VOI formula in Gaussian case

For a Gaussian variable **w** with mean **m** and variance $\mathbf{r^2}$:

$$
\begin{aligned}
PoV(x) &= E\left(max\{0,w\}\right) \\
&= \int max\{0,w\}p(w)dw \\
&= \int_0^\infty wp(w)dw \\
&= \int_{-\frac{m}{r}}^\infty (m+rz)\phi(z)dz \\
&= m\int_{-\frac{m}{r}}^\infty \phi(z)dz + r\int_{-\frac{m}{r}}^\infty z\phi(z)dz \\
&= m(1-\Phi(-\frac{m}{r})) + r\phi(-\frac{m}{r}) \\
&= m\Phi mr + r\phi mr
\end{aligned}
$$

Since here it uses another standard Gaussian variable z to transform the given w. Therefore, according to standard Gaussian,

$$
\begin{aligned}
\Phi(z) &= \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} \exp^{-\frac{x^2}{2}} dx \\
\phi(z) &= \frac{1}{\sqrt{2\pi}} \exp^{-\frac{x^2}{2}}
\end{aligned}
$$

Additonal notes to understand the second part of the equation

$$
\begin{aligned}
r\int_{-\frac{m}{r}}^\infty z\phi(z)dz &= \int_{-\frac{m}{r}}^\infty z\exp(-\frac{z^2}{2})dz \\
&= \left[-\frac{z}{z}\exp^{-\frac{z^2}{2}}\right]_{-\frac{m}{r}}^\infty - \int_{-\frac{m}{r}}^\infty \exp^{-\frac{z^2}{2}}(-z)dz \\
&= \left[-\frac{z}{z}\exp^{-\frac{z^2}{2}}\right]_{-\frac{m}{r}}^\infty - \int_{-\frac{m}{r}}^\infty \exp^{-\frac{z^2}{2}} d(-\frac{z^2}{2}) \\
&= \left[-\frac{z}{z}\exp^{-\frac{z^2}{2}}\right]_{-\frac{m}{r}}^\infty - \left[\exp^{-\frac{z^2}{2}}\right]_{-\frac{m}{r}}^\infty
\end{aligned}
$$

$$
\begin{aligned}
r\int_{-\frac{m}{r}}^\infty z\phi(z)dz &= \int_{-\frac{m}{r}}^\infty z\exp(-\frac{z^2}{2})dz \\
&= -\int_{-\frac{m}{r}}^\infty \exp(-\frac{z^2}{2})d(-\frac{z^2}{2}) \\
&= \left[\exp^{-\frac{z^2}{2}}\right]_{-\frac{m}{r}}^\infty
\end{aligned}
$$

VOI for gaussian perfect information

Prior model for profits:

$$p(x) = N(0,\Sigma),\ \Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \tag{1}$$

Now, assume we know perfect infomormation about 1 project, thus it becomes:

$$\begin{aligned}
y &= x_1 \\
p(x_1) &= N(0,1) \\
E(x_1) &= x_1 \\
E(x_2|x_1) &= E(x_1) + \Sigma_{x_2x_1}\Sigma_{x_2}^{-1}(x_1 - E(x_1)) = 0 + \rho \cdot 1 \cdot (1-0) = \rho x_1 \\
Var(x_1) &= 0 \\
Var(x_2|x_1) &= \Sigma_{x_2} - \Sigma_{x_2x_1}\Sigma_{x_2}^{-1}\Sigma_{x_1x_2} = 1 - \rho \cdot 1 \cdot \rho = 1 - \rho^2
\end{aligned} \tag{2}$$

0.1 Imperfect information

As for imperfect information, measurement error τ^2 feed into the system by adding additional variance, therefore, it becomes:

$$\begin{aligned}
y &= x + N(0,\tau^2\boldsymbol{I}) \\
p(y) &= N(\boldsymbol{0},\tau^2\boldsymbol{I}+\Sigma) = N(\boldsymbol{0},\boldsymbol{C}) \\
E(x|y) &= \Sigma\boldsymbol{C}^{-1}\boldsymbol{y} \\
PoV &= E\{max\{E(x_i|y),0\}\} = E(max(u_i,0)) \\
u &= \Sigma\boldsymbol{C}^{-1}y \sim N(0,\Sigma\boldsymbol{C}^{-1}\boldsymbol{C}\boldsymbol{C}^{-1}\Sigma) \sim N(0,\Sigma\boldsymbol{C}^{-1}\Sigma) \\
\boldsymbol{S} &= \begin{pmatrix} S_{1,1} & S_{1,2} \\ S_{2,1} & S_{2,2} \end{pmatrix} \\
u_1 &\sim N(0,S_{11}) \\
u_1 &\sim N(0,S_{22}) \\
PoV(y) &= \Sigma_{i=1}^{n}(\mu_i)\Phi(\frac{\mu_i}{\sqrt{r_{ii}}}) + \sqrt{r_{ii}}\phi(\frac{mu_i}{\sqrt{r_{ii}}}) = 0 + r\phi(0) = 0 + S_{ii}/\sqrt{2\pi} = \frac{\sqrt{S_{11}}+\sqrt{S_{22}}}{\sqrt{2\pi}}
\end{aligned} \tag{3}$$