

INSTRUCTOR'S EDITION

F I F T H E D I T I O N

# Linear Algebra and Its Applications

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#### Library of Congress Cataloging-in-Publication Data

Lay, David C.

Linear algebra and its applications / David C. Lay, University of Maryland, College Park, Steven R. Lay, Lee University, Judi J. McDonald, Washington State University. – Fifth edition.

pages cm

Includes index.

ISBN 978-0-321-98238-4

ISBN 0-321-98238-X

1. Algebras, Linear—Textbooks. I. Lay, Steven R., 1944- II. McDonald, Judi. III. Title.

QA184.2.L39 2016

512'.5—dc23

2014011617



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# Contents

*Photocredits*   **P1**

## **Chapter 10**   Finite-State Markov Chains (Online)   **1**

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**INTRODUCTORY EXAMPLE:** Googling and Markov Chains   **1**

10.1   Introduction and Examples   **2**

10.2   The Steady-State Vector and Google's PageRank   **13**

10.3   Communication Classes   **25**

10.4   Classification of States and Periodicity   **33**

10.5   The Fundamental Matrix   **42**

10.6   Markov Chains and Baseball Statistics   **54**

## **Appendixes**

---

1   Proof of Theorem 1   **A1**

2   Probability   **A5**

*Answers to Exercises*   **A1**

# Photo Credits

## Chapter 1

*Page 1* 30th October 1973: Wassily Leontief, Russian-born American winner of the Nobel Prize for Economics: Keystone/Hulton Archive/Getty Images.

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## Chapter 3

*Page 165* Physicist Richard Feynman: AP Images.

## Chapter 4

*Page 191* Dryden Observes 31st Anniversary of STS-1 Mission: NASA.

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## Chapter 5

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## Chapter 6

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## **Chapter 7**

*Page 395* Landsat Satellite: Landsat Data/U.S. Geological Survey.

*Page 396* Spectral band 1: Landsat Data/U.S. Geological Survey; Spectral band 4: Landsat Data/U.S. Geological Survey; Spectral band 7: Landsat Data/U.S. Geological Survey; Principal component 1: Landsat Data/U.S. Geological Survey; Principal component 2: Landsat Data/U.S. Geological Survey; Principal component 3: Landsat Data/U.S. Geological Survey.

*Page 414* Small wood bridge with railings: Dejangasparin/Fotolia; Wheel loader machine unloading sand: Dmitry Kalinovsky/Shutterstock; Young family having a picnic by the river: Viki2win/Shutterstock.

## **Chapter 8**

*Page 437* School of Athens fresco: The Art Gallery Collection/Alamy.

## **Chapter 9**

Chapter 9 (Online): *Page 1* Berlin airlift: Everett Collection Inc/Alamy.

## **Chapter 10**

Chapter 10 (Online): *Page 1* Larry Page: Kim Kulish/Corbis.

# 10

## Finite-State Markov Chains

### WEB

#### INTRODUCTORY EXAMPLE

### Googling Markov Chains

Google means many things: it is an Internet search engine, the company that produces the search engine, and a verb meaning to search on the Internet for a piece of information. Although it may seem hard to believe, there was a time before people could “google” to find the capital of Botswana, or a recipe for deviled eggs, or other vitally important matters. Users of the Internet depend on trustworthy search engines—the amount of available information is so vast that the searcher relies on the search engine not only to find those webpages that contain the terms of the search, but also to return first those webpages most likely to be relevant to the search. Early search engines had no good way of determining which pages were more likely to be relevant. Searchers had to check the returned pages one by one, which was a tedious and frustrating process. This situation improved markedly in 1998, when search engines began to use the information contained in the hyperlinked structure of the World Wide Web to help to rank pages. Foremost among this new generation of search engines was Google, a project of two computer science graduate students at Stanford University: Sergey Brin and Lawrence Page.



Brin and Page reasoned that a webpage was important if it had hyperlinks to it from other important pages. They used the idea of the random surfer: a web surfer moving from webpage to webpage merely by choosing at random which hyperlink to follow. The motion of the surfer among the webpages can be modeled using Markov chains, which were introduced in Section 4.9. The pages that this random surfer visits more often ought to be more important, and thus more relevant, if their content matches the terms of a search. Although Brin and Page did not know it at the time, they were attempting to find the steady-state vector for a particular Markov chain whose transition matrix modeled the hyperlinked structure of the web. After some important modifications of this impressively large matrix (detailed in Section 10.2), a steady-state vector can be found, and its entries can be interpreted as the amount of time a random surfer will spend at each webpage. The calculation of this steady-state vector is the basis for Google’s PageRank algorithm.

So the next time you google the capital of Botswana, know that you are using the results of this chapter to find just the right webpage.

Even though the number of webpages is huge, it is still finite. When the link structure of the World Wide Web is modeled by a Markov chain, each webpage is a state of the Markov chain. This chapter continues the study of Markov chains begun in Section 4.9,

focusing on those Markov chains with a finite number of states. Section 10.1 introduces useful terminology and develops some examples of Markov chains: signal transmission models, diffusion models from physics, and random walks on various sets. Random walks on directed graphs will have particular application to the PageRank algorithm. Section 10.2 defines the steady-state vector for a Markov chain. Although every Markov chain has a steady-state vector, not every Markov chain converges to a steady-state vector. When the Markov chain converges to a steady-state vector, that vector can be interpreted as telling the amount of time the chain will spend in each state. This interpretation is necessary for the PageRank algorithm, so the conditions under which a Markov chain converges to a steady-state vector will be developed. The model for the link structure of the World Wide Web will then be modified to meet these conditions, forming what is called the Google matrix. Sections 10.3 and 10.4 discuss Markov chains that do not converge to steady-state vectors. These Markov chains can be used to model situations in which the chain eventually becomes confined to one state or a set of states. Section 10.5 introduces the fundamental matrix. This matrix can be used to calculate the expected number of steps it takes the chain to move from one state to another, as well as the probability that the chain ends up confined to a particular state. In Section 10.6, the fundamental matrix is applied to a model for run production in baseball: the number of batters in a half inning and the state in which the half inning ends will be of vital importance in calculating the expected number of runs scored.

## 10.1 INTRODUCTION AND EXAMPLES

Recall from Section 4.9 that a **Markov chain** is a mathematical model for movement between **states**. A process starts in one of these states and moves from state to state. The moves between states are called **steps** or **transitions**. The terms “chain” and “process” are used interchangeably, so the chain can be said to move between states and to be “at a state” or “in a state” after a certain number of steps.

The state of the chain at any given step is not known; what is known is the probability that the chain moves from state  $j$  to state  $i$  in one step. This probability is called a **transition probability** for the Markov chain. The transition probabilities are placed in a matrix called the **transition matrix**  $P$  for the chain by entering the probability of a transition from state  $j$  to state  $i$  at the  $(i, j)$ -entry of  $P$ . So if there were  $m$  states named  $1, 2, \dots, m$ , the transition matrix would be the  $m \times m$  matrix

$$P = \begin{array}{c} \begin{array}{ccc} & \text{From:} & \\ & \begin{array}{ccc} 1 & j & m \end{array} & \\ \begin{bmatrix} & & \\ & \vdots & \\ & \downarrow & \\ p_{ij} & \cdots & \end{bmatrix} & \begin{array}{c} \text{To:} \\ 1 \\ \\ i \\ m \end{array} \end{array} \end{array}$$

The probabilities that the chain is in each of the possible states after  $n$  steps are listed in a **state vector**  $\mathbf{x}_n$ . If there are  $m$  possible states, the state vector would be

$$\mathbf{x}_n = \begin{bmatrix} a_1 \\ \vdots \\ a_j \\ \vdots \\ a_m \end{bmatrix} \leftarrow \text{Probability that the chain is at state } j \text{ after } n \text{ steps}$$

State vectors are **probability vectors** since their entries must sum to 1. The state vector  $\mathbf{x}_0$  is called the **initial probability vector**.

Notice that the  $j^{\text{th}}$  column of  $P$  is a probability vector—its entries list the probabilities of a move from state  $j$  to the states of the Markov chain. The transition matrix is thus a **stochastic matrix** since all of its columns are probability vectors.

The state vectors for the chain are related by the equation

$$\mathbf{x}_{n+1} = P\mathbf{x}_n \quad (1)$$

for  $n = 1, 2, \dots$ . Notice that Equation (1) may be used to show that

$$\mathbf{x}_n = P^n \mathbf{x}_0 \quad (2)$$

Thus any state vector  $\mathbf{x}_n$  may be computed from the initial probability vector  $\mathbf{x}_0$  and an appropriate power of the transition matrix  $P$ .

This chapter concerns itself with Markov chains with a finite number of states—that is, those chains for which the transition matrix  $P$  is of finite size. To use a finite-state Markov chain to model a process, the process must have the following properties, which are implied by Equations (1) and (2).

1. Since the values in the vector  $\mathbf{x}_{n+1}$  depend only on the transition matrix  $P$  and on  $\mathbf{x}_n$ , the state of the chain before time  $n$  must have no effect on its state at time  $n + 1$  and beyond.
2. Since the transition matrix  $P$  does not change with time, the probability of a transition from one state to another must not depend on how many steps the chain has taken.

Even with these restrictions, Markov chains may be used to model an amazing variety of processes. Here is a sampling.

## Signal Transmission

Consider the problem of transmitting a signal along a telephone line or by radio waves. Each piece of data must pass through a multi-stage process to be transmitted, and at each stage there is a probability that a transmission error will cause the data to be corrupted. Assume that the probability of an error in transmission is not affected by transmission errors in the past and does not depend on time, and that the number of possible pieces of data is finite. The transmission process may then be modeled by a Markov chain. The object of interest is the probability that a piece of data goes through the entire multi-stage process without error. Here is an example of such a model.

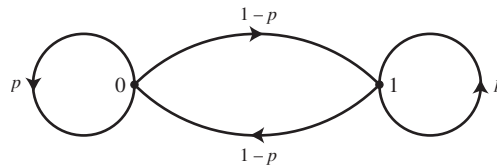
**EXAMPLE 1** Suppose that each bit of data is either a 0 or a 1, and at each stage there is a probability  $p$  that the bit will pass through the stage unchanged. Thus the probability is  $1 - p$  that the bit will be transposed. The transmission process is modeled by a Markov chain, with states 0 and 1 and transition matrix

$$P = \begin{array}{cc} \begin{array}{c} \text{From:} \\ 0 \quad 1 \end{array} & \begin{array}{c} \text{To:} \\ 0 \\ 1 \end{array} \\ \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix} \end{array}$$

It is often easier to visualize the action of a Markov chain by representing its transition probabilities graphically, as in Figure 1. The points are the states of the chain, and the arrows represent the transitions.

Suppose that  $p = .99$ . Find the probability that the signal 0 will still be a 0 after a two-stage transmission process.





**FIGURE 1** Transition diagram for signal transmission.

**SOLUTION** Since the signal begins as 0, the probability that the chain begins at 0 is 100%, or 1; that is, the initial probability vector is  $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ . To find the probability of a two-step transition, compute

$$\mathbf{x}_2 = P^2 \mathbf{x}_0 = \begin{bmatrix} .99 & .01 \\ .01 & .99 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .9802 & .0198 \\ .0198 & .9802 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .9802 \\ .0198 \end{bmatrix}$$

The probability that the signal 0 will still be a 0 after the two-stage process is thus .9802. Notice that this is not the same as the probability that the 0 is transmitted without error; that probability would be  $(.99)^2 = .9801$ . Our analysis includes the very small probability that the 0 is erroneously changed to 1 in the first step, then back to 0 in the second step of transmission. ■

## Diffusion

Consider two compartments filled with different gases which are separated only by a membrane that allows molecules of each gas to pass from one container to the other. The two gases will then diffuse into each other over time, so that each container will contain some mixture of the gases. The major question of interest is what mixture of gases is in each container at a time after the containers are joined. A famous mathematical model for this process was described originally by the physicists Paul and Tatyana Ehrenfest. Since their preferred term for “container” was *urn*, the model is called the **Ehrenfest urn model** for diffusion.

Label the two urns A and B, and place  $k$  molecules of gas in each urn. At each time step, select one of the  $2k$  molecules at random and move it from its urn to the other urn, and keep track of the number of molecules in urn A. This process can be modeled by a finite-state Markov chain: the number of molecules in urn A after  $n + 1$  time steps depends only on the number in urn A after  $n$  time steps, the transition probabilities do not change with time, and the number of states is finite.

**EXAMPLE 2** For this example, let  $k = 3$ . Then the two urns contain a total of 6 molecules, and the possible states for the Markov chain are 0, 1, 2, 3, 4, 5, and 6. Notice first that if there are 0 molecules in urn A at time  $n$ , then there must be 1 molecule in urn A at time  $n + 1$ , and if there are 6 molecules in urn A at time  $n$ , then there must be 5 molecules in urn A at time  $n + 1$ . In terms of the transition matrix  $P$ , this means that the columns in  $P$  corresponding to states 0 and 6 are

$$\mathbf{p}_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{p}_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

If there are  $i$  molecules in urn A at time  $n$ , with  $0 < i < 6$ , then there must be either  $i - 1$  or  $i + 1$  molecules in urn A at time  $n + 1$ . In order for a transition from  $i$  to  $i - 1$  molecules to occur, one of the  $i$  molecules in urn A must be selected to move; this event happens with probability  $i/6$ . Likewise, a transition from  $i$  to  $i + 1$  molecules occurs when one of the  $6 - i$  molecules in urn B is selected, and this occurs with probability  $(6 - i)/6$ . Allowing  $i$  to range from 1 to 5 creates the columns of  $P$  corresponding to these states, and the transition matrix for the Ehrenfest urn model with  $k = 3$  is thus

$$P = \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{cccccc} \left[ \begin{array}{cccccc} 0 & 1/6 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 5/6 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 2/3 & 0 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 5/6 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1/6 & 0 \end{array} \right] \end{array} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

Figure 2 shows a transition diagram of this Markov chain. Other models for diffusion will be considered in the Exercises for this section. ■

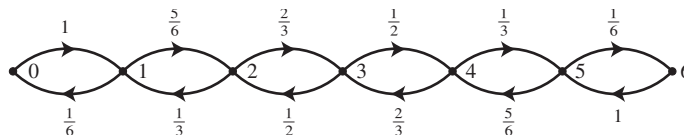


FIGURE 2 Transition diagram of the Ehrenfest urn model.

