A new kinematical conservation law for glaciers

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ABSTRACT. We derive a new form of the kinematical equation for glacier surfaces, one which applies to any fluid layer which has a moving surface which experiences accumulation and/or ablation. This kinematical conservation law says that over any interval of time, and in any map-plane region, the change in the *square* of the ice thickness is equal to certain integrals of surface velocity, surface slope, and climatic (surface) mass balance rate, but weighted by ice thickness. The law holds on the ice, across the ice margin, and in ice-free areas, and it has no lateral boundary condition at the ice margin, so glacier outlines are not needed for its applications. The law is derived from a complementarity principle combining the standard surface kinematical equation with an inequality which applies in ice-free areas. We illustrate two applications of the discretized form of the law, the first being attribution to ice dynamics or surface balance of modeled thickness changes in numerical ice sheet model runs, and the second being the inversion of kinematic data, here from a synthetic test case, to determine time-dependent surface balance and glacier outlines.

INTRODUCTION

Glaciology, and classical mechanics generally, distinguishes between kinematics and dynamics. While both are addressed quantitatively, kinematics describes the motion itself while dynamics explains that motion in terms of forces (stresses). A glacier's surface elevation changes through an imbalance between addition and removal of ice mass, a function of local atmospheric processes and runoff, and the ice flow velocity at the surface, and thus the manner in which a glacier interacts with the atmospheric climate is substantially kinematic. Specifically, the surface kinematical equation (SKE; equation (4))[1] describes changes in the glacier's surface elevation independently of the stresses within the ice.

On the other hand the SKE only applies on the surface of the ice. Though Eulerian (Greve and Blatter, 2009) in its usual form, the SKE ceases to apply in map-plane locations from which a glacier has retreated, and it appears as a new equation which must be satisfied in a re-glaciated area. By contrast, the more fundamental mass, momentum, and energy conservation equations apply everywhere, even when or where there is no ice.

In this paper we propose an integral equation restatement of the SKE, a differential equation. The new form might be regarded as more fundamental because it applies everywhere, but we do not claim it has the deeper status of the mass, momentum, and energy conservation principles, or indeed any different status than the SKE itself. The new integral form, which applies whether or not ice is present, thus disregarding uncertainties in ice margin position, may prove useful in the automatic processing of noisy surface observations available through remote sensing.

To set notation for the new form, suppose s is the ice surface elevation, b is the bed elevation, h is the ice thickness, and $\mathbf{u}$ is the 3D velocity within the ice. Denote by a the climatic mass balance (CMB; Cogley and others, 2011), the local, typically-annualized, difference between precipitation and runoff. In these terms the following kinematical conservation law (KCL) holds for any times $t_0 < t_1$ and any map-plane region Ω:

$$\begin{aligned}\frac{1}{2}\int_\Omega h(t_1,\mathbf{x})^2\,d\mathbf{x} &= \frac{1}{2}\int_\Omega h(t_0,\mathbf{x})^2\,d\mathbf{x} \\ &\quad + \int_{t_0}^{t_1}\int_\Omega \mathbf{u}|_s(t,\mathbf{x})\cdot\mathbf{n}_s(t,\mathbf{x})\,h(t,\mathbf{x})\,d\mathbf{x}\,dt \\ &\quad + \int_{t_0}^{t_1}\int_\Omega a\,(t,\mathbf{x})\;h(t,\mathbf{x})\,d\mathbf{x}\,dt\end{aligned} \tag{1}$$

Here $\mathbf{x} = (x_1, x_2)$ denotes horizontal, geoid-parallel coordinates. (We will use z to denote the vertical coordinate.) Under the assumption that the ice surface is smooth, $\mathbf{n}_s = \langle -\nabla_\mathbf{x} s, 1\rangle$ denotes the upward surface normal, and $\mathbf{u}|_s$ denotes the 3D velocity evaluated at the ice surface.

