Report

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1 Introduction

In the past few years, the study of chaotic effects in quantum systems has been the subject of much attention. However, a fundamental issue still remains at the heart of this study: the concept of "quantum chaos" is one that is not rigorously defined.

Classically, chaos is defined by an extreme sensitivity to infinitesimally small changes in initial conditions, resulting into an exponential deviation in phase-space trajectories. Yet, this definition unfortunately does not transfer smoothly into the quantum realm because those trajectories are not well-defined in the first place. Thus, a new definition is required.

A potential definition, colloquially known as "scrambling", is motivated by the butterfly effect. Scrambling is directly related to out-of-time order correlators (OTOCs), whose exponential growths in an array of quantum systems have been proposed to indicate "quantum chaos". Yet, it has recently been shown that the exponential growth of OTOCs does not necessarily imply chaos. In a classical phase-space, the presence of unstable saddle points can lead to such an exponential growth even in the absence of chaos.

This report is divided into 3 main parts. The first one deals with reproducing a numerical analysis of the exponential growth of the infinite-temperature OTOC of the integrable Lipkin-Meshkov-Glick(LMG) model in the semi-classical limit. The second part is a classical analysis of the Euler Top model, demonstrating how the presence of unstable saddle points emulates chaotic operator growth even in the absence of chaos. The final subsection is an attempt to analytically find the Krylov-complexity growth rate in the classical limit of the Euler Top model. (It has been previously proven the K-complexity growth rate to be related to the Lanczos linear coefficient which in turn serves as a bound to the OTOC lyapunov exponent)

2 Numerical Analysis of the OTOC in the LMG model

Let us consider the LMG model with Hamiltonian $H = x + 2z^2$ (where x, y, z form a classical SU(2) spin satisfying $x^2 + y^2 + z^2 = 1$ and x, y = z, etc.) in the classical limit.

By linearizing the equations of motion near the unstable saddle point $(0, 0, 1)$ (the exact method is explained in the next section using the Euler Top model), we obtain that near (0,0,1) the system can be approximated by the ordinary differential equation:

$$\frac{da}{dt} = \sqrt{3}a$$

implying we get a classical lyapunov exponent of $\sqrt{3}$. The numerics are consistent with this value, returning an exponent value of 1.714 (see figure 1 below).

For the numerics, we consider the quantization of the above Hamiltonian. that is $\hat{H} = \hat{x} + 2\hat{z}^2$ where $\hat{x}, \hat{z} = \hat{S}_x/25, \hat{S}_z/25$ are rescaled SU(2) spin operator with spin S=25. Then the OTOC is defined at infinite temperature, with respect to the operator $\hat{O} = \hat{z}$:

$$C(t) := \frac{S^2 \mathrm{Tr}([\hat{O}(t), \hat{O}]^\dagger), [\hat{O}, \hat{O}(t)])}{\mathrm{Tr}(\mathbb{1})}$$

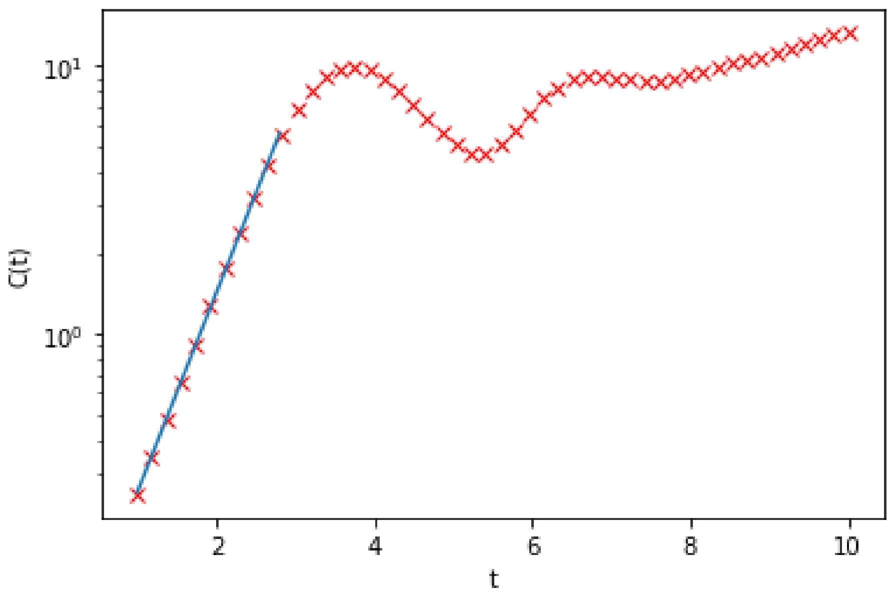

Figure 1: Extended exponential growth of the infinite temperature OTOC of the integrable LMG model with spin S=25 in the semi-classical limit. The gradient of the blue line is 1.714