## Random Walks on $\{1, \dots, n\}$

Molecular motion has long been an important issue in physics. Einstein and others investigated Brownian motion, which is a mathematical model for the motion of a molecule exposed to collisions with other molecules. The analysis of Brownian motion turns out to be quite complicated, but a discrete version of Brownian motion called a **random walk** provides an introduction to this important model. Think of the states  $\{1, 2, \dots, n\}$  as lying on a line. Place a molecule at a point that is not on the end of the line. At each step the molecule moves left one unit with probability  $p$  and right one unit with probability  $1 - p$ . See Figure 3. The molecule thus “walks randomly” along the line. If  $p = 1/2$ , the walk is called **simple**, or **unbiased**. If  $p \neq 1/2$ , the walk is said to be **biased**.

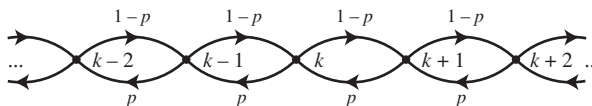


FIGURE 3 A graphical representation of a random walk.

The molecule must move to either the left or the right at the states  $2, \dots, n - 1$ , but it cannot do this at the endpoints 1 and  $n$ . The molecule’s possible movements at the endpoints 1 and  $n$  must be specified. One possibility is to have the molecule stay at an endpoint forever once it reaches either end of the line. This is called a **random walk with absorbing boundaries**, and the endpoints 1 and  $n$  are called **absorbing states**. Another possibility is to have the molecule bounce back one unit when an endpoint is reached. This is called a **random walk with reflecting boundaries**.

**EXAMPLE 3** A random walk on  $\{1, 2, 3, 4, 5\}$  with absorbing boundaries has a transition matrix of

$$P = \begin{array}{ccccc} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{bmatrix} 1 & p & 0 & 0 & 0 \\ 0 & 0 & p & 0 & 0 \\ 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 1-p & 0 & 0 \\ 0 & 0 & 0 & 1-p & 1 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \end{array}$$

since the molecule at state 1 has probability 1 of staying at state 1, and a molecule at state 5 has probability 1 of staying at state 5. A random walk on  $\{1, 2, 3, 4, 5\}$  with reflecting boundaries has a transition matrix of

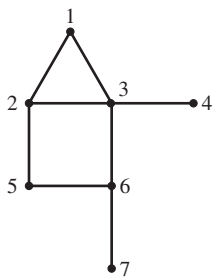
$$P = \begin{array}{ccccc} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{bmatrix} 0 & p & 0 & 0 & 0 \\ 1 & 0 & p & 0 & 0 \\ 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 1-p & 0 & 1 \\ 0 & 0 & 0 & 1-p & 0 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \end{array}$$

since the molecule at state 1 has probability 1 of moving to state 2, and a molecule at state 5 has probability 1 of moving to state 4. ■

In addition to their use in physics, random walks also occur in problems related to gambling and its more socially acceptable variants: the stock market and the insurance industry.

**EXAMPLE 4** Consider a very simple casino game. A gambler (who still has some money left with which to gamble) flips a fair coin and calls heads or tails. If the gambler is correct, he wins a dollar; if he is wrong, he loses a dollar. Suppose that the gambler will quit the game when he has either won  $n$  dollars or lost all of his money.

Suppose that  $n = 7$  and the gambler starts with \$4. Notice that the gambler's winnings move either up or down \$1 for each coin flip, and once the gambler's winnings reach 0 or 7, they do not change any more since the gambler has quit the game. Thus the gambler's winnings may be modeled by a random walk on  $\{0, 1, 2, 3, 4, 5, 6, 7\}$  with absorbing boundaries. Since a move up or down is equally likely in this case,  $p = 1/2$  and the walk is simple. ■



**FIGURE 4**  
A graph with seven vertices.

## Random Walks on Graphs

It is useful to perform random walks on geometrical objects other than the one-dimensional line. For example, a **graph** is a collection of points and lines connecting some of the points. The points of a graph are called vertices, and the lines connecting the vertices are called the edges. In Figure 4, the vertices are labeled with the numbers 1 through 7.

To define a simple random walk on a graph, allow the chain to move from vertex to vertex on the graph. At each step, the chain is equally likely to move along any of the edges attached to the vertex. For example, if the molecule is at state 5 in Figure 4, it has probability  $1/2$  of moving to state 2 and probability  $1/2$  of moving to state 6. This Markov chain is called a **simple random walk on a graph**.

**EXAMPLE 5** The simple random walk on the graph in Figure 4 has transition matrix

$$P = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix} & \begin{bmatrix} 0 & 1/3 & 1/4 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/4 & 0 & 1/2 & 0 & 0 \\ 1/2 & 1/3 & 0 & 1 & 0 & 1/3 & 0 \\ 0 & 0 & 1/4 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 1/4 & 0 & 1/2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 0 \end{bmatrix} \end{matrix}$$

Find the probability that the chain in Figure 4 moves from state 6 to state 2 in exactly three steps.

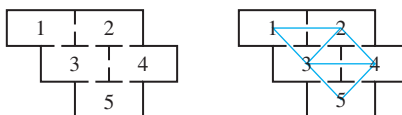
**SOLUTION** Compute

$$\mathbf{x}_3 = P^3 \mathbf{x}_0 = P^3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .0833 \\ .0417 \\ .4028 \\ 0 \\ .2778 \\ 0 \\ .1944 \end{bmatrix}$$

Thus the probability of moving from state 6 to state 2 in exactly three steps is .0417. ■

Sometimes interpreting a random process as a random walk on a graph can be useful.

**EXAMPLE 6** Suppose a mouse runs through the five-room maze at left in Figure 5. The mouse moves to a different room at each time step. When the mouse is in a particular room, it is equally likely to choose any of the doors out of the room. Note that a Markov chain can model the motion of the mouse. Find the probability that a mouse starting in room 3 returns to that room in exactly five steps.



**FIGURE 5** Five-room maze with overlaid graph.

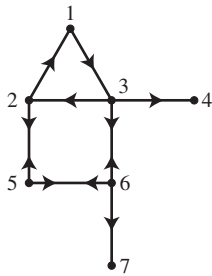
**SOLUTION** A graph is overlaid on the maze, as shown at right in Figure 5. Notice that the motion of the mouse is identical to a simple random walk on the graph, so the transition matrix is

$$P = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 1/3 & 1/4 & 0 & 0 \\ 1/2 & 0 & 1/4 & 1/3 & 0 \\ 1/2 & 1/3 & 0 & 1/3 & 1/2 \\ 0 & 1/3 & 1/4 & 0 & 1/2 \\ 0 & 0 & 1/4 & 1/3 & 0 \end{bmatrix} \end{matrix}$$

and

$$\mathbf{x}_5 = P^5 \mathbf{x}_0 = P^5 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} .1507 \\ .2143 \\ .2701 \\ .2143 \\ .1507 \\ 0 \\ 0 \end{bmatrix}$$

Thus the probability of a return to room 3 in exactly five steps is .2701. ■



**FIGURE 6**  
A directed graph with seven vertices.

Another interesting object on which to walk randomly is a **directed graph**. A directed graph is a graph in which the vertices are joined not by lines but by arrows. See Figure 6.

To perform a simple random walk on a directed graph, allow the chain to move from vertex to vertex on the graph but only in the directions allowed by the arrows. At each step the walker is equally likely to move away from its current state along any of the arrows pointing away from the vertex. For example, if the molecule is at state 6 in Figure 6, it has probability  $1/3$  of moving to state 3, state 5, or state 7.

The PageRank algorithm which Google uses to rank the importance of pages on the World Wide Web (see the Introductory Example) begins with a simple random walk on a directed graph. The Web is modeled as a directed graph in which the vertices are the pages and an arrow is drawn from page  $j$  to page  $i$  if there is a hyperlink from page  $j$  to page  $i$ . A person surfs randomly in the following way: when the surfer gets to a page, he or she chooses a link from the page so that it is equally probable to choose any of the possible “outlinks.” The surfer then follows the link to arrive at another page. The person surfing in this way is performing a simple random walk on the directed graph that is the World Wide Web.

**EXAMPLE 7** Consider a set of seven pages hyperlinked by the directed graph in Figure 6. If the random surfer starts at page 5, find the probability that the surfer will be at page 3 after four clicks.

**SOLUTION** The transition matrix for the simple random walk on the directed graph is

$$P = \begin{array}{ccccccc|c} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\ \begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} & \begin{array}{c} 1/2 \\ 0 \\ 0 \\ 0 \\ 1/2 \\ 0 \\ 0 \end{array} & \begin{array}{c} 0 \\ 1/3 \\ 0 \\ 1/3 \\ 0 \\ 1/3 \\ 0 \end{array} & \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{array} & \begin{array}{c} 0 \\ 1/2 \\ 0 \\ 0 \\ 0 \\ 1/2 \\ 0 \end{array} & \begin{array}{c} 0 \\ 0 \\ 1/3 \\ 0 \\ 1/3 \\ 0 \\ 1/3 \end{array} & \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{array} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \end{array}$$

Notice that there are no arrows coming from either state 4 or state 7 in Figure 6. If the surfer clicks on a link to either of these pages, there is no link to click on next.<sup>1</sup> For this reason, the transition probabilities  $p_{44}$  and  $p_{77}$  are set equal to 1—the chain must stay

<sup>1</sup> Using the “Back” key is not allowed: the state of the chain before time  $n$  must have no effect on its state at time  $n + 1$  and beyond.

at state 4 or state 7 forever once it enters either of these states. Computing  $\mathbf{x}_4$  gives

$$\mathbf{x}_4 = P^4 \mathbf{x}_0 = \begin{bmatrix} .1319 \\ .0833 \\ .0880 \\ .1389 \\ .2199 \\ .0833 \\ .2546 \end{bmatrix}$$

so the probability of being at page 3 after exactly four clicks is .0880. ■

States 4 and 7 are absorbing states for the Markov chain in the previous example. In technical terms, they are called **dangling nodes** and are quite common on the Web; data pages in particular usually have no links leading from them. Dangling nodes will appear in the next section, where the PageRank algorithm will be explained.

As noted in Section 4.9, the most interesting questions about Markov chains concern their long-term behavior—that is, the behavior of  $\mathbf{x}_n$  as  $n$  increases. This study will occupy a large portion of this chapter. The foremost issues in our study will be whether the sequence of vectors  $\{\mathbf{x}_n\}$  is converging to some limiting vector as  $n$  increases, and how to interpret this limiting vector if it exists. Convergence to a limiting vector will be addressed in the next section.

### PRACTICE PROBLEMS

1. Fill in the missing entries in the stochastic matrix.

$$P = \begin{bmatrix} .1 & * & .2 \\ * & .3 & .3 \\ .6 & .2 & * \end{bmatrix}$$

2. In the signal transmission model in Example 1, suppose that  $p = .97$ . Find the probability that the signal “1” will be a “0” after a three-stage transmission process.

## 10.1 EXERCISES

In Exercises 1 and 2, determine whether  $P$  is a stochastic matrix. If  $P$  is not a stochastic matrix, explain why not.

1. a.  $P = \begin{bmatrix} .3 & .4 \\ .7 & .6 \end{bmatrix}$       b.  $P = \begin{bmatrix} .3 & .7 \\ .4 & .6 \end{bmatrix}$

2. a.  $P = \begin{bmatrix} 1 & .5 \\ 0 & .5 \end{bmatrix}$       b.  $P = \begin{bmatrix} .2 & 1.1 \\ .8 & -.1 \end{bmatrix}$

In Exercises 3 and 4, compute  $\mathbf{x}_3$  in two ways: by computing  $\mathbf{x}_1$  and  $\mathbf{x}_2$ , and by computing  $P^3$ .

3.  $P = \begin{bmatrix} .6 & .5 \\ .4 & .5 \end{bmatrix}, \mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

4.  $P = \begin{bmatrix} .3 & .8 \\ .7 & .2 \end{bmatrix}, \mathbf{x}_0 = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$

In Exercises 5 and 6, the transition matrix  $P$  for a Markov chain with states 0 and 1 is given. Assume that in each case the chain starts in state 0 at time  $n = 0$ . Find the probability that the chain will be in state 1 at time  $n$ .

5.  $P = \begin{bmatrix} 1/3 & 3/4 \\ 2/3 & 1/4 \end{bmatrix}, n = 3$

6.  $P = \begin{bmatrix} .4 & .2 \\ .6 & .8 \end{bmatrix}, n = 5$

In Exercises 7 and 8, the transition matrix  $P$  for a Markov chain with states 0, 1, and 2 is given. Assume that in each case the chain starts in state 0 at time  $n = 0$ . Find the probability that the chain will be in state 1 at time  $n$ .