[1]The SKE is often called the kinematic boundary condition (Greve and Blatter, 2009; van der Veen, 2013). However, unlike the stress boundary conditions of the Stokes model, for example, it is not the boundary condition of any partial differential equation.

abbrev.	*expansion*	*equation*
CMB	climatic mass balance	
KCL	kinematical conservation law	(1)
NCP	nonlinear complementarity problem	(7)
SKE	surface kinematical equation	(4)

Table 1. Abbreviations used in this article.

To state equation (1) in words, the square of the ice thickness is a conserved quantity in any map-plane region Ω. This quantity changes over time only according to two sources, the ice flow at the surface which is normal to the surface ($\mathbf{u}|_s \cdot \mathbf{n}_s$) and the CMB, each weighted by the ice thickness.

We emphasize that equation (1) holds at *all* times and places. If Ω covers an unglaciated and never-glaciated valley then equation (1) says "$0 = 0$", but it is valid. More significantly, it applies in a region Ω where the glacier margin moves or is uncertain because of debris cover, for example. Note that we consider only land-based glaciers for simplicity; marine ice sheets have more-complicated kinematics because of the non-trivial mass flux through a calving front.

The main purpose of this work is to introduce a framework for glacier kinematics in which precise glacier outlines are not needed as inputs to mass balance analysis. The next section derives equation (1), after which we discuss two applications. The second application, namely inversion for CMB from kinematical observations, pioneered by Gudmundsson and Bauder (1999) using the SKE, can be extended using the KCL to a method which generates both time-dependent CMB values and glacier outlines as outputs.

Table 1 lists our few abbreviations.

DERIVATION

Our brief derivation of Equation (1) goes via a system formed from the SKE and certain inequalities which are unsurprising to glaciologists. Though this system appears in the glaciers literature, the inequalities are rarely emphasized.

Let $b(\mathbf{x})$ be the continuous-differentiable bed elevation function, assumed time-independent for simplicity. The ice surface elevation $s(t,\mathbf{x})$, assumed to be continuously-differentiable on the ice, can be defined everywhere in Ω as a continuous function, namely by extending with $s = b$ in ice-free areas. An obvious inequality holds everywhere in Ω and at all times:

$$s \ge b. \tag{2}$$

Note that $\mathbf{n}_s = \langle -\nabla_{\mathbf{x}} s, 1\rangle$ is defined almost everywhere on Ω, but not necessarily at margin locations where the surface gradient $\nabla_{\mathbf{x}} s$ may be discontinuous. In mathematical language, we assume from here on that the glacier margin is a continuous map-plane curve with measure zero.

Let z be the vertical, positive-upward coordinate and denote the three-dimensional ice velocity by $\mathbf{u}(t,\mathbf{x},z)$. We extend the surface value of the velocity to all of Ω in the obvious manner:

$$\mathbf{u}|_s(t,\mathbf{x}) = \begin{cases} \mathbf{u}(t,\mathbf{x},s(t,\mathbf{x})), & s(t,\mathbf{x}) > b(\mathbf{x}), \\ \mathbf{0}, & \text{otherwise.} \end{cases} \tag{3}$$

An integrable discontinuity of $\mathbf{u}|_s$ is allowed at the ice margin.

Let $a(t,\mathbf{x})$ be the value of the CMB on the ice surface. The well-known surface kinematical equation (SKE; Greve and Blatter, 2009, equation (5.21)) now describes the motion of the ice surface:

$$\frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s = a \qquad \textit{(on the ice)}. \tag{4}$$

On the glacier surface the CMB function $a(t,\mathbf{x})$, the sum of accumulation and ablation, is measured by placing a stake at map-plane location $\mathbf{x}$, resetting it to location $\mathbf{x}$ to compensate for horizontal ice motion. Here the CMB is described in ice-equivalent thickness per time units. If the CMB also depends on surface elevation z the symbol $a(t,\mathbf{x})$ used herein should be replaced by $a|_s(t,\mathbf{x}) = a(t,\mathbf{x},s(t,\mathbf{x}))$, that is, by a evaluated on the ice surface.

