous in φ and for $p_t > 0$ strictly increasing in φ; (ii) if $0 < x_t < 1$, then $\int v_{t+1}(\varphi', z)F(d\varphi'|x_t) = 0$; (iii) v_t is uniformly bounded.*

PROOF: (i) Continuity follows from standard dynamic programming arguments using continuity and boundedness of π and continuity of F. Strict monotonicity in φ for $p_t > 0$ follows from A.2. (ii) follows from (i) and the continuity of F. (iii) follows from discounting and the uniform boundedness of prices.   Q.E.D.

New firms will enter the market until expected discounted profits net of the entry cost is zero. For a potential entrant, expected discounted profits are given by

$$(5) \qquad v_t^e(z) = \int v_t(\varphi, z)\nu(d\varphi).$$

Let M_t denote the mass of entrants in period t. Free entry implies that in equilibrium $v_t^e(z) \leqslant c_e$, with equality if $M_t > 0$.

The entry and exit rules imply an evolution for the state of the industry μ_t which satisfies for each $\varphi' \in [0, 1]$

$$(6) \qquad \mu_{t+1}([0, \varphi')) = \int_{\varphi \geqslant x_t} F(\varphi'|\varphi)\mu_t(d\varphi) + M_{t+1}G(\varphi')$$

where G is the distribution function corresponding to ν. An alternative and convenient expression for equation (6) can be obtained as follows. For all borel sets A in S define

$$\hat{P}_t(\varphi, A) = \begin{cases} \int_A F(ds|\varphi) & \text{if } \varphi \geqslant x_t \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\hat{P}_t$ defines a bounded linear operator on the space of positive bounded measures defined by $\hat{P}_t\mu_t(A) = \int \hat{P}_t(\varphi, A)\mu_t(d\varphi)$ for all borel sets A in S, with operator norm $\|P_t\| \leqslant 1$. Using this we rewrite (6) as follows:

$$(7) \qquad \mu_{t+1} = \hat{P}_t\mu_t + M_{t+1}\nu.$$

The last term in this equation gives the contribution of new entrants to μ_{t+1}. Note that this embodies the assumption that the distribution from which entrants draw their initial shock is also the distribution of realized values across new entrants.

A *competitive equilibrium* for the industry consists of bounded sequences $\{p_t^*\}, \{w_t^*\}, \{Q_t^*\}, \{N_t^*\}, \{M_t^*\}, \{x_t^*\}, \{\mu_t^*\}$ such that:

(i) (a) $p_t^* = D(Q_t^*)$ and $w_t^* = W(N_t^*)$;
   (b) $Q_t^* = Q^s(\mu_t^*, p_t^*, w_t^*)$ and $N_t^* = N^d(\mu_t^*, p_t^*, w_t^*)$.

(ii) x_t^* satisfies (4).

(iii) $v_t^e(z^*) \leqslant c_e$ with equality if $M_t^* > 0$, where $z^* = \{p_t^*, w_t^*\}$.

(iv) μ_t^* is defined recursively by (7) given μ_0, M_t^*, and x_t^*.

Condition (i) says that prices are market clearing. Condition (ii) says that the exit rule is chosen optimally. Condition (iii) says that there are no further incentives to enter the industry, and (iv) that the sequence of distributions are consistent with the law of motion generated by the equilibrium exit rules and mass of entrants.

Equilibria in this model maximize net discounted surplus. This is a familiar result in models of industry equilibria. Lucas and Prescott (1971) first established this in a model where firms' technology exhibits constant returns to scale, all shocks are aggregate and there is no entry and exit. The same is proved by Jovanovic (1982), where all shocks are firm specific and follow a more specific but nonstationary process. Both of these results are extended in Hopenhayn (1990) to a general setup that also covers the case considered here. Theorems 1 and 2 in Hopenhayn (1990) can be used to establish existence and uniqueness of a competitive equilibrium starting from an arbitrary measure μ_0. Since the focus of this paper is on the stationary equilibria, for which a specific existence proof is provided in Section 4, we give here only a brief outline of the existence argument for the general case. For more details see Hopenhayn (1990).

Starting from μ_0 the *feasible set* $\Gamma(\mu_0)$ is defined as the set of all sequences $\{N_t, Q_t\}$ such that there exist sequences $\{\mu_t, M_t, x_t^*\}$, where μ_t is derived from μ_0 and the entry and exit rules as given by equation (7), and feasible input output plans for the firms in each period that aggregate to N_t and Q_t. In Hopenhayn (1990) it is established that the feasible set is closed and convex.[6]

Starting from an initial distribution μ, the sequential problem solved by the equilibrium allocations is given by

$$(8) \qquad V(\mu) = \max \sum_{t=0}^{\infty} \beta^t \big[R(Q_t) - C(N_t) - c_e M_t - c_f \mu_t(S) \big]$$

$$\text{subject to} \qquad \{N_t, Q_t\} \in \Gamma(\mu)$$

where $R(Q) = \int_0^Q D(x)\, dx$ and $C(N) = \int_0^N W(x)\, dx$.

---

[6] The convexity is obtained allowing firms with the *same shocks* to choose *different actions*. Though this is ruled out here, it is without loss of generality since by Assumption A.2 the profit maximizing output choice is unique and Assumptions A.3 and A.5 imply that the set of firms with $\varphi = x_t^*$ has measure zero.

Without loss of generality function D may be assumed bounded. It is easy to show that the period return is uniformly bounded above on the feasible set. In consequence (N_t, Q_t) can be restricted to a compact subset Y of $\mathbb{R}^2$ and the objective in (8) is bounded and continuous—in the product topology—on Y^∞. Since Y^∞ is compact, it follows that the maximum in (8) exists. Given the correspondence between competitive equilibria and solutions to (8) proved in Hopenhayn (1990), this also implies existence of a competitive equilibrium. Furthermore, as shown in the following theorem, (8) has a unique solution, so the competitive equilibrium is unique.

THEOREM 1: *Given any initial distribution there exists a unique equilibrium.*

PROOF: Existence follows from the above argument. To establish uniqueness note that if (Q_0, N_0) and (Q_1, N_1) are two input output vectors such that $D(Q_0) \neq D(Q_1)$ or $W(N_0) \neq W(N_1)$ and $\lambda \in (0, 1)$, then

$$R(Q^\lambda) - C(N^\lambda) > \lambda [R(Q_0) - C(N_0)] + (1 - \lambda)[R(Q_1) - C(N_1)]$$

where Q^λ and N^λ are the obvious convex combinations. This, together with the convexity of the feasible set, implies that if two distinct equilibrium allocations exist they must have the same prices. An immediate implication is that exit rules will coincide for all equilibria. Since D is strictly decreasing, aggregate output will also be the same for all equilibria. Now it is easy to see that starting from the initial distribution, the mass of entrants for the first period will coincide for all equilibria and, recursively, so will it be in the following periods.   Q.E.D.

REMARK: The sequential problem in (8) can be written as a dynamic programming problem which by the previous theorem has a unique solution. The state of this programming problem is μ_t and its solution implies a dynamical system on the space of bounded positive measures given by $\mu_{t+1} = H(\mu_t)$, where H is a nonlinear map.

The rest of the paper is concerned with a *stationary equilibrium*, which is a vector $(p^*, w^*, Q^*, N^*, x^*, M^*, \mu^*)$ such that for $p_t = p^*$, $w_t = w^*$, $Q_t = Q^*$, $N_t = N^*$, $x_t = x^*$, $M_t = M^*$, $\mu_t = \mu^*$, $\{p_t, w_t, Q_t, N_t, M_t, x_t, \mu_t\}$ is an equilibrium from $\mu_0 = \mu^*$.