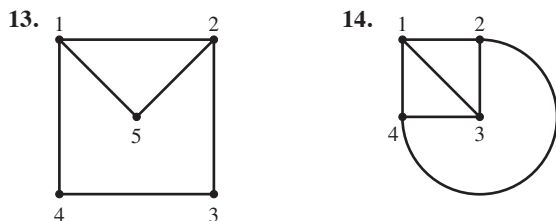
7.  $P = \begin{bmatrix} 1/3 & 1/4 & 1/2 \\ 1/3 & 1/2 & 1/4 \\ 1/3 & 1/4 & 1/4 \end{bmatrix}, n = 2$

8.  $P = \begin{bmatrix} .1 & .2 & .4 \\ .6 & .3 & .4 \\ .3 & .5 & .2 \end{bmatrix}, n = 3$

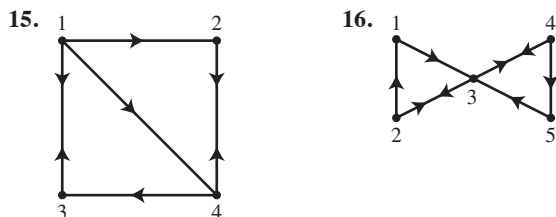
9. Consider a pair of Ehrenfest urns labeled A and B. There are currently 3 molecules in urn A and 1 in urn B. What is the probability that the exact same situation will apply after
  - a. 4 selections?
  - b. 5 selections?

10. Consider a pair of Ehrenfest urns labeled A and B. There are currently no molecules in urn A and 5 in urn B. What is the probability that the exact same situation will apply after
- 4 selections?
  - 5 selections?
11. Consider an unbiased random walk on the set  $\{1, 2, 3, 4\}$ . What is the probability of moving from 2 to 3 in exactly 3 steps if the walk has
- reflecting boundaries?
  - absorbing boundaries?
12. Consider a biased random walk on the set  $\{1, 2, 3, 4\}$  with probability  $p = .2$  of moving to the left. What is the probability of moving from 2 to 3 in exactly 3 steps if the walk has
- reflecting boundaries?
  - absorbing boundaries?

In Exercises 13 and 14, find the transition matrix for the simple random walk on the given graph.

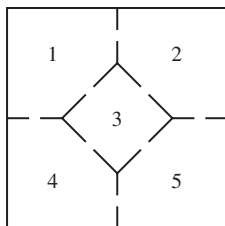


In Exercises 15 and 16, find the transition matrix for the simple random walk on the given directed graph.



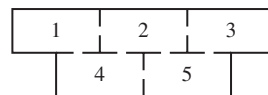
In Exercises 17 and 18, suppose a mouse wanders through the given maze. The mouse must move into a different room at each time step and is equally likely to leave the room through any of the available doorways.

17. The mouse is placed in room 2 of the maze shown below.
- Construct a transition matrix and an initial probability vector for the mouse's travels.
  - What are the probabilities that the mouse will be in each of the rooms after 3 moves?



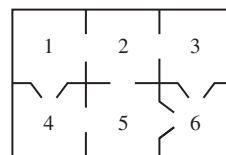
18. The mouse is placed in room 3 of the maze shown below.
- Construct a transition matrix and an initial probability vector for the mouse's travels.

- What are the probabilities that the mouse will be in each of the rooms after 4 moves?

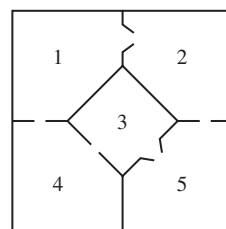


In Exercises 19 and 20, suppose a mouse wanders through the given maze, some of whose doors are "one-way": they are just large enough for the mouse to squeeze through in only one direction. The mouse still must move into a different room at each time step if possible. When faced with accessible openings into two or more rooms, the mouse chooses them with equal probability.

19. The mouse is placed in room 1 of the maze shown below.
- Construct a transition matrix and an initial probability vector for the mouse's travels.
  - What are the probabilities that the mouse will be in each of the rooms after 4 moves?



20. The mouse is placed in room 1 of the maze shown below.
- Construct a transition matrix and an initial probability vector for the mouse's travels.
  - What are the probabilities that the mouse will be in each of the rooms after 3 moves?



In Exercises 21 and 22, mark each statement True or False. Justify each answer.

21. a. The columns of a transition matrix for a Markov chain must sum to 1.  
 b. The transition matrix  $P$  may change over time.  
 c. The  $(i, j)$ -entry in a transition matrix  $P$  gives the probability of a move from state  $j$  to state  $i$ .
22. a. The rows of a transition matrix for a Markov chain must sum to 1.  
 b. If  $\{\mathbf{x}_n\}$  is a Markov chain, then  $\mathbf{x}_{n+1}$  must depend only on the transition matrix and  $\mathbf{x}_n$ .  
 c. The  $(i, j)$ -entry in  $P^3$  gives the probability of a move from state  $i$  to state  $j$  in exactly three time steps.

23. The weather in Charlotte, North Carolina, can be classified as sunny, cloudy, or rainy on a given day. Climate data from 2003<sup>2</sup> reveal that

- If a day is sunny, then the next day will be sunny with probability .65, cloudy with probability .1, and rainy with probability .25.
- If a day is cloudy, then the next day will be sunny with probability .25, cloudy with probability .25, and rainy with probability .5.
- If a day is rainy, then the next day will be sunny with probability .25, cloudy with probability .15, and rainy with probability .60.

Suppose it is cloudy on Monday. Use a Markov chain to find the probabilities of the different kinds of weather on Friday.

24. Suppose that whether it rains in Charlotte tomorrow depends on the weather conditions for today and yesterday. Climate data from 2003<sup>2</sup> show that

- If it rained yesterday and today, then it will rain tomorrow with probability .58.
- If it rained yesterday but not today, then it will rain tomorrow with probability .29.
- If it rained today but not yesterday, then it will rain tomorrow with probability .47.
- If it did not rain yesterday or today, then it will rain tomorrow with probability .31.

Even though the weather depends on the last two days in this case, we can create a Markov chain model using the states

- 1 it rained yesterday and today
- 2 it rained yesterday but not today
- 3 it rained today but not yesterday
- 4 it did not rain yesterday or today

So, for example, the probability of a transition from state 1 to state 1 is .58, and the transition from state 1 to state 3 is 0.

- a. Complete the creation of the transition matrix for this Markov chain.
  - b. If it rains on Tuesday and is clear on Wednesday, what is the probability of no rain on the next weekend?
25. Consider a set of four webpages hyperlinked by the directed graph in Exercise 15. If a random surfer starts at page 1, what is the probability that the surfer will be at each of the pages after 3 clicks?
26. Consider a set of five webpages hyperlinked by the directed graph in Exercise 16. If a random surfer starts at page 2, what is the probability that the surfer will be at each of the pages after 4 clicks?
27. Consider a model for signal transmission in which data is sent as two-bit bytes. Then there are four possible bytes, 00, 01,

10, and 11, which are the states of the Markov chain. At each stage there is a probability  $p$  that each bit will pass through the stage unchanged.

- a. Construct the transition matrix for the model.
  - b. Suppose that  $p = .99$ . Find the probability that the signal “01” will still be “01” after a three-stage transmission.
28. Consider a model for signal transmission in which data is sent as three-bit bytes. Construct the transition matrix for the model.
29. Another version of the Ehrenfest model for diffusion starts with  $k$  molecules of gas in each urn. One of the  $2k$  molecules is picked at random just as in the Ehrenfest model in the text. The chosen molecule is then moved to the other urn with a fixed probability  $p$  and is placed back in its urn with probability  $1 - p$ . (Note that the Ehrenfest model in the text is this model with  $p = 1$ .)
- a. Let  $k = 3$ . Find the transition matrix for this model.
  - b. Let  $k = 3$  and  $p = 1/2$ . If there are currently no molecules in urn A, what is the probability that there will be 3 molecules in urn A after 5 selections?
30. Another model for diffusion is called the **Bernoulli-Laplace model**. Two urns (urn A and urn B) contain a total of  $2k$  molecules. In this case,  $k$  of the molecules are of one type (called type I molecules) and  $k$  are of another type (type II molecules). In addition,  $k$  molecules must be in each urn at all times. At each time step, a pair of molecules is selected, one from urn A and one from urn B, and these molecules change urns. Let the Markov chain model the number of type I molecules in urn A (which is also the number of type II molecules in urn B).
- a. Suppose that there are  $j$  type I molecules in urn A with  $0 < j < k$ . Explain why the probability of a transition to  $j - 1$  type I molecules in urn A is  $(j/k)^2$ , and why the probability of a transition to  $j + 1$  type I molecules in urn A is  $((k - j)/k)^2$ .
  - b. Let  $k = 5$ . Use the result in part (a) to set up the transition matrix for the Markov chain which models the number of type I molecules in urn A.
  - c. Let  $k = 5$  and begin with all type I molecules in urn A. What is the distribution of type I molecules after 3 time steps?
31. To win a game in tennis, one player must score four points and must also score at least two points more than his or her opponent. Thus if the two players have scored an equal number of points (four or more), which is called “deuce” in tennis jargon, one player must then score two points in a row to win the game. Suppose that players A and B are playing a game of tennis which is at deuce. If A wins the next point it is called “advantage A,” while if B wins the point it is “advantage B.” If the game is at advantage A and player A wins the next point, then player A wins the game. If player B wins the point at advantage A, the game is back at deuce.

<sup>2</sup> <http://www.wunderground.com/history/airport/KCLT/2003/1/1/MonthlyHistory.html>



- a. Suppose the probability of player A winning any point is  $p$ . Model the progress of a tennis game starting at deuce using a Markov chain with the following five states.
- 1 deuce
  - 2 advantage A
  - 3 advantage B
  - 4 A wins the game
  - 5 B wins the game
- Find the transition matrix for this Markov chain.
- b. Let  $p = .6$ . Find the probability that the game will be at “advantage B” after three points starting at deuce.
32. Volleyball uses two different scoring systems in which a team must win by at least two points. In both systems, a *rally* begins with a serve by one of the teams and ends when the ball goes out of play or touches the floor or a player commits a fault. The team that wins the rally gets to serve for the next rally. Games are played to 15, 25, or 30 points.
- a. In *rally point scoring*, the team that wins a rally is awarded a point no matter which team served for the rally. Assume that team A has probability  $p$  of winning a rally for which it serves, and that team B has probability  $q$  of winning a rally for which it serves. Model the progress of a volleyball game using a Markov chain with the following six states.
- 1 tied – A serving
  - 2 tied – B serving
  - 3 A ahead by 1 point – A serving
  - 4 B ahead by 1 point – B serving
  - 5 A wins the game
  - 6 B wins the game
- Find the transition matrix for this Markov chain.
- b. Suppose that team A and team B are tied 15-15 in a 15-point game and that team A is serving. Let  $p = q = .6$ . Find the probability that the game will not be finished after three rallies.
- c. In *side out scoring*, the team that wins a rally is awarded a point only when it served for the rally. Assume that team A has probability  $p$  of winning a rally for which it serves, and that team B has probability  $q$  of winning a rally for which it serves. Model the progress of a volleyball game using a Markov chain with the following eight states.
- 1 tied – A serving
  - 2 tied – B serving
  - 3 A ahead by 1 point – A serving
  - 4 A ahead by 1 point – B serving
  - 5 B ahead by 1 point – A serving
  - 6 B ahead by 1 point – B serving
  - 7 A wins the game
  - 8 B wins the game
- Find the transition matrix for this Markov chain.
- d. Suppose that team A and team B are tied 15-15 in a 15-point game and that team A is serving. Let  $p = q = .6$ . Find the probability that the game will not be finished after three rallies.
33. Suppose that  $P$  is a stochastic matrix all of whose entries are greater than or equal to  $p$ . Show that all of the entries in  $P^n$  are greater than or equal to  $p$  for  $n = 1, 2, \dots$

### SOLUTIONS TO PRACTICE PROBLEMS

1. Since a stochastic matrix must have columns that sum to 1,

$$P = \begin{bmatrix} .1 & .5 & .2 \\ .3 & .3 & .3 \\ .6 & .2 & .5 \end{bmatrix}$$

2. The transition matrix for the model is

$$P = \begin{bmatrix} .97 & .03 \\ .03 & .97 \end{bmatrix}$$

Since the signal begins as “1,” the initial probability vector is

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

To find the probability of a three-step transition, compute

$$\mathbf{x}_2 = P^3 \mathbf{x}_0 = \begin{bmatrix} .9153 & .0847 \\ .0847 & .9153 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} .0847 \\ .9153 \end{bmatrix}$$

The probability of a change to “0” is thus .0847.

## 10.2 THE STEADY-STATE VECTOR AND GOOGLE'S PAGERANK

As was seen in Section 4.9, the most interesting aspect of a Markov chain is its long-range behavior: the behavior of  $\mathbf{x}_n$  as  $n$  increases without bound. In many cases, the sequence of vectors  $\{\mathbf{x}_n\}$  is converging to a vector which is called the **steady-state vector** for the Markov chain. This section will review how to compute the steady-state vector of a Markov chain, explain how to interpret this vector if it exists, and offer an expanded version of Theorem 18 in Section 4.9, which describes the circumstances under which  $\{\mathbf{x}_n\}$  converges to a steady-state vector. This theorem will be applied to the Markov chain model used for the World Wide Web in the previous section and will show how the PageRank method for ordering the importance of webpages is derived.

### Steady-State Vectors

In many cases, the Markov chain  $\mathbf{x}_n$  and the matrix  $P^n$  change very little for large values of  $n$ .

**EXAMPLE 1** To begin, recall Example 3 in Section 4.9. That example concerned

a Markov chain with transition matrix  $P = \begin{bmatrix} .5 & .2 & .3 \\ .3 & .8 & .3 \\ .2 & 0 & .4 \end{bmatrix}$  and initial probability

vector  $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ . The vectors  $\mathbf{x}_n$  were seen to be converging to the vector  $\mathbf{q} = \begin{bmatrix} .3 \\ .6 \\ .1 \end{bmatrix}$ .