According to any energy-balance or parameterized model of precipitation, melt, and runoff, the details of which are not important here, $a(t,\mathbf{x})$ can be extended to ice-free locations. That is, the function $a(t,\mathbf{x})$ is taken here to be either the measurable CMB or its potential value at $(t,\mathbf{x})$, supposing ice were present then and there. For example, at a warm ice-free location like a tropical, low-altitude desert, $a(t,\mathbf{x})$ would have a very-negative value evaluated by applying an energy balance model to an entirely-hypothetical surface of ice at that location.

With these understandings about $\mathbf{u}|_s$, $\mathbf{n}_s$, and a, the inequality

$$\frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s - a \ge 0 \tag{5}$$

applies everywhere in Ω. In ice-free locations, because $s = b$ is time-independent and $\mathbf{u}|_s = 0$, inequality (5) reduces to $a \le 0$. Logically we may also say that if $a(t,\mathbf{x}) > 0$ at some time-space location $(t,\mathbf{x})$, and if a is a continuous function, then ice must be present at that location ($s(t,\mathbf{x}) > b(\mathbf{x})$)

and, since SKE (4) will then hold on the ice surface, so (5) applies there as well.

The combination of inequalities (2) and (5), along with the SKE (4), forms a *nonlinear complementarity problem* (NCP; Bueler, 2021; Facchinei and Pang, 2003), a mathematical framework that has been studied for the shallow ice approximation (Bueler, 2016; Calvo and others, 2002). An essentially-equivalent variational inequality form, again for the shallow ice approximation, has also been considered (Jouvet and Bueler, 2012).

By definition, a finite-dimensional NCP on a vector space $\mathcal{V} = \mathbb{R}^k$, for a function $f : \mathcal{V} \to \mathcal{V}$, is a system combining the three statements

$$v \geq 0, \quad f(v) \geq 0, \quad vf(v) = 0, \tag{6}$$

for all $v \in \mathcal{V}$. Note that, because of the first two inequalities, the third statement of (6) can be regarded either entry-wise ($v_i f(v)_i = 0$ for $i = 1, \dots, k$) or as an inner product ($\langle v, f(v)\rangle = 0$); they are equivalent. The equation $vf(v) = 0$ is called *complementarity* because it says that either an entry of v is zero or the corresponding entry of $f(v)$ is zero.

The surface kinematic NCP for glaciers applies in a space of surface elevation functions on a map-plane region Ω. By inequalities (2), (5), and equality (4), we have

$$s - b \geq 0, \tag{7a}$$

$$\frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s - a \geq 0, \tag{7b}$$

$$(s-b)\left(\frac{\partial s}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s - a\right) = 0 \tag{7c}$$

at almost every point of Ω. Equation (7c) is key; it says that either there is no ice at a location ($s = b$) or there is ice and the SKE (4) applies. Inequalities (7a) and (7b) can be regarded as obvious statements of glaciological consistency, but the reasons for (7b) are not often explicated in glaciers literature. In fact, a little thought should convince the reader that system (7) fully-expresses the usual, contextual meaning of the SKE in glaciology. It is understood that at an ice margin, equivalently when and where ice transport into an ablation area no longer sustains a glacier, the SKE itself will cease to apply, but then $s = b$ and $a \leq 0$ do hold. In other words, in an ice-free location the NCP system (7) says that the conditions are not suitable for the current existence (or initiation) of a glacier.

By itself equation (4), while essential to glaciology, does not apply universally. By contrast, all the statements in NCP system (7) *are* universal as they apply both at glacier surfaces and at ice-free locations. However, NCP (7) is not a conservation law; it is stated using inequalities. Furthermore, the numerical application of NCPs is not familiar for glacier modellers, though certain tools for this do exist (Bueler, 2016). Instead we seek an *equality* which is as universal as system (7), and the result is equation (1). The derivation of (1) is quite simple: we integrate complementarity statement (7c) in time and space, as follows.

