Stability Analysis of an Euler Disk Rolling on a Plane

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Special Project

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Introduction

In this project I studied the equations of motion of an Euler Disk rolling without slippage on a plane in order to find the minimal spinning velocity needed for the disk to remain stable. Our first intuition indicates that when we lay a disk on a floor without rolling it, it will immediately fall. However, if we'll give the disk enough angular velocity (by rolling it about it's own axis or spinning it about the vertical axis), the disk wouldn't fall and keep spinning even if we start spinning it with some inaccuracy. To find the marginal angular velocity needed to keep the disk stable, I found the equations of motion with the Newtonian and the Lagrangian approaches, analyzed these equations and finally made some numeric simulations to support my analysis. This report concludes my work on this subject.

1 The Newtonian Approach

In this part we'll find the disk's equations of motion by writing it's angular momentum. To formulate the angular momentum we'll denote the following set of coordinates (which are the Euler's angles) $q = \begin{bmatrix} \psi & \theta & \phi \end{bmatrix}^T$. In figure 1 we can see the directions that define the disk's orientation. Angle ψ

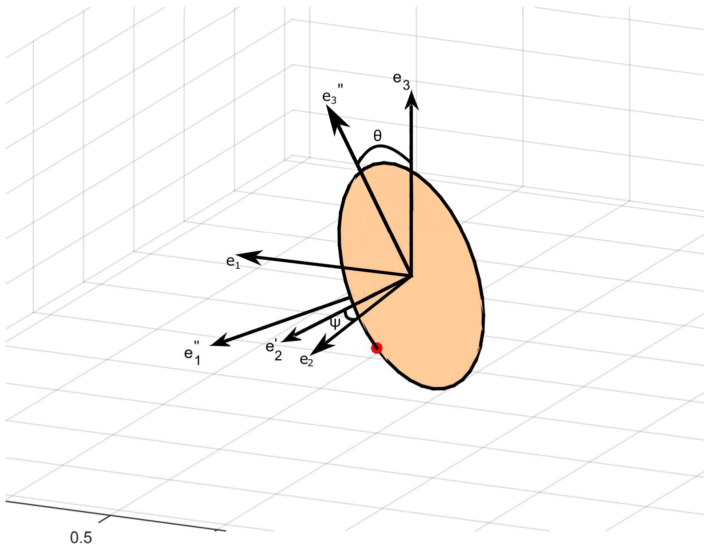


Figure 1: The unit vectors that define the disk's orientation

defines the direction of the unit vector e_2' (with respect to the vectors e_1 and e_2), θ is the nutation angle and is defined with respect to e_3. Finally, ϕ is the angle that defines the roll of the disk about it's own axis, namely about e_1". Let's define the relevant coordinates systems

$$\begin{aligned} e_1' &= \cos(\psi)\, e_1 - \sin(\psi)\, e_2 & e_1'' &= \cos(\theta)\, e_1' - \sin(\theta)\, e_3' \\ e_2' &= \sin(\psi)\, e_1 + \cos(\psi)\, e_2 & e_2'' &= e_2' \\ e_3' &= e_3 & e_3'' &= \sin(\theta)\, e_1' + \cos(\theta)\, e_3' \end{aligned}$$

when e_i' spins with $\Omega' = \dot{\psi} e_3$ and defines the rolling direction of the disk. e_i" spins with the angular velocity $\Omega" = \dot{\psi} e_3 + \dot{\theta} e_2'$ and defines the rolling and the nutation of the disk.

1.1 Angular Momentum About the Fixed Point A

We'll derive the angular momentum equation about a fixed point that is located at the contact point of the disk with the floor. For a general point the angular momentum equation is

$$\dot{H}_A + m r_{c/A} \times a_A = M_A \tag{1}$$

A is a fixed point thus $a_A = 0$. Furthermore, for a general point we know that

$$H_A = r_{c/A} \times m v_{c/A} + H_c$$

differentiating with respect to time yields

$$\dot{H}_A = r_{c/A} \times m a_{c/A} + \dot{H}_c$$

Let's derive each term in this eqaution.

We'll find $\dot{H}_c$ by using the e_i" coordinate system. e_i" is not attached to the disk because it does not spin about e_1" as the disk. Even so, because the symmetry of the disk in this direction we can use $H_c = I_c\omega$. The tensor of inertia of the disk in e_i" coordiante system is

$$I_c = \alpha \frac{mR^2}{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

the angular velocity of the disk is

$$\omega = \dot{\psi} e_3^{'} + \dot{\theta} e_2^{'} - \dot{\phi} e_1^{"} = \left(-\dot{\phi} - \dot{\psi} \sin(\theta)\right) e_1^{"} + \dot{\theta} e_2^{"} + \dot{\psi} \cos(\theta)\, e_3^{"}$$

where α defines the mass distribution of the disk. If $\alpha = 1/2$ the disk has a uniform mass distribution and if $\alpha = 1$ all of the disk's mass is at the circumference of the disk, namely the disk becomes a hoop. For a uniform mass distribution, the angular momentum about the center of mass is

$$H_c = \frac{mR^2}{4} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\dot{\phi} - \dot{\psi} \sin(\theta) \\ \dot{\theta} \\ \dot{\psi} \cos(\theta) \end{bmatrix} = \begin{bmatrix} -\frac{R^2\, m\, \left(\dot{\phi} + \dot{\psi}\, \sin(\theta)\right)}{2} \\ \frac{R^2\, \dot{\theta}\, m}{4} \\ \frac{R^2\, \dot{\psi}\, m\, \cos(\theta)}{4} \end{bmatrix}$$

