2DBI00 Linear algebra and applications: Class 14

12 June 2018

Announcements

- Final online (Cirrus) test 3 open! (deadline 19 June 23:00)

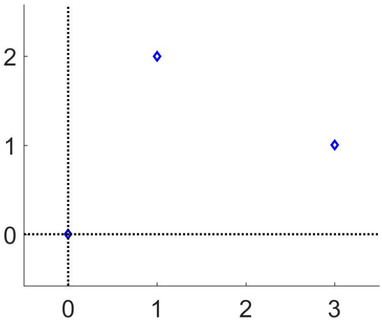
Previous class:

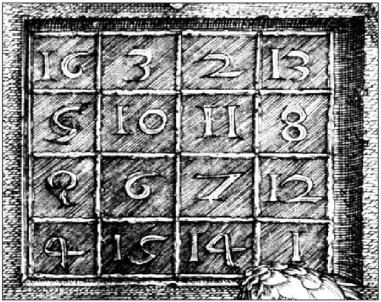
- Least squares $A\mathbf{x} \approx \mathbf{b}$, with A size $m \times n$, $m > n$
- This means $\min_{\mathbf{x}} \|A\mathbf{x} - \mathbf{b}\|$
- More equations than variables, more rows than columns
- Linear system $A\mathbf{x} = \mathbf{b}$ usually has no solution
- Instead solve: $A^T A\mathbf{x} = A^T\mathbf{b}$ (normal equations), is an $n \times n$ linear system
- (Proof normal equations by calculus or geometric argument)
- Often gives a nice fit to data
- May be more robust against (measurement) errors
- Other name (in statistics): regression

Today:

- Splines: application of linear systems
- Image compression: application of SVD

Today! Last 2 topics ☹





How to:
elegantly connect these points?

Splines

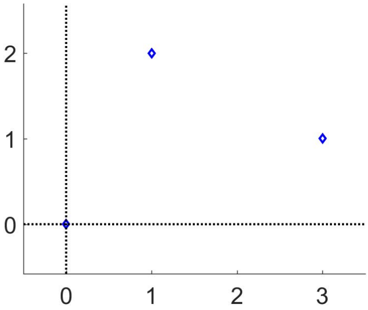
Application of $A\mathbf{x} = \mathbf{b}$

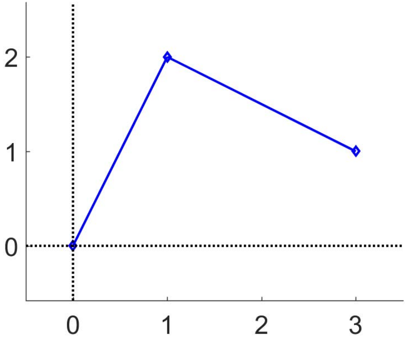
How to:
compress this image?

Truncated SVD

Application of $A = U\Sigma V^T$

Curves fitting points: piecewise linear



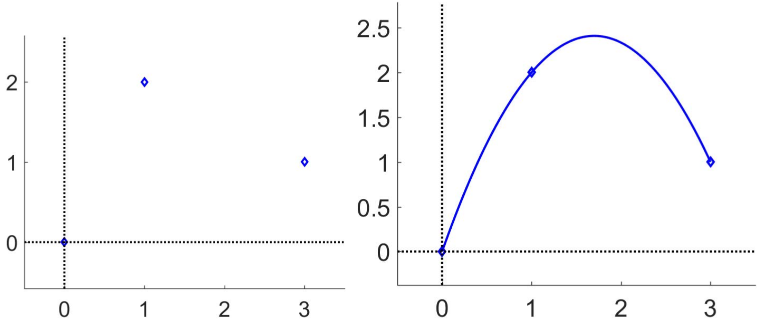


Not very subtle!

Splines are a popular alternative

- Important in computer graphics and computer aided design!
- 1991: "If you want to get rich, study splines"
- 2003: "If you want to get rich, study nonlinear least squares"

Curves fitting points: quadratic



Quadratic polynomial is somewhat more elegant, but:

- no control over initial and end slope
- cannot easily be combined with more points

Recall from Calculus: differentiable

Are the following piecewise defined function continuous? Differentiable?

$$f(x) = \begin{cases} \frac{1}{2}\left(x^2 - 4\right) & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases} \qquad \text{(exam 15.01.26)}$$

$$g(x) = \begin{cases} 1 - |x| - \frac{\sin(x)\,\cos(x)}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \qquad \text{(exam 17.01.30)}$$

f is not continuous in $x = 2$: $\lim_{x \to 2} f(x) = 0 \neq f(2)$

So f is certainly not differentiable

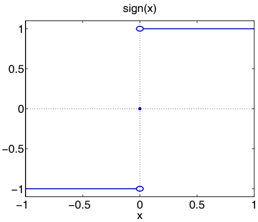
g is continuous in $x = 0$: $\lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} g(x) = 0 = g(0)$

g is not differentiable in $x = 0$: $1 = \lim_{x \to 0^-} g'(x) \neq \lim_{x \to 0^+} g'(x) = -1$

