

# Finding $\alpha$ for the Euler top

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Consider a system with Hamiltonian  $H = J_x x^2 + J_y y^2$ . We seek to analytically find the K-complexity growth rate  $\alpha$ . We will assume that  $J_x > J_y > 0$ .

First consider the Hamiltonian. Rewriting the equation we get

$$y^2 = \frac{H - J_x x^2}{J_y} \quad (1)$$

We know (from the 'Euler Top' paper) that  $z'(t) = 2(J_x - J_y)xy$ . Since  $x^2 + y^2 + z^2 = 1$ ,

$$x = \sqrt{1 - y^2 - z^2} \quad (2)$$

Replacing (1) in (2) and doing some algebraic manipulation, we get

$$x^2 = \frac{J_y - H - z^2}{J_y - J_x} \quad (3)$$

We then replace (3) in  $x^2 + y^2 + z^2 = 1$  to obtain

$$y^2 = 1 - z^2 - \frac{J_y - H - z^2}{J_y - J_x} \quad (4)$$

Finally, we replace (3) and (4) in our equation for  $z'(t)$  to obtain

$$z'(t) = 2(J_x - J_y) \sqrt{(1 - z^2) \left( \frac{J_y - H - z^2}{J_y - J_x} \right) - \left( \frac{J_y - H - z^2}{J_y - J_x} \right)^2} \quad (5)$$

$$\text{Put } y_E(z) = 2(J_x - J_y) \sqrt{(1 - z^2) \left( \frac{J_y - H - z^2}{J_y - J_x} \right) - \left( \frac{J_y - H - z^2}{J_y - J_x} \right)^2}$$

$$\text{We obtain } t = \int \frac{1}{2(J_x - J_y) \sqrt{(1 - z^2) \left( \frac{J_y - H - z^2}{J_y - J_x} \right) - \left( \frac{J_y - H - z^2}{J_y - J_x} \right)^2}} dz = \int \frac{1}{y_E(z)}$$

Now we need to find the zeroes of  $y_E(z)$ :

$$2(J_x - J_y) \sqrt{(1 - z^2) \left( \frac{J_y - H - z^2}{J_y - J_x} \right) - \left( \frac{J_y - H - z^2}{J_y - J_x} \right)^2} = 0$$

By inspection, we can already observe that  $z = \pm \sqrt{J_y - H}$  are roots.

We find the other 2 roots as follows:

$$(1 - z^2) \left( \frac{J_y - H - z^2}{J_y - J_x} \right) - \left( \frac{J_y - H - z^2}{J_y - J_x} \right)^2 = 0$$

$$(1 - z^2)(J_y - H - z^2)(J_y - J_x) = (J_y - H - z^2)^2$$

$$(1 - z^2)(J_y - J_x) = J_y - H - z^2$$

$$J_y - J_x + z^2(J_x - J_y) = J_y - H - z^2$$

$$z^2(J_x - J_y + 1) = J_x - H$$

$$z = \pm \sqrt{\frac{J_x - H}{J_x - J_y + 1}} \text{ are roots}$$

To find the root with the greatest real part, we simplify things by setting  $J_x$  to be 1. Then it follows that  $0 < H < 1$  and  $0 < J_y < 1$ .

The roots become  $\pm\sqrt{J_y - H}$  and  $\pm\sqrt{\frac{1-H}{2-J_y}}$ .

Observe that for  $H > J_y$ ,  $\sqrt{J_y - H}$  is completely imaginary, while  $\sqrt{\frac{1-H}{2-J_y}}$  is real. Then, it follows that  $\sqrt{\frac{1-H}{2-J_y}}$  has the largest real part in this case. For  $J_y \geq H$ , we plot the roots to find which root has the greatest real part.

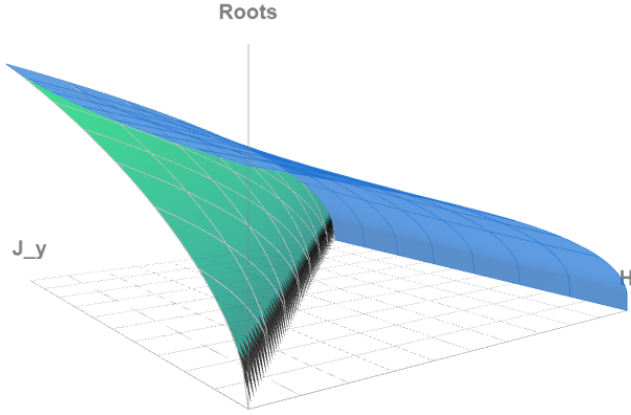


Figure 1: graph of roots in real plane. Blue graph is  $\sqrt{\frac{1-H}{2-J_y}}$  while green graph is  $\sqrt{J_y - H}$

As shown in Figure 1, the real part of  $\sqrt{\frac{1-H}{2-J_y}}$  is always greater or equal to that of  $\sqrt{J_y - H}$ .