The stationary equilibrium can also be defined by singly imposing the stationarity requirement $\mu_t = \mu^*$ since, as established in Section 4, given μ^* there exist unique equilibrium prices p^* and w^* that satisfy condition (i) of the definition of equilibrium. It follows that the stationary equilibrium corresponds to the steady state of the dynamical system defined in the Remark to Theorem 1.

4. STATIONARY EQUILIBRIA: EXISTENCE, ENTRY, AND EXIT

This section addresses the existence and uniqueness of a stationary equilibrium with entry and exit. The arguments developed in this section are also used

in the comparative statics analysis of the following section. The method of proof is algorithmic, providing guidelines for the computation of equilibria.

Given any distribution of firms μ with $\mu(S) > 0$ there exist a unique aggregate input-output vector (N, Q) and prices (p, w) that satisfy (i) in the definition of equilibrium (Lemma 3). Let $p^e(\mu)$ and $w^e(\mu)$ denote these equilibrium prices. That is, p^e and w^e are defined by

$$p^e(\mu) = D[Q(\mu, p^e(\mu), w^e(\mu))], \quad \text{and}$$

$$w^e(\mu) = W[N(\mu, p^e(\mu), w^e(\mu))].$$

Define $\tilde{\pi}(\varphi, \mu)$ as the current profits for a firm of type φ in a market where prices are $p^e(\mu)$ and $w^e(\mu)$. Similarly, define $v(\varphi, \mu)$ as its present discounted value. This function is the unique solution to

$$(9) \qquad v(\varphi, \mu) = \tilde{\pi}(\varphi, \mu) + \max\left\{0, \beta \int v(\varphi', \mu) F(d\varphi'|\varphi)\right\}.$$

All properties of v are derived from corresponding properties of $\tilde{\pi}$ and F. The key properties of $\tilde{\pi}$ are given in the following Proposition, proved in the Appendix.

PROPOSITION 2: *The function $\tilde{\pi}$ is jointly continuous, strictly increasing in φ, and decreasing in μ.*

The next Proposition gives the key properties of v used in the paper.

PROPOSITION 3 (Properties of v): *There exists a unique continuous solution v to (9) and (i) it is strictly increasing in φ and decreasing in μ; (ii) the integral on (9) is strictly increasing in φ.*

PROOF: Existence and continuity of v follow immediately from Proposition 2, A.3(a), and weak* convergence, applying standard dynamic programming arguments. (i) follows from the properties of $\tilde{\pi}$ and Assumption A.3(b). (ii) is a consequence of (i) and A.3(b).   *Q.E.D.*

We now express the equilibrium conditions for exit and entry using the above. A consequence of Proposition 3 is that for given μ the exit point x satisfying equation (4) for prices $p(\mu)$ and $w(\mu)$ is unique. Furthermore, if $0 < x < 1$ then

$$(10) \qquad \int v(\varphi', \mu) F(d\varphi'|x) = 0.$$

The following condition is necessary for an equilibrium with invariant measure μ to exhibit positive entry:

$$(11) \qquad \int v(\varphi, \mu) \nu(d\varphi) = c_e.$$

Now suppose $m(x, M)$ is an invariant measure for exit rule x and entry mass M. Letting $\hat{P}_x$ be the transition operator for exit rule x as defined in Section 3 and $\mu = m(x, M)$,

$$(12) \qquad \mu = \hat{P}_x \mu + M\nu.$$

A stationary equilibrium with positive entry is then given by (x, M, μ) that satisfy equations (10)–(12).

The steps of the existence proof are then as follows. Prove that $m(x, M)$ is well defined, jointly continuous, decreasing in x, and increasing in M (Lemmas 4 and 5). For a fixed exit rule $x \in (0, 1]$ define $M_1(x)$ by $\int v[\varphi, m(x, M_1(x))] F(d\varphi | x) = 0$, i.e. $M_1(x)$ is the entry rule with the property that for the invariant measure $m(x, M_1(x))$, the exit rule x is optimal. M_1 is well-defined, continuous, and strictly increasing (Lemma 6). These properties are derived from those of function v given in Proposition 3.

Define $M_2(x)$ for $x \in (0, 1]$ by

$$\int v(\varphi, m(x, M_2(x))) \nu(d\varphi) = c_e,$$

i.e. if the exit rule is x then $M_2(x)$ is the mass of entrants that are needed so that expected discounted profits for entrants are equal to the cost of entry. M_2 is well defined, continuous, and nondecreasing (Lemma 7).

An equilibrium with positive entry exists if and only if there is an $x^* \in (0, 1]$ such that $M_1(x^*) = M_2(x^*)$.

Figure 1 depicts the graph of functions M_1 and M_2. Since $v(\varphi, m(1, M))$ has a maximum at $\varphi = 1$ for all $M > 0$, it follows that $M_1(1) > M_2(1)$. This leaves two possibilities: (a) there exists some point such that $M_1 < M_2$, and thus an equilibrium exists—as in Figure 1; (b) M_1 is always above M_2.

Though no equilibrium with $M > 0$ exists in case (b), there does exist an equilibrium with $M = 0$. For this case the equilibrium price vector has to be such that (i) there are no incentives to enter the industry, and (ii) firms with state $\varphi = 0$ prefer staying in the industry rather than exiting. This leaves room
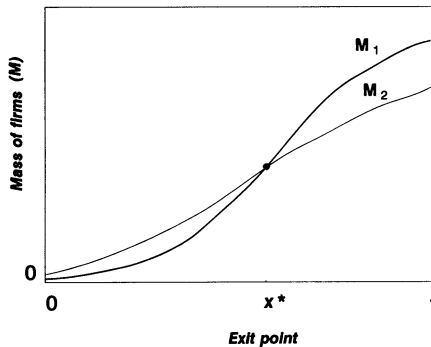

FIGURE 1.—Existence of stationary equilibrium.

for the following indeterminacy: Denoting by μ the unique invariant probability distribution for the Markov process given by F, the stationary equilibria are given by $\lambda\mu$ for some $\lambda \in [\underline{\lambda}, \overline{\lambda}]$. If $\underline{\lambda} = 0$, then $\mu^* = 0$ and thus an empty industry is an equilibrium. This occurs when costs of entry are prohibitively high. For $\lambda > 0$, it must be the case that $x = 0$, so the firms that are already in the industry remain there forever.

From the above analysis we conclude the following:

THEOREM 2: *There exists a stationary competitive equilibrium for the industry.*

Under what conditions will the equilibria have positive entry and exit? Intuitively, the cost of entry acts as an entry barrier. So existence of this type of equilibria depends on the height of this barrier relative to demand. This is summarized in the following result:

THEOREM 3: *Given the technological and demand assumptions there is a real number $c^* > 0$ such that a stationary equilibrium with positive entry exists if and only if $c_e < c^*$.*

The comparative statics results presented in the next section are justified when the stationary equilibrium is unique. This requires curves M_1 and M_2 to intersect only once. To derive conditions that rule out multiple crossings consider hypothetically the case where there are two intersections at points x_1 and x_2, where $x_2 > x_1$. Let μ_1 and μ_2 be the corresponding measures. Then $v(x_1, \mu_2) < v(x_2, \mu_2) = 0$, and $v(x_1, \mu_1) = 0$. Thus, there must exist some φ such that $\pi(\varphi, \mu_2) < \pi(\varphi, \mu_1)$. But since $\int v(\varphi, \mu_1)\nu(d\varphi) = \int v(\varphi, \mu_2)\nu(d\varphi)$, while profits at μ_2 are lower for some level of φ, they cannot be lower for all of them. The following conditions imply that profits for all productivity states move in the same direction, excluding this possible behavior.