to find $\dot{H}_c$ let's use the general operator

	$e_1^{''}$	$e_2^{''}$	$e_3^{''}$
$\Omega^{''}$	$-\dot{\psi} \sin(\theta)$	$\dot{\theta}$	$\dot{\psi} \cos(\theta)$
H_c	$\frac{R^2\, m\, \left(\dot{\phi}+\dot{\psi}\, \sin(\theta)\right)}{2}$	$\frac{R^2\, \dot{\theta}\, m}{4}$	$\frac{R^2\, \dot{\psi}\, m\, \cos(\theta)}{4}$
$\Omega^{''} \times H_c$	0	$\frac{R^2\, \dot{\psi}^2\, m\, \cos(\theta)\, \sin(\theta)}{4} - \frac{R^2\, \dot{\psi}\, m\, \cos(\theta)\, \left(\dot{\phi}+\dot{\psi}\, \sin(\theta)\right)}{2}$	$\frac{R^2\, \dot{\theta}\, m\, \left(\dot{\phi}+\dot{\psi}\, \sin(\theta)\right)}{2} - \frac{R^2\, \dot{\psi}\, \dot{\theta}\, m\, \sin(\theta)}{4}$
$\frac{\delta H_c}{\delta t}$	$-\frac{R^2\, \ddot{\phi}\, m}{2} - \frac{R^2\, \ddot{\psi}\, m\, \sin(\theta)}{2} - \frac{R^2\, \dot{\psi}\, \dot{\theta}\, m\, \cos(\theta)}{2}$	$\frac{R^2\, \ddot{\theta}\, m}{4}$	$\frac{R^2\, \ddot{\psi}\, m\, \cos(\theta)}{4} - \frac{R^2\, \dot{\psi}\, \dot{\theta}\, m\, \sin(\theta)}{4}$
$\dot{H}_c$	$-\frac{R^2\, m\, \left(\ddot{\phi}+\ddot{\psi}\, \sin(\theta)+\dot{\psi}\, \dot{\theta}\, \cos(\theta)\right)}{2}$	$-\frac{R^2\, m\, \left(\frac{\sin(2\,\theta)\, \dot{\psi}^2}{2}+2\, \dot{\phi}\, \cos(\theta)\, \dot{\psi}-\ddot{\theta}\right)}{4}$	$\frac{R^2\, m\, \left(2\, \dot{\phi}\, \dot{\theta}+\ddot{\psi}\, \cos(\theta)\right)}{4}$

To find a_c we'll use the following kinematic relation (when p is a material point of the disk that is instantaneously in contact with the floor)

$$v_p = v_c + \omega \times r_{p/c} = 0 \tag{2}$$

therefore

$$\begin{aligned} v_c = -\omega \times r_{p/c} &= \left(\left(-\dot{\phi} - \dot{\psi} \sin(\theta)\right) e_1^{"} + \dot{\theta} e_2^{"} + \dot{\psi} \cos(\theta)\, e_3^{"}\right) \times R e_3^{"} \\ &= R\left(\dot{\phi} + \dot{\psi} \sin(\theta)\right) e_2^{"} + R\dot{\theta} e_1^{"} \end{aligned}$$

using the general operator again to differentiate

	$e_1^{"}$	$e_2^{"}$	$e_3^{"}$
$\Omega^{"}$	$-\dot{\psi} \sin(\theta)$	$\dot{\theta}$	$\dot{\psi} \cos(\theta)$
v_c	$R\dot{\theta}$	$R\dot{\phi} + R\dot{\psi} \sin(\theta)$	0
$\Omega^{"} \times v_c$	$-R\dot{\phi}\dot{\psi} \cos(\theta) - R\dot{\psi}^2 \sin(\theta) \cos(\theta)$	$R\dot{\theta}\dot{\psi} \cos(\theta)$	$-R\dot{\psi}^2 \sin(\theta)^2 - R\dot{\phi}\dot{\psi} \sin(\theta) - R\dot{\theta}^2$
$\frac{\delta v_c}{\delta t}$	$R\ddot{\theta}$	$R\ddot{\phi} + R\ddot{\psi} \sin(\theta) + R\dot{\psi}\dot{\theta} \cos(\theta)$	0
a_c	$-R\dot{\phi}\dot{\psi} \cos(\theta) - R\dot{\psi}^2 \sin(\theta) \cos(\theta) + R\ddot{\theta}$	$R\ddot{\phi} + R\ddot{\psi} \sin(\theta) + 2R\dot{\psi}\dot{\theta} \cos(\theta)$	$-R\dot{\psi}^2 \sin(\theta)^2 - R\dot{\phi}\dot{\psi} \sin(\theta) - R\dot{\theta}^2$