Consider map-plane location $\mathbf{x}$ and choose an interval of time $[t_0, t_1]$. For notational simplicity in this paragraph, let $\phi = \mathbf{u}|_s \cdot \mathbf{n}_s + a$. Noting that b is time-independent, for ice thickness $h = s - b$ we have $\partial h/\partial t = \partial s/\partial t$. Now integrate (7c) over time, by parts:

$$\begin{aligned} 0 &= \int_{t_0}^{t_1} h(t,\mathbf{x}) \left[\frac{\partial s}{\partial t}(t,\mathbf{x}) - \phi(t,\mathbf{x})\right] dt \qquad (8) \\ &= \int_{t_0}^{t_1} \frac{\partial}{\partial t}\left[\frac{1}{2} h(t,\mathbf{x})^2\right] dt - \int_{t_0}^{t_1} h(t,\mathbf{x})\phi(t,\mathbf{x})\, dt \\ &= \frac{1}{2} h(t_1,\mathbf{x})^2 - \frac{1}{2} h(t_0,\mathbf{x})^2 - \int_{t_0}^{t_1} h(t,\mathbf{x})\phi(t,\mathbf{x})\, dt. \end{aligned}$$

Equation (8) applies almost everywhere, but recall that $\mathbf{u}|_s$ and $\mathbf{n}_s$ may be discontinuous at an ice margin. Therefore we integrate over any map-plane region Ω, yielding equation (1).

The logic can be reversed to show that KCL (1) implies equation (7c). Taking $t_1 = t$ and computing time derivatives of both sides of (1), then using the fundamental theorem of calculus, and then collecting on the left we find

$$\int_\Omega h \left(\frac{\partial h}{\partial t} - \mathbf{u}|_s \cdot \mathbf{n}_s - a\right) d\mathbf{x} = 0. \tag{9}$$

As Ω is arbitrary in (9), the integrand must be zero everywhere. Noting again that $\partial h/\partial t = \partial s/\partial t$, complementarity equation (7c) follows, and in icy areas $\{\mathbf{x} : h(t,\mathbf{x}) > 0\}$, the SKE (4) also follows.

APPLICATIONS

FIXME two suggested applications; formulas here and then results in the next section; our second application of inversion for CMB is not new but parallels (Gudmundsson and Bauder, 1999); the form of the equations and the ability to ignore the glacier outline is different, however

1: Attribution of ice geometry changes in model runs

FIXME form of (1) suggests it; use one of Andy's long-time Greenland runs to illustrate?

2A: Inversion of kinematic observations for CMB (fully-observed surface velocity)

In this application we start from what is observable through remote sensing. From optical, radar, or laser altimetry the surface elevation (DEM) is generally observable. Furthermore, through optical image-comparison techniques, or via interferometric synthetic aperature radar the surface velocity may also be observable. We will also suppose that the bed elevation is observed, e.g. through ice-penetrating radar, and thus the ice thickness is observed as well.

The vertical component of the surface velocity is more difficult to measure by remote sensing than the horizontal components. In the current application we assume that the full 3D surface velocity vector is observed, with consequent simplificaton of the equations, but in application 2B we suppose that only the horizontal components are observed. In that case we reconstruct the vertical velocity from a simplified model of ice flow, and adapt formulas accordingly.

In this context, wherein the variables s, h, and $\mathbf{u}|_s$ are observed, we describe how to invert for the CMB. KCL equation (1) can be rewritten

$$\begin{aligned}\int_{t_0}^{t_1}\int_\Omega a(t,\mathbf{x})\,h(t,\mathbf{x})\,d\mathbf{x}\,dt &= \frac{1}{2}\int_\Omega h(t_1,\mathbf{x})^2 - h(t_0,\mathbf{x})^2\,dx \\ &\quad - \int_{t_0}^{t_1}\int_\Omega \mathbf{u}|_s(t,\mathbf{x})\cdot\mathbf{n}_s(t,\mathbf{x})\,h(t,\mathbf{x})\,d\mathbf{x}\,dt\end{aligned} \tag{10}$$

In this form the observed quantities are on the right and the unknown $a(t,\mathbf{x})$ is on the left, with a known thickness coefficient.