CONDITION U.1: *The industry is price taker in the input markets, i.e. it operates on a region where from the point of view of the industry there is unlimited supply of inputs at price $\underline{w}$.*

CONDITION U.2: *The profit function is separable in the following form $\pi(\varphi, p, w) = h(\varphi)g(p, w)$, for some functions h and g.*

THEOREM 4: *If either U.1 or U.2 are satisfied and if there exists an equilibrium with entry and exit, the equilibrium is unique.*

Note that one standard case considered in the literature, namely that where firm's technology is given by a cost function of the form $c(\varphi, q)$, satisfies U.1. Production functions of the form $f(\varphi, n) = \varphi g(n)$, where g is homogeneous of degree $k \in (0, 1)$ generate profit functions with $\pi(\varphi_2) = (\varphi_2/\varphi_1)^{1/(1-k)}\pi(\varphi_1)$ for

all φ_1 and φ_2 in S, which obviously satisfy Condition U.2. Production functions that are homogeneous of degree one in the vector of inputs and shocks will also satisfy this assumption.[7]

An important step in the existence argument is proving that for any exit rule $x > 0$ and entry $M > 0$ there exists a unique invariant measure, i.e. $m(x, M)$ is well defined. This invariant measure is given by the fixed point of the mapping defined by equation (12). In the Appendix we establish that as a consequence of Assumption A.4 the inverse of the operator $(I - \hat{P}_x)$ exists, so

$$(13) \qquad \mu = M\left(I - \hat{P}_x\right)^{-1}\nu = M \sum_{t=0}^{\infty} \hat{P}_x^t \nu$$

where $\hat{P}_x^t$ is the composition of $\hat{P}_x$ with itself t times and $\hat{P}_x^0 = I$, the identity operator. The role of Assumption A.4 (recurrence) is to guarantee that the series in equation (13) converges. For a fixed exit rule x, letting λ_t be the probability that a firm is still in the industry t periods after entry and $\tilde{\mu}_t$ the corresponding probability distribution for its shock φ,

$$(14) \qquad \sum_{t=0}^{\infty} \hat{P}_x^t \nu(A) = \sum_{t=0}^{\infty} \lambda_t \tilde{\mu}_t(A)$$

for any borel set A. For $A = [0, 1]$, the right-hand side of (14) equals the expectation of the stopping time derived from exit rule x, i.e. the average age at exit. Existence of a stationary equilibrium with positive entry and exit is thus equivalent to the existence of a stopping rule with finite expectation and a mass of entrants such that for the (stationary) prices that correspond to the associated invariant distribution this stopping rule is optimal and the expected discounted profits of entrants are equal to the cost of entry. The analysis in this section could have been carried out without the recurrence assumption replacing 0 as the lower point in Figure 1 by the infimum of $\{x \in S | E\tau_x < \infty\}$, where τ_x is the stopping time associated with a stopping rule x.

This also suggests that the idea of a stationary equilibrium with entry and exit can be extended to a setup where firms' shocks follow a nonstationary process. Jovanovic's selection model (1982) is one of this class. However, he proves there

---

[7] One special case that allows for firm's states to be a vector of the form $(\varphi_1, \ldots, \varphi_k)$, is the homogeneous CES function:

$$q = \left[\left(\varphi_1/\hat{D}_1\right)^{1-(1/\sigma)} + \cdots + \left(\varphi_k/\hat{D}_k\right)^{1-(1/\sigma)} + \left(n_1/D_1\right)^{1-(1/\sigma)} \right.$$
$$\left. + \cdots + \left(n_j/D_j\right)^{1-(1/\sigma)}\right]^{\sigma/(\sigma-1)}$$

with profit function

$$\Pi = \left[p^{1-\sigma} - \sum_{i=1}^{j} (D_i w_i)^{1-\sigma}\right]^{1/(1-\sigma)} \left[\sum_{i=1}^{k} \left(\varphi_i/\hat{D}_i\right)^{1/(1-\sigma)}\right]^{\sigma/(1-\sigma)}$$

provided $p^{1-\sigma} > \sum_{i=1}^{j}(D_i w_i)^{1-\sigma}$ (see McFadden (1978)).

is no entry and exit in the limit and thus no stationary equilibrium with entry and exit. This occurs precisely because the stopping time derived in the model has positive probability at infinity, and thus infinite expectation.

5. PROPERTIES OF THE STATIONARY EQUILIBRIUM

This section is divided in three parts. The first one, focuses on the life cycle (birth, growth, and death) of firms. The second one gives comparative statics results. The third one discusses implications of the model for the distribution of firm values and profits.

Birth, Growth, and Death of Firms

Given the production function and stationary equilibrium prices, the dynamics of firm size is driven by the evolution of the productivity shock. Here, this process is specified by the pair (ν, F), which are exogenously given. Since the analysis that follows depends only on some general properties of this process, it can also be of interest for models where the process is derived endogenously so that ν and F depend on the equilibrium prices. For instance, if firm's can take costly actions that improve the conditional distribution for their shocks, as in Ericson and Pakes (1989), at the equilibrium prices firm evolution will also be given by a Markov process on φ. If regardless of the level of prices, the derived process satisfies our general conditions, the implications for firm dynamics will follow.

Our model implies an evolution for the size distribution of firms by age cohorts. A striking regularity found in empirical studies is that this size distribution is stochastically increasing in age. Under what conditions on (ν, F) will our model lead to such behavior? The size of a firm measured either by inputs or output is an increasing function of its productivity shock φ. So we seek conditions under which $\tilde{\mu}_t$, the distribution of shocks for firms of age t, is increasing in t.

Fix an exit point $x \in S$ and for any probability measure λ on S let $H_x\lambda$ denote the conditional probability of λ on the set $[x, 1]$. H_x is the linear operator defined by

$$(H_x\lambda)(A) = \lambda(A \cap [x, 1])/\lambda([x, 1]), \quad \text{for all borel sets} \quad A \subset S.$$

Let T denote the linear operator on measures associated to the conditional distribution function F defined by

$$(T\mu)([0, z)) = \int F(z, \varphi)\mu(d\varphi), \quad \text{for all} \quad z \in [0, 1].$$

Note that for all t, $\tilde{\mu}_{t+1} = (T \circ H_x)\tilde{\mu}_t$. So if $T \circ H_x$ is a *monotone operator*, i.e. $(T \circ H_x)\mu_2 \succeq (T \circ H_x)\mu_1$ whenever $\mu_2 \succeq \mu_1$ and $(T \circ H_x)$ *increases* ν, i.e. $\tilde{\mu}_1 = (T \circ H_x)\nu \succeq \nu$—where the ordering considered on measures is the first

stochastic dominance—it follows by induction that the sequence $\{\tilde{\mu}_t\}$ will be increasing. What properties of ν and F give this result?

Assumption A.3(b) implies that T is monotone. Is it enough just to assume additionally that it increases ν? For a positive measure μ, it is easy to check that $H_x \mu \succcurlyeq \mu$. In particular, $H_x \nu \succcurlyeq \nu$ and since T is monotone, $\tilde{\mu}_1 = T \circ H_x \nu = T(H_x \nu) \succcurlyeq T\nu$. So if $T\nu \succcurlyeq \nu$, then by transitivity of the stochastic order $\tilde{\mu}_1 \succcurlyeq \nu$. Now suppose $\tilde{\mu}_t \succcurlyeq \nu$. Since $H_x \tilde{\mu}_t \succcurlyeq \tilde{\mu}_t$ it follows that $\tilde{\mu}_{t+1} = T(H_x \tilde{\mu}_t) \succcurlyeq T\tilde{\mu}_t \succcurlyeq T\nu \succcurlyeq \nu$. This proves the following proposition.

PROPOSITION 3: *Assume A.3(b) and that T increases ν. Then $\tilde{\mu}_t \succcurlyeq \nu$ for all t.*

Note that by Proposition 3 the expectation of any increasing function will be at its minimum for the most recent entrants. This implies not only that the average value of firms and profits will be at a minimum but that hazard rates will be highest for this cohort. The latter is consistent with survival data for manufacturing reported by Dunne, Roberts, and Samuelson.