Because A is a fixed point we know that $a_c = a_{c/A}$ hence

$$\begin{aligned}
mr_{c/A} \times a_c = Re_3^{"} \times &\left(\left(-R\dot{\phi}\dot{\psi}\cos(\theta) - R\dot{\psi}^2 \sin(\theta)\cos(\theta) + R\ddot{\theta}\right) e_1^{"} + \right.\\
&... \left(R\ddot{\phi} + R\ddot{\psi}\sin(\theta) + 2R\dot{\psi}\dot{\theta}\cos(\theta)\right) e_2^{"}\\
&... + \left.\left(-R\dot{\psi}^2 \sin(\theta)^2 - R\dot{\phi}\dot{\psi}\sin(\theta) - R\dot{\theta}^2\right) e_3^{"}\right)\\
&= mR^2 \left(\left(-\dot{\phi}\dot{\psi}\cos(\theta) - \dot{\psi}^2 \sin(\theta)\cos(\theta) + \ddot{\theta}\right) e_2^{"}\right.\\
&... - \left.\left(\ddot{\phi} + \ddot{\psi}\sin(\theta) + 2\dot{\psi}\dot{\theta}\cos(\theta)\right) e_1^{"}\right)
\end{aligned}$$

therfore the change in the angular momentum with respect to time is

$$\begin{aligned}
\dot{H}_A = &-R\,m\,\left(R\,\ddot{\psi}\,\sin(\theta) + 2\,R\,\dot{\psi}\,\dot{\theta}\,\cos(\theta)\right) - R^2\,\alpha\,m\,\left(\ddot{\phi} + \ddot{\psi}\,\sin(\theta) + \dot{\psi}\,\dot{\theta}\,\cos(\theta)\right) e_1^{"}\\
&... + \left(R\,m\,\left(R\,\ddot{\theta} - \frac{R\,\dot{\psi}^2\,\sin(2\,\theta)}{2}\right) - \frac{R^2\,\alpha\,m\,\left(\frac{\sin(2\,\theta)\,\dot{\psi}^2}{2} + 2\,\dot{\phi}\,\cos(\theta)\,\dot{\psi} - \ddot{\theta}\right)}{2}\right) e_2^{"}\\
&... + \left(\frac{R^2\,\alpha\,m\,\left(2\,\dot{\phi}\,\dot{\theta} + \ddot{\psi}\,\cos(\theta)\right)}{2}\right) e_3^{"}
\end{aligned}$$

Finally, an expression for the moments about point A is

$$M_A = Re_3^{"} \times -mge_3 = Re_3^{"} \times -mg\left(-\sin(\theta)\,e_1^{"} + \cos(\theta)\,e_3^{"}\right) = Rmg\sin(\theta)\,e_2^{"}$$

Substituting these expressions to 1 gives (after simplifying with MATLAB)

$$\begin{aligned}
0 &= \ddot{\phi} + \alpha\,\ddot{\phi} + \ddot{\psi}\,\sin(\theta) + 2\,\dot{\psi}\,\dot{\theta}\,\cos(\theta) + \alpha\,\ddot{\psi}\,\sin(\theta) + \alpha\,\dot{\psi}\,\dot{\theta}\,\cos(\theta)\\
2\,R\,\ddot{\theta}\,(\alpha + 2) &= 4\,g\,\sin(\theta) + 2\,R\,\dot{\psi}^2\,\sin(2\,\theta) + R\,\alpha\,\dot{\psi}^2\,\sin(2\,\theta) + 4\,R\,\dot{\phi}\,\dot{\psi}\,\cos(\theta) + 4\,R\,\alpha\,\dot{\phi}\,\dot{\psi}\,\cos(\theta)\\
0 &= 2\,\dot{\phi}\,\dot{\theta} + \ddot{\psi}\,\cos(\theta)
\end{aligned}$$

Solving these equations for $\ddot{\theta}$, $\ddot{\psi}$ and $\ddot{\phi}$ gives

$$\begin{aligned}
\ddot{\theta} &= \frac{4\,g\,\sin(\theta) + 2\,R\,\dot{\psi}^2\,\sin(2\,\theta) + R\,\alpha\,\dot{\psi}^2\,\sin(2\,\theta) + 4\,R\,\dot{\phi}\,\dot{\psi}\,\cos(\theta) + 4\,R\,\alpha\,\dot{\phi}\,\dot{\psi}\,\cos(\theta)}{2\,R\,(\alpha + 2)}\\
\ddot{\psi} &= -\frac{2\,\dot{\phi}\,\dot{\theta}}{\cos(\theta)}\\
\ddot{\phi} &= \frac{\dot{\theta}\,\left(2\,\dot{\phi}\,\sin(\theta) - 2\,\dot{\psi}\cos(\theta)^2 + 2\,\alpha\,\dot{\phi}\,\sin(\theta) - \alpha\,\dot{\psi}\cos(\theta)^2\right)}{\cos(\theta)\,(\alpha + 1)}
\end{aligned} \tag{3}$$

It should be noted that we could get the same equations by writing the linear and angular momentum of the disk about it's center of mass. If we would choose to do so, we would get a set of 6 equations with $\ddot{\theta}$, $\ddot{\psi}$, $\ddot{\phi}$ and 3 contact forces as 6 unknowns.

