Lecture 12: Reinforcement Learning

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Recap: Motivation for Machine Learning

- Dynamic programming faces the curse of dimensionality
  - Parameterization problem
    - Classical functional forms (linear, polynomials, splines)
    - Number of basis terms grows rapidly with state dimension
    - Strong restrictions and poor capture of interactions
  - Grid / data problem
    - Grid-based DP requires N^d state points
    - Computation and memory grow exponentially
    - Sparse grids mitigate but do not eliminate the problem

How Machine Learning Addresses the Curse of Dimensionality

- Machine learning relaxes both constraints
  - Neural networks for parameterization
    - Flexible, high-dimensional function approximation
    - Captures nonlinearities and interactions
  - Stochastic simulation for data collection
    - Learn from simulated trajectories, not full grids
    - Scales to high-dimensional state spaces

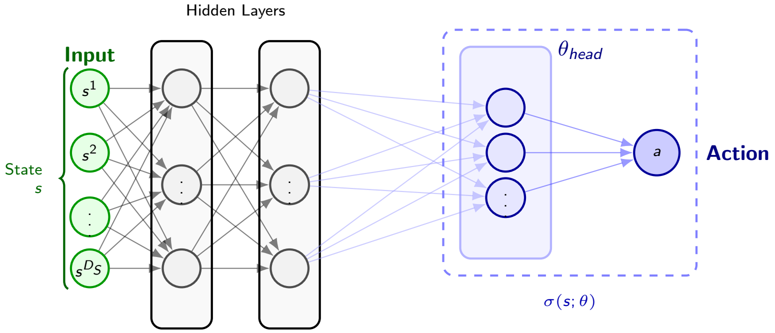
Overview

1. Network Architecture
   - Actor: policy network $\sigma(s;\theta)$ maps states to actions
   - Critic: value network $Q(s,a;\phi)$ evaluates state–action pairs
2. Forward Pass
   - Batch of states $S \in \mathbb{R}^{N\times D_S}$
   - Actor produces actions; critic produces value estimates
3. Objective Functions
   - Actor: maximize expected return via policy gradient
   - Critic: minimize Bellman residual (temporal difference error)
4. Backward Pass and Parameter Updates
   - Compute gradients $\nabla_\theta \mathcal{L}_\theta$ and $\nabla_\phi \mathcal{L}_\phi$
   - Update parameters via gradient descent
5. Implementation Caveats
   - Target networks for training stability
   - Gradient detachment to separate actor and critic updates

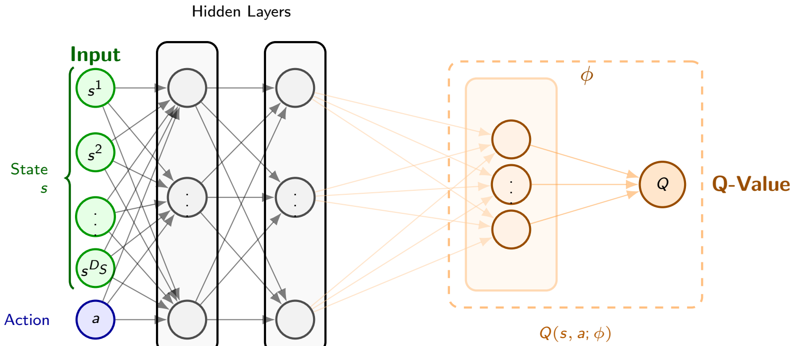
Actor–Critic Methods: Introduction

- Actor–critic methods are a general framework for reinforcement learning (RL) problem
- Particularly suited for Large or continuous state spaces and action spaces
- The framework decomposes learning into policy learning (actor) and value learning (critic)
  - Actor: a parameterized policy $\sigma(a \mid s; \theta)$ that selects actions
  - Critic: a parameterized value function $Q(s, a; \phi)$ that provides an evaluation signal guiding policy improvement
  - The notation $\sigma(a \mid s; \theta)$ denotes a (possibly stochastic) policy: the probability of choosing action a in state s given parameters θ; deterministic policies are a special case that assign probability one to a single action (often written simply as $\sigma(s)$)

Actor Network Architecture



Critic Network Architecture



Forward Pass

- Suppose a batch of states available: $\{s_i\}_{i=1}^N$ where $s_i \in \mathbb{R}^{D_S}$, stacked as $S \in \mathbb{R}^{N \times D_S}$
- Two neural networks:
  - Actor: $\sigma(s; \theta)$
  - Critic: $Q(s, a; \phi)$
- Forward Pass:

$$\begin{aligned} S \in \mathbb{R}^{N \times D_S} &\quad \xrightarrow{\theta} \quad A \in \mathbb{R}^{N \times D_A} \\ (S, A) \in \mathbb{R}^{N \times (D_S + D_A)} &\quad \xrightarrow{\phi} \quad Q \in \mathbb{R}^{N} \end{aligned}$$