FIXME assume noisy, time-dependent data on $[0,T]\times\Omega$ for ice thickness and ice surface velocity, with observations at times t_q for $\ell = 0,1,\dots,Q$

FIXME need to interpolate data in time for $h(t,\mathbf{x})$, $s(t,\mathbf{x})$, $\mathbf{u}|_s(t,\mathbf{x})$, $\mathbf{n}_s(t,\mathbf{x})$ on intervals $[t_q,t_{q+1}]$; interpolated fields are denoted with hats

FIXME assume form for time- and space-dependence for a; we compare two modes:

I. $a_j(t) = c_j + d_j t$

II. $$a(t,x) = \sum_{k=0}^{K-1}\sum_{r=0}^{R-1} c_{k,r}\cos\left(\frac{\pi r t}{T}\right)\cos\left(\frac{\pi k x}{2L}\right)$$

with J cells; mode I has given form independently for each cell ω_j while mode II is "spatially-correllated"

FIXME get $n = 2J$ total unknowns $\{c_j, d_j\}$ in mode I and $n = KR$ total unknowns $\{c_{k,r}\}$ in mode II

FIXME $m = JQ$ applications of (10) will be done over spacetime cells $[t_q,t_{q+1}]\times\omega_j$ for $q = 0,\dots,Q-1$ and $j = 0,\dots,J-1$

FIXME need overdetermined system so want $m \gg n$

FIXME in mode I set up linear system $M\mathbf{a} = \mathbf{d}$ for each cell ω_j; the matrix M for a cell has Q rows and 2 columns and entries

$$M_{q,0} = \int_{t_q}^{t_{q+1}}\int_{\omega_j}\hat h(t,\mathbf{x})\,d\mathbf{x}\,dt \tag{11}$$

$$M_{q,1} = \int_{t_q}^{t_{q+1}}\int_{\omega_j} t\,\hat h(t,\mathbf{x})\,d\mathbf{x}\,dt \tag{12}$$

and right-hand side

$$\begin{aligned}d_j &= \frac{1}{2}\int_{\omega_j}\hat h(t_{q+1},\mathbf{x})^2 - \hat h(t_q,\mathbf{x})^2\,dx \\ &\quad - \int_{t_q}^{t_{q+1}}\int_{\omega_j}\hat h(t,\mathbf{x})\hat{\mathbf{u}}|_s(t,\mathbf{x})\cdot\hat{\mathbf{n}}_s(t,\mathbf{x})\,d\mathbf{x}\,dt\end{aligned} \tag{13}$$

FIXME in mode II ...

FIXME mode I overdetermined system with $M \in \mathbb{R}^{Q\times 2}$

$$M\mathbf{a} = \mathbf{d} \tag{14}$$

assume M full rank (easy); solve by SVD

FIXME null space of $M^\top$ is like a map of unglaciated area; explain procedure of projecting spacetime cell indicator functions onto null space of $M^\top$; this gives spacetime cells which are ice free

2B: Inversion of kinematic observations for CMB (observed horizontal velocity only)

FIXME will show how to separate the vertical surface velocity; this is done by (Gudmundsson and Bauder, 1999) but our integral form looks different

FIXME now need model of glacier interior, not just surface; assume incompressibility and negligible basal melt, thus $w|_b(t,\mathbf{x}) = 0$

$$\begin{aligned}w|_s(t,\mathbf{x}) &= \int_{b(t,\mathbf{x})}^{s(t,\mathbf{x})}\frac{\partial w}{\partial z}(t,\mathbf{x},z)\,dz \\ &= -\int_{b(t,\mathbf{x})}^{s(t,\mathbf{x})}\frac{\partial u}{\partial x}(t,\mathbf{x},z) + \frac{\partial v}{\partial y}(t,\mathbf{x},z)\,dz\end{aligned}$$

FIXME assume vertical profile form

$$\begin{aligned}\frac{\partial u}{\partial x}(t,\mathbf{x},z) &= \phi\left(\frac{z-b(t,\mathbf{x})}{h(t,\mathbf{x})}\right)\frac{\partial u|_s}{\partial x}(t,\mathbf{x}) \\ \frac{\partial v}{\partial y}(t,\mathbf{x},z) &= \phi\left(\frac{z-b(t,\mathbf{x})}{h(t,\mathbf{x})}\right)\frac{\partial v|_s}{\partial y}(t,\mathbf{x})\end{aligned}$$

for some increasing profile function $\phi(\zeta)$ on $0 \le \zeta \le 1$ such that $\phi(1) = 1$