Unfortunately the assumptions of Proposition 3 are not sufficient to obtain the stronger result, namely that the sequence $\tilde{\mu}_t$ is increasing. The problem is that though T is monotone, $T \circ H_x$ is not. This happens because H_x is not a monotone operator.[8] To get stronger results stronger monotonicity conditions are needed so that T increases the distribution even after the truncation resulting from exit. We now define a stronger order that survives this truncation.

For two distributions μ_1 and μ_2, we will say that $\mu_2 \succcurlyeq_{mcd} \mu_1$ (read μ_2 is greater than μ_1 in the monotone conditional dominance (*mcd*) order) if $H_z \mu_2 \succcurlyeq H_z \mu_1$ for all $z \in [0,1]$. Note that for $z = 0$ this condition implies that $\mu_2 \succcurlyeq \mu_1$, so this order is stronger than first stochastic dominance. In fact it requires that first stochastic dominance hold when conditioning on any *increasing* set.[9] This is weaker than likelihood ratio ordering, which requires stochastic dominance when conditioning on any *borel* set.[10] We will say that T is monotone in the *mcd* order if $T\mu_2 \succcurlyeq_{mcd} T\mu_1$ whenever $\mu_2 \succcurlyeq_{mcd} \mu_1$.

PROPOSITION 4: *Assume that T is monotone in the mcd order and that $T\nu \succcurlyeq_{mcd} \nu$. Then $\tilde{\mu}_{t+1} \succcurlyeq_{mcd} \tilde{\mu}_t$ for all t.*

PROOF: For any measure μ and $z \in [0,1]$, it is easy to check that $H_z \mu \succcurlyeq_{mcd} \mu$ and in particular $H_x \nu \succcurlyeq_{mcd} \nu$ for the exit point x. Letting $z = x$, by the monotonicity of T, $\tilde{\mu}_1 = T \circ H_x \nu = T(H_x \nu) \succcurlyeq_{mcd} T\nu \succcurlyeq_{mcd} \nu$. It is easy to check that the *mcd* order is transitive, so $\tilde{\mu}_1 \succcurlyeq_{mcd} \nu$. Now suppose that $\tilde{\mu}_{t+1} \succcurlyeq_{mcd} \tilde{\mu}_t$.

---

[8] For example, consider the case where $\mu_1(\{0\}) = 0.9$, $\mu_1(\{1\}) = 0.1$, $\mu_2(\{0.5\}) = 0.8$, and $\mu_2(\{1\}) = 0.2$. Then for $x \in (0, 0.5)$, $H\mu_1$ has all its mass at $\{1\}$ and thus $H\mu_1 \succ H\mu_2 = \mu_2$. Similar examples can obviously be constructed with continuous distributions.

[9] This ordering belongs to the class of uniform conditional stochastic orderings (UCSO) as defined in Whitt (1980).

[10] An interesting extension to the multidimensional case is developed in Pakes and Ericson (1990) under a similar set of assumptions but involving likelihood ratio ordering.

Since T is monotone in the *mcd* order, $\bar{\mu}_{t+2} = TH_x\bar{\mu}_{t+1} \succcurlyeq_{mcd} TH_x\bar{\mu}_t = \bar{\mu}_{t+1}$, where the inequality follows from the definition of *mcd* order for $z = x$.   *Q.E.D.*

COROLLARY: *Under the hypotheses of this theorem, the distribution of firms' shocks is increasing in the age of the cohort, and so is the integral of any increasing function. In particular, the rate of survival will be higher for older firms and so will average size, profits and value of firms.*

This corollary implies that hazard rates will be lower for older firms. It is also true—almost by definition—that hazard rates in a given period will be lower for larger firms. A stronger result can be proved for T monotone in the *mcd* order. For a given exit rule x, let $\lambda_t(\varphi)$ denote the probability that a firm with current shock φ will still be in the industry after t periods and let $\bar{\mu}_t(\cdot\,|\varphi)$ denote its conditional distribution.

PROPOSITION 5: *Assume T is monotone in the mcd order. If $\varphi_2 \succcurlyeq \varphi_1$ then $\lambda_t(\varphi_2) \geqslant \lambda_t(\varphi_1)$ and $\bar{\mu}_t(\cdot\,|\varphi_2) \succcurlyeq_{mcd} \bar{\mu}_t(\cdot\,|\varphi_1)$.*

PROOF: Same argument as in the proof of Proposition 4.

This Proposition implies that larger firms are longer lived on average and will tend to remain larger before they exit.

We derived implications for the size of firms and hazard rates. Other characteristics of firm growth depend on more specific features of the stochastic process for shocks and the production function. Empirical studies show some regression to the mean in size but higher persistence for larger and also for older firms (see Evans (1987), Dunne, Roberts, and Samuelson (1989a), and Leonard (1987)). Our model is consistent with the first two observations for appropriately chosen F. But since size is a sufficient statistic for φ, age has no extra predictive role. However age effects can be introduced in the model at no analytical cost by allowing production to be affected also by a purely temporary shock ε_t. Though production decisions and size will now depend on the two shocks, exit decisions will still depend only on φ_t. The implications of this extension are not studied here.

Comparative Statics

This sections analyzes the effect that changes in some of the parameters of the industry have on the equilibrium. For this purpose it is necessary to assume that there exists a unique stationary equilibrium with entry and exit, e.g. that Assumptions A.1–A.5 and U.1 or U.2 are satisfied.

An increase in the cost of entry shifts curve M_2 downwards: to match the higher entry cost discounted profits need to be higher. Since curve M_1 is increasing, this implies that the equilibrium x^* decreases and M^* too (see Figure 1). What happens to the rate of turnover, $M^*/\mu^*(S)$? Equations (13) and (14) imply that $\mu^*(S) = M^*\sum_{t=0}^{\infty}\lambda_t$, where λ_t is the probability that a firm is

in the industry t periods after entry. In consequence, the rate of turnover equals $(\sum_{t=0}^{\infty} \lambda_t)^{-1}$. Since λ_t is decreasing in x^*, higher cost of entry leads to a lower turnover rate.

The lower value of x^* implies less selection and higher expected lifetime of firms. The cost of entry acts as a barrier to entry, hence higher costs of entry protect the incumbent firms. The higher cost of entry reduces the mass of entrants and the rate of entry. This is consistent with the evidence reported by Orr (1974) for the Canadian industry. Also higher costs of entry imply higher profits for large firms but not necessarily for small ones; this certainly would be the case for highly persistent processes where profits of exiting firms would be approximately zero regardless of the cost of entry. Notice also that for any fixed mass of the largest firms, the market share will be higher in the high entry cost case. These two observations imply that, as reported by Demsetz (1973), more concentrated industries could show higher profits for large firms and not for small ones.

The effect on size distribution is not obvious. The increase in c_e has a *price effect* and a *selection effect*. Consider the case where input prices are fixed. Output price increases with c_e leading to higher employment and output for each φ. But since x^* decreases, the fraction of firms with lower shocks (and thus lower output and employment) increases. The strength of each of these effects depends on properties of the stochastic process for shocks and the production function.

A related question is the effect changes in the distribution of entrants have on the stationary equilibrium. For example, suppose the distribution of entrants stochastically decreases. In contrast to the previous case, there is a direct and indirect effect: at the same market prices the stochastically lower distribution of entrants reduces their expected profits. But for fixed entry and exit rules the invariant distribution stochastically decreases, with a positive effect on expected profits of entrants. If the net effect were negative, then while curve M_1 in Figure 1 would shift upwards, curve M_2 would shift downwards, resulting in lower values for M^* and x^*. As in the previous case, this would imply lower rates of turnover. This occurs under assumptions U.1 or U.2, since the increase in profits for some states—necessary to compensate for the lower ν—implies that profits will increase for all states, which in turn leads to a reduction in x^*.

