# C180-as5

ymubarak

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## 1 Abstract

In this assignment, we will solve the Navier-Stokes equation given the velocity field. Using FENICs, which is an open source FEM library, we will solve the equation on a heart valve flap's geometric mesh. It is interesting to see how the pressure will be affected from the velocity field on something as important as a heart valve.

## 2 Introduction

The Navier Stoke's Equation is given as follows :

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\nabla P + \nu \nabla^2 \vec{u} \tag{1}$$

With Incompressibility giving :

$$\nabla \cdot \vec{u} = 0 \tag{2}$$

We are given $\vec{u}$. This makes our equation much easier to solve because we only have a Ordinary differential equation. We have solved that countless times before using the Semi-Galerkin dynamic Finite Element Method. We will do that now, using FENICS.

## 3 Method

### 3.1 Mathematics

Consider the known part of our Navier Stokes equation, which is :

$$\vec{b} = \nu \nabla^2 \vec{u} - \frac{\partial \vec{u}}{\partial t} - \vec{u} \cdot \nabla \vec{u} \tag{3}$$

Then we take the Diveregeance of a 0 valued equation, and set it to 0 as well.

$$\nabla \cdot (\nabla P - b) = 0 \tag{4}$$

The equation above is then put into Weak form as follows :

$$\int_{\Omega} w(\nabla \cdot (\nabla P - b)) d\Omega = 0 \tag{5}$$

Then Integrating by Parts we get :

$$\int_{\Omega} \nabla w \cdot \nabla P d\Omega = \int_{\Omega} \nabla w \cdot b d\Omega + \int_{\Gamma} w(\nabla P - b) \cdot d\Gamma$$

Then because $w(\nabla P - b) = w * 0 = 0$ we get the following :

$$\int_{\Omega} \nabla w \cdot \nabla P d\Omega = \int_{\Omega} \nabla w \cdot b d\Omega \tag{6}$$

There is another approach to this equation if we are not given $\vec{u}$. That is that given :

$$\begin{aligned} \nabla \cdot \frac{\partial \vec{u}}{\partial t} = \frac{\partial \nabla \cdot \vec{u}}{\partial t} = \frac{\partial 0}{\partial t} = 0 \\ \nabla \cdot \nu \nabla^2 \vec{u} = \nu \nabla^2 (\nabla \cdot \vec{u}) = \nu \nabla^2 0 = 0 \end{aligned} \tag{7}$$

That gives the weak form to look as follows :

$$\int_{\Omega} w(\nabla \cdot (\nabla P)) d\Omega + \int_{\Omega} w(\nabla \cdot (\vec{u} \cdot \nabla \vec{u})) d\Omega = 0 \tag{8}$$

Then after Integration by Parts to the first part of the equation we get what in the assignment is equation 6 below :

$$\int_{\Gamma} w(\nabla P) \cdot n d\Gamma - \int_{\Omega} \nabla w \cdot \nabla P d\Omega + \int_{\Omega} (\nabla \cdot (\vec{u} \cdot \nabla \vec{u})) d\Omega = 0 \tag{9}$$

### 3.2 Procedure

As usual, we need to discretize our space with basis function. Our regular $u^h$ is now pressure. Our driving function is now also a nodal discrete one instead of a continuous one. It is already given in nodal values. Moreover, our $\vec{u}$ is already given at each node in our input files. However, its divergence and gradient will be taken with respect to our shape functions, as usual. Its gradient is a second order tensor and its divergeance is a scalar value.

Mathematically, taking the gradient of our space with respect to $\vec{x}$ will be done the same way (by mapping to our natural coordinates and using the Jacobian determinent).

We then solve for $p$ at every time step which we have a different $\vec{u}(\vec{x})$.