One can see that $M_A \cdot e_3 = 0$ and because $a_A = 0$ the angular momentum about point A in this direction is conserved

$$H_A \cdot e_3 = Const = \sin(\theta)\,\left(R^2\,m\,\left(\dot{\phi} + \dot{\psi}\,\sin(\theta)\right) + R^2\,\alpha\,m\,\left(\dot{\phi} + \dot{\psi}\,\sin(\theta)\right)\right) + \frac{R^2\,\alpha\,\dot{\psi}\,m\cos(\theta)^2}{2} \tag{4}$$

Another conservation law is the conservation of energy in the system. As long as the disk does not slip, the velocity at the contact point is zero therefore the friction forces don't make work. In figure 2

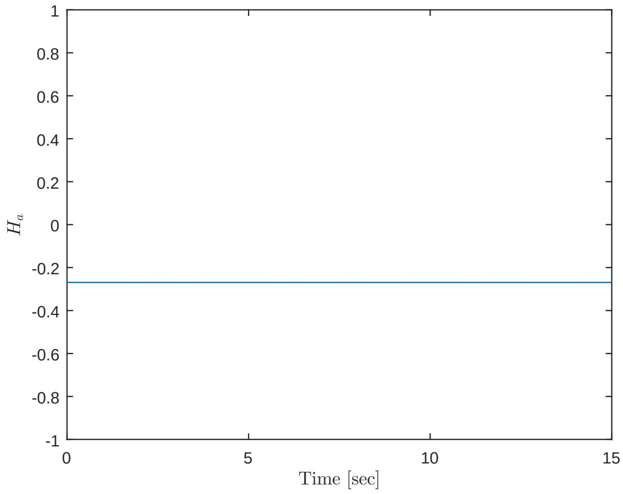


Figure 2: $H_A \cdot e_3$

we can see the result of 4 from a numerical simulation.

2 The Lagrangian Approach With Non-Holonomic Constraints

For the Lagrangian approach we'll use the following set of coordinates

$$q = \begin{bmatrix} x & y & \psi & \theta & \phi \end{bmatrix}^T$$

where x and y are the coordinates of the contact point. Because the disk is free to roll on a plane, the non-slip constraints become non-holonomic.

The center of mass velocity is

$$\begin{aligned} v_c = &\left(\dot{\mathrm{x}}\,\cos(\psi) + \dot{\mathrm{y}}\,\sin(\psi) + R\,\dot{\theta}\,\cos(\theta)\right) e_1^{'} + \left(\dot{\mathrm{y}}\,\cos(\psi) - \dot{\mathrm{x}}\,\sin(\psi) + R\,\dot{\psi}\,\sin(\theta)\right) e_2^{'} + \\ &... \left(-R\,\dot{\theta}\,\sin(\theta)\right) e_3^{'} \end{aligned}$$

The angular velocity of the disk is

$$\omega = \left(\dot{\phi} - \dot{\psi}\sin(\theta)\right) e_1^{''} + \left(\dot{\theta}\right) e_2^{''} + \left(\dot{\psi}\cos(\theta)\right) e_3^{''}$$

therfore the kinetic energy of the disk is

$$\begin{aligned} T &= \frac{1}{2} m v_c \cdot v_c^T + \frac{1}{2}\omega I_c \omega^T \\ &= \frac{1}{2} m \left(\dot{\mathrm{x}} + R\,\dot{\theta}\,\cos(\psi)\,\cos(\theta) - R\,\dot{\psi}\,\sin(\psi)\,\sin(\theta)\right)^2 \\ &... \frac{1}{2} m \big(\dot{\mathrm{y}} + R\,\dot{\psi}\,\cos(\psi)\,\sin(\theta) + R\,\dot{\theta}\,\cos(\theta)\,\sin(\psi)\big)^2 + R^2\,\dot{\theta}^2\,\sin(\theta)^2 + \frac{1}{16} m\,R^2\,\dot{\theta}^2 \\ &... \frac{1}{16} m R^2\,\dot{\psi}^2\cos(\theta)^2 + \frac{1}{4} m R^2\,\left(\dot{\phi} + \dot{\psi}\,\sin(\theta)\right)\,\left(\frac{\dot{\phi}}{2} + \frac{\dot{\psi}\,\sin(\theta)}{2}\right) \end{aligned}$$

The potential energy of the disk

$$V = -\left(-mge_3\right) \cdot r_c = R\,g\,m\,\cos(\theta)$$

Deriving the non-holonomic constraints in equation 2 gives

$$\begin{aligned} 0 &= \dot{\mathrm{x}}\,\cos(\psi) + \dot{\mathrm{y}}\,\sin(\psi) \\ 0 &= \dot{\mathrm{y}}\,\cos(\psi) - R\,\dot{\phi} - \dot{\mathrm{x}}\,\sin(\psi) \end{aligned} \tag{5}$$

we can write these equation as $W\dot{q} = 0$ where W is

$$W = \frac{\partial v_i}{\partial \dot{q}_j} = \begin{bmatrix} \cos(\psi) & \sin(\psi) & 0 & 0 & 0 \\ -\sin(\psi) & \cos(\psi) & 0 & 0 & -R \end{bmatrix}$$

Writing the constrained Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = W^T \lambda$$

where λ are the contact forces that dictate the constraints. Writing the equations of motion in a matrix form gives

$$M\,(q)\,\ddot{q} + B\,(q,\,\dot{q}) + G\,(q) = W^T \lambda$$

where

$$M\,(q) = \frac{\partial^2 T}{\partial^2 \dot{q}};\; B\,(q,\,\dot{q}) = \sum \frac{\partial^2 T}{\partial \dot{q} \partial q}\dot{q} - \frac{\partial T}{\partial q};\; G\,(q) = \frac{\partial V}{\partial q}$$