What effect does a mean preserving spread for the state of entering firms have on the stationary equilibrium? To answer this question, assume the profit function is strictly convex in φ and that *F preserves convexity*, i.e. $\int f(y) F(dy|x)$ is a convex function of x whenever f is convex.[11] Then, by standard dynamic programming arguments, the value function is also strictly convex in φ. So without changes in equilibrium prices expected profits for entrants would increase. Again under assumption U.1 or U.2 this implies that the equilibrium value for x^* increases, and so does the rate of turnover.

---

[11] For example, this condition holds for linear AR1 processes where $y = \rho x + \varepsilon$, since $\int f(y) F(dy|x) = \int f(\rho x + \varepsilon) \Psi(d\varepsilon)$ is convex in x for convex f.

The effect of higher demand on the stationary equilibrium depends on whether the input supply price to the industry is fixed or increasing. In the first case, the *long run* supply curve of the industry is horizontal, so higher demand only affects the mass of entrants.[12] In this case entry increases with the size of the industry as also reported by Orr (1974). When input supply prices are increasing to the industry changes in aggregate demand will have price effects. It is easy to show that the real wage increases together with the increase of aggregate demand.

What happens to turnover? When the profit function is separable in the state of the firm and market prices, as defined in assumption U.2 the zero profit condition for entry implies that profits must remain the same for all states. In this case changes in aggregate demand are neutral on all life cycle properties and on the rate of turnover in the industry, causing only changes in the total number of firms and the market price for the good in the industry. Note also that due to the presence of higher real wages, under these hypotheses higher demand implies smaller firms. This in turn implies that the market share of a given mass m of largest firms decreases.[13] The same qualitative results discussed in this paragraph are obtained for an increase in an exogenously given wage rate.

Given the connection between the level of fixed costs and the degree of economies of scale, it is interesting to analyze the effect of higher fixed costs. This shifts both the M_1 and M_2 curves downward, suggesting the possibility of an ambiguous answer. The following proposition provides a condition under which the higher fixed costs lead to stochastically larger distribution of sizes. An example is then presented that indicates the possibility of a change in the opposite direction.

PROPOSITION 6: *Under assumption U.2 an increase in the fixed cost, all other things equal, leads to a stationary equilibrium with higher x^*.*

PROOF: Let $\hat{v}(\varphi, \mu) = v(\varphi, \mu)/c_f$. Rewriting equation (9) for the stationary case and using U.2 the following is obtained:

$$\hat{v}(\varphi, \mu) = \frac{h(\varphi) g(p^e(\mu), w^e(\mu))}{c_f}$$

$$- 1 + \max\left\{0, \beta \int \hat{v}(\varphi', \mu) F(d\varphi'|\varphi)\right\}.$$

[12] This is connected to the fact that cost of entry is independent of the number of firms and that all potential entrants are identical. If either of these assumptions were changed, e.g. if cost of entry is an increasing function of the number of entrants, then the equilibrium price of the output would increase while the exit point would decrease, leading to higher expected life of firms and lower *rate* of turnover.

[13] If concentration is measured instead by the Gini coefficient, the effect of demand changes on concentration depend on the relative values of the employment price elasticity of different firms. In particular, if $-f_{21}(\varphi, n)/f_{22}(\varphi, n) \cdot n$ is increasing (decreasing) in n for all (φ, n), the Gini coefficient will decrease (increase).

Consider two industries, one with a lower fixed cost than the other, denoted respectively by subscripts 1 and 2. Let x_1 and x_2 be the corresponding equilibrium exit points and μ_1 and μ_2 the corresponding distributions. If $x_2 \leqslant x_1$, then

$$\int \hat{v}(\varphi, \mu_2) F(d\varphi | x_1) \geqslant \int \hat{v}(\varphi, \mu_2) F(d\varphi | x_2)$$

$$= 0 = \int \hat{v}(\varphi, \mu_1) F(d\varphi | x_1).$$

Using standard dynamic programming arguments it is easy to see that this can only happen when

$$\frac{g(p^e(\mu_1), w^e(\mu_1))}{c_{f1}} \leqslant \frac{g(p^e(\mu_2), w^e(\mu_2))}{c_{f2}},$$

which in turn implies that $\hat{v}(\varphi, \mu_1) \leqslant \hat{v}(\varphi, \mu_2)$. But since fixed costs are higher in industry 2, $v(\varphi, \mu_1) < v(\varphi, \mu_2)$ which in turn implies higher expected discounted profits for entrants in industry 2, contradicting condition (iii) of the definition of equilibrium. Hence $x_1^* < x_2^*$.   Q.E.D.

The following example shows that even under assumption U.1 the above result need not hold.

EXAMPLE: Let $\nu(\{0\}) = .9$ and $\nu(\{1\}) = .1$; $P = I$; $\beta = 0$ and $c_e = 0.1$. The production function is $f(\varphi, n) = \varphi \ln(\max\{1, n\})$. With these assumptions the following entry and exit condition are obtained:

(entry) $0.1 \cdot \pi(1, p) - c_e = 0$,

(exit) $\pi(x^*, p) = 0$.

Numerical results for different levels of fixed cost are reported in the following table.

Equilibrium values	$c_f = 0 \cdot 1$	$c_f = 0 \cdot 5$	$c_f = 1 \cdot 0$
p	4.3	6.7	9.1
x^*	6.52	4.76	3.95
$q(x^*)$	6.75	5.51	5.04
$n(x^*)$	2.82	3.18	3.59
$q(x^*)/n(x^*)$	2.40	1.73	1.41

As fixed cost increases, the size—measured by output—of the marginal firm decreases, and so does its productivity. So this example shows that higher economies of scale can, in theory, increase the survival rates for smaller firms! This suggests some caution in connecting cross industry differences in the survival rates for a given size class to the degree of economies of scale.

Though the environment used for this example is rather degenerate, using continuity arguments it is easy to see that this result is not exceptional. The key feature needed to obtain this behavior is that increases in c_f require very large

increases in p for the entry condition to be satisfied, which outweigh the ones needed to maintain the exit point. Loosely speaking, this requires a technology where high φ leads to high profits but not high elasticity of profits.

Profits and Value of Firms

We now study the effect that selection resulting from exit has on average profits and value of firms. Incumbent firms have the option of staying in the industry without the need of paying a new entry cost. Because of this option value it is likely that $\pi(x^*, \mu^*) < 0$ and thus the support of the distribution of profits will contain some negative values. The following Proposition shows that in spite of this, a positive lower bound can be obtained for average industry profits. Let $\rho = M^*/\mu^*(S)$, the rate of turnover.

PROPOSITION 7: *At the stationary equilibrium,* $\bar{\pi}(\mu^*) \equiv \int \tilde{\pi}(\varphi, \mu^*)\mu^*(d\varphi) \geqslant \rho c_e$.

PROOF: Let $V_t = \int v(\varphi, \mu^*)\lambda_t \tilde{\mu}_t(d\varphi)$ and $\Pi_t = \int \tilde{\pi}(\varphi, \mu^*)\lambda_t \tilde{\mu}_t(d\varphi)$. It follows that $V_t = \Pi_t + \beta V_{t+1}$ for all t, and since V_t/λ_t is the average expected value of a surviving firm, $V_t \geqslant 0$ for all t. In consequence, $V_t \leqslant \Pi_t + V_{t+1}$ and inductively noting that $\lambda_t \to 0$ implies $V_t \to 0$, $V_t \leqslant \sum_{s=t}^{\infty} \Pi_s$. In particular, $c_e = v^e(\mu^*) = V_0 \leqslant \sum_{t=0}^{\infty} \Pi_t = \sum_{t=0}^{\infty} \lambda_t \bar{\pi}(\mu^*)$, where the last equality follows from the fact that $\mu^* = \sum_{t=0}^{\infty} \lambda_t \tilde{\mu}_t / \sum_{t=0}^{\infty} \lambda_t$. Since $\rho = (\sum_{t=0}^{\infty} \lambda_t)^{-1}$, dividing through by $\sum_{t=0}^{\infty} \lambda_t$ the inequality is obtained. $\qquad$ Q.E.D.

