Finding α for the Euler top

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Consider a system with Hamiltonian $H = J_x x^2 + J_y y^2$. We seek to analytically find the K-complexity growth rate α. We will assume that $J_x > J_y > 0$.

First consider the Hamiltonian. Rewriting the equation we get
$y^2 = \frac{H - J_x x^2}{J_y}$ -(1)

We know (from the 'Euler Top' paper) that $z'(t) = 2(J_x - J_y)xy$. Since $x^2 + y^2 + z^2 = 1$,
$x = \sqrt{1 - y^2 - z^2}$ -(2)

Replacing (1) in (2) and doing some algebraic manipulation, we get
$x^2 = \frac{J_y - H - z^2}{J_y - J_x}$ - (3)
We then replace (3) in $x^2 + y^2 + z^2 = 1$ to obtain
$y^2 = 1 - z^2 - \frac{J_y - H - z^2}{J_y - J_x}$ -(4)

Finally, we replace (3) and (4) in our equation for $z'(t)$ to obtain
$z'(t) = 2(J_x - J_y)\sqrt{(1 - z^2)(\frac{J_y - H - z^2}{J_y - J_x}) - (\frac{J_y - H - z^2}{J_y - J_x})^2}$ -(5)

Put $y_E(z) = 2(J_x - J_y)\sqrt{(1 - z^2)(\frac{J_y - H - z^2}{J_y - J_x}) - (\frac{J_y - H - z^2}{J_y - J_x})^2}$
We obtain t = $\int \frac{1}{2(J_x - J_y)\sqrt{(1 - z^2)(\frac{J_y - H - z^2}{J_y - J_x}) - (\frac{J_y - H - z^2}{J_y - J_x})^2}} dz = \int \frac{1}{y_E(z)}$ -(6)

Now we need to find the zeroes of $y_E(z)$:
$2(J_x - J_y)\sqrt{(1 - z^2)(\frac{J_y - H - z^2}{J_y - J_x}) - (\frac{J_y - H - z^2}{J_y - J_x})^2} = 0$
By inspection, we can already observe that $z = \pm\sqrt{J_y - H}$ are roots.
We find the other 2 roots as follows:
$(1 - z^2)(\frac{J_y - H - z^2}{J_y - J_x}) - (\frac{J_y - H - z^2}{J_y - J_x})^2 = 0$
$(1 - z^2)(J_y - H - z^2)(J_y - J_x) = (J_y - H - z^2)^2$
$(1 - z^2)(J_y - J_x) = J_y - H - z^2$
$J_y - J_x + z^2(J_x - J_y) = J_y - H - z^2$
$z^2(J_x - J_y + 1) = J_x - H$
$z = \pm\sqrt{\frac{J_x - H}{J_x - J_y + 1}}$ are roots
To find the root with the greatest real part, we simplify things by setting J_x to be 1. Then it follows that $0 < H < 1$ and $0 < J_y < 1$.

The roots become $\pm\sqrt{J_y - H}$ and $\pm\sqrt{\frac{1-H}{2-J_y}}$.

Observe that for $H > J_y, \sqrt{J_y - H}$ is completely imaginary, while $\sqrt{\frac{1-H}{2-J_y}}$ is real. Then, it follows that $\sqrt{\frac{1-H}{2-J_y}}$ has the largest real part in this case. For $J_y \geq H$, we plot the roots to find which root has the greatest real part.

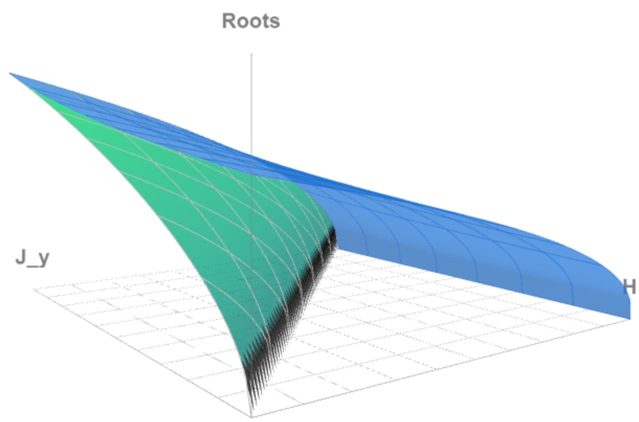

Figure 1: graph of roots in real plane with $0 < J_y < 1$ and $0 < H < 1$. Blue graph is $\sqrt{\frac{1-H}{2-J_y}}$ while green graph is $\sqrt{J_y - H}$

As shown in Figure 1, the real part of $\sqrt{\frac{1-H}{2-J_y}}$ is always greater or equal to that of $\sqrt{J_y - H}$.

We are interested in the case where we get complex roots.

We rewrite the integral 6 as

$\sigma = \frac{1}{2(J_x - J_y)} \int_{z_0}^{\infty} \frac{1}{\sqrt{(z^2 + (H - J_y))(z^2 - (\frac{1-H}{2-J_y}))}} dz$ where $z_0 = \sqrt{\frac{1-H}{2-J_y}}$ - (7). From 212.00 in Byrd and Friedman, the integral evaluates to

$\sigma = \frac{1}{2(J_x - J_y)\sqrt{(H-J_y) + (\frac{1-H}{2-J_y})}} K(\frac{H-J_y}{(H-J_y)+(\frac{1-H}{2-J_y})})$ where K is a complete integral of the first kind. - (8)

Then, we relate σ to the singularities of the auto-correlation function

$C(t) = < z(0)z(t) >$ by the time average $C(t) = \frac{1}{T}\int z(s + t/2)z(s - t/2)ds$ where the integral is over a period T.

Now if z(s) is analytical in the strip $\{s : \zeta(s) \leq \sigma\}$, C(t) is analytical in the strip $\{t : \zeta(t) \leq \tau = 2\sigma\}$, which has twice the width.

Then from "A Universal Operator Growth Hypothesis"and (8), we have

$$\alpha = \frac{\pi}{2\tau} = \frac{\pi}{4\sigma} = \frac{2(J_x - J_y)\pi\sqrt{(H-J_y)+(\frac{1-H}{2-J_y})}}{4K(\frac{H-J_y}{(H-J_y)+(\frac{1-H}{2-J_y})})}$$