And for the disk problem we get

$$M = \begin{bmatrix} m & 0 & -R\,m\,\sin(\psi)\,\sin(\theta) & R\,m\,\cos(\psi)\,\cos(\theta) & 0 \\ 0 & m & R\,m\,\cos(\psi)\,\sin(\theta) & R\,m\,\cos(\theta)\,\sin(\psi) & 0 \\ -R\,m\,\sin(\psi)\,\sin(\theta) & R\,m\,\cos(\psi)\,\sin(\theta) & \frac{R^2\,m\,\left(5\sin(\theta)^2+1\right)}{4} & 0 & \frac{R^2\,m\,\sin(\theta)}{2} \\ R\,m\,\cos(\psi)\,\cos(\theta) & R\,m\,\cos(\theta)\,\sin(\psi) & 0 & \frac{9\,R^2\,m}{8} & 0 \\ 0 & 0 & \frac{R^2\,m\,\sin(\theta)}{2} & 0 & \frac{R^2\,m}{2} \end{bmatrix}$$

$$G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -R\,g\,m\,\sin(\theta) \\ 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -R\,m\,\left(\cos(\psi)\,\sin(\theta)\,\dot{\psi}^2 + 2\,\cos(\theta)\,\sin(\psi)\,\dot{\psi}\,\dot{\theta} + \cos(\psi)\,\sin(\theta)\,\dot{\theta}^2\right) \\ -R\,m\,\left(\sin(\psi)\,\sin(\theta)\,\dot{\psi}^2 - 2\,\cos(\psi)\,\cos(\theta)\,\dot{\psi}\,\dot{\theta} + \sin(\psi)\,\sin(\theta)\,\dot{\theta}^2\right) \\ \frac{R^2\,\dot{\theta}\,m\,\left(\frac{5\,\dot{\psi}\,\sin(2\,\theta)}{4} + \frac{1}{2}\dot{\phi}\,\cos(\theta)\right)}{4} \\ -\frac{R^2\,\dot{\psi}\,m\,\left(2.5\,\dot{\psi}\,\sin(2\,\theta) + 2\,\dot{\phi}\,\cos(\theta)\right)}{4} \\ \frac{R^2\,\dot{\psi}\,\dot{\theta}\,m\,\cos(\theta)}{2} \end{bmatrix}$$

differentiating $W\dot{q} = 0$ gives

$$\dot{W}\dot{q} + W\ddot{q} = 0$$

Therefore we can write

$$\begin{bmatrix} M & -W^T \\ W & 0_{2\times 2} \end{bmatrix} \begin{bmatrix} \ddot{q} \\ \lambda \end{bmatrix} = \begin{bmatrix} -B - G \\ -\dot{W}\dot{q} \end{bmatrix}$$

and get $\ddot{q}$ and λ.

The explicit solution (using MATLAB) is

$$\ddot{x} = -\frac{2\,\dot{\psi}\,\dot{\mathrm{y}}\cos(\psi)^2\,\cos(\theta) - 2\,\dot{\psi}\,\dot{\mathrm{x}}\,\sin(\psi)\,\cos(\psi)\,\cos(\theta) - 5\,R\,\dot{\psi}\,\dot{\theta}\,\sin(\psi)\,\cos(\theta)^2}{3\,\cos(\theta)} +$$

$$\dots\frac{\dot{\psi}\,\dot{\mathrm{y}}\,\cos(\theta) + 6\,R\,\dot{\phi}\,\dot{\theta}\,\sin(\psi)\,\sin(\theta)}{3\,\cos(\theta)}$$

$$\ddot{y} = -\frac{2\,\dot{\psi}\,\dot{\mathrm{x}}\cos(\psi)^2\,\cos(\theta) + 5\,R\,\dot{\psi}\,\dot{\theta}\,\cos(\psi)\,\cos(\theta)^2 + 2\,\dot{\psi}\,\dot{\mathrm{y}}\,\sin(\psi)\,\cos(\psi)\,\cos(\theta)}{3\,\cos(\theta)} +$$

$$\dots\frac{-6\,R\,\dot{\phi}\,\dot{\theta}\,\sin(\theta)\,\cos(\psi) - 3\,\dot{\psi}\,\dot{\mathrm{x}}\,\cos(\theta)}{3\,\cos(\theta)}$$

$$\ddot{\psi} = -\frac{2\,\dot{\phi}\,\dot{\theta}}{\cos(\theta)}$$

$$\ddot{\theta} = \frac{4\,g\,\sin(\theta) + 5\,R\,\dot{\psi}^2\,\cos(\theta)\,\sin(\theta) + 2\,R\,\dot{\phi}\,\dot{\psi}\,\cos(\theta) + 4\,\dot{\psi}\,\dot{\mathrm{y}}\,\cos(\psi)\,\cos(\theta) - 4\,\dot{\psi}\,\dot{\mathrm{x}}\,\cos(\theta)\,\sin(\psi)}{5\,R}$$

$$\ddot{\phi} = -\frac{5\,R\,\dot{\psi}\,\dot{\theta}\cos(\theta)^2 - 6\,R\,\dot{\phi}\,\dot{\theta}\,\sin(\theta) + 2\,\dot{\psi}\,\dot{\mathrm{x}}\,\cos(\psi)\,\cos(\theta) + 2\,\dot{\psi}\,\dot{\mathrm{y}}\,\cos(\theta)\,\sin(\psi)}{3\,R\,\cos(\theta)}$$

and by substituting $\dot{x}$ and $\dot{y}$ (known from 5) we can get the same expressions as equation 3.

3 Stability Analysis

In this part we'll analyze the stability of the disk under small perturbation from a steady state solution.