We now turn to the average value of firms, $\bar{v}(\mu) = \int v(\varphi, \mu)\mu(d\varphi)$. With no entry and exit $\bar{v}(\mu) = \bar{\pi}(\mu)/(1 - \beta)$, since the total value of the industry portfolio is the discounted flow of its profits. A gap appears when there is entry and exit since to support this constant flow of profits entry costs must be borne. The following Proposition introduces this correction.

PROPOSITION 8: *The average value of firms* $\bar{v}(\mu^*) = \bar{\pi}(\mu^*) - \beta \rho c_e/(1 - \beta) \geqslant \rho c_e$.

PROOF: $\bar{v}(\mu^*)\Sigma\lambda_t = \sum_{t=0}^{\infty} V_t = \sum_{t=0}^{\infty} \Pi_t + \beta \sum_{t=1}^{\infty} V_t = \sum_{t=0}^{\infty} \Pi_t + \beta \sum_{t=0}^{\infty} V_t - \beta c_e$. Dividing through by $\Sigma\lambda_t$ the equality is obtained. The inequality follows from the bound given in Proposition 7. $\qquad$ Q.E.D.

Note that the formula given in Proposition 8 offers an indirect way of measuring implicit entry (sunk) costs to different industries. Also since ρ is equal to the inverse of the expected lifetime of a firm, the formula implies that the gap between the average value of firms and the discounted stream of average profits will be smaller the higher the average age of firms. Of course in the limit, when $\rho = 0$, this gap disappears.

A stronger result can be obtained under the assumptions of Proposition 3. A consequence of this Proposition is that $\mu^* \geqslant \nu$, which in particular implies $\bar{v}(\mu^*) \geqslant c_e$. In the absence of other sources of investment c_e measures the capital of a firm, so this inequality also implies that for the industry the average value for Tobin's q ratio exceeds one. Comparative statics results on average industry q ratios are developed in Hopenhayn (1992).

Do industries with higher fixed cost or entry costs have higher average profits and value? We now analyze this question.

Increase in entry cost. This has two effects: a *value* effect, given by the increase in profits, and a *selection* effect given by the reduction in x^*. The first effect is positive but the second one negative, so no general conclusion can be obtained. If the density of firms near the exit point is very small, the first effect will dominate and the average value and profits increase with c_e.

Increase in fixed cost. This also has value and selection effects. By Proposition 6, under assumption U.2 the latter effect is positive. We now show that if a regularity condition holds the value effect is also positive.

It is useful to write v^e as the weighted sum of profits:

$$(15) \qquad v^e(\mu) = \sum_{t=0}^{\infty} \beta^t \lambda_t \int \pi[\varphi, p^e(\mu), w^e(\mu)] \tilde{\mu}_t(d\varphi).$$

Similarly one can express $\bar{v}$ by[14]

$$(16) \qquad \bar{v}(\mu) = \sum_{t=0}^{\infty} \lambda_t (1 - \beta^t) \int \pi[\varphi, p^e(\mu), w^e(\mu)] \tilde{\mu}_t(d\varphi) / (1 - \beta) \sum_{t=0}^{\infty} \lambda_t.$$

As the fixed cost increases, the profit function changes, making profits lower for each pair (φ, μ). To maintain the equality in (15) μ must change to reestablish profitability. Is this compensating effect for entrants enough to compensate the average firm too? Both (15) and (16) give weighted averages of profits of firms of different ages. But because of discounting, (16) gives more weight to older firms. Under the assumptions of Proposition 4 these are also firms with higher φ's. So if the price effect is higher for firms with higher shocks $\bar{v}$ will increase. Since $\pi_2(\varphi, p, w) = q(\varphi, p, w)$ which is increasing in φ, if w is constant the price effect will indeed be higher for firms with higher shocks. The following Proposition is proved in Hopenhayn (1992).

PROPOSITION 9: *Under the assumptions of Proposition 4 and if U.1 holds, an increase in c_f leads to an increase in $\bar{v}$.*

Similar results are obtained for average industry profits using exactly the same argument, since these are proportional to the right side of equation (15) but without discounting.

---

[14] This formula is developed in Hopenhayn (1992).

6. FINAL REMARKS

This paper has developed the stationary equilibrium analysis of an industry equilibrium model. Entry and exit are part of the limiting behavior of the industry and not only part of the adjustment to a steady state, as occurs in much of the previous literature. The concept of a stationary equilibrium thus extends standard long run industry equilibrium theory to account for entry, exit, and firm dynamics.

The model developed here has abstracted from some interesting features. In particular, the only dynamic decision faced by firms—as in Jovanovic—is the exit decision, a stopping time problem, and the stochastic structure is a fairly simple one. But the idea of a stationary equilibrium that we develop extends easily to much more general setups. In particular, investment decisions can be easily introduced, either expanding the state vector in the dynamic problem solved by firms or allowing these to affect the conditional distribution for shocks, as in Ericson and Pakes. We conjecture that many results developed here will extend and even if the comparative statics analysis is not as simple as it is here, at the very least the numerical computation will remain a very simple one.

We have emphasized the importance of the process of resource reallocation that takes place through firm and job turnover. Many countries have policies, such as firing costs, that affect this process. Stationary equilibrium analysis can provide a useful tool to study the impact of such policies. As an example, a version of the model developed here is used in Hopenhayn and Rogerson (1991) as the productive sector of a general equilibrium model to study the effect of labor firing costs on job turnover, productivity, and welfare. The steady state analysis provides a means of evaluating long run impact of these policies.

Graduate School of Business, Stanford University, Stanford, CA 94305, U.S.A.

APPENDIX

1. *Preliminaries*

Let M be the set of positive bounded borel measures on $S = [0, 1]$ endowed with the weak* topology. For a linear operator $T: M \to M$ we will denote by $\|T\|$ its norm in the strong topology. Unless otherwise specified, continuity of functions with more than one argument will be in the product topology. *Orderings*: Let the order $\succcurlyeq$ on M be defined by $\mu_2 \succcurlyeq \mu_1$ iff $\int f d\mu_2 \geq \int f d\mu_1$ for *every nondecreasing and bounded borel measurable function f*. When μ_2 and μ_1 are probability measures this coincides with standard first order dominance. This defines a partial order with closed graph (see Kamae and Krengel (1978)). When the inequality is strict for all *strictly increasing f* we will say that $\mu_2 \succ \mu_1$. This order is stronger than $\succcurlyeq$ but weaker than the canonical order on M. All statements of (strict) monotonicity involving measures will refer to the order $(\succ) \succcurlyeq$.

2. *Lemmas and Proofs*

LEMMA 1: *Let Z be a metric space and h: $S \times Z \to \mathbb{R}$ a continuous function. Assume $\mu_n \to \mu$ and $z_n \to z$, where z_n and z are in Z. Then $\int h(\varphi, z_n)\mu_n(d\varphi) \to \int h(\varphi, z)\mu(d\varphi)$.*

PROOF: Follows from Theorem 5.5 in Billingsley by letting $h_n = h(\cdot, z_n)$, using joint continuity of h and the fact that since $\mu_n \to \mu$, $h_n \in Y \subset \mathbb{R}$, a bounded set, for all n.   Q.E.D.

LEMMA 2: Q^s and N^d are continuous functions.

PROOF: Follows immediately from Lemma 1 and the continuity of function q and n.   Q.E.D.

LEMMA 3: Functions p^e and w^e are well defined and continuous.