- Training vs. evaluation:
  - In *training mode*, the forward pass constructs a computational graph and records gradients with respect to the network parameters
  - In *evaluation mode*, the same forward mappings are applied, but no gradients are recorded

Backward Propagation: Intuition

- In training mode, once a scalar loss $\mathcal{L}$ is defined, the gradient $\nabla_\theta \mathcal{L}$ is computed automatically by back-propagation in any ML package.
- Backpropagation applies the chain rule through the computational graph constructed in the forward pass
- The actor and critic would be updated via gradient descent (although in practice, more stochastic methods are used) as:

$$\theta \leftarrow \theta - \alpha_\theta \nabla_\theta \mathcal{L}_\theta$$

$$\phi \leftarrow \phi - \alpha_\phi \nabla_\phi \mathcal{L}_\phi$$

- The question is how to construct the loss function $\mathcal{L}$ for the actor and critic.

Backward Propagation: Actor Objective (1/2)

- The actor aims to maximize expected discounted returns:

$$J(\theta) = \mathbb{E}_{\tau \sim P_\theta}\big[R(\tau)\big], \qquad R(\tau) \equiv \sum_{k=0}^{\infty} \gamma^k r_k$$

- The policy gradient can be written as:

$$\nabla_\theta J(\theta) = \mathbb{E}\left[\sum_{t=0}^{\infty} \nabla_\theta \log \sigma(a_t \mid s_t; \theta)\, \gamma^t G_t\right], \qquad G_t \equiv \sum_{k=t}^{\infty} \gamma^{k-t} r_k$$

- Realized returns G_t are noisy and depend on the entire future trajectory
- For a given policy $\sigma(a \mid s; \theta)$, the state–action value satisfies:

$$Q^\sigma(s_t, a_t) = \mathbb{E}[\, G_t \mid s_t, a_t \,]$$

Backward Propagation: Actor Objective (2/2)

- By iterated expectations, replacing G_t with $Q^\sigma(s_t, a_t)$ is exact in expectation:

$$\mathbb{E}[\nabla_\theta \log \sigma(a_t \mid s_t; \theta)\, G_t] = \mathbb{E}[\nabla_\theta \log \sigma(a_t \mid s_t; \theta)\, Q^\sigma(s_t, a_t)]$$

- In practice, the true Q^σ is unknown; we use the current critic estimate $Q(s, a; \phi)$
- Sample-based actor loss (depends on current critic parameters ϕ):

$$\mathcal{L}_\theta = -\frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T_i} \log \sigma(a_{i,t} \mid s_{i,t}; \theta)\, \gamma^t Q(s_{i,t}, a_{i,t}; \phi)$$

- The critic $Q(s, a; \phi)$ provides a lower-variance learning signal compared to realized returns

Policy Gradient Derivation* (1/4: Likelihood-Ratio Reformulation)

- Start from the RL objective in integral form:

$$J(\theta) = \mathbb{E}_{\tau \sim P_\theta}[R(\tau)] = \int R(\tau)\, P_\theta(\tau)\, d\tau, \quad \text{where} \quad R(\tau) \equiv \sum_{k=0}^{\infty} \gamma^k r_k$$

- Differentiate under the integral sign:

$$\nabla_\theta J(\theta) = \int R(\tau)\, \nabla_\theta P_\theta(\tau)\, d\tau$$

- Use $\nabla_\theta P_\theta(\tau) = P_\theta(\tau) \nabla_\theta \log P_\theta(\tau)$:

$$\nabla_\theta J(\theta) = \int R(\tau)\, P_\theta(\tau)\, \nabla_\theta \log P_\theta(\tau)\, d\tau$$

- Convert to expectation:

$$\nabla_\theta J(\theta) = \mathbb{E}_{\tau \sim P_\theta}\left[\nabla_\theta \log P_\theta(\tau)\ R(\tau)\right]$$

This is the likelihood-ratio (score-function) trick.

Policy Gradient Derivation* (2/4: Trajectory Likelihood Factorization)

- Trajectory density factorizes as

$$P_\theta(\tau) = p(s_0) \prod_{t=0}^{\infty} \sigma(a_t \mid s_t; \theta)\, P(s_{t+1} \mid s_t, a_t)$$

- Only the policy depends on θ, hence

$$\nabla_\theta \log P_\theta(\tau) = \sum_{t=0}^{\infty} \nabla_\theta \log \sigma(a_t \mid s_t; \theta)$$

Environment transition terms drop out of the gradient.

