

Finding α for the Euler top

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Consider a system with Hamiltonian $H = J_x x^2 + J_y y^2$. We seek to analytically find the K-complexity growth rate α . We will assume that $J_x > J_y > 0$.

First consider the Hamiltonian. Rewriting the equation we get

$$y^2 = \frac{H - J_x x^2}{J_y} \quad (1)$$

We know (from the 'Euler Top' paper) that $z'(t) = 2(J_x - J_y)xy$. Since $x^2 + y^2 + z^2 = 1$,

$$x = \sqrt{1 - y^2 - z^2} \quad (2)$$

Replacing (1) in (2) and doing some algebraic manipulation, we get

$$x^2 = \frac{J_y - H - z^2}{J_y - J_x} \quad (3)$$

We then replace (3) in $x^2 + y^2 + z^2 = 1$ to obtain

$$y^2 = 1 - z^2 - \frac{J_y - H - z^2}{J_y - J_x} \quad (4)$$

Finally, we replace (3) and (4) in our equation for $z'(t)$ to obtain

$$z'(t) = 2(J_x - J_y) \sqrt{(1 - z^2) \left(\frac{J_y - H - z^2}{J_y - J_x} \right) - \left(\frac{J_y - H - z^2}{J_y - J_x} \right)^2} \quad (5)$$

$$\text{Put } y_E(z) = 2(J_x - J_y) \sqrt{(1 - z^2) \left(\frac{J_y - H - z^2}{J_y - J_x} \right) - \left(\frac{J_y - H - z^2}{J_y - J_x} \right)^2}$$

$$\text{We obtain } t = \int \frac{1}{2(J_x - J_y) \sqrt{(1 - z^2) \left(\frac{J_y - H - z^2}{J_y - J_x} \right) - \left(\frac{J_y - H - z^2}{J_y - J_x} \right)^2}} dz = \int \frac{1}{y_E(z)} \quad (6)$$

Now we need to find the zeroes of $y_E(z)$:

$$2(J_x - J_y) \sqrt{(1 - z^2) \left(\frac{J_y - H - z^2}{J_y - J_x} \right) - \left(\frac{J_y - H - z^2}{J_y - J_x} \right)^2} = 0$$

By inspection, we can already observe that $z = \pm \sqrt{J_y - H}$ are roots.

We find the other 2 roots as follows:

$$(1 - z^2) \left(\frac{J_y - H - z^2}{J_y - J_x} \right) - \left(\frac{J_y - H - z^2}{J_y - J_x} \right)^2 = 0$$

$$(1 - z^2)(J_y - H - z^2)(J_y - J_x) = (J_y - H - z^2)^2$$

$$(1 - z^2)(J_y - J_x) = J_y - H - z^2$$

$$J_y - J_x + z^2(J_x - J_y) = J_y - H - z^2$$

$$z^2(J_x - J_y + 1) = J_x - H$$

$$z = \pm \sqrt{\frac{J_x - H}{J_x - J_y + 1}} \text{ are roots}$$

To find the root with the greatest real part, we simplify things by setting J_x to be 1. Then it follows that $0 < H < 1$ and $0 < J_y < 1$.

The roots become $\pm\sqrt{J_y - H}$ and $\pm\sqrt{\frac{1-H}{2-J_y}}$.

Observe that for $H > J_y$, $\sqrt{J_y - H}$ is completely imaginary, while $\sqrt{\frac{1-H}{2-J_y}}$ is real. Then, it follows that $\sqrt{\frac{1-H}{2-J_y}}$ has the largest real part in this case. For $J_y \geq H$, we plot the roots to find which root has the greatest real part.

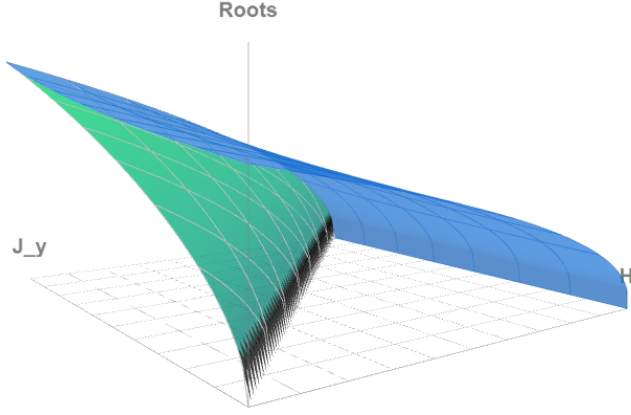


Figure 1: graph of roots in real plane with $0 < J_y < 1$ and $0 < H < 1$. Blue graph is $\sqrt{\frac{1-H}{2-J_y}}$ while green graph is $\sqrt{J_y - H}$

As shown in Figure 1, the real part of $\sqrt{\frac{1-H}{2-J_y}}$ is always greater or equal to that of $\sqrt{J_y - H}$.

We are interested in the case where we get complex roots.

We rewrite the integral 6 as

$$t = \frac{1}{2(J_x - J_y)} \int_{z_0}^{\infty} \frac{1}{\sqrt{(z^2 + (H - J_y))(z^2 - (\frac{1-H}{2-J_y}))}} dz \text{ where } z_0 = \sqrt{\frac{1-H}{2-J_y}} - (7).$$

From 212.00 in Byrd and Friedman, the integral evaluates to

$$\sigma = \frac{1}{\sqrt{(H - J_y) + (\frac{1-H}{2-J_y})}} K\left(\frac{H - J_y}{(H - J_y) + (\frac{1-H}{2-J_y})}\right) \text{ where } K \text{ is a complete integral of the first kind.}$$