## 4 Results

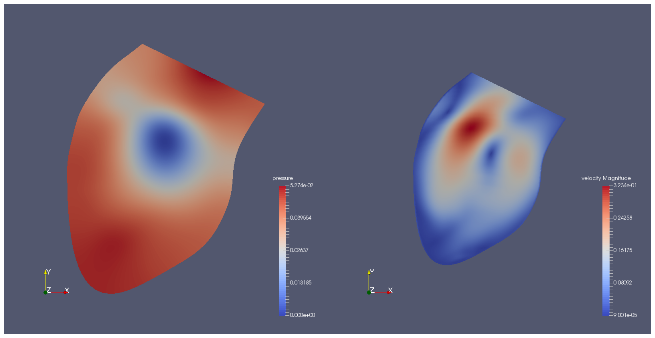


Figure 1: Pressure and Velocity Heatmaps at t = 0

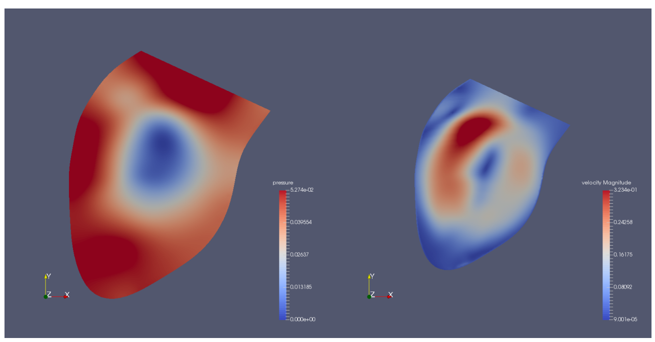


Figure 2: Pressure and Velocity Heatmaps at t = 6

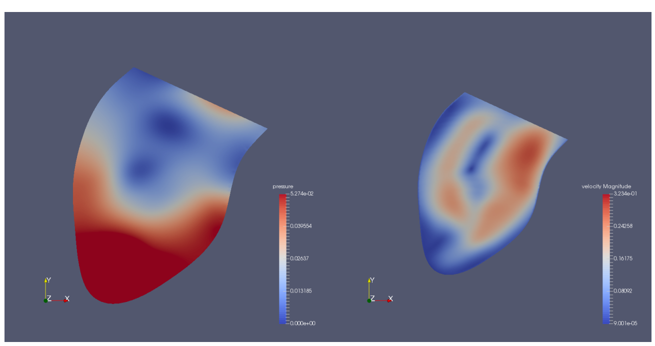


Figure 3: Pressure and Velocity Heatmaps at t = 16

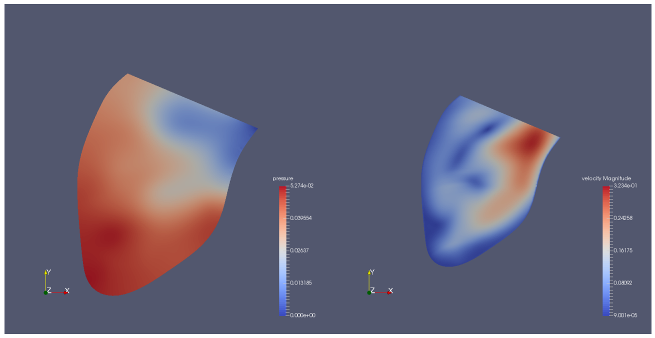


Figure 4: Pressure and Velocity Heatmaps at t = 30

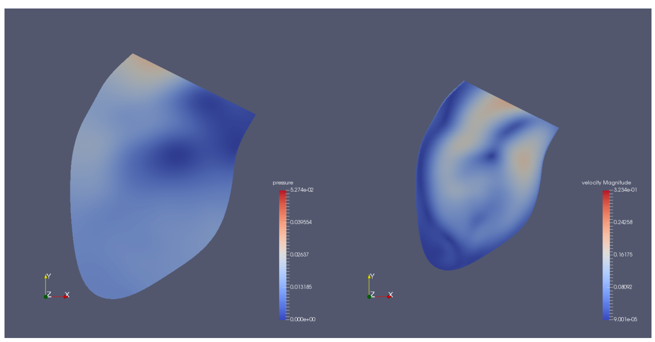


Figure 5: Pressure and Velocity Heatmaps at t = 48

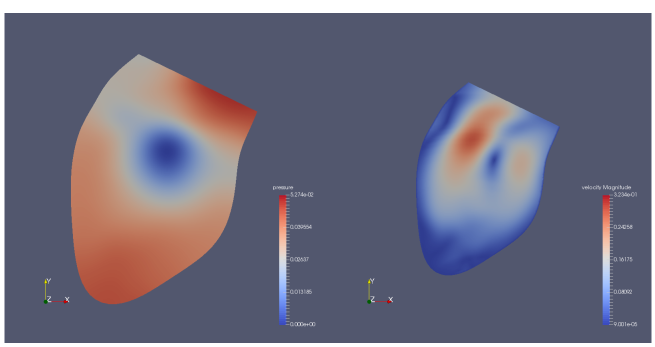


Figure 6: Pressure and Velocity Heatmaps at t = 55

## 5 Discussion

In the case where we have no driving force, all our flow is pressure driven. Therefore as basics conservation of momentum will give $\nabla P$ will be proportional to velocity. Moreover, as is said in elementary fluid mechanics, the fluid will move to places with lower pressure. That is seen in our figures, where changes in pressure are where we have a noticeable flow field in the fluid, or velocity is non-zero.

In Figure 1 you see that the pressure is high where the fluid is at stagnation and the part where there is high velocity, you see a following circular area of low pressure to where the fluid is going. The same can be said for Figure 2. In Figure 3, it can be seen that there are two linear-like ellipses of flows going upwards in the flap where the pressure is lower. The same can be said for Figure 4 but it is the remanents of Figure 4. In Figure 5 you see a plot where there is no variation in neither pressure of velocity, which makes sense that they both come hand in hand. In Figure 6, you see that the variation is coming back and velocity is accompanied by the same pattern of pressure seen at time 0 in Figure 1 .