3.1 Equations of Motion in a State Space Form

As a preperation for the stability analysis using linearization of the equations of motion lets denote the following state space coordinates

$$X = \left[\begin{array}{cccc} \theta & v_\theta & v_\psi & v_\phi \end{array} \right]^T$$

We can write the equations of motion in the following way $\dot{X} = f(X)$ and get

$$\begin{aligned}
\dot{\theta} &= \mathrm{v}_\theta \\
\dot{\mathrm{v}}_\theta &= \frac{4\,g\,\sin(\theta) + 2\,R\,{\mathrm{v}_\psi}^2\,\sin(2\,\theta) + R\,\alpha\,{\mathrm{v}_\psi}^2\,\sin(2\,\theta) + 4\,R\,\mathrm{v}_\phi\,\mathrm{v}_\psi\,\cos(\theta) + 4\,R\,\alpha\,\mathrm{v}_\phi\,\mathrm{v}_\psi\,\cos(\theta)}{2\,R\,(\alpha+2)} \\
\dot{\mathrm{v}}_\psi &= -\frac{2\,\mathrm{v}_\phi\,\mathrm{v}_\theta}{\cos(\theta)} \\
\dot{\mathrm{v}}_\phi &= \frac{\mathrm{v}_\theta\,\left(2\,\mathrm{v}_\phi\,\sin(\theta) - 2\,\mathrm{v}_\psi\cos(\theta)^2 + 2\,\alpha\,\mathrm{v}_\phi\,\sin(\theta) - \alpha\,\mathrm{v}_\psi\cos(\theta)^2\right)}{\cos(\theta)\,(\alpha+1)}
\end{aligned} \tag{6}$$

3.2 Equilibria of the Disk

In order to linearize the eqautions of motion, we'll have to find the states on which the disk is in equilibrium

$$\dot{X} = \left[\begin{array}{cccc} v_\theta & \dot{v}_\theta & \dot{v}_\psi & \dot{v}_\phi \end{array} \right]^T = 0_{4\times 1}$$

we can see that because $v_\theta = 0$ the third on fourth state equations (equation 6) zero out and don't give further information. The second equation can zero with the right choice of θ^*, v_ψ^* and v_ϕ^*.At the general case of equilibrium the disk will spin in circles with radii $\rho = R \cdot v_\phi^*/v_\psi^*$ about a fixed point (as seen in section 3.5). There are three special cases that causes the disk to act differently. The first case is when $\theta = 0$, $v_\psi = 0$ and $\rho \to \infty$. In this case, the disk spins about it's axis and rolles in a straight line. The second case is when $\theta = 0$, $v_\phi = 0$ and $\rho \to 0$. In this case the disk spins about the vertical axis like a top. The third case is when $\theta = 0$, $v_\psi = 0$ and $v_\phi = 0$. In this case the second equation is in the form of a pendulum

$$\dot{\mathrm{v}}_\theta = \ddot{\theta} = \frac{4g\sin(\theta)}{2R\,(\alpha+2)}$$

One can see that when the disk does not spin at all it is in an unstable equilibrium.

3.3 The Disk Rolling About It's Symmetry Axis

3.3.1 Analytical derivation

In this part we'll want to find the minimal angular velocity needed to keep the disk in a stable motion. Suppose that the initial conditions are

$$X(0) = \left[\begin{array}{cccc} 0 & \varepsilon & 0 & \dot{\phi}_0 \end{array} \right]^T$$

Linearization about the equilibrium gives

$$\begin{aligned}
\dot{\mathrm{v}}_\theta &= \frac{2\,g\,\theta + 2\,R\,\dot{\phi}_0\,\mathrm{v}_\psi + 2\,R\,\alpha\,\dot{\phi}_0\,\mathrm{v}_\psi}{R\,(\alpha+2)} \\
\dot{\mathrm{v}}_\psi &= -2\,\dot{\phi}_0\,\mathrm{v}_\theta \\
\dot{\mathrm{v}}_\phi &= 0
\end{aligned}$$

Differentiating $\dot{v}_\theta$ and substituting $\dot{v}_\psi$ gives

$$\ddot{\mathrm{v}}_\theta = \mathrm{v}_\theta \frac{2g - 4R\dot{\phi}_0^2 - 4R\alpha\dot{\phi}_0^2}{R\,(\alpha+2)}$$

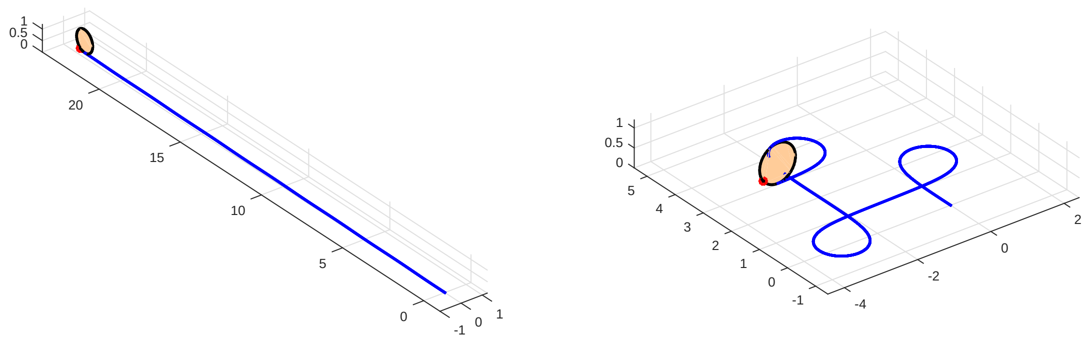
We have a second order ODE. As seen, for a uniform mass distribution ($\alpha = 1/2$) choosing $\dot{\phi}_0^2 > \frac{g}{3R}$ will give a harmonic solution for the ODE thus the disk will continue to oscillate in the absence of damping. On the other hand, if we'll choose $\dot{\phi}_0^2 < \frac{g}{3R}$ the ODE's solution will be exponential and the disk would fall. For a hoop, ($\alpha = 1$) the ODE will have a stable solution if $\dot{\phi}_0^2 > \frac{g}{4R}$.