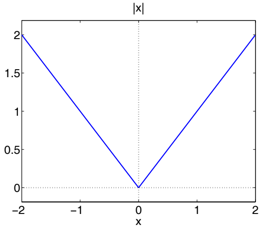
For cubic splines we demand that:

function value and first + second derivatives match

Differentiable



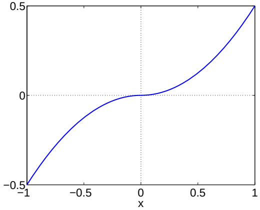
Not continuous



Continuous

f' not continuous in 0

(f not differentiable in 0)



Continuous

f' continuous

f'' not continuous in 0

3rd function: $$f(x) = \begin{cases} \frac{1}{2}x^2 & x \geq 0 \\ -\frac{1}{2}x^2 & x < 0 \end{cases}$$

The 2nd function is the derivative of the 3rd,
and the 1st can be viewed as the derivative of the 2nd (only $x = 0$ problematic)

In the 3rd, one supposedly can see the non-differentiability in the hood of a car

Cubic spline: smoother than 3rd function: f'' should be continuous

Spline for our example

8 dofs (degrees of freedom):

$$p(t) = x_1 + x_2 t + x_3 t^2 + x_4 t^3 \quad \text{on } [0,1]$$
$$q(t) = x_5 + x_6 t + x_7 t^2 + x_8 t^3 \quad \text{on } [1,3]$$

8 requirements:

First point $t = 0$ $\quad p(0) = 0$

Second point $t = 1$ $\quad p(1) = 2, \;\; q(1) = 2$

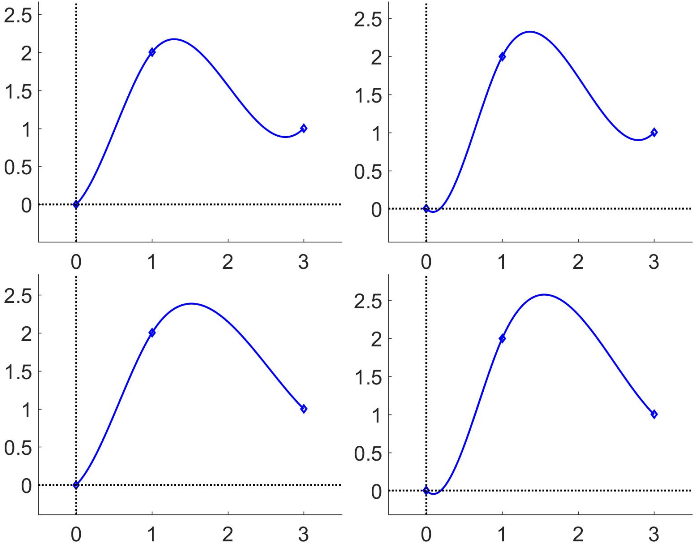
Third point $t = 3$ $\quad q(3) = 1$

Smooth connection $\quad p'(1) = q'(1), \;\; p''(1) = q''(1)$

Freedom initial/end slopes $\quad p'(0) = \alpha, \;\; q'(3) = \beta$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 & 9 & 27 \\ 0 & 0 & 0 & 0 & 0 & 1 & 6 & 27 \\ 0 & 1 & 2 & 3 & 0 & -1 & -2 & -3 \\ 0 & 0 & 2 & 6 & 0 & 0 & -2 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ \alpha \\ 2 \\ 1 \\ \beta \\ 0 \\ 0 \end{bmatrix}$$

Curves fitting points: cubic spline, $p'(0) = \alpha$, $q'(3) = \beta$



Resp. $(\alpha, \beta) = (1,1),\ \ (-1,1),\ \ (1,-1),\ \ (-1,-1)$

but any α and β are possible

Splines

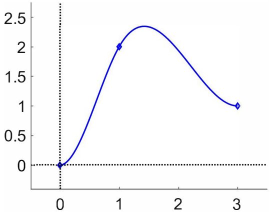
Definition:

- Piecewise degree k polynomials
- $k-1$ "smooth connections":
  this means that in a connection point t, we want
  $p'(t) = q'(t),\ \ p''(t) = q''(t),\ \ldots,\ \ p^{(k-1)}(t) = q^{(k-1)}(t)$
- Standard spline: $k = 3$: cubic polynomial,
  twice differentiable: $p'(t) = q'(t)$ and $p''(t) = q''(t)$
- Two dofs: slopes at beginning and end point

