Project 1: Markov chain Monte Carlo techniques

Please send your solutions to andrea.riebler@ntnu.no. **Deadline:** February 17th.

1 Metropolis-Hastings (MH) for bivariate densities

We will consider three different bivariate target densities for $\boldsymbol{x} = (x_1, x_2)^t$:

1. Standard Gaussian distribution with correlation:

$$\pi(\boldsymbol{x}) = \frac{1}{2\pi|\boldsymbol{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}\boldsymbol{x}^t\boldsymbol{\Sigma}^{-1}\boldsymbol{x}\right),$$

   where $\boldsymbol{\Sigma}$ has 1 on the diagonal and $\rho = 0.9$ on the off-diagonal.

2. A multimodal density:

$$\pi(\boldsymbol{x}) = \sum_{i=1}^{3} w_i \frac{1}{2\pi|\boldsymbol{\Sigma}_i|^{1/2}} \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_i)^t\boldsymbol{\Sigma}_i^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_i)\right),$$

   which is a mixture of Gaussian densities where weights are $w_1 = 1/3$, $w_2 = 1/3$ and $w_3 = 1/3$, means are $\boldsymbol{\mu}_1 = (-1.5, -1.5)^t$, $\boldsymbol{\mu}_2 = (1.5, 1.5)^t$ and $\boldsymbol{\mu}_3 = (-2, 2)^t$, and covariance matrices $\boldsymbol{\Sigma}_i$ all have correlation 0 and variance $\sigma_1^2 = 1$, $\sigma_2^2 = 1$ and $\sigma_3^2 = 0.8$.

3. A volcano-shaped density:

$$\pi(\boldsymbol{x}) \propto \frac{1}{2\pi} \exp\left(-\frac{1}{2}\boldsymbol{x}^t\boldsymbol{x}\right)(\boldsymbol{x}^t\boldsymbol{x} + 0.25),$$

We will explore each one with random walk Metropolis–Hastings (MH) algorithms, Langevin MH and Hamiltonian MH.

1.1 Plotting

Visualize the three densities on a grid covering $[-5, 5] \times [-5, 5]$. The grid spacing could be 0.1, which gives $101 \times 101 =$ grid cells. Note that the volcano-density in 3. is not normalized, but the relative levels are still representative.

1.2 Random walk MH

- Implement a random walk MH sampler for the Gaussian density in 1. above. Try tuning parameter $\sigma = 0.5$ in the random walk proposal, and then experiment with a few others. Keep track of the mean acceptance rate. Inspect the autocorrelation of the resulting Markov chain, and use this along with trace plots of x_1 and x_2 to evaluate the suitability of the different levels of the tuning parameter.

- Implement a random walk MH sampler for the multimodal density in 2. above. Repeat the same algorithm evaluation as for the Gaussian density above.

- Implement a random walk MH sampler for the volcano-shaped density in 3. above. Repeat the same algorithm evaluation as above.

1.3 Langevin MH

- Implement a Langevin MH sampler for the Gaussian density in 1. above. Try tuning parameter $\sigma = 0.5$ in the Langevin proposal, and then experiment with a few others. Keep track of the mean acceptance rate. Inspect the autocorrelation of the resulting Markov chain, and use this along with trace plots of x_1 and x_2 to evaluate the suitability of the different levels of the tuning parameter. Compare also with the Random Walk MH in the previous section. Keep in mind that each iteration of the Langevin MH sampler relies on the evaluations of both target and derivative.

- Implement a Langevin MH sampler for the multimodal density in 2. above. Repeat the same algorithm evaluation as for the Gaussian density above.

- Implement a Langevin MH sampler for the volcano-shaped density in 3. above. Repeat the same algorithm evaluation as above.

1.4 Hamiltonian MH

- Implement a Hamiltonian MH sampler for the Gaussian density in 1. above. Set the momentum proposal to $\boldsymbol{z} \sim N(0, \boldsymbol{I})$. Try tuning parameter $\epsilon = 0.1$ in the leap-frog scheme for $T = 10$ steps, and then experiment with a few other settings. Keep track of the mean acceptance rate. Approximate the integrated autocorrelation of the resulting Markov chain, and use this along with trace plots of x_1 and x_2 to evaluate the suitability of the different levels of the tuning parameter. Compare also with the Random Walk and Langevin MH in the previous sections. Keep in mind that each iteration of the Hamiltonian MH sampler requires several evaluations of both target and derivative in the leap-frog calculation.

- Implement a Hamiltonian MH sampler for the multimodal density in 2. above. Repeat the same algorithm evaluation as for the Gaussian density above.

- Implement a Hamiltonian MH sampler for the volcano-shaped density in 3. above. Repeat the same algorithm evaluation as above.

2 RStan: Gamma-Poisson Hierarchical model

We consider an example from George, et al. (Scandinavian Journal of Statistics, 20:147–156, 1993) concerning the number of failures of ten power plants. The data are as follows:

Pump	1	2	3	4	5	6	7	8	9	10
y	5	1	5	14	3	19	1	1	4	22
t	94.3	15.7	62.9	126.0	5.24	31.4	1.05	1.05	2.1	0.5

Here, y_i is the number of times that pump i failed and t_i is the operation time of the pump (in 1000s of hourse). Pump failures are modelled as:

$$y_i \mid \lambda_i \sim \text{Poisson}(\lambda_i t_i)$$

Conjugate prior for λ_i:

$$\lambda_i \mid \alpha, \beta \sim \text{Gamma}(\alpha, \beta)$$

Hyper-prior on α and β:

$$\alpha \sim \text{Exp}(1.0) \qquad\qquad \beta \sim \text{Gamma}(0.1, 1.0)$$

- Install RStan on your computer. For instructions see `https://github.com/stan-dev/rstan/wiki/RStan-Getting-Started`
- Write a Stan program for the model presented above and save it in "pump.stan". Be sure that your Stan programs ends in a blank line without any characters including spaces and comments.
- Prepare the data in R and fit the model in R using the function `stan` and save the results in an object called `fit`.
- Check the effective sample size (ESS) estimates and traceplots to assess convergence. Inspect and interpret your posterior output.