3.3.2 Numerical solution

For the numerical simulation let's choose a disk with mass $m = 1\,Kg$ uniformly distributed, radius of $R = 0.5\,m$ and $g = 9.81\,{}^{\mathrm{m}}/_{\mathrm{sec}^2}$. The initial condition for the simulation is

$$X\,(0) = \begin{bmatrix} 0 & 0.01 & 0 & \dot{\phi}_0 \end{bmatrix}^T$$

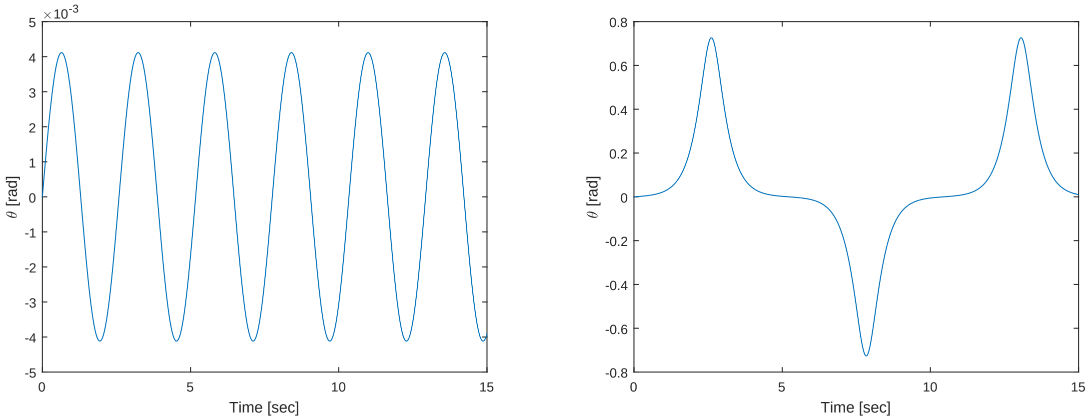
When on the first simulation we'll pick $9 = \dot{\phi}_0^2 > \frac{g}{3R} \cong 6.540$ and on the second simulation $4 = \dot{\phi}_0^2 < \frac{g}{3R}$. In figure 3 it is shown that the disk's response for the initial conditions is significantly different. In figure



(a) Disk's path for $\dot{\phi}_0 = 3\ {}^{\mathrm{rad}}/_{\mathrm{sec}}$

(b) Disk's path for $\dot{\phi}_0 = 2\ {}^{\mathrm{rad}}/_{\mathrm{sec}}$

Figure 3: Disk's path



(a) $\dot{\phi}_0 = 3$ rad/sec

(b) $\dot{\phi}_0 = 2$ rad/sec

Figure 4: θ as function of time

4 it is shown that the ratio of θ between both of the cases is two orders of magnitude.

3.4 The Disk Spinning About the Vertical Axis

3.4.1 Analytical derivation

Similarly to subsection 3.3, we'll check the conditions for stability when the initial condition is

$$X(0) = \begin{bmatrix} 0 & \varepsilon & \dot{\psi}_0 & 0 \end{bmatrix}^T$$

Linearization of the equations of motion about the equilibrium gives

$$\begin{aligned} \dot{\mathrm{v}}_\theta &= \frac{2\,g\,\theta + 2\,R\,\mathrm{v}_\phi\,\dot{\psi}_0 + 2\,R\,\dot{\psi}_0^2\,\theta + R\,\alpha\,\dot{\psi}_0^2\,\theta + 2\,R\,\alpha\,\mathrm{v}_\phi\,\dot{\psi}_0}{R\,(\alpha+2)} \\ \dot{\mathrm{v}}_\psi &= 0 \\ \dot{\mathrm{v}}_\phi &= -\frac{\dot{\psi}_0\,\mathrm{v}_\theta\,(\alpha+2)}{\alpha+1} \end{aligned}$$

Differentiating $\dot{v}_\theta$ and substituting $\dot{v}_\phi$ gives

$$\ddot{v}_\theta = -\frac{\mathrm{v}_\theta\,\left(2\,R\,\dot{\psi}_0^2 - 2\,g + R\,\alpha\,\dot{\psi}_0^2\right)}{R\,(\alpha+2)}$$

Therfore, a uniformly mass distributed disk will be stable if $\dot{\psi}_0^2 > \frac{2g}{2.5R}$. A hoop will be stable for $\dot{\psi}_0^2 > \frac{2g}{3R}$.