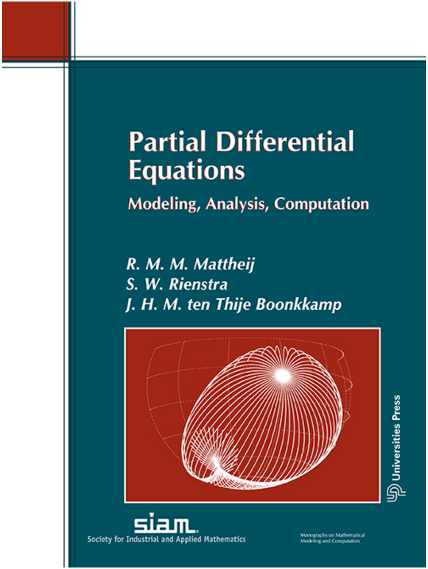
$p'(1) = q'(1)$ and $p''(1) = q''(1)$

$(\alpha, \beta) = (0, 0)$

so $p(0) = 0$ and $q(0) = 0$



Application of smooth curves . . . (not for exam)



The meaning of this image has been explained in class ("videocollege" 14 . . .)

Cubic spline: plotting (not for exam)

Plot: $A\mathbf{x} = \mathbf{b}$ provides the coefficients $x_1, \ldots, x_8$

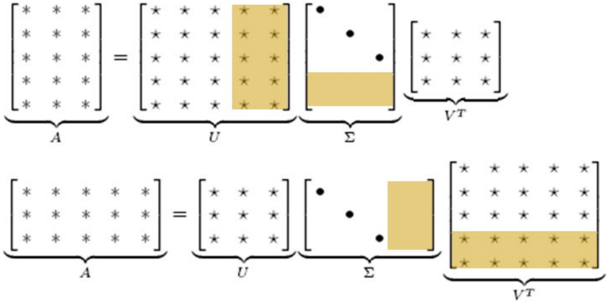
How to plot the spline? Sample Matlab code:

```
t = linspace(0,1,100)                        (100 points in [0,1])
y = x(1)+x(2)*t+x(3)*t.^2+x(4)*t.^3          (evaluate p)
plot(t,y)
```

```
t = linspace(1,3,100)                        (100 points in [1,3])
y = x(5)+x(6)*t+x(7)*t.^2+x(8)*t.^3          (evaluate q)
hold on; plot(t,y)
```

Next: second new topic: truncated SVD

Recall: SVD $A = U\Sigma V^T$ in a picture



Colored areas: not important since multiplied with 0s

2 options for $A = U\Sigma V^T$, A size $m \times n$, $m > n$:

- $U\Sigma V^T$: $(m \times m)(m \times n)(n \times n)$
- $U\Sigma V^T$: $(m \times n)(n \times n)(n \times n)$

No difference, but second option more economical (store less)

TSVD: truncated SVD

Suppose A is 3×3 with singular value 100, 50, 1:

$$A = \begin{bmatrix} \vdots & \vdots & \vdots \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} 100 & & \\ & 50 & \\ & & \mathbf{1} \end{bmatrix} \begin{bmatrix} \vdots & \vdots & \vdots \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ \vdots & \vdots & \vdots \end{bmatrix}^T$$

$$= \begin{bmatrix} \vdots & \vdots & \vdots \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} 100 & & \\ & 50 & \\ & & \mathbf{1} \end{bmatrix} \begin{bmatrix} \cdots & \mathbf{v}_1^T & \cdots \\ \cdots & \mathbf{v}_2^T & \cdots \\ \cdots & \mathbf{v}_3^T & \cdots \end{bmatrix}$$

$$\approx \begin{bmatrix} \vdots & \vdots & \vdots \\ \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} 100 & & \\ & 50 & \\ & & \mathbf{0} \end{bmatrix} \begin{bmatrix} \cdots & \mathbf{v}_1^T & \cdots \\ \cdots & \mathbf{v}_2^T & \cdots \\ \cdots & \mathbf{v}_3^T & \cdots \end{bmatrix}$$

Idea:

- The "**1**" is not so important compared to 100 and 50: replace by a **0**
- But then we do not need $\mathbf{u}_3$ and $\mathbf{v}_3$
- We can save storage space!
- Ideal for image compression

TSVD: truncated SVD

A truncated SVD is a matrix approximation $A \approx A_k = U_k \Sigma_k V_k^T$

Sizes:

- U_k: size $m \times k$
- Σ_k: size $k \times k$
- V_k: size $n \times k$ so V_k^T: size $k \times n$
- so $A_k = U_k \Sigma_k V_k^T$: size $m \times n$: same as original A