This result may be written as  $\lim_{n \rightarrow \infty} \mathbf{x}_n = \mathbf{q}$ . Increasing powers of the transition matrix  $P$  may also be computed, as follows:

$$\begin{aligned} P^2 &= \begin{bmatrix} .3700 & .2600 & .3300 \\ .4500 & .7000 & .4500 \\ .1800 & .0400 & .2200 \end{bmatrix} & P^3 &= \begin{bmatrix} .3290 & .2820 & .3210 \\ .5250 & .6500 & .5250 \\ .1460 & .0680 & .1540 \end{bmatrix} \\ P^4 &= \begin{bmatrix} .3133 & .2914 & .3117 \\ .5625 & .6250 & .5625 \\ .1242 & .0836 & .1258 \end{bmatrix} & P^5 &= \begin{bmatrix} .3064 & .2958 & .3061 \\ .5813 & .6125 & .5813 \\ .1123 & .0917 & .1127 \end{bmatrix} \\ P^{10} &= \begin{bmatrix} .3002 & .2999 & .3002 \\ .5994 & .6004 & .5994 \\ .1004 & .0997 & .1004 \end{bmatrix} & P^{15} &= \begin{bmatrix} .3000 & .3000 & .3000 \\ .6000 & .6000 & .6000 \\ .1000 & .1000 & .1000 \end{bmatrix} \end{aligned}$$

so the sequence of matrices  $\{P^n\}$  also seems to be converging to a matrix as  $n$  increases, and this matrix has the unusual property that all of its columns equal  $\mathbf{q}$ . The example also showed that  $P\mathbf{q} = \mathbf{q}$ . This equation forms the definition of the steady-state vector and is a straightforward way to calculate it. ■

### DEFINITION

If  $P$  is a stochastic matrix, then a **steady-state vector** (or **equilibrium vector** or **invariant probability vector**) for  $P$  is a probability vector  $\mathbf{q}$  such that

$$P\mathbf{q} = \mathbf{q}$$

Exercises 36 and 37 will show that every stochastic matrix  $P$  has a steady-state vector  $\mathbf{q}$ . Notice that 1 must be an eigenvalue of any stochastic matrix, and the steady-state vector is a probability vector which is also an eigenvector of  $P$  associated with the eigenvalue 1.

Although the definition of the steady-state vector makes the calculation of  $\mathbf{q}$  straightforward, it has the major drawback that there are Markov chains which have a steady-state vector  $\mathbf{q}$  but for which  $\lim_{n \rightarrow \infty} \mathbf{x}_n \neq \mathbf{q}$ : the definition is not sufficient for  $\mathbf{x}_n$  to converge. Examples 3 to 5 below will show different ways in which  $\mathbf{x}_n$  can fail to converge. Later in this section, the conditions under which  $\lim_{n \rightarrow \infty} \mathbf{x}_n = \mathbf{q}$  will be restated. For now, consider what  $\mathbf{q}$  means when  $\lim_{n \rightarrow \infty} \mathbf{x}_n = \mathbf{q}$ , as it does in the example above. When  $\lim_{n \rightarrow \infty} \mathbf{x}_n = \mathbf{q}$ , there are two ways to interpret this vector.

- Since  $\mathbf{x}_n$  is approximately equal to  $\mathbf{q}$  for large  $n$ , the entries in  $\mathbf{q}$  approximate the probability that the chain is in each state after  $n$  time steps. Thus in the example above, no matter what the value of the initial probability vector is, after many steps the probability that the chain will be in state 1 is approximately  $q_1 = .3$ . Likewise, the probability that the chain will be in state 2 in the distant future is approximately  $q_2 = .6$ , and the probability that the chain will be in state 3 in the distant future is approximately  $q_3 = .1$ . So the entries in  $\mathbf{q}$  give **long-run probabilities**.
- When  $N$  is large,  $\mathbf{q}$  approximates  $\mathbf{x}_n$  for almost all values of  $n \leq N$ . Thus the entries in  $\mathbf{q}$  approximate the proportion of time steps that the chain spends in each state. In the example above, the chain will end up spending .3 of the time steps in state 1, .6 of the time steps in state 2, and .1 of the time steps in state 3. So the entries in  $\mathbf{q}$  give the proportion of the time steps spent in each state, which are called the **occupation times** for each state.

**EXAMPLE 2** For an application of computing  $\mathbf{q}$ , consider the mouse-in-the-maze example (Example 6, Section 10.1). In this example, the position of a mouse in a five-room maze is modeled by a Markov chain with states  $\{1, 2, 3, 4, 5\}$  and transition matrix

$$P = \begin{array}{ccccc} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{bmatrix} 0 & 1/3 & 1/4 & 0 & 0 \\ 1/2 & 0 & 1/4 & 1/3 & 0 \\ 1/2 & 1/3 & 0 & 1/3 & 1/2 \\ 0 & 1/3 & 1/4 & 0 & 1/2 \\ 0 & 0 & 1/4 & 1/3 & 0 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \end{array}$$

The steady-state vector may be computed by solving the system  $P\mathbf{q} = \mathbf{q}$ , which is equivalent to the homogeneous system  $(P - I)\mathbf{q} = \mathbf{0}$ . Row reduction yields

$$\begin{bmatrix} -1 & 1/3 & 1/4 & 0 & 0 & 0 \\ 1/2 & -1 & 1/4 & 1/3 & 0 & 0 \\ 1/2 & 1/3 & -1 & 1/3 & 1/2 & 0 \\ 0 & 1/3 & 1/4 & -1 & 1/2 & 0 \\ 0 & 0 & 1/4 & 1/3 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -3/2 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & -3/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so a general solution is

$$q_5 \begin{bmatrix} 1 \\ 3/2 \\ 2 \\ 3/2 \\ 1 \end{bmatrix}$$

Letting  $q_5$  be the reciprocal of the sum of the entries in the vector results in the steady-state vector

$$\mathbf{q} = \frac{1}{7} \begin{bmatrix} 1 \\ 3/2 \\ 2 \\ 3/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/7 \\ 3/14 \\ 2/7 \\ 3/14 \\ 1/7 \end{bmatrix} \approx \begin{bmatrix} .142857 \\ .214286 \\ .285714 \\ .214286 \\ .142857 \end{bmatrix}$$

There are again two interpretations for  $q$ : long-run probabilities and occupation times. After many moves, the probability that the mouse will be in room 1 at a given time is approximately  $1/7$  no matter where the mouse began its journey. Put another way, the mouse is expected to be in room 1 for  $1/7$  (or about 14.3%) of the time.

Again notice that taking high powers of the transition matrix  $P$  gives matrices whose columns are converging to  $\mathbf{q}$ ; for example,

$$P^{10} = \begin{bmatrix} .144169 & .141561 & .142613 & .144153 & .142034 \\ .212342 & .216649 & .214286 & .211922 & .216230 \\ .285226 & .285714 & .286203 & .285714 & .285226 \\ .216230 & .211922 & .214286 & .216649 & .212342 \\ .142034 & .144153 & .142613 & .141561 & .144169 \end{bmatrix}$$

The columns of  $P^{10}$  are very nearly equal to each other, and each column is also nearly equal to  $\mathbf{q}$ . ■

## Interpreting the Steady-State Vector

As noted above, every stochastic matrix will have a steady-state vector, but in some cases steady-state vectors cannot be interpreted as vectors of long-run probabilities or of occupation times. The following examples show some difficulties.

**EXAMPLE 3** Consider an unbiased random walk on  $\{1, 2, 3, 4, 5\}$  with absorbing boundaries. The transition matrix is

$$P = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 1 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1 \end{bmatrix} \end{matrix}$$

Notice that only two long-term possibilities exist for this chain: it must end up in state 1 or state 5. Thus the probability that the chain is in state 2, 3, or 4 becomes smaller and smaller as  $n$  increases, as  $P^n$  illustrates:

$$P^{20} = \begin{bmatrix} 1 & .74951 & .49951 & .24951 & 0 \\ 0 & .00049 & 0 & .00049 & 0 \\ 0 & 0 & .00098 & 0 & 0 \\ 0 & .00049 & 0 & .00049 & 0 \\ 0 & .24951 & .49951 & .74951 & 1 \end{bmatrix}$$

$$P^{30} = \begin{bmatrix} 1 & .749985 & .499985 & .249985 & 0 \\ 0 & .000015 & 0 & .000015 & 0 \\ 0 & 0 & .000030 & 0 & 0 \\ 0 & .000015 & 0 & .000015 & 0 \\ 0 & .249985 & .499985 & .749985 & 1 \end{bmatrix}$$

It seems that  $P^n$  converges to the matrix

$$\begin{bmatrix} 1 & .75 & .5 & .25 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & .25 & .5 & .75 & 1 \end{bmatrix}$$

as  $n$  increases. But the columns of this matrix are not equal; the probability of ending up either at 1 or at 5 depends on where the chain begins. Although the chain has steady-state vectors, they cannot be interpreted as in Example 1. Exercise 31 confirms that if  $0 \leq q \leq 1$  the vector

$$\begin{bmatrix} q \\ 0 \\ 0 \\ 0 \\ 1 - q \end{bmatrix}$$

is a steady-state vector for  $P$ . This matrix then has an infinite number of possible steady-state vectors, which shows in another way that  $\mathbf{x}_n$  cannot be expected to have convergent behavior which does not depend on  $\mathbf{x}_0$ . ■

**EXAMPLE 4** Consider an unbiased random walk on  $\{1, 2, 3, 4, 5\}$  with reflecting boundaries. The transition matrix is

$$P = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 1/2 & 0 & 0 & 0 \\ 1 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1 \\ 0 & 0 & 0 & 1/2 & 0 \end{bmatrix} \end{matrix}$$

If the chain  $\mathbf{x}_n$  starts at state 1, notice that it can return to 1 only when  $n$  is even, while the chain can be at state 2 only when  $n$  is odd. In fact, the chain must be at an even-numbered site when  $n$  is odd and at an odd-numbered site when  $n$  is even. If the chain were to start at state 2, however, this situation would be reversed: the chain must be at an odd-numbered site when  $n$  is odd and at an even-numbered site when  $n$  is even. Therefore,  $P^n$  cannot converge to a unique matrix since  $P^n$  looks very different depending on whether  $n$  is even or odd, as shown:

$$P^{20} = \begin{bmatrix} .2505 & 0 & .2500 & 0 & .2495 \\ 0 & .5005 & 0 & .4995 & 0 \\ .5000 & 0 & .5000 & 0 & .5000 \\ 0 & .4995 & 0 & .5005 & 0 \\ .2495 & 0 & .2500 & 0 & .2505 \end{bmatrix}$$

$$P^{21} = \begin{bmatrix} 0 & .2502 & 0 & .2498 & 0 \\ .5005 & 0 & .5000 & 0 & .4995 \\ 0 & .5000 & 0 & .5000 & 0 \\ .4995 & 0 & .5000 & 0 & .5005 \\ 0 & .2498 & 0 & .2502 & 0 \end{bmatrix}$$

Even though  $P^n$  does not converge to a unique matrix,  $P$  does have a steady-state vector.

In fact,

$$\begin{bmatrix} 1/8 \\ 1/4 \\ 1/4 \\ 1/4 \\ 1/8 \end{bmatrix}$$

is a steady-state vector for  $P$  (see Exercise 32). This vector **can** be interpreted as giving long-run probabilities and occupation times in a sense that will be made precise in Section 10.4. ■

**EXAMPLE 5** Consider a Markov chain on  $\{1, 2, 3, 4, 5\}$  with transition matrix

$$P = \begin{array}{ccccc} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 1/4 & 1/3 & 1/2 & 0 & 0 \\ 1/4 & 1/3 & 1/4 & 0 & 0 \\ 1/2 & 1/3 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 3/4 \\ 0 & 0 & 0 & 2/3 & 1/4 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix}$$

If this Markov chain begins at state 1, 2, or 3, then it must always be at one of those states. Likewise if the chain starts at state 4 or 5, then it must always be at one of those states. The chain splits into two separate chains, each with its own steady-state vector. In this case  $P^n$  converges to a matrix whose columns are not equal. The vectors

$$\begin{bmatrix} 4/11 \\ 3/11 \\ 4/11 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 9/17 \\ 8/17 \end{bmatrix}$$

both satisfy the definition of steady-state vector (Exercise 33). The first vector gives the limiting probabilities if the chain starts at state 1, 2, or 3, and the second does the same for states 4 and 5. ■

## Regular Matrices

Examples 1 and 2 show that in some cases a Markov chain  $\mathbf{x}_n$  with transition matrix  $P$  has a steady-state vector  $\mathbf{q}$  for which

$$\lim_{n \rightarrow \infty} P^n = [\mathbf{q} \quad \mathbf{q} \quad \cdots \quad \mathbf{q}]$$

In these cases,  $\mathbf{q}$  can be interpreted as a vector of long-run probabilities or occupation times for the chain. These probabilities or occupation times do not depend on the initial probability vector; that is, for any probability vector  $\mathbf{x}_0$ ,

$$\lim_{n \rightarrow \infty} P^n \mathbf{x}_0 = \lim_{n \rightarrow \infty} \mathbf{x}_n = \mathbf{q}$$

Notice also that  $\mathbf{q}$  is the only probability vector which is also an eigenvector of  $P$  associated with the eigenvalue 1.

Examples 3, 4, and 5 do not have such a steady-state vector  $\mathbf{q}$ . In Examples 3 and 5, the steady-state vector is not unique; in all three examples the matrix  $P^n$  does not converge to a matrix with equal columns as  $n$  increases. The goal is then to find some property of the transition matrix  $P$  that leads to these different behaviors, and to show that this property causes the differences in behavior.

