Euler top, general Hamiltonian

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May 2022

For this exercise, we will work with a quantum version of the Lipkin-Meshkov-Glick (LMG) model with Hamiltonian $H = J_x x^2 + J_y y^2$. Note that the curly brackets we use are Poisson brackets

The Hamiltonian equation of motion are given as follows:

$$
\begin{aligned}
\frac{dx}{dt} = \{x, H\} &= \{x, J_x x^2 + J_y y^2\} \\
&= J_x\{x, x^2\} + J_y\{x, y^2\} \\
&= -J_y\{y^2, x\} \\
&= -J_y(\{y, x\}y + y\{y, x\}) \\
&= -J_y(-2zy) \\
&= 2J_y yz
\end{aligned}
$$

$$
\begin{aligned}
\frac{dy}{dt} = \{y, H\} &= \{y, J_x x^2 + J_y y^2\} \\
&= J_x\{y, x^2\} + J_y\{y, y^2\} \\
&= -J_x(\{x, y\}x + x\{x, y\}) \\
&= -2J_x xz
\end{aligned}
$$

$$
\begin{aligned}
\frac{dz}{dt} = \{z, H\} &= \{z, J_x x^2 + J_y y^2\} \\
&= -J_x\{x^2, z\} - J_y\{y^2, z\} \\
&= -J_x(\{x, z\}x + x\{x, z\}) - J_y(\{y, z\}y + y\{y, z\}) \\
&= (2J_x - 2J_y)xy
\end{aligned}
$$

The derivatives define a system of first order differential equations. The saddle points are located at (0,1,0) and (0,-1,0) The Jacobian is

$$
\begin{bmatrix}
0 & 2J_y z & 2J_y y \\
-2J_x z & 0 & -2J_x x \\
2(J_x - J_y)y & 2(J_x - J_y)x & 0
\end{bmatrix}
$$

Evaluating the Jacobian at the saddle point (0,1,0) gives us

$$
\begin{bmatrix}
0 & 0 & 2J_y \\
0 & 0 & 0 \\
2(J_x - J_y) & 0 & 0
\end{bmatrix}
$$

The characteristic equation then gives the eigenvalues $2\sqrt{J_y}\sqrt{J_x - J_y}$, 0, $-2\sqrt{J_y}\sqrt{J_x - J_y}$, implying that λ is $2\sqrt{J_y}\sqrt{J_x - J_y}$. Note that we would get the same eigenvalues if we evaluated the Jacobian at the saddle point (0,-1,0) instead.