Numerical Optimisation Project Proposal

Subsampled Magnetic Resonance Imaging (MRI): Problem Setting

Keshav Iyengar

To start the formulation of the forward problem, the static magnetic field strength

$$\vec{B_o}(\vec{r}) = B_o(\vec{r})\widehat{k}$$

Induces a net magnetization of

$$\vec{M} = \vec{M}_x\widehat{i} + \vec{M}_y\widehat{j} + \vec{M}_z\widehat{k}$$

Where $r = (\vec{x}, y, z)$ is a 3D spatial position. However, generally, the static magnetic field also varies in time and position resulting in:

$$\vec{M}(\vec{r}, t) = \vec{M}_x(\vec{r}, t)\widehat{i} + \vec{M}_y(\vec{r}, t)\widehat{j} + \vec{M}_z(\vec{r}, t)\widehat{k}$$

The MRI procedure itself consists of two main parts, the excitation and readout. During excitation the static magnetic field $\vec{B_o}$ is designed to move the magnetization $\vec{M}(\vec{r}, t)$ over some transverse slice away from the original equilibrium position. This is represented as a complex number for convenience.

$$\vec{M}(\vec{r}, t) = \vec{M}_x(\vec{r}, t)\widehat{i} + i\vec{M}_y(\vec{r}, t)\widehat{j}$$

During the readout, the applied static field is manipulated to make the transverse magnetization clear. This relationship is defined by the Bloch equations and not described in this report. The main points are the precession and transverse relaxation. The precession frequency is described in the Larmar relation:

$$w = \gamma\left|\vec{B}\right|$$

Where w is the precession frequency and γ is the gyromagnetic ratio. Also during the readout,

$$\vec{B}(\vec{r}, t) = B_z(\vec{r}, t)\widehat{k}$$

Thus combining the above equation with the Larmar relation we have that,

$$w(\vec{r}, t) = B_z(\vec{r}, t)\widehat{k}$$

The precession frequency is dependant on the magnetic field strength and changes both in position and time dimensions.

Define the evolution of the transverse magnetization in time during readout in the following definition.

$$\vec{M}(\vec{r}, t) = f(\vec{r})e^{-t/T_2{}^*(\vec{r})}exp(\ -i\gamma\int_0^t B_z(\vec{r}, t)dt\ )$$

Where $f(\vec{r}) = \vec{M}(\vec{r}, 0)$ is the transverse magnetization just after excitation. Because the magnetization will reduce over time due to individual hydrogen atom spins becoming out of phase, the $T_2{}^*$ term is used. Generally this term is in order of milliseconds and phase variations are in MHz.

For data acquisition, coil(s) are used to capture the electric potential generated by the magnetic field in the following model.

$$v(t) = real(\int c(\vec{r})M(\vec{r}, t)d\vec{r} \,)$$

Where $c(\vec{r})$ is the coil response and reduces with increased distance from the coil itself. The coil response is amplified and modulated at preferably the precession frequency $w_o = \gamma B_o$. The signal can then defined in the following as a compitation of in-phase and quadrature signals.

$$s(t) = I(t) + iQ(t)$$

In practice, the quadrature and inphase signals are sampled separately.

$$s(t) = I(m_{\delta T}) + iQ(m_{\delta T}) \;\; m = 1, 2...n_d$$

Where n_d is the number of samples.

Applying a low pass filter of the electric potential attenuated at the precession frequency, we have:

$$s(t) = e^{iw_o t}(\int c(\vec{r})M(\vec{r}, t)d\vec{r} \,)$$

The forward model can now be defined by combining the above equation with temporal evolution of the transverse magnetization.

$$s_l(t) = \int c_l(\vec{r})f(\vec{r})e^{-t/T_2{}^*(\vec{r})}e^{-i\phi(\vec{r},t))}d\vec{r} \,)$$

$$\phi(\vec{r}, t') = \int_0^{t'} (\gamma B_z(\vec{r}, t') - w_o)dt'$$

where $l = 1, 2..L$ and L is the total number of coils.

The measurement model is as follows.

$$y_{li} = s_l(t_i) + \varepsilon_{li} \;\; i = 1, 2..n_d \;\; l = 1, 2..L$$

Where y represents the sample measurement, and ε is white Gaussian noise.
The process estimating $f(\vec{r})$ from measurements y_{li} is known as linear construction. This is an ill-posed problem because $f(\vec{r})$ is continuous whereas y_{li} are discrete samples. We can approximate $f(\vec{r})$ with a finite series expansion.

$$f(\vec{r}) \;=\; \sum_{j=1}^{N} f_j b(\vec{r} - \vec{r_j})$$

Where $b(\bullet)$ is the basis function. $\vec{r_j}$ is the center of the jth translated basis function and N is the number of parameters. We can define the basis function as a rectangular function defined as

$$rect(t) = \{0 \; if \;\; |t| > 1/2, 1/2 \; if \;\; |t| = 1/2, \;\; 1 \; if \;\; |t| < 1/2 \; \}$$

Then having square pixels of dimension Δ, $b(\vec{r}) = rect(\vec{r}/\Delta)$. Thus we have

$$a_{lij} = \int b(\vec{r}-\vec{r}_j)c_l(\vec{r})e^{-t_i/T_2{}^*(\vec{r})}e^{-i\phi(\vec{r},t))}d\vec{r}$$