PROOF: For given μ and using the arguments for the proof of Theorem 1 the allocations $Q(\mu, p^e(\mu), w^e(\mu))$ and $N(\mu, p^e(\mu), w^e(\mu))$ maximize consumer surplus. Since D is strictly decreasing, also by the argument used in the proof of Theorem 1 the aggregates Q and N are unique, and so are the corresponding equilibrium prices. To establish continuity note that $p^e(\mu)$ and $w^e(\mu)$ are the minimizers of $|p - D(Q^s(\mu, p, w))| + |w - W(N^d(\mu, p, w))|$, so by the theorem of the maximum and using Lemma 2 p^e and w^e are continuous.   Q.E.D.

PROOF OF PROPOSITION 2: Continuity follows from Assumption A.2(b) and Lemma 3. Since $p^e(\mu) > 0$, $\tilde{\pi}$ is strictly increasing in φ also by Assumption A.2(b).

We now establish that $\tilde{\pi}$ is decreasing in μ. Let $\mu_2 \succ \mu_1$. Let p_j and w_j, $j = 1, 2$ denote the respective equilibrium prices. Suppose by way of contradiction that $p_2 \leqslant p_1$. Then $w_2 > w_1$ for otherwise $q(\varphi, p_2, w_2) > q(\varphi, p_1, w_1)$ and since q is strictly increasing in φ, $\int q(\varphi, p_2, w_2)\mu_2(d\varphi) > \int q(\varphi, p_1, w_1)\mu_1(d\varphi)$ and thus $p_2 < p_1$. But $w_2 > w_1$ implies that $N_2 > N_1$ and since $\mu_2 \succ \mu_1$, this is not consistent with a decrease in aggregate output. In consequence $p_2 < p_1$. It also follows easily that $w_2/p_2 > w_1/p_1$ and thus, letting $\lambda = p_2/p_1$, $\pi(\varphi, p_2, w_2) \leqslant \pi(\varphi, \lambda p_1, \lambda w_1) = \lambda \pi(\varphi, p_1, w_1) < \pi(\varphi, p_1, w_1)$.   Q.E.D.

LEMMA 4: For any $x \in (0, 1]$ the operator $(I - \hat{P}_x)$ has an inverse.

PROOF: Let $\hat{P}_x^n$ denote the composition of $\hat{P}_x$ n times with itself and $\hat{P}_x^0 = I$. Assumption A.4 implies that $\|\hat{P}_x^n\| < 1$ and thus $(I - \hat{P}_x^n)^{-1} = \sum_{t=0}^{\infty} \hat{P}_x^{nt}$, as follows from Kolmogorov and Fomin (1970, Theorem 4, pg. 231). Since $\|\hat{P}_x^n\|$ is nonincreasing in n, this also implies that $(I - \hat{P}_x)^{-1}$ exists and satisfies equation (13).   Q.E.D.

LEMMA 5: Function m is jointly continuous, decreasing in x, and strictly increasing in M.

PROOF: We prove continuity in the strong topology. Since $m(x, M) = M(I - \hat{P}_x)^{-1}\nu$ it obviously suffices to prove continuity in x for any $M > 0$. Let $x_n \to x \in (0, 1]$. Let $B(x, \varepsilon)$ be an epsilon neighborhood of x in $(0, 1]$. Let $P(\varphi, A)$ be the transition function corresponding to F. For $x_n \in B(x, \varepsilon)|\hat{P}_{x_n}\nu(A) - \hat{P}_x\nu(A)| \leqslant \int_{B(x, \varepsilon)}P(\varphi, A)\nu(d\varphi) \leqslant \nu(B(x, \varepsilon))$ for any borel set A. By Assumption A.5 $\nu(B(x, \varepsilon)) \to 0$ as $\varepsilon \to 0$, so $\hat{P}_{x_n}\nu \to \hat{P}_x\nu$ in the norm topology. We now prove the same for $\hat{P}_n^t$. Suppose that $\hat{P}_x^{t-1}\nu$ is also continuous in x and let $\mu_n = \hat{P}_{x_n}^{t-1}\nu$. For any borel set A and $x_n \in B(x, \varepsilon)$,

$$\left|\hat{P}_{x_n}^t\nu(A) - \hat{P}_x^t\nu(A)\right| = \left|\int_{\varphi \geqslant x_n}P(\varphi, A)\mu_n(d\varphi) - \int_{\varphi \geqslant x}P(\varphi, A)\mu(d\varphi)\right|$$

$$\leqslant \left|\int_{\varphi \geqslant x_n}P(\varphi, A)\mu_n(d\varphi) - \int_{\varphi \geqslant x}P(\varphi, A)\mu_n(d\varphi)\right|$$

$$+ \left|\int_{\varphi \geqslant x}P(\varphi, A)\mu_n(d\varphi) - \int_{\varphi \geqslant x}P(\varphi, A)\mu(d\varphi)\right|$$

$$\leqslant \mu(B(x, \varepsilon)) + 2\|\mu_n - \mu\|$$

which converges to $\mu(B(x, \varepsilon))$ as $n \to \infty$. Assumptions A.3(a) and A.5 imply that μ is nonatomic so as $\varepsilon \to 0$, $\mu(B(x, \varepsilon)) \to 0$ too, proving strong continuity of $\hat{P}_x^{t-1}\nu$ in x. For any finite T, $\sum\hat{P}_x^t\nu$ is also strongly continuous in x and since $x > 0$, by Assumption A.4 the same is true for $T = \infty$, so m is continuous in x on $(0, 1]$.

For $x \in (0, 1]$ and $M > 0$, m is strictly increasing in M and nonincreasing in x in the canonical order. Hence m is strictly increasing in M and weakly decreasing in x.   Q.E.D.

LEMMA 6: *Function $M_1: (0, 1] \to \mathbb{R}_+$ is well defined, continuous, and strictly increasing.*

PROOF: For fixed $x \in (0, 1]$ and $M > 0$, let $H(\varphi, x, M) \equiv \int v[\varphi', m(x, M)] F(d\varphi' | \varphi)$. H is continuous by Lemma 5, Proposition 3, Assumption A.3.a, and Lemma 1. Also it is easy to check that H is strictly increasing in φ, nondecreasing in x, and strictly decreasing in M. Furthermore, as $M \to \infty$ output price must decrease to zero, for otherwise aggregate output Q^s must remain bounded and so must employment. But in that case w/p will be bounded above and thus $q(\varphi, p, w) \geqslant q > 0$ for all φ, which in turn implies that Q^s will grow without bound. In consequence, as $M \to \infty$, $\pi(\varphi, \cdot)$ decreases to $-c_f$ for all φ so $H(\varphi, x, M)$ decreases to $-\beta c_f$. On the other hand, as $M \downarrow 0$ so does $m(x, M)$ so by Assumption A.2(c) and letting $\mu = m(x, M)$, $\pi[0, p^e(\mu), w^e(\mu)]$ become positive for small M and so does $H(\varphi, x, M)$. Thus, for each $x > 0$ and φ there exists a unique M such that $H(\varphi, x, M) = 0$. $M_1(x)$ is the unique M for which $H(x, x, M) = 0$. Since $H(x, x, M)$ is strictly increasing in x and strictly decreasing in M it follows that M_1 is strictly increasing. Continuity of M_1 is immediate from the continuity of H.   Q.E.D.

LEMMA 7: *Assume $v^e(0) > c_e$. Then M_2 is well defined, continuous, and nondecreasing.*

PROOF: As $M \to \infty$ $v^e(m(x, M)) \to 0$ and as $M \to 0$ $v^e(m(x, \mu)) \to v^e(0) > c_e$. Since $v^e(m(x, \cdot))$ is continuous and strictly decreasing, $M_2(x)$ is well defined. Furthermore since $M_2(x)$ is the minimizer of $|v^e(m(x, M)) - c_e|$ which is a continuous function, it is continuous. From the monotonicity properties of m it easily follows that M_2 is nondecreasing.   Q.E.D.