FIXME the value of $\phi(0)$ is the fraction which is sliding; let $\bar\phi = \int_0^1 \phi(\zeta)\,d\zeta$; expect $\frac{1}{2} < \bar\phi \le 1$

FIXME let $\mathbf{U}|_s = \langle u|_s, v|_s\rangle$ be the horizontal velocity at the surface, and denote its horizontal divergence by $\nabla\cdot_{\mathbf{x}}\mathbf{U}|_s = \frac{\partial u|_s}{\partial x} + \frac{\partial v|_s}{\partial y}$

FIXME suppressing $t, \mathbf{x}$ for clarity, the above assumptions and the substitution $\zeta = (z-b)/h$ give

$$w|_s = -(\nabla \cdot_{\mathbf{x}} \mathbf{U}|_s) \int_b^s \phi\left(\frac{z-b}{h}\right)\, dz = -\bar\phi h\, \nabla \cdot_{\mathbf{x}} \mathbf{U}|_s$$

FIXME now integrating by parts (Green's theorem),

$$\begin{aligned}\int_\Omega w|_s h\, d\mathbf{x} &= -\bar\phi \int_\Omega h^2 (\nabla \cdot_{\mathbf{x}} \mathbf{U}|_s)\, d\mathbf{x} \\ &= -\bar\phi \int_\gamma h^2 \mathbf{U}|_s \cdot d\boldsymbol{\ell} + \bar\phi \int_\Omega \nabla_{\mathbf{x}}(h^2) \cdot \mathbf{U}|_s\, d\mathbf{x}\end{aligned}$$

where γ is the oriented bounding curve of Ω, with length element $d\boldsymbol{\ell}$

FIXME this allows us to rewrite (10) using only the horizontal surface velocity on the right side

$$\begin{aligned}\int_{t_0}^{t_1} \int_\Omega & a(t,\mathbf{x})\, h(t,\mathbf{x})\, d\mathbf{x}\, dt \qquad (15) \\ &= \frac{1}{2}\int_\Omega h(t_1,\mathbf{x})^2 - h(t_0,\mathbf{x})^2\, dx \\ &\quad - \int_{t_0}^{t_1}\int_\Omega \mathbf{U}|_s(t,\mathbf{x}) \cdot \mathbf{S}(t,\mathbf{x})\, h(t,\mathbf{x})\, d\mathbf{x}\, dt \\ &\quad + \bar\phi \int_{t_0}^{t_1}\int_\gamma h(t,\mathbf{x})^2 \mathbf{U}|_s(t,\mathbf{x}) \cdot d\boldsymbol{\ell}\, dt\end{aligned}$$

where

$$\mathbf{S} = 2\bar\phi \nabla_{\mathbf{x}} h - \nabla_{\mathbf{x}} s = (2\bar\phi - 1)\nabla_{\mathbf{x}} s - 2\bar\phi \nabla_{\mathbf{x}} b$$

is a weighted combination of the surface and thickness gradients

FIXME expect (15) to be more sensitive that (10) to data noise for two reasons; first ∇b is involved; second $\int_\gamma$ is not as averaging as $\int_\Omega$

FIXME (15) imposes a "model error" not present in (10), from assumed incompressibility and vertical profile

RESULTS

FIXME for application 1 use PISM Greenland run?

FIXME for applications 2A (modes I and II) and 2B (modes I and II) we use a test case with synthetic geometry and 1D space applying the Bueler profile (Greve and Blatter, 2009; van der Veen, 2013) with added time variation; discretize $R = [-L, L]$ into J cells ω_j

FIXME for application 2B (mode I) we use a real-data case from Landsat and Alaska? (need help with this)

DISCUSSION AND CONCLUSION

FIXME

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FIXME Thomas Frank lent a generous ear

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Appendix. Formulas for synthetic test case

FIXME see PDF: *A synthetic, time-dependent glacier for testing surface kinematical inversion*; in `projects/2022/kinematic/` directory of `https://github.com/bueler/mccarthy`