3 Finding the "Lyapunov exponent" of the classical general Euler Top model

Consider the classical Euler Top model with Hamiltonian $H = J_x x^2 + J_y y^2$ where $J_x > J_y > 0$. . Note that the curly brackets we will use are Poisson brackets

The Hamiltonian equation of motion are given as follows:

$$\begin{aligned}\frac{dx}{dt} = \{x, H\} &= \{x, J_x x^2 + J_y y^2\}\\ &= J_x\{x, x^2\} + J_y\{x, y^2\}\\ &= -J_y\{y^2, x\}\\ &= -J_y(\{y, x\}y + y\{y, x\})\\ &= -J_y(-2zy)\\ &= 2J_y yz\end{aligned}$$

$$\begin{aligned}\frac{dy}{dt} = \{y, H\} &= \{y, J_x x^2 + J_y y^2\}\\ &= J_x\{y, x^2\} + J_y\{y, y^2\}\\ &= -J_x(\{x, y\}x + x\{x, y\})\\ &= -2J_x xz\end{aligned}$$

$$\begin{aligned}\frac{dz}{dt} = \{z, H\} &= \{z, J_x x^2 + J_y y^2\}\\ &= -J_x\{x^2, z\} - J_y\{y^2, z\}\\ &= -J_x(\{x, z\}x + x\{x, z\}) - J_y(\{y, z\}y + y\{y, z\})\\ &= (2J_x - 2J_y)xy\end{aligned}$$

The derivatives define a system of first order differential equations. For this system, we know that the unstable saddle points are located at (0,1,0) and (0,-1,0).

The Jacobian of the system is

$$\begin{bmatrix} 0 & 2J_y z & 2J_y y \\ -2J_x z & 0 & -2J_x x \\ 2(J_x - J_y)y & 2(J_x - J_y)x & 0 \end{bmatrix}$$

Evaluating the Jacobian at the saddle point (0,1,0) gives us

$$\begin{bmatrix} 0 & 0 & 2J_y \\ 0 & 0 & 0 \\ 2(J_x - J_y) & 0 & 0 \end{bmatrix}$$

The characteristic equation then gives the eigenvalues $2\sqrt{J_y}\sqrt{J_x - J_y}$, 0, $-2\sqrt{J_y}\sqrt{J_x - J_y}$. From the study of ODEs, we know that the solutions at the saddle points will be of the form $e^{\lambda t}$. From the eigenvalues, we know that λ is $2\sqrt{J_y}\sqrt{J_x - J_y}$. Note that we would get the same eigenvalues if we evaluated the Jacobian at the saddle point (0,-1,0) instead.

The solution to the system of ODEs produces exponential growth with a classical "Lyapunov exponent" $\lambda_L = 2\sqrt{J_y}\sqrt{J_x - J_y}$. From the paper "Does scrambling equal chaos?", we know that the presence of these saddle points mimicks chaotic operator growth even when the system involved is not chaotic.

4 Finding the microcanonical K-complexity growth rate for the general Euler Top model in the classical limit

Once again we consider the general Euler Top model with hamiltonian $H = J_x x^2 + J_y y^2$ where $J_x > J_y > 0$. Rewriting the equation we get

$$y^2 = \frac{E - J_x x^2}{J_y} \tag{1}$$

We know (from the previous section) that $z'(t) = 2(J_x - J_y)xy$. Since $x^2 + y^2 + z^2 = 1$,

$$x = \sqrt{1 - y^2 - z^2} \tag{2}$$

Replacing (1) in (2) and doing some algebraic manipulation, we get

$$x^2 = \frac{J_y - E - z^2}{J_y - J_x} \tag{3}$$

We then replace (3) in $x^2 + y^2 + z^2 = 1$ to obtain

$$y^2 = 1 - z^2 - \frac{J_y - E - z^2}{J_y - J_x} \tag{4}$$

Finally, we replace (3) and (4) in our equation for $z'(t)$ to obtain

$$z'(t) = 2(J_x - J_y)\sqrt{(1 - z^2)(\frac{J_y - E - z^2}{J_y - J_x}) - (\frac{J_y - E - z^2}{J_y - J_x})^2} \tag{5}$$

Put $y_E(z) = 2(J_x - J_y)\sqrt{(1 - z^2)(\frac{J_y - E - z^2}{J_y - J_x}) - (\frac{J_y - E - z^2}{J_y - J_x})^2}$
We obtain

$$t = \int \frac{1}{2(J_x - J_y)\sqrt{(1 - z^2)(\frac{J_y - E - z^2}{J_y - J_x}) - (\frac{J_y - E - z^2}{J_y - J_x})^2}} dz = \int \frac{1}{y_E(z)} \tag{6}$$

Now we need to find the zeroes of $y_E(z)$:
$2(J_x - J_y)\sqrt{(1 - z^2)(\frac{J_y - E - z^2}{J_y - J_x}) - (\frac{J_y - E - z^2}{J_y - J_x})^2} = 0$
By inspection, we can already observe that $z = \pm\sqrt{J_y - E}$ are roots.