THEOREM 2: *There exists a stationary competitive equilibrium for the industry.*

PROOF: Without loss of generality assume $v^e(0) > c_e$, for otherwise an empty industry is an equilibrium. Thus functions M_1 and M_2 are well defined. Looking at Figure 1 there are two cases: (i) there exists some $x \in (0, 1]$ such that $M_1(x) \leqslant M_2(x)$, so there is an interior stationary equilibrium; (ii) $M_1(x) > M_2(x)$ for all $x \in (0, 1]$. In this case $v^e(m(x, M_1(x)) < c_e$ for all x. As x_n decreases to 0, let $\mu_n = m(x_n, M_1(x_n))$ and $b_n = \mu_n(S)$. Note that since $v(\varphi, \cdot)$ is decreasing, b_n must be a decreasing sequence. It converges to some $b > 0$ for otherwise $v^e(\mu_n) > c_e$ for large n. Also $\{\mu_n/b_n\}$ is a sequence of probability measures which is easily seen to be decreasing in $\succeq$. By Proposition 1 in Hopenhayn and Prescott (1991) this sequence also converges and in consequence $\mu_n \to \mu$ a nonzero measure. It is not hard to show that μ is a fixed point for the operator defined by the transition function P that corresponds to the conditional cdf F.[15] By continuity, $v^e(\mu) \leqslant c_e$ and $v(0, \mu) = 0$ so μ and the respective prices are an equilibrium for $M = 0$. Define λ_0 by $v^e(\lambda_0 \mu) = c_e$. If $\lambda_0 < 1$, then for all $\lambda_0 < \lambda < 1$, $v^e(\lambda \mu) < c_e$ and $v(0, \mu) > 0$ and thus $\lambda \mu$ and $M = 0$ also define a stationary equilibrium.   Q.E.D.

PROOF OF THEOREM 3: Let μ be the measure as defined in the proof of the previous theorem. Let $c^* = v^e(\mu)$. If $c_e \geqslant c^*$ there is obviously no equilibrium with entry and exit. If $c_e < c^*$ then $c_e < v^e(\mu_n)$ for sufficiently large n, where μ_n is the sequence given in the above proof. In that case there exists $x > 0$ such that $M_1(x) < M_2(x)$, so there is an equilibrium with entry and exit. Finally note that $v^e(\mu) > v(0, \mu) \geqslant \bar{\pi}(0, \mu)(1 - \beta)^{-1}$, which by Assumption A.2(c) is strictly positive, so $c^* > 0$.   Q.E.D.

REFERENCES

ADELMAN, I. (1958): "A Stochastic Analysis of the Size Distribution of Firms," *American Statistical Association Journal*, 53, 893–900.
BILLINGSLEY, P. (1968): *Convergence of Probability Measures*. New York: John Wiley & Sons.
BROCK, W. A. (1972): "On Models of Expectations that Arise from Maximizing Behaviour of Economic Agents over Time," *Journal of Economic Theory*, 5, 348–376.

[15] The proof is contained in a previous draft and available by request.

DEMSETZ, H. (1973): "Industry Structure, Rivalry and Public Policy," *Journal of Law and Economics*, 16, 1–9.

DAVIS, S., AND J. HALTIWANGER (1991): "Gross Job Creation, Gross Job Destruction and Employment Reallocation," Working Paper Series, Research Department, Federal Reserve Bank of Chicago.

DUNNE, T., M. ROBERTS, AND L. SAMUELSON (1989a): "The Growth and Failure of US Manufacturing Plants," *Quarterly Journal of Economics*, 104, 671–698.

——— (1989b): "Plant Turnover and Gross Employment Flows in the US Manufacturing Sector," *Journal of Labor Economics*, 7, 48–71.

ERICSON, R., AND A. PAKES (1989): "An Alternative Theory of Firm and Industry Dynamics," Working Paper, Yale University.

EVANS, D. S. (1987): "The Relationship Between Firm Growth, Size and Age: Estimates for 100 Manufacturing Industries," *Journal of Industrial Economics*, 35, 567–581.

FELDMAN, M., AND C. GILLES (1985): "An Expository Note on Individual Risk without Aggregate Uncertainty," *Journal of Economic Theory*, 35, 26–32.

GREEN, E. (1989): "Individual-Level Randomness in a Nonatomic Population," Working Paper No. 227, Department of Economics, University of Pittsburgh.

HALL, B. (1987): "The Relationship Between Firm Growth, Size and Age: Estimates for 100 Manufacturing Industries," *Journal of Industrial Economics*, 35, 583–606.

HART, P. E., AND S. J. PRAIS (1956): "The Analysis of Business Concentration: A Statistical Approach," *Journal of the Royal Statistical Society*, 119, pt. 2, 150–191.

HOPENHAYN, H. (1990): "Industry Equilibrium Dynamics: A General Competitive Theory," Unpublished manuscript, Stanford.

——— (1992): "Exit, Selection and the Value of Firms," *Journal of Economic Dynamics and Control*, forthcoming.

HOPENHAYN, H., AND E. C. PRESCOTT (1991): "Stochastic Monotonicity and Stationary Distributions for Dynamic Economies," Research Paper Series No. 1169, Graduate School of Business, Stanford University.

HOPENHAYN, H., AND R. ROGERSON (1991): "Job Turnover and Policy Evaluation: A General Equilibrium Analysis," Unpublished manuscript, University of Minnesota.

JOVANOVIC, B. (1982): "Selection and the Evolution of Industry," *Econometrica*, 50, 649–670.

JUDD, K. L. (1985): "The Law of Large Numbers with a Continuum of IID Random Variables," *Journal of Economic Theory*, 35, 19–25.

KAMAE, T., AND U. KRENGEL (1978): "Stochastic Partial Ordering," *The Annals of Probability*, 6, 1044–1049.

KOLMOGOROV, A. N., AND S. V. FOMIN (1970): *Introductory Real Analysis*. Translated and edited by Richard A. Silverman. New York: Dover Publications Inc.

LEONARD, J. (1987): "In the Wrong Place at the Wrong Time: The Extent of Frictional and Structural Unemployment," in *Unemployment & the Structure of Labor Markets*, edited by K. Lang and J. Leonard. New York: Basil Blackwell.

LIPPMAN, S. A., AND R. P. RUMMELT (1982): "Uncertain Imitability: An Analysis of Interfirm Differences in Efficiency Under Competition," *Bell Journal of Economics*, 13, 418–438.

LUCAS, R. (1978): "On the Size Distribution of Business Firms," *Bell Journal of Economics*, 9, 508–523.

LUCAS, R., AND E. C. PRESCOTT (1971): "Investment Under Uncertainty," *Econometrica*, 39, 659–681.

MCFADDEN, D. (1978): "Cost, Revenue and Profit Functions," in *Production Economics: A Dual Approach to Theory and Applications*, Vol. 1, Edited by Melvyn Fuss and Daniel McFadden. Amsterdam: North Holland.

ORR, D. (1974): "The Determinants of Entry: A Study of the Canadian Manufacturing Industries," *Review of Economics and Statistics*, 56, 58–66.

PAKES, A., AND R. ERICSON (1990): "Empirical Implications of Alternative Models of Firm Dynamics," Working Paper, Yale University, 1990.

SIMON, H. E., AND C. P. BONINI (1958): "The Size Distribution of Business Firms," *American Economic Review*, 48, 607–617.

SMITH, V. (1974): "Optimal Costly Firm Entry in General Equilibrium," *Journal of Economic Theory*, 9, 397–417.

UHLIG, H. (1987): "A Law of Large Numbers for Large Economies," Working Paper No. 342, Federal Reserve Bank of Minneapolis.

WHITT, W. (1980): "Uniform Conditional Stochastic Order," *Journal of Applied Probability*, 17, 112–123.