Because the model is highly localized, $\vec{r}-\vec{r}_j = 1$ and and $T_2{}^*$ is much larger than the total readout time, this exponential term can be approximated to a constant and added to the image $f(\vec{r})$. Now we can approximate to the center of the pixel as,

$$a_{lij} = c_l(\vec{r})e^{-t_i/T_2{}^*(\vec{r})}e^{-i\phi(\vec{r},t))}$$

In matrix form:

$y_l = A_l f + \varepsilon_l$, stacking all the equations together with respect to the coils

$$y = Af + \varepsilon$$

The longitudinal component of applied magnetic field has three main components.

$$B_z(\vec{r},t) = B_o + \Delta B_o(\vec{r},t) + \vec{G}(t)\cdot\vec{r}$$

Where B_o is the main static field, $\Delta B_o(\vec{r},t)$ is the change of the field in position and time and $\vec{G}(t)$ is the field gradient that is user controlled. The $\phi(\vec{r},t)$ term can then be expanded.

$$\phi(\vec{r},t) = \int_0^t (\gamma\Delta B_o(\vec{r},t) + \gamma\vec{G}(t)\cdot\vec{r})dt$$

$$e^{i\phi(\vec{r},t)} = e^{-i\Delta w_o(\vec{r})t}e^{-i2\pi\vec{k}(t)\cdot\vec{r}}$$

The first term $e^{-i\Delta w_o(\vec{r})t}$ is off resonance and undesirable. The second term $\vec{k}$ is defined as the k-space trajectory.

$$\vec{k}(t) = \frac{1}{2\pi}\int_0^t \gamma\vec{G}(t)dt$$

If we define $z(\vec{r})$ as the rate map combination of the relaxation and field map as

$$z(\vec{r}) = \frac{1}{T_2{}^*(\vec{r})} + i\Delta w_0(\vec{r})$$

Then our matrix can be approximated to

$$a_{lij} = c_l(\vec{r})e^{-z(\vec{r}_j)t_i}e^{-i2\pi\vec{k}(\vec{t}_i)\cdot\vec{r}_j}$$

If the rate map is assumed to be zero (ie. relaxation and off-resonance is ignored) then the matrix will have a fourier encoding matrix with elements $e^{-i2\pi\vec{k}(\vec{t}_i)\cdot\vec{r}_j}$ a diagonal sensitivity matrix of $c_l(\vec{r})$. If we use cartesian spacing for the k-space trajectory, then fast-fourier-transforms can be used to multiply out the A matrix or find its transpose. A non-cartesian approach requires the use of non-uniform fast-fourier-transforms.

When the rate map is non-zero, it can be approximated as,

$$e^{-z(\vec{r}_j)t_i} = \sum_k b_{ik}c_{kj}$$

Rewrite the matrix multiplication as

$$[A_l f]_i = \sum_k b_{ik}\sum_{j=1}^{N} c_{kj}c_l(\vec{r}_j)f_j e^{-i2\pi\vec{k}(\vec{t}_i)\cdot\vec{r}_j}$$

The inner sum can be done with a fast-fourier transform or non-uniform-fast-fourier-transform depending on the k-space trajectory.

With a linear model specified in the form $y = Ax$, because there is noise in the system, the estimation of f can be done via regularized least-squares cost function with regularization.

$$\hat{f} = arg_f min\ \Psi(f)\ ,\ \ \Psi(f) = \|y - Af\|^2 + \beta R(f)$$

The standard inversion method assumes the following:

- Equally spaced grid k-space (cartesian k-space trajectory)
- Rate map $z(\vec{r}) = 0$, ignore relaxation and inhomogeneity
- Single coil $L = 1$,
- $c(\vec{r}) = 1$ coil sensitivity pattern is uniform

Then the system matrix A is orthogonal. No regularization is needed and the solution is simply,

$$\hat{f} = \frac{1}{N} A'y$$

This can be evaluated via a inverse fast-fourier-transform. However, if these conditions are not met, then A is ill-conditioned and the least-squares can amplify the noise. This is when regularization is needed. When subsampling is done, the number of measurements is less than the number of unknown voxels. This leads to an under-sampled k-space data set and fine artifacts caused by the missing data in k-space and depend on the undersampling pattern.
Two main methods used widely today are partial fourier imaging and parallel imaging. In partial fourier imaging, since high resolution phase information is not needed in radiology, the property of a fourier transform of a purely real function has complex conjugate symmetry in k-space meaning only half of the k-space in phase encoding direction is needed. In practice it is about 60%. This reduces acquisition time greatly. In parallel imaging, the use of multiple coils allows for their sensitivity terms to be used to stabilize the undersampled image construction.

The case study proposed is to evaluate two different regularizers with multiple coils (parallel imaging). The first regularizer is the Tikhonov regularization.

$$R(f) = \sum_{j=2}^{N} \left\| f - \bar{f} \right\|^2$$

Where $\bar{f}$ is some reference image. This regularizer is differentiable, convex and constrained so conjugate gradient algorithm can be used. This type of regularizer will bias towards zero when the reference image is set to zero causing some contrast issues.
The second regularizer is a type of total variation method with a potential function of the form

$$R(f) = \sum_{j=2}^{N} \psi(f_j - f_{j-1})$$

$$\psi(x) = \sqrt{1 - |x/\delta|^2} - 1$$

Where the choice of the potential function will be a hyperbola to ensure that the function is strictly convex. This type of regularizer will biases towards a piecewise smooth image, preserving image edges.

References:

Fessler, Jeffrey A. "MODEL-BASED IMAGE RECONSTRUCTION FOR MRI." *IEEE signal processing magazine*, U.S. National Library of Medicine, 1 July 2010, www.ncbi.nlm.nih.gov/pmc/articles/PMC2996730/.