A little calculation shows that in Examples 3, 4, and 5, every matrix of the form  $P^k$  has some zero entries. In Examples 1 and 2, however, some power of  $P$  has all positive entries. As was mentioned in Section 4.9, this is exactly the property that is needed.

### DEFINITION

A stochastic matrix  $P$  is **regular** if some power  $P^k$  contains only strictly positive entries.

Since the matrix  $P^k$  contains the probabilities of a  $k$ -step move from one state to another, a Markov chain with a regular transition matrix has the property that, for some  $k$ , it is possible to move from any state to any other in exactly  $k$  steps. The following theorem expands on the content of Theorem 18 in Section 4.9. One idea must be defined before the theorem is presented. The limit of a sequence of  $m \times n$  matrices is the  $m \times n$  matrix (if one exists) whose  $(i, j)$ -entry is the limit of the  $(i, j)$ -entries in the sequence of matrices. With that understanding, here is the theorem.

### THEOREM 1

If  $P$  is a regular  $m \times m$  transition matrix with  $m \geq 2$ , then the following statements are all true.

- There is a stochastic matrix  $\Pi$  such that  $\lim_{n \rightarrow \infty} P^n = \Pi$ .
- Each column of  $\Pi$  is the same probability vector  $\mathbf{q}$ .
- For any initial probability vector  $\mathbf{x}_0$ ,  $\lim_{n \rightarrow \infty} P^n \mathbf{x}_0 = \mathbf{q}$ .
- The vector  $\mathbf{q}$  is the unique probability vector which is an eigenvector of  $P$  associated with the eigenvalue 1.
- All eigenvalues  $\lambda$  of  $P$  other than 1 have  $|\lambda| < 1$ .

A proof of Theorem 1 is given in Appendix 1. Theorem 1 is a special case of the Perron-Frobenius Theorem, which is used in applications of linear algebra to economics, graph theory, and systems analysis. Theorem 1 shows that a Markov chain with a regular transition matrix has the properties found in Examples 1 and 2. For example, since the transition matrix  $P$  in Example 1 is regular, Theorem 1 justifies the conclusion that  $P^n$

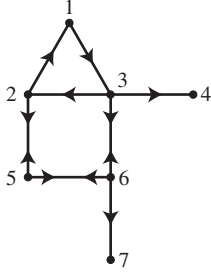
converges to a stochastic matrix all of whose columns equal  $\mathbf{q} = \begin{bmatrix} .3 \\ .6 \\ .1 \end{bmatrix}$ , as numerical evidence seemed to indicate.

## PageRank and the Google Matrix

In Section 10.1, the notion of a simple random walk on a graph was defined. The World Wide Web can be modeled as a directed graph, with the vertices representing the webpages and the arrows representing the links between webpages. Let  $P$  be the huge transition matrix for this Markov chain. If the matrix  $P$  were regular, then Theorem 1 would show that there is a steady-state vector  $\mathbf{q}$  for the chain, and that the entries in  $\mathbf{q}$  can be interpreted as occupation times for each state. In terms of the model, the entries in  $\mathbf{q}$  would tell what fraction of the random surfer's time was spent at each webpage. The founders of Google, Sergey Brin and Lawrence Page, reasoned that "important" pages had links coming from other "important" pages. Thus the random surfer would spend more time at more important pages and less time at less important pages. But the amount of time spent at each page is just the occupation time for each state in the Markov chain.

This observation is the basis for PageRank, which is the model that Google uses to rank the importance of all webpages it catalogs:

The importance of a webpage may be measured by the relative size of the corresponding entry in the steady-state vector  $\mathbf{q}$  for an appropriately chosen Markov chain.



**FIGURE 1**  
A seven-page Web.

Unfortunately, a simple random walk on the directed graph model for the Web is not the appropriate Markov chain, because the matrix  $P$  is not regular. Thus Theorem 1 will not apply. For example, consider the seven-page Web modeled in Section 10.1 using the directed graph in Figure 1. The transition matrix is

$$P = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{l} \left[ \begin{array}{ccccccc} 0 & 1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 1/3 & 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 1/3 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 1 \end{array} \right] \end{array} \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \end{array}$$

Pages 4 and 7 are dangling nodes, and so are absorbing states for the chain. Just as in Example 3, the presence of absorbing states implies that the state vectors  $\mathbf{x}_n$  do not approach a unique limit as  $n \rightarrow \infty$ . To handle dangling nodes, an adjustment is made to  $P$ :

**ADJUSTMENT 1:** If the surfer reaches a dangling node, the surfer will pick any page in the Web with equal probability and will move to that page. In terms of the transition matrix  $P$ , if state  $j$  is an absorbing state, replace column  $j$  of  $P$  with the vector

$$\begin{bmatrix} 1/n \\ 1/n \\ \vdots \\ 1/n \end{bmatrix}$$

where  $n$  is the number of rows (and columns) in  $P$ .

In the seven-page example, the transition matrix is now

$$P_* = \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{l} \left[ \begin{array}{ccccccc} 0 & 1/2 & 0 & 1/7 & 0 & 0 & 1/7 \\ 0 & 0 & 1/3 & 1/7 & 1/2 & 0 & 1/7 \\ 1 & 0 & 0 & 1/7 & 0 & 1/3 & 1/7 \\ 0 & 0 & 1/3 & 1/7 & 0 & 0 & 1/7 \\ 0 & 1/2 & 0 & 1/7 & 0 & 1/3 & 1/7 \\ 0 & 0 & 1/3 & 1/7 & 1/2 & 0 & 1/7 \\ 0 & 0 & 0 & 1/7 & 0 & 1/3 & 1/7 \end{array} \right] \end{array} \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \end{array}$$

Yet even this adjustment is not sufficient to ensure that the transition matrix is regular: while dangling nodes are no longer possible, it is still possible to have “cycles” of pages. If page  $j$  linked only to page  $i$  and page  $i$  linked only to page  $j$ , a random



surfer entering either page would be condemned to spend eternity linking from page  $i$  to page  $j$  and back again. Thus the columns of  $P_*^k$  corresponding to these pages would always have zeros in them, and the transition matrix  $P_*$  would not be regular. Another adjustment is needed.

**ADJUSTMENT 2:** Let  $p$  be a number between 0 and 1. Assume the surfer is now at page  $j$ . With probability  $p$  the surfer will pick from among all possible links from page  $j$  with equal probability and will move to that page. With probability  $1 - p$ , the surfer will pick *any* page in the Web with equal probability and will move to that page. In terms of the transition matrix  $P_*$ , the new transition matrix will be

$$G = pP_* + (1 - p)K$$

where  $K$  is an  $n \times n$  matrix all of whose columns are<sup>1</sup>

$$\begin{bmatrix} 1/n \\ 1/n \\ \vdots \\ 1/n \end{bmatrix}$$

The matrix  $G$  is called the **Google matrix**, and  $G$  is now a regular matrix since all entries in  $G^1 = G$  are positive. Although any value of  $p$  between 0 and 1 is allowed, Google is said to use a value of  $p = .85$  for their PageRank calculations. In the seven-page Web example, the Google matrix is thus

$$\begin{aligned} G &= .85 \begin{bmatrix} 0 & 1/2 & 0 & 1/7 & 0 & 0 & 1/7 \\ 0 & 0 & 1/3 & 1/7 & 1/2 & 0 & 1/7 \\ 1 & 0 & 0 & 1/7 & 0 & 1/3 & 1/7 \\ 0 & 0 & 1/3 & 1/7 & 0 & 0 & 1/7 \\ 0 & 1/2 & 0 & 1/7 & 0 & 1/3 & 1/7 \\ 0 & 0 & 1/3 & 1/7 & 1/2 & 0 & 1/7 \\ 0 & 0 & 0 & 1/7 & 0 & 1/3 & 1/7 \end{bmatrix} \\ &+ .15 \begin{bmatrix} 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 \\ 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 \\ 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 \\ 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 \\ 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 \\ 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 \\ 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 \end{bmatrix} \\ &= \begin{bmatrix} .021429 & .446429 & .021429 & .142857 & .021429 & .021429 & .142857 \\ .021429 & .021429 & .304762 & .142857 & .446429 & .021429 & .142857 \\ .871429 & .021429 & .021429 & .142857 & .021429 & .304762 & .142857 \\ .021429 & .021429 & .304762 & .142857 & .021429 & .021429 & .142857 \\ .021429 & .446429 & .021429 & .142857 & .021429 & .304762 & .142857 \\ .021429 & .021429 & .304762 & .142857 & .446429 & .021429 & .142857 \\ .021429 & .021429 & .021429 & .142857 & .021429 & .304762 & .142857 \end{bmatrix} \end{aligned}$$

<sup>1</sup> PageRank really uses a  $K$  that has all its columns equal to a probability vector  $\mathbf{v}$ , which could be linked to an individual searcher or group of searchers. This modification also makes it easier to police the Web for websites attempting to generate Web traffic. See *Google's PageRank and Beyond: The Science of Search Engine Rankings* by Amy N. Langville and Carl D. Meyer (Princeton: Princeton University Press, 2006) for more information.

It is now possible to find the steady-state vector by the methods of this section:

$$\mathbf{q} = \begin{bmatrix} .116293 \\ .168567 \\ .191263 \\ .098844 \\ .164054 \\ .168567 \\ .092413 \end{bmatrix}$$

so the most important page according to PageRank is page 3, which has the largest entry in  $\mathbf{q}$ . The complete ranking is 3, 2 and 6, 5, 1, 4, and 7.

### NUMERICAL NOTE

The computation of  $\mathbf{q}$  is not a trivial task, since the Google matrix has more than 8 billion rows and columns. Google uses a version of the power method introduced in Section 5.8 to compute  $\mathbf{q}$ . While the power method was used in that section to estimate the eigenvalues of a matrix, it can also be used to provide estimates for eigenvectors. Since  $\mathbf{q}$  is an eigenvector of  $G$  corresponding to the eigenvalue 1, the power method applies. It turns out that only between 50 and 100 iterations of the method are needed to get the vector  $\mathbf{q}$  to the accuracy that Google needs for its rankings. It still takes days for Google to compute a new  $\mathbf{q}$ , which it does every month.

### PRACTICE PROBLEM

1. Consider the Markov chain on  $\{1, 2, 3\}$  with transition matrix

$$P = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

- Show that  $P$  is a regular matrix.
- Find the steady-state vector for this Markov chain.
- What fraction of the time does this chain spend in state 2? Explain your answer.

## 10.2 EXERCISES

In Exercises 1 and 2, consider a Markov chain on  $\{1, 2\}$  with the given transition matrix  $P$ . In each exercise, use two methods to find the probability that, in the long run, the chain is in state 1. First, raise  $P$  to a high power. Then directly compute the steady-state vector.

1.  $P = \begin{bmatrix} .2 & .4 \\ .8 & .6 \end{bmatrix}$

2.  $P = \begin{bmatrix} 1/4 & 2/3 \\ 3/4 & 1/3 \end{bmatrix}$

In Exercises 3 and 4, consider a Markov chain on  $\{1, 2, 3\}$  with the given transition matrix  $P$ . In each exercise, use two methods to find the probability that, in the long run, the chain is in state 1.

First, raise  $P$  to a high power. Then directly compute the steady-state vector.

3.  $P = \begin{bmatrix} 1/3 & 1/4 & 0 \\ 1/3 & 1/2 & 1 \\ 1/3 & 1/4 & 0 \end{bmatrix}$

4.  $P = \begin{bmatrix} .1 & .2 & .3 \\ .2 & .3 & .4 \\ .7 & .5 & .3 \end{bmatrix}$

In Exercises 5 and 6, find the matrix to which  $P^n$  converges as  $n$  increases.

5.  $P = \begin{bmatrix} 1/4 & 2/3 \\ 3/4 & 1/3 \end{bmatrix}$

6.  $P = \begin{bmatrix} 1/4 & 3/5 & 0 \\ 1/4 & 0 & 1/3 \\ 1/2 & 2/5 & 2/3 \end{bmatrix}$

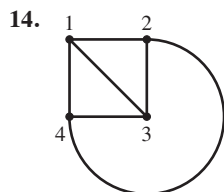
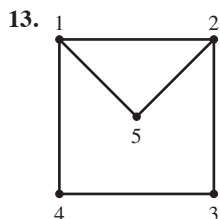
In Exercises 7 and 8, determine whether the given matrix is regular. Explain your answer.

$$7. P = \begin{bmatrix} 1/3 & 0 & 1/2 \\ 1/3 & 1/2 & 1/2 \\ 1/3 & 1/2 & 0 \end{bmatrix}$$

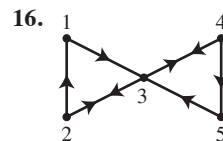
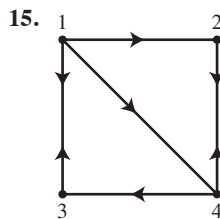
$$8. P = \begin{bmatrix} 1/2 & 0 & 1/3 & 0 \\ 0 & 2/5 & 0 & 3/7 \\ 1/2 & 0 & 2/3 & 0 \\ 0 & 3/5 & 0 & 4/7 \end{bmatrix}$$

9. Consider a pair of Ehrenfest urns with a total of 4 molecules divided between them.
- Find the transition matrix for the Markov chain which models the number of molecules in urn A, and show that this matrix is not regular.
  - Assuming that the steady-state vector may be interpreted as occupation times for this Markov chain, in what state will this chain spend the most steps?
10. Consider a pair of Ehrenfest urns with a total of 5 molecules divided between them.
- Find the transition matrix for the Markov chain which models the number of molecules in urn A, and show that this matrix is not regular.
  - Assuming that the steady-state vector may be interpreted as occupation times for this Markov chain, in what state will this chain spend the most steps?
11. Consider an unbiased random walk with reflecting boundaries on  $\{1, 2, 3, 4\}$ .
- Find the transition matrix for the Markov chain and show that this matrix is not regular.
  - Assuming that the steady-state vector may be interpreted as occupation times for this Markov chain, in what state will this chain spend the most steps?
12. Consider a biased random walk with reflecting boundaries on  $\{1, 2, 3, 4\}$  with probability  $p = .2$  of moving to the left.
- Find the transition matrix for the Markov chain and show that this matrix is not regular.
  - Assuming that the steady-state vector may be interpreted as occupation times for this Markov chain, in what state will this chain spend the most steps?