Policy Gradient Derivation* (3/4: Causality)

- Substitute the log-policy sum and expand $R(\tau)$:

$$\nabla_\theta J(\theta) = \mathbb{E}_{\tau \sim P_\theta}\left[\left(\sum_{t=0}^{\infty} \nabla_\theta \log \sigma(a_t \mid s_t; \theta)\right)\left(\sum_{k=0}^{\infty} \gamma^k r_k\right)\right]$$

- Swap the order of summation:

$$\nabla_\theta J(\theta) = \mathbb{E}_{\tau \sim P_\theta}\left[\sum_{t=0}^{\infty}\sum_{k=0}^{\infty} \nabla_\theta \log \sigma(a_t \mid s_t; \theta)\, \gamma^k r_k\right]$$

- **Causality:** For each t, terms with $k < t$ have zero expectation, since rewards before t do not depend on a_t. Therefore, restrict inner sum to $k \geqslant t$:

$$= \mathbb{E}\left[\sum_{t=0}^{\infty} \nabla_\theta \log \sigma(a_t \mid s_t; \theta)\ \sum_{k=t}^{\infty} \gamma^k r_k\right]$$

- Define the return from t: $G_t \equiv \sum_{k=t}^{\infty} \gamma^{k-t} r_k$, so $\sum_{k=t}^{\infty} \gamma^k r_k = \gamma^t G_t$

Policy Gradient Derivation* (4/4: Surrogate Objective)

- Substitute G_t and collect terms:

$$\nabla_\theta J(\theta) = \mathbb{E}\left[\sum_{t=0}^{\infty} \nabla_\theta \log \sigma(a_t \mid s_t;\theta)\, \gamma^t G_t\right]$$

- **Surrogate objective:**

$$\max_\theta \ \mathbb{E}\left[\sum_{t=0}^{\infty} \log \sigma(a_t \mid s_t;\theta)\, \gamma^t G_t\right]$$

- This surrogate is constructed so that its gradient with respect to θ equals the true policy gradient $\nabla_\theta J(\theta)$.
- We maximize the surrogate because it is differentiable from sampled data and enables gradient-based optimization of the policy.
- Sample-based approximation:

$$\widehat{J}(\theta) = \frac{1}{N}\sum_{i=1}^{N}\sum_{t=1}^{T_i} \log \sigma(a_{i,t} \mid s_{i,t};\theta)\, \gamma^t G_{i,t}$$

Backward Propagation: Critic Objective

- The critic aims to approximate the state–action value function $Q^{\sigma}(s, a)$
- The true Q^{σ} satisfies the Bellman equation:

$$Q^{\sigma}(s_t, a_t) = r_t + \gamma\, \mathbb{E}_{s_{t+1}}\big[Q^{\sigma}(s_{t+1}, a_{t+1})\big]$$

- Sample-based critic loss (mean squared Bellman error):

$$\mathcal{L}_{\phi} = \frac{1}{N}\sum_{i=1}^{N}\big(Q(s_i, a_i; \phi) - (r_i + \gamma\mathbb{E}_{s_{i+1}}[Q(s_{i+1}, a_{i+1}; \phi)])\big)^2$$

Implementation Caveat 1: Target Networks

- Problem: The critic loss depends on its own predictions

$$\mathcal{L}_\phi = \frac{1}{N}\sum_{i=1}^{N}\left(Q(s_i, a_i; \phi) - \underbrace{(r_i + \gamma Q(s_{i+1}, a_{i+1}; \phi))}_{\text{target uses same } \phi}\right)^2$$

- This creates a *moving target* problem $\Rightarrow$ training instability
- Solution: Use separate *target networks* $\phi_{\text{target}}, \theta_{\text{target}}$

$$y_i = r_i + \gamma Q(s_{i+1}, \sigma(s_{i+1}; \theta_{\text{target}}); \phi_{\text{target}})$$

- Target networks are updated slowly:
  - Hard update: Copy parameters every K steps
  - Soft update (Polyak averaging): $\phi_{\text{target}} \leftarrow \tau\phi + (1-\tau)\phi_{\text{target}}$, with $\tau \ll 1$
- Result: Stable targets $\Rightarrow$ more stable learning

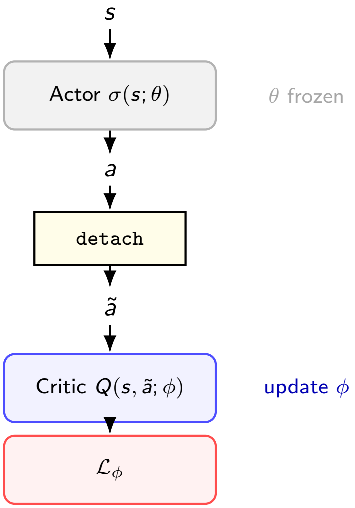
Implementation Caveat 2: Detaching the Action

The `detach` operation creates a copy of a tensor that is disconnected from the computational graph, blocking gradient propagation.

- The action $a = \sigma(s; \theta)$ is produced by the actor
- During critic updates, the action should be treated as fixed input data
- Detaching prevents gradients from flowing back to θ

Implementation:

- `a_detached = a.detach()`
- Pass `a_detached` as input to $Q(s, a; \phi)$



The critic update optimizes ϕ only; the actor is updated separately via policy gradient.