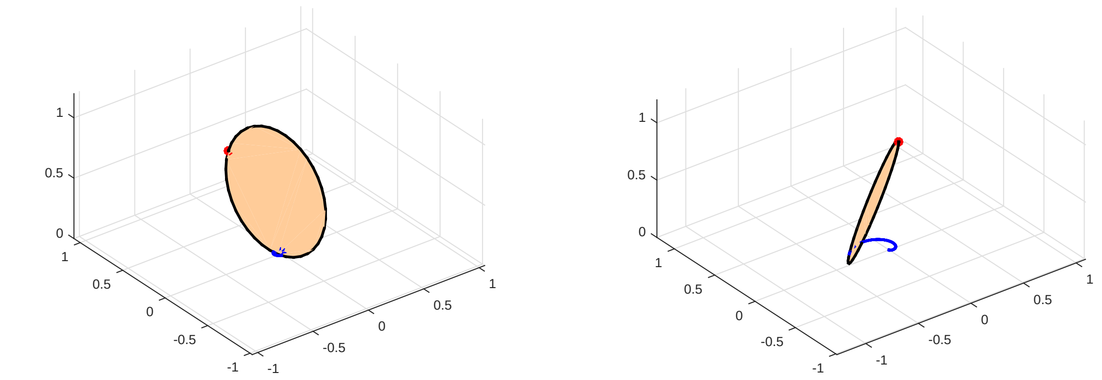
3.4.2 Numerical solution

For the same parameters we chose in part 3.3.2 and the following initial condition

$$X(0) = \begin{bmatrix} 0 & 0.1 & \dot{\psi}_0 & 0 \end{bmatrix}^T$$

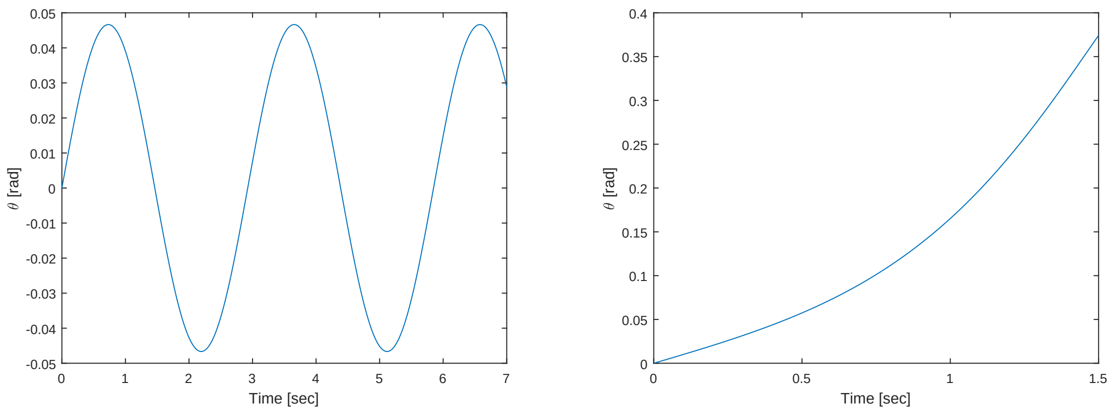
we'll chose $\sqrt{\frac{2g}{2.5R}} < \dot{\psi}_0 = 4.5$ rad/sec and $\dot{\psi}_0 = 3.5 < \sqrt{\frac{2g}{2.5R}}$ to simulate the disk's motion.

In figure 5 the disk's paths for each simulation is shown. In figure



(a) Disk's path for $\dot{\psi}_0 = 4.5$ rad/sec (b) Disk's path for $\dot{\psi}_0 = 3.5$ rad/sec

Figure 5: Disk's path



(a) $\dot{\psi}_0 = 4.5$ rad/sec (b) $\dot{\psi}_0 = 3.5$ rad/sec

Figure 6: θ as function of time

6 one can see that for $\dot{\psi}_0 = 4.5$ rad/sec the solution is harmonic and does not diverge.

3.5 The Disk Circles a Fixed Point

In order to make the disk follow a circular paths we'll need to solve the second equation (from equations6) for $\dot{v}_\theta = 0$. Given the nutation angle θ^* and the angular velocity v_ψ^* at equilibrium, the angular velocity v_ϕ^* needed to keep the disk at equilibrium is

$$v_\phi^* = -\frac{4\,g\,\sin(\theta^*) + 2\,R\,{v_\psi^*}^2\,\sin(2\,\theta^*) + R\,\alpha\,{v_\psi^*}^2\,\sin(2\,\theta)}{4\,R\,v_\psi^*\,\cos(\theta)\,\left(\alpha+1\right)}$$

In example, choosing $\theta^* = {}^{10}\!/_{180} \cdot \pi$ rad and $v_\psi^* = 0.5$ rad/sec we'll get

$$v_\phi^* = -4.6851 \text{ rad/sec}$$

Simulation of these values gives the following path

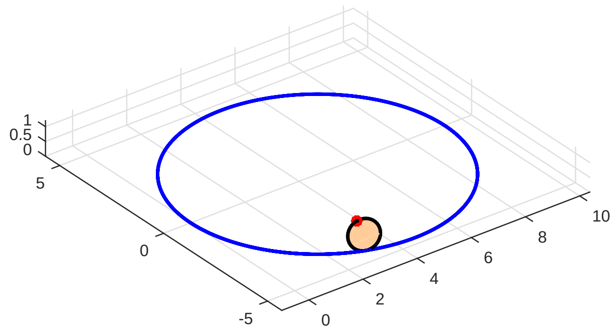


Figure 7: Simulation of the disk following a circular path