We find the other 2 roots as follows:

$$(1-z^2)(\frac{J_y-E-z^2}{J_y-J_x})-(\frac{J_y-E-z^2}{J_y-J_x})^2=0 \tag{7}$$
$$(1-z^2)(J_y-E-z^2)(J_y-J_x)=(J_y-E-z^2)^2 \tag{8}$$
$$(1-z^2)(J_y-J_x)=J_y-E-z^2 \tag{9}$$
$$J_y-J_x+z^2(J_x-J_y)=J_y-E-z^2 \tag{10}$$
$$z^2(J_x-J_y+1)=J_x-E \tag{11}$$
$$z=\pm\sqrt{\frac{J_x-E}{J_x-J_y+1}} \tag{12}$$

$z=\pm\sqrt{\frac{J_x-E}{J_x-J_y+1}}$ are roots

To find the root with the greatest real part, we simplify things by setting J_x to be 1. Then it follows that $0<E<1$ and $0<J_y<1$.

The roots become $\pm\sqrt{J_y-H}$ and $\pm\sqrt{\frac{1-E}{2-J_y}}$.

Observe that for $E>J_y$,$\sqrt{J_y-E}$ is completely imaginary, while $\sqrt{\frac{1-E}{2-J_y}}$ is real. Then, it follows that $\sqrt{\frac{1-E}{2-J_y}}$ has the largest real part in this case. For $J_y\geq E$, we plot the roots to find which root has the greatest real part.

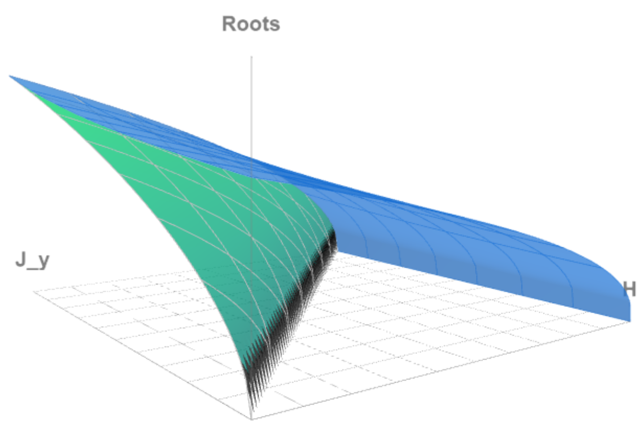

Figure 2: graph of roots in real plane with $0<J_y<1$ and $0<E<1$. Blue graph is $\sqrt{\frac{1-E}{2-J_y}}$ while green graph is$\sqrt{J_y-E}$

As shown in Figure 1, the real part of $\sqrt{\frac{1-E}{2-J_y}}$ is always greater or equal to that of $\sqrt{J_y - E}$.

We are interested in the case where we get complex roots.

We rewrite the integral (6) as

$$\sigma = \frac{1}{2(J_x - J_y)} \int_{z_0}^{\infty} \frac{1}{\sqrt{(z^2 + (E - J_y))(z^2 - (\frac{1-E}{2-J_y}))}} dz \tag{13}$$

where $z_0 = \sqrt{\frac{1-E}{2-J_y}}$ From 212.00 in Byrd and Friedman, the integral evaluates to:

$$\sigma = \frac{1}{2(J_x - J_y)\sqrt{(H - J_y) + (\frac{1-E}{2-J_y})}} K(\frac{E - J_y}{(E - J_y) + (\frac{1-E}{2-J_y})}) \tag{14}$$

where K is a complete integral of the first kind.

Then, we relate σ to the singularities of the auto-correlation function $C(t) =< z(0)z(t) >$ by the time average $C(t) = \frac{1}{T} \int z(s + t/2)z(s - t/2)ds$ where the integral is over a period T.

Now if z(s) is analytical in the strip $\{s : \zeta(s) \leq \sigma\}$, C(t) is analytical in the strip $\{t : \zeta(t) \leq \tau = 2\sigma\}$, which has twice the width.

Then from "A Universal Operator Growth Hypothesis" and (14), we have

$$\alpha = \frac{\pi}{2\tau} = \frac{\pi}{4\sigma} = \frac{(1 - J_y)\pi\sqrt{(E - J_y) + (\frac{1-E}{2-J_y})}}{2K(\frac{E-J_y}{(E-J_y)+(\frac{1-E}{2-J_y})})} \tag{15}$$

5 Conclusion

In this report, we performed a numerical analysis of a special case of the LMG model. Then we found the value of the exponential factor that mimicks chaotic operator growth in the general Euler Top model. We also derived a value for the microcanonical K-complexity growth rate for the general Euler Top model in the classical limit. While this report deals with specific models, we wish to point out that the methods used in this report can be applied to other models with unstable saddle points. We finally would like to add that the derived value of the microcanonical K-complexity growth is only theoretical in the sense that it has not been cross-checked against some known values. We do not know if the derived value is correct in practice. This requires some further investigation.