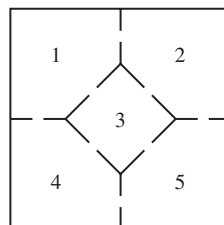
In Exercises 13 and 14, consider a simple random walk on the given graph. In the long run, what fraction of the time will the walk be at each of the various states?



In Exercises 15 and 16, consider a simple random walk on the given directed graph. In the long run, what fraction of the time will the walk be at each of the various states?

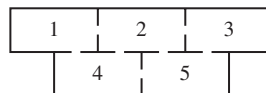


17. Consider the mouse in the following maze from Section 10.1, Exercise 17.



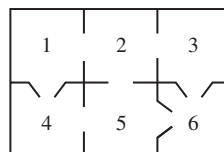
The mouse must move into a different room at each time step and is equally likely to leave the room through any of the available doorways. If you go away from the maze for a while, what is the probability that the mouse will be in room 3 when you return?

18. Consider the mouse in the following maze from Section 10.1, Exercise 18.



What fraction of the time does it spend in room 3?

19. Consider the mouse in the following maze, which includes “one-way” doors, from Section 10.1, Exercise 19.

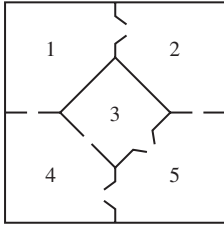


Show that

$$\mathbf{q} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

is a steady-state vector for the associated Markov chain, and interpret this result in terms of the mouse’s travels through the maze.

20. Consider the mouse in the following maze, which includes “one-way” doors.

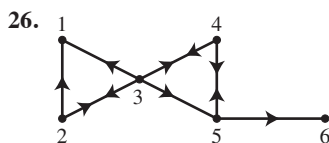
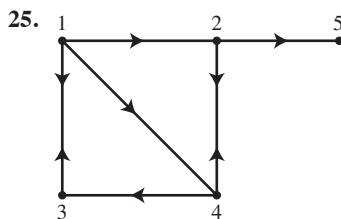


What fraction of the time does the mouse spend in each of the rooms in the maze?

In Exercises 21 and 22, mark each statement True or False. Justify each answer.

21. a. Every stochastic matrix has a steady-state vector.  
 b. If its transition matrix is regular, then the steady-state vector gives information on long-run probabilities of the Markov chain.  
 c. If  $\lambda = 1$  is an eigenvalue of a matrix  $P$ , then  $P$  is regular.
22. a. Every stochastic matrix is regular.  
 b. If  $P$  is a regular stochastic matrix, then  $P^n$  approaches a matrix with equal columns as  $n$  increases.  
 c. If  $\lim_{n \rightarrow \infty} \mathbf{x}_n = \mathbf{q}$ , then the entries in  $\mathbf{q}$  may be interpreted as occupation times.
23. Suppose that the weather in Charlotte is modeled using the Markov chain in Section 10.1, Exercise 23. Over the course of a year, about how many days in Charlotte are sunny, cloudy, and rainy according to the model?
24. Suppose that the weather in Charlotte is modeled using the Markov chain in Section 10.1, Exercise 24. Over the course of a year, about how many days in Charlotte are rainy according to the model?

In Exercises 25 and 26, consider a set of webpages hyperlinked by the given directed graph. Find the Google matrix for each graph and compute the PageRank of each page in the set.



27. A genetic trait is often governed by a pair of genes, one inherited from each parent. The genes may be of two types,

often labeled A and a. An individual then may have three different pairs: AA, Aa (which is the same as aA), or aa. In many cases the AA and Aa individuals cannot be otherwise distinguished; in these cases gene A is *dominant* and gene a is *recessive*. Likewise, an AA individual is called *dominant* and an aa individual is called *recessive*. An Aa individual is called a *hybrid*.

- a. Show that if a dominant individual is mated with a hybrid, the probability of an offspring being dominant is  $1/2$  and the probability of an offspring being a hybrid is  $1/2$ .
- b. Show that if a recessive individual is mated with a hybrid, the probability of an offspring being recessive is  $1/2$  and the probability of an offspring being a hybrid is  $1/2$ .
- c. Show that if a hybrid individual is mated with another hybrid, the probability of an offspring being dominant is  $1/4$ , the probability of an offspring being recessive is  $1/4$ , and the probability of an offspring being a hybrid is  $1/2$ .
28. Consider beginning with an individual of known type and mating it with a hybrid, then mating an offspring of this mating with a hybrid, and so on. At each step, an offspring is mated with a hybrid. The type of the offspring can be modeled by a Markov chain with states AA, Aa, and aa.
- a. Find the transition matrix for this Markov chain.
- b. If the mating process in Exercise 27 is continued for an extended period of time, what percent of the offspring will be of each type?
29. Consider the variation of the Ehrenfest urn model of diffusion studied in Section 10.1, Exercise 29, where one of the  $2k$  molecules is chosen at random and is then moved between the urns with a fixed probability  $p$ .
- a. Let  $k = 3$  and suppose that  $p = 1/2$ . Show that the transition matrix for the Markov chain that models the number of molecules in urn A is regular.
- b. Let  $k = 3$  and suppose that  $p = 1/2$ . In what state will this chain spend the most steps, and what fraction of the steps will the chain spend at this state?
- c. Does the answer to part (b) change if a different value of  $p$  with  $0 < p < 1$  is used?
30. Consider the Bernoulli-Laplace diffusion model studied in Section 10.1, Exercise 30.
- a. Let  $k = 5$  and show that the transition matrix for the Markov chain that models the number of type I molecules in urn A is regular.
- b. Let  $k = 5$ . In what state will this chain spend the most steps, and what fraction of the steps will the chain spend at this state?

31. Let  $0 \leq q \leq 1$ . Show that  $\begin{bmatrix} q \\ 0 \\ 0 \\ 0 \\ 1 - q \end{bmatrix}$  is a steady-state vector for the Markov chain in Example 3.

32. Consider the Markov chain in Example 4.

a. Show that  $\begin{bmatrix} 1/8 \\ 1/4 \\ 1/4 \\ 1/4 \\ 1/8 \end{bmatrix}$  is a steady-state vector for this Markov chain.

- b. Compute the average of the entries in  $P^{20}$  and  $P^{21}$  given in Example 4. What do you find?

33. Show that  $\begin{bmatrix} 4/11 \\ 3/11 \\ 4/11 \\ 0 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 9/17 \\ 8/17 \end{bmatrix}$  are steady-state vectors

for the Markov chain in Example 5. If the chain is equally likely to begin in each of the states, what is the probability of being in state 1 after many steps?

34. Let  $0 \leq p, q \leq 1$ , and define

$$P = \begin{bmatrix} p & 1-q \\ 1-p & q \end{bmatrix}$$

- Show that 1 and  $p + q - 1$  are eigenvalues of  $P$ .
- By Theorem 1, for what values of  $p$  and  $q$  will  $P$  fail to be regular?
- Find a steady-state vector for  $P$ .

35. Let  $0 \leq p, q \leq 1$ , and define

$$P = \begin{bmatrix} p & q & 1-p-q \\ q & 1-p-q & p \\ 1-p-q & p & q \end{bmatrix}$$

- For what values of  $p$  and  $q$  is  $P$  a regular stochastic matrix?
- Given that  $P$  is regular, find a steady-state vector for  $P$ .

36. Let  $A$  be an  $m \times m$  stochastic matrix, let  $\mathbf{x}$  be in  $\mathbb{R}^m$ , and let  $\mathbf{y} = A\mathbf{x}$ . Show that

$$|y_1| + \cdots + |y_m| \leq |x_1| + \cdots + |x_m|$$

with equality holding if and only if all of the nonzero entries in  $\mathbf{x}$  have the same sign.

37. Show that every stochastic matrix has a steady-state vector using the following steps.

- Let  $P$  be a stochastic matrix. By Exercise 30 in Section 4.9,  $\lambda = 1$  is an eigenvalue for  $P$ . Let  $\mathbf{v}$  be an eigenvector of  $P$  associated with  $\lambda = 1$ . Use Exercise 36 to conclude that the nonzero entries in  $\mathbf{v}$  must have the same sign.
- Show how to produce a steady-state vector for  $P$  from  $\mathbf{v}$ .

38. Consider a simple random walk on a finite connected graph. (A graph is connected if it is possible to move from any vertex of the graph to any other along the edges of the graph.)

- Explain why this Markov chain must have a regular transition matrix.
- Use the results of Exercises 13 and 14 to hypothesize a formula for the steady-state vector for such a Markov chain.

39. By Theorem 1(e), all eigenvalues  $\lambda$  of a regular matrix other than 1 have the property that  $|\lambda| < 1$ ; that is, the eigenvalue 1 is a *strictly dominant eigenvalue*. Suppose that  $P$  is an  $n \times n$  regular matrix with eigenvalues  $\lambda_1 = 1, \dots, \lambda_n$  ordered so that  $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_n|$ . Suppose that  $\mathbf{x}_0 = c_1\mathbf{q} + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$  is a linear combination of eigenvectors of  $P$ .

- Use Equation (2) in Section 5.8 to derive an expression for  $\mathbf{x}_k = P^k\mathbf{x}_0$ .
- Use the result of part (a) to derive an expression for  $\mathbf{x}_k - c_1\mathbf{q}$ , and explain how the value of  $|\lambda_2|$  affects the speed with which  $\{\mathbf{x}_k\}$  converges to  $c_1\mathbf{q}$ .

### SOLUTION TO PRACTICE PROBLEM

1. a. Since

$$P^2 = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

$P$  is regular by the definition with  $k = 2$ .

- b. Solve the equation  $P\mathbf{q} = \mathbf{q}$ , which may be rewritten as  $(P - I)\mathbf{q} = \mathbf{0}$ . Since

$$P - I = \begin{bmatrix} -1/2 & 0 & 1/2 \\ 1/2 & -1/2 & 0 \\ 0 & 1/2 & -1/2 \end{bmatrix}$$

and row reducing the augmented matrix gives

$$\left[ \begin{array}{cccc} -1/2 & 0 & 1/2 & 0 \\ 1/2 & -1/2 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \end{array} \right] \sim \left[ \begin{array}{cccc} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

the general solution is  $q_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ . Since  $\mathbf{q}$  must be a probability vector, set  $q_3 = 1/(1 + 1 + 1) = 1/3$  and compute that

$$\mathbf{q} = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

- c. The chain will spend  $1/3$  of its time in state 2 since the entry in  $\mathbf{q}$  corresponding to state 2 is  $1/3$ , and we can interpret the entries as occupation times.

## 10.3 COMMUNICATION CLASSES

Section 10.2 showed that if the transition matrix for a Markov chain is regular, then  $\mathbf{x}_n$  converges to a unique steady-state vector for any choice of initial probability vector. That is,  $\lim_{n \rightarrow \infty} \mathbf{x}_n = \mathbf{q}$ , where  $\mathbf{q}$  is the unique steady-state vector for the Markov chain. Examples 3, 4, and 5 in Section 10.2 illustrated that, even though every Markov chain has a steady-state vector, not every Markov chain has the property that  $\lim_{n \rightarrow \infty} \mathbf{x}_n = \mathbf{q}$ . The goal of the next two sections is to study these examples further, and to show that Examples 3, 4, and 5 in Section 10.2 describe all the ways in which Markov chains fail to converge to a steady-state vector. The first step is to study which states of the Markov chain can be reached from other states of the chain.

### Communicating States

Suppose that state  $j$  and state  $i$  are two states of a Markov chain. If state  $j$  can be reached from state  $i$  in a finite number of steps and state  $i$  can be reached from state  $j$  in a finite number of steps, then states  $j$  and  $i$  are said to **communicate**. If  $P$  is the transition matrix for the chain, then the entries in  $P^k$  give the probabilities of going from one state to another in  $k$  steps:

$$P^k = \begin{array}{ccc} \text{From:} & & \\ \begin{array}{c} 1 \quad j \quad m \\ \vdots \\ \downarrow \\ p_{ij} \quad \cdots \end{array} & \rightarrow & \begin{array}{c} \text{To:} \\ 1 \\ i \\ m \end{array} \end{array}$$

and powers of  $P$  can be used to make the following definition.

#### DEFINITION

Let  $i$  and  $j$  be two states of a Markov chain with transition matrix  $P$ . Then state  $i$  **communicates** with state  $j$  if there exist nonnegative integers  $m$  and  $n$  such that the  $(j, i)$ -entry of  $P^m$  and the  $(i, j)$ -entry of  $P^n$  are both strictly positive. That is, state  $i$  communicates with state  $j$  if it is possible to go from state  $i$  to state  $j$  in  $m$  steps and from state  $j$  to state  $i$  in  $n$  steps.