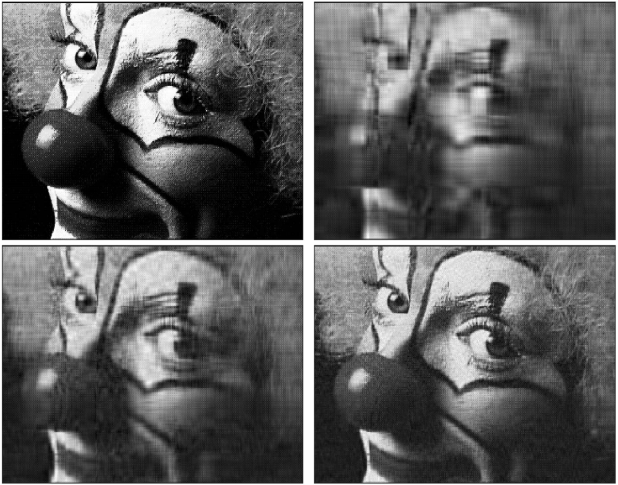
Main idea:

- Keep largest singular values $\sigma_1, \ldots, \sigma_k$
  and corresponding vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ and $\mathbf{v}_1, \ldots, \mathbf{v}_k$
- Discard rest: big savings in space:
  We discard $\sigma_{k+1}, \ldots, \sigma_\ell$ $(\ell = \min(m, n))$
  but more importantly: do not need corresponding $\mathbf{u}_j$'s and $\mathbf{v}_j$'s

Used in many, many applications, one of which: image compression

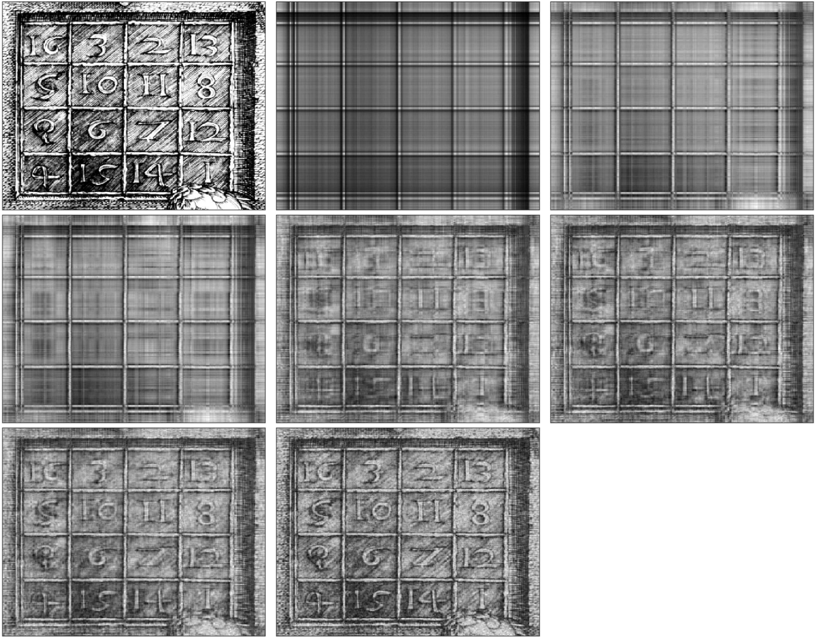
TSVD of picture: $k = 10, 20, 50$

Ex: Picture `clown`, $200 \times 320 = 64000$ pixels,
so $64000 \cdot 8 = 512000$ bytes $= 512$ kB



Storage $k = 20$: $(20 \cdot 200 + 20 \cdot 320 + 20) \cdot 8 = 83.4$ kB

TSVD of 359×371 picture: $k = 1, 3, 5, 15, 20, 30, 50$



This is . . .

THE END of the new material ☺☹

- Thanks a lot for your attention!
- Hope you learned some fun and useful topics!
- 2 more classes and instructions on Thu and Tue: recap, summary, nice quiz, requests, questions, etc!

Advertisement

If you liked this course . . .

- 2IV60 Computer Graphics
- 2ID90 Artificial intelligence
- 2IIG0 Data mining and machine learning
- 2WN40 Numerical Linear Algebra (math)
- 2IMW30 Foundations of data mining (master)

Final exam: recall

- 25 questions, 5 choices (A–E), grade: $\frac{1}{2}(\text{nr of correct} - 5)$
- Exam passed with 15 correct; still need 5.5 average
- Only pen+paper, no calculator
- Optional: **hand in full answers**,
  which are graded in case ≤ 14 correct
- Strong advice: stay full 3 hours, and check answers again and again if possible
- Test exam on Canvas: answers available but no explanation: please read slides, look at homework, and ask instructors

Homework: class 14: see `homework-2DBI00`