This definition implies three properties that will allow the states of a Markov chain to be placed into groups called **communication classes**. First, the definition allows the integers  $m$  and  $n$  to be zero, in which case the  $(i, i)$ -entry of  $P^0 = I$  is 1, which is

positive. This ensures that every state communicates with itself. Because both  $(i, j)$  and  $(j, i)$  are included in the definition, it follows that if state  $i$  communicates with state  $j$  then state  $j$  communicates with state  $i$ . Finally, you will show in Exercise 36 that if state  $i$  communicates with state  $j$  and state  $j$  communicates with state  $k$ , then state  $i$  communicates with state  $k$ . These three properties are called, respectively, the **reflexive**, **symmetric**, and **transitive** properties:

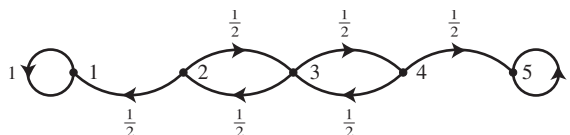
- (Reflexive Property) Each state communicates with itself.
- (Symmetric Property) If state  $i$  communicates with state  $j$ , then state  $j$  communicates with state  $i$ .
- (Transitive Property) If state  $i$  communicates with state  $j$ , and state  $j$  communicates with state  $k$ , then state  $i$  communicates with state  $k$ .

A relation with these three properties is called an **equivalence relation**. The communication relation is an equivalence relation on the state space for the Markov chain. Using the properties listed above simplifies determining which states communicate.

**EXAMPLE 1** Consider an unbiased random walk with absorbing boundaries on  $\{1, 2, 3, 4, 5\}$ . Find which states communicate.

**SOLUTION** The transition matrix  $P$  is given below, and Figure 1 shows the transition diagram for this Markov chain.

$$P = \begin{array}{ccccc|c} & 1 & 2 & 3 & 4 & 5 \\ \begin{bmatrix} 1 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1 \end{bmatrix} & 1 & 2 & 3 & 4 & 5 \end{array}$$



**FIGURE 1** Unbiased random walk with absorbing boundaries.

First note that, by the reflexive property, each state communicates with itself. It is clear from the diagram that states 2, 3, and 4 communicate with each other. The same conclusion may be reached using the definition by finding that the  $(2, 3)$ -,  $(3, 2)$ -,  $(3, 4)$ -, and  $(4, 3)$ -entries in  $P$  are positive, thus states 2 and 3 communicate, as do states 3 and 4. States 2 and 4 must also communicate by the transitive property. Now consider state 1 and state 5. If the chain starts in state 1, it cannot move to any state other than itself. Thus it is not possible to go from state 1 to any other state in any number of steps, and state 1 does not communicate with any other state. Likewise, state 5 does not communicate with any other state. In summary,

- State 1 communicates with state 1.
- State 2 communicates with state 2, state 3, and state 4.
- State 3 communicates with state 2, state 3, and state 4.
- State 4 communicates with state 2, state 3, and state 4.
- State 5 communicates with state 5.

Notice that even though states 1 and 5 do not communicate with states 2, 3, and 4, it is possible to go **from** these states to either state 1 or state 5 in a finite number of steps: this is clear from the diagram, or by confirming that the appropriate entries in  $P$ ,  $P^2$ , or  $P^3$  are positive. ■

In Example 1, the state space  $\{1, 2, 3, 4, 5\}$  can now be divided into the classes  $\{1\}$ ,  $\{2, 3, 4\}$ , and  $\{5\}$ . The states in each of these classes communicate only with the other members of their class. This division of the state space occurs because the communication relation is an equivalence relation. The communication relation **partitions** the state space into **communication classes**. Each state in a Markov chain communicates only with the members of its communication class. For the Markov chain in Example 1, the communication classes are  $\{1\}$ ,  $\{2, 3, 4\}$ , and  $\{5\}$ .

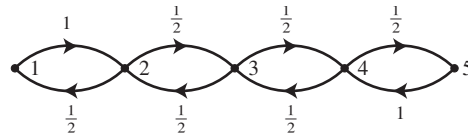
**EXAMPLE 2** Consider an unbiased random walk with reflecting boundaries on  $\{1, 2, 3, 4, 5\}$ . Find the communication classes for this Markov chain.

**SOLUTION** The transition matrix  $P$  for this chain, as well as  $P^2$ ,  $P^3$ , and  $P^4$ , is shown below.

$$P = \begin{array}{ccccc} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 1/2 & 0 & 0 & 0 \\ 1 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1 \\ 0 & 0 & 0 & 1/2 & 0 \end{bmatrix} \end{array} \quad P^2 = \begin{array}{ccccc} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 1/2 & 0 & 1/4 & 0 & 0 \\ 0 & 3/4 & 0 & 1/4 & 0 \\ 1/2 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1/4 & 0 & 3/4 & 0 \\ 0 & 0 & 1/4 & 0 & 1/2 \end{bmatrix} \end{array}$$

$$P^3 = \begin{array}{ccccc} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 3/8 & 0 & 1/8 & 0 \\ 3/4 & 0 & 1/2 & 0 & 1/4 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 1/4 & 0 & 1/2 & 0 & 3/4 \\ 0 & 1/8 & 0 & 3/8 & 0 \end{bmatrix} \end{array} \quad P^4 = \begin{array}{ccccc} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 3/8 & 0 & 1/4 & 0 & 1/8 \\ 0 & 5/8 & 0 & 3/8 & 0 \\ 1/2 & 0 & 1/2 & 0 & 1/2 \\ 0 & 3/8 & 0 & 5/8 & 0 \\ 1/8 & 0 & 1/4 & 0 & 3/8 \end{bmatrix} \end{array}$$

The transition diagram for this Markov chain is given in Figure 2.



**FIGURE 2** Unbiased random walk with reflecting boundaries.

Notice that the  $(i, j)$ -entry in at least one of these matrices is positive for any choice of  $i$  and  $j$ . Thus every state is reachable from any other state in four steps or fewer, and every state communicates with every state. There is only one communication class:  $\{1, 2, 3, 4, 5\}$ . ■

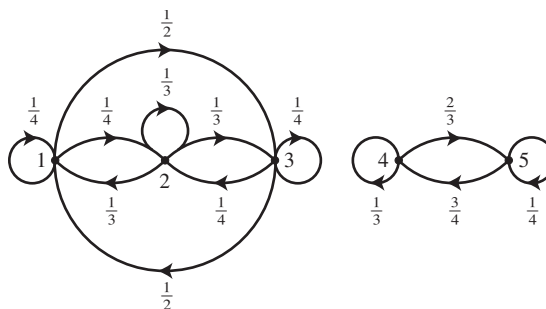


**EXAMPLE 3** Consider the Markov chain given in Example 5 in Section 10.2. Find the communication classes for this Markov chain.

**SOLUTION** The transition matrix for this Markov chain is

$$P = \begin{array}{ccccc} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{bmatrix} 1/4 & 1/3 & 1/2 & 0 & 0 \\ 1/4 & 1/3 & 1/4 & 0 & 0 \\ 1/2 & 1/3 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 3/4 \\ 0 & 0 & 0 & 2/3 & 1/4 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \end{array}$$

and a transition diagram is shown in Figure 3.



**FIGURE 3** Transition diagram for Example 3.

It is impossible to move from any of the states 1, 2, or 3 to either of the states 4 or 5, so these states must be in separate communication classes. In addition, state 1, state 2, and state 3 communicate; state 4 and state 5 also communicate. Thus the communication classes for this Markov chain are  $\{1, 2, 3\}$  and  $\{4, 5\}$ . ■

The Markov chains in Examples 1 and 3 have more than one communication class, while the Markov chain in Example 2 has only one communication class. This distinction leads to the following definitions.

### DEFINITION

A Markov chain with only one communication class is **irreducible**. A Markov chain with more than one communication class is **reducible**.

Thus the Markov chains in Examples 1 and 3 are reducible, while the Markov chain in Example 2 is irreducible. Irreducible Markov chains and regular transition matrices are connected by the following theorem.

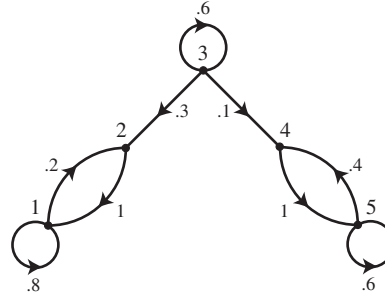
### THEOREM 2

If a Markov chain has a regular transition matrix, then it is irreducible.

**PROOF** Suppose that  $P$  is a regular transition matrix for a Markov chain. Then, by definition, there is a  $k$  such that  $P^k$  is a positive matrix. That is, for any states  $i$  and  $j$ , the  $(i, j)$ - and  $(j, i)$ -elements in  $P^k$  are strictly positive. Thus there is a positive probability of moving from  $i$  to  $j$  and from  $j$  to  $i$  in exactly  $k$  steps, and so  $i$  and  $j$  communicate with each other. Since  $i$  and  $j$  were arbitrary states and must be in the same communication class, there can be only one communication class for the chain, so the Markov chain must be irreducible. ■

Example 2 shows that the converse of Theorem 2 is not true, because the Markov chain in this example is irreducible, but its transition matrix is not regular.

**EXAMPLE 4** Consider the Markov chain whose transition diagram is given in Figure 4. Determine whether this Markov chain is reducible or irreducible.



**FIGURE 4** Transition diagram for Example 4.

**SOLUTION** The diagram shows that states 1 and 2 communicate, as do states 4 and 5. Notice that states 1 and 2 cannot communicate with states 3, 4, or 5 since the probability of moving from state 2 to state 3 is 0. Likewise states 4 and 5 cannot communicate with states 1, 2, or 3 since the probability of moving from state 4 to state 3 is 0. Finally, state 3 cannot communicate with any state other than itself since it is impossible to return to state 3 from any other state. Thus the communication classes for this Markov chain are  $\{1, 2\}$ ,  $\{3\}$ , and  $\{4, 5\}$ . Since there is more than one communication class, this Markov chain is reducible. ■

## Mean Return Times

Let  $\mathbf{q}$  be the steady-state vector for an irreducible Markov chain. It can be shown using advanced methods in probability theory that the entries in  $\mathbf{q}$  may be interpreted as occupation times; that is,  $q_i$  is the fraction of time steps that the chain will spend at state  $i$ . For example, consider a Markov chain on  $\{1, 2, 3\}$  with steady-state vector

$$\mathbf{q} = \begin{bmatrix} .2 \\ .5 \\ .3 \end{bmatrix}.$$

In the long run, the chain will spend about half of its steps in state 2. If the chain is currently in state 2, it should take about two  $(1/.5)$  steps to return to state 2. Likewise, since the chain spends about  $1/5$  of its time in state 1, it should visit state 1 once every five steps.

Given a Markov chain and states  $i$  and  $j$ , a quantity of considerable interest is the number of steps  $n_{ij}$  that it will take for the system to first visit state  $i$  given that it starts in state  $j$ . The value of  $n_{ij}$  cannot be known—it could be any positive integer depending on how the Markov chain evolves. Such a quantity is known as a **random variable**. Since  $n_{ij}$  is unknowable, the **expected value** of  $n_{ij}$  is studied instead. The expected value of a random variable functions as a type of average value of the random variable. The following definition will be used in subsequent sections.

**DEFINITION**

The **expected value of a random variable**  $X$  which takes on the values  $x_1, x_2, \dots$  is

$$E[X] = x_1 P(X = x_1) + x_2 P(X = x_2) + \dots = \sum_{k=1}^{\infty} x_k P(X = x_k) \quad (1)$$

where  $P(X = x_k)$  denotes the probability that the random variable  $X$  equals the value  $x_k$ .

Now let  $t_{ii} = E[n_{ii}]$  be the expected value of  $n_{ii}$ , which is the expected number of steps it will take for the system to return to state  $i$  given that it starts in state  $i$ . Unfortunately, Equation (1) will not be helpful at this point. Instead, proceeding intuitively, the system should spend one step in state  $i$  for each  $t_{ii}$  steps on average. It seems reasonable to say that the system will, over the long run, spend about  $1/t_{ii}$  of the time in state  $i$ . But that quantity is  $q_i$ , so the expected number of time steps needed to return, or **mean return time** to a state  $i$ , is the reciprocal of  $q_i$ . This informal argument may be made rigorous using methods from probability theory; see Appendix 2 for a complete proof.

**THEOREM 3**

Consider an irreducible Markov chain with a finite state space, let  $n_{ij}$  be the number of steps until the chain first visits state  $i$  given that the chain starts in state  $j$ , and let  $t_{ii} = E[n_{ii}]$ . Then

$$t_{ii} = \frac{1}{q_i} \quad (2)$$

where  $q_i$  is the entry in the steady-state vector  $\mathbf{q}$  corresponding to state  $i$ .

The example above matches Equation 2:  $t_{11} = 1/.2 = 5$ ,  $t_{22} = 1/.5 = 2$ , and  $t_{33} = 1/.3 = 10/3$ . Recall that the mean return time is an expected value, so the fact that  $t_{33}$  is not an integer ought not be troubling. Section 10.5 will include a discussion of  $t_{ij} = E[n_{ij}]$ , where  $i \neq j$ .

**PRACTICE PROBLEM**

1. Consider the Markov chain on  $\{1, 2, 3, 4\}$  with transition matrix

$$P = \begin{bmatrix} 1/4 & 1/3 & 1/2 & 0 \\ 0 & 1/3 & 0 & 1/3 \\ 3/4 & 0 & 1/2 & 1/3 \\ 0 & 1/3 & 0 & 1/3 \end{bmatrix}$$

Determine the communication classes for this chain.

## 10.3 EXERCISES

In Exercises 1–6, consider a Markov chain with state space with  $\{1, 2, \dots, n\}$  and the given transition matrix. Find the communication classes for each Markov chain, and state whether the Markov chain is reducible or irreducible.

1. 
$$\begin{bmatrix} 1/4 & 0 & 1/3 \\ 1/2 & 1 & 1/3 \\ 1/4 & 0 & 1/3 \end{bmatrix}$$

2. 
$$\begin{bmatrix} 1/4 & 1/2 & 1/3 \\ 1/2 & 1/2 & 1/3 \\ 1/4 & 0 & 1/3 \end{bmatrix}$$

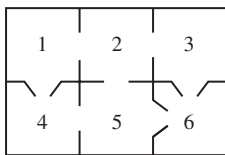
3. 
$$\begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

4. 
$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1/3 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 0 & 0 & 0 \\ 0 & 1/4 & 2/3 & 0 & 0 \\ 0 & 3/4 & 1/3 & 0 & 0 \end{bmatrix}$$

5. 
$$\begin{bmatrix} 0 & 0 & .4 & 0 & .8 & 0 \\ 0 & 0 & 0 & .7 & 0 & .5 \\ .3 & 0 & 0 & 0 & .2 & 0 \\ 0 & .1 & 0 & 0 & 0 & .5 \\ .7 & 0 & .6 & 0 & 0 & 0 \\ 0 & .9 & 0 & .3 & 0 & 0 \end{bmatrix}$$

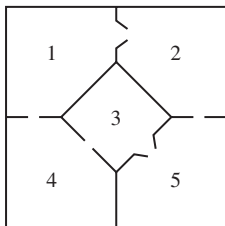
6. 
$$\begin{bmatrix} 0 & 1/3 & 0 & 2/3 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 1/3 & 0 \\ 0 & 2/3 & 0 & 1/3 & 0 & 0 & 2/5 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3/5 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2/3 & 0 \end{bmatrix}$$

7. Consider the mouse in the following maze from Section 10.1, Exercise 19.



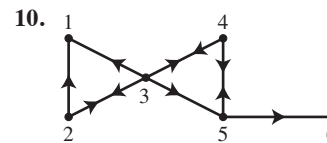
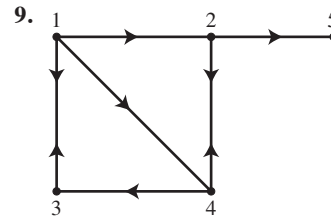
Find the communication classes for the Markov chain that models the mouse's travels through this maze. Is this Markov chain reducible or irreducible?

8. Consider the mouse in the following maze from Section 10.1, Exercise 20.



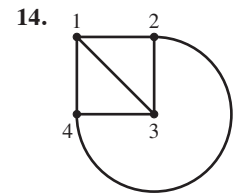
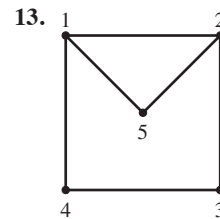
Find the communication classes for the Markov chain that models the mouse's travels through this maze. Is this Markov chain reducible or irreducible?

In Exercises 9 and 10, consider the set of webpages hyperlinked by the given directed graph. Find the communication classes for the Markov chain that models a random surfer's progress through this set of webpages. Use the transition matrix derived from the graph itself instead of the Google matrix.

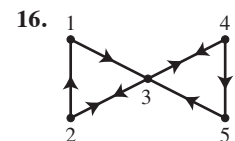
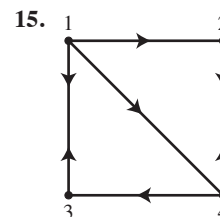


11. Consider an unbiased random walk with reflecting boundaries on  $\{1, 2, 3, 4\}$ . Find the communication classes for this Markov chain and determine whether it is reducible or irreducible.
12. Consider an unbiased random walk with absorbing boundaries on  $\{1, 2, 3, 4\}$ . Find the communication classes for this Markov chain and determine whether it is reducible or irreducible.

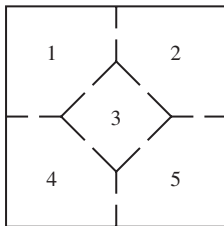
In Exercises 13 and 14, consider a simple random walk on the given graph. Show that the Markov chain is irreducible and calculate the mean return times for each state.



In Exercises 15 and 16, consider a simple random walk on the given directed graph. Show that the Markov chain is irreducible and calculate the mean return times for each state.

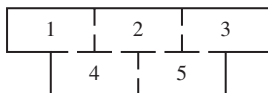


17. Consider the mouse in the following maze from Section 10.1, Exercise 17.



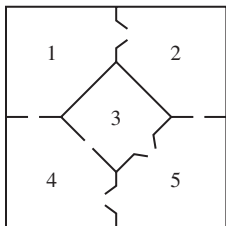
If the mouse starts in room 3, how long on average will it take the mouse to return to room 3?

18. Consider the mouse in the following maze from Section 10.1, Exercise 18.



If the mouse starts in room 2, how long on average will it take the mouse to return to room 2?

In Exercises 19 and 20, consider the mouse in the following maze from Section 10.2, Exercise 20.



19. If the mouse starts in room 1, how long on average will it take the mouse to return to room 1?
20. If the mouse starts in room 4, how long on average will it take the mouse to return to room 4?

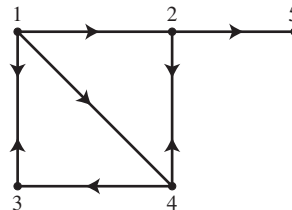
In Exercises 21 and 22, mark each statement True or False. Justify each answer.

21. a. If it is possible to go from state  $i$  to state  $j$  in  $n$  steps, where  $n \geq 0$ , then states  $i$  and  $j$  communicate with each other.
- b. If a Markov chain is reducible, then it cannot have a regular transition matrix.
- c. The entries in the steady-state vector are the mean return times for each state.
22. a. An irreducible Markov chain must have a regular transition matrix.
- b. If the  $(i, j)$ - and  $(j, i)$ -entries in  $P^k$  are positive for some  $k$ , then the states  $i$  and  $j$  communicate with each other.
- c. If state  $i$  communicates with state  $j$  and state  $j$  communicates with state  $k$ , then state  $i$  communicates with state  $k$ .

23. Suppose that the weather in Charlotte is modeled using the Markov chain in Section 10.1, Exercise 23. About how many days elapse in Charlotte between rainy days?

24. Suppose that the weather in Charlotte is modeled using the Markov chain in Section 10.1, Exercise 24. About how many days elapse in Charlotte between consecutive rainy days?

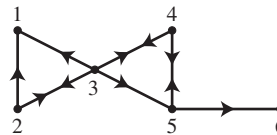
25. The following set of webpages hyperlinked by the directed graph was studied in Section 10.2, Exercise 25.



Consider randomly surfing on this set of webpages using the Google matrix as the transition matrix.

- a. Show that this Markov chain is irreducible.
- b. Suppose the surfer starts at page 1. How many mouse clicks on average must the surfer make to get back to page 1?

26. The following set of webpages hyperlinked by the directed graph was studied in Section 10.2, Exercise 26.



Repeat Exercise 25 for this set of webpages.

27. Consider the pair of Ehrenfest urns studied in Section 10.2, Exercise 9. Suppose that there are now 2 molecules in urn A. How many steps on average will be needed until there are again 2 molecules in urn A?
28. Consider the pair of Ehrenfest urns studied in Section 10.2, Exercise 10. Suppose that urn A is now empty. How many steps on average will be needed until urn A is again empty?
29. A variation of the Ehrenfest model of diffusion was studied in Section 10.2, Exercise 29. Consider this model with  $k = 3$  and  $p = 1/2$  and suppose that there are now 3 molecules in urn A. How many draws on average will be needed until there are again 3 molecules in urn A?
30. Consider the Bernoulli-Laplace model of diffusion studied in Section 10.2, Exercise 30. Let  $k = 5$ . Suppose that all of the type I molecules are now in urn A. How many draws on average will be needed until all of the type I molecules are again in urn A?
31. A Markov chain model for scoring a tennis game was studied in Section 10.1, Exercise 31. What are the communication classes for this Markov chain?

32. A Markov chain model for the rally point method for scoring a volleyball game was studied in Section 10.1, Exercise 32. What are the communication classes for this Markov chain?

In Exercises 33 and 34, consider the Markov chain on  $\{1, 2, 3, 4, 5\}$  with transition matrix

$$P = \begin{bmatrix} 0 & 0 & 0 & 1/2 & 1 \\ 1/3 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 0 & 0 & 0 \\ 0 & 2/5 & 1/5 & 1/2 & 0 \\ 0 & 3/5 & 4/5 & 0 & 0 \end{bmatrix}$$

33. Show that this Markov chain is irreducible.
34. Suppose the chain starts in state 1. What is the expected number of steps until it is in state 1 again?
35. How does the presence of dangling nodes in a set of hyperlinked webpages affect the communication classes of the associated Markov chain?
36. Show that the communication relation is transitive. *Hint:* Show that the  $(i, k)$ -entry of  $P^{n+m}$  must be greater than or equal to the product of the  $(i, j)$ -entry of  $P^m$  and the  $(j, k)$ -entry of  $P^n$ .

### SOLUTION TO PRACTICE PROBLEM

1. First note that states 1 and 3 communicate with each other, as do states 2 and 4. However, there is no way to proceed from either state 1 or state 3 to either state 2 or state 4, so the communication classes are  $\{1, 3\}$  and  $\{2, 4\}$ .

## 10.4 CLASSIFICATION OF STATES AND PERIODICITY

The communication classes of a Markov chain have important properties which help determine whether the state vectors converge to a unique steady-state vector. These properties are studied in this section, and it will be shown that Examples 3, 4, and 5 in Section 10.2 are examples of all the ways that the state vectors of a Markov chain can fail to converge to a unique steady-state vector.

### Recurrent and Transient States

One way to describe the communication classes is to determine whether it is possible for the Markov chain to leave the class once it has entered it.

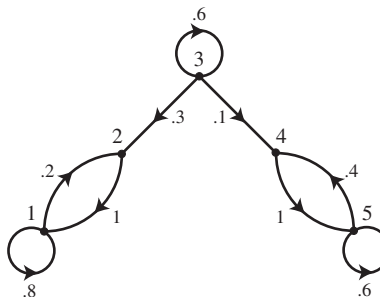
#### DEFINITION

Let  $C$  be a communication class of states for a Markov chain, and let  $j$  be a state in  $C$ . If there is a state  $i$  not in  $C$  and  $k > 0$  such that the  $(i, j)$ -entry in  $P^k$  is positive, then the class  $C$  is called a **transient class** and each state in  $C$  is a **transient state**. If a communication class is not transient, it is called a **recurrent class** and each state in the class is a **recurrent state**.

Suppose that  $C$  is a transient class. Notice that once the system moves from  $C$  to another communication class  $D$ , the system can never return to  $C$ . This is true because  $D$  cannot contain a state  $i$  from which it is possible to move to a state in  $C$ . If  $D$  did contain such a state  $i$ , then the transitive property of the communication relation would imply that every state in  $C$  communicates with every state in  $D$ . This is impossible.

**EXAMPLE 1** Consider the Markov chain on  $\{1, 2, 3, 4, 5\}$  studied in Example 4 in Section 10.3. Its transition diagram is given in Figure 1. Determine whether each of the communication classes is transient or recurrent.

**SOLUTION** The communication classes were found to be  $\{1, 2\}$ ,  $\{3\}$ , and  $\{4, 5\}$ . First consider class  $\{3\}$ . There is a positive probability of a transition from state 3 to state



**FIGURE 1** Transition diagram for Example 1.

2, so applying the definition with  $k = 1$  shows that class  $\{3\}$  is a transient class. Now consider class  $\{1, 2\}$ . The probability of a one-step transition from either state 1 or state 2 to any of states 3, 4, or 5 is zero, and this is also true for any number of steps. If the system starts in state 1 or 2, it will always stay in state 1 or 2. Class  $\{1, 2\}$  is thus a recurrent class. A similar argument shows that class  $\{4, 5\}$  is also a recurrent class. ■

**EXAMPLE 2** Consider the random walk with reflecting boundaries studied in Example 2 in Section 10.3. Determine whether each of the communication classes is transient or recurrent.

**SOLUTION** This Markov chain is irreducible: the single communication class for this chain is  $\{1, 2, 3, 4, 5\}$ . By the definition, this class cannot be transient. Thus the communication class must be recurrent. ■

The result of the preceding example may be generalized to any irreducible Markov chain.

**Remark:** All states of an irreducible Markov chain are recurrent.

Suppose that a reducible Markov chain has two transient classes  $C_1$  and  $C_2$  and no recurrent classes. Since  $C_1$  is transient, there must be a state in  $C_2$  which can be reached from a state in  $C_1$ . Since  $C_2$  is transient, there must be a state in  $C_1$  which can be reached from  $C_2$ . Thus all states in  $C_1$  and  $C_2$  communicate, which is impossible. Thus the Markov chain must have at least one recurrent class. This argument can be generalized to refer to any reducible Markov chain with any number of transient classes, which along with the previous remark proves the following.

**Remark:** Every Markov chain must have at least one recurrent class.

**EXAMPLE 3** Consider the Markov chain studied in Example 3 in Section 10.3. Determine whether each of the communication classes is transient or recurrent.

**SOLUTION** The transition matrix for this Markov chain is

$$P = \begin{array}{ccccc} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{bmatrix} 1/4 & 1/3 & 1/2 & 0 & 0 \\ 1/4 & 1/3 & 1/4 & 0 & 0 \\ 1/2 & 1/3 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 3/4 \\ 0 & 0 & 0 & 2/3 & 1/4 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \end{array}$$

and the two communication classes are  $\{1, 2, 3\}$  and  $\{4, 5\}$ . The matrix  $P$  may be written as the partitioned matrix  $P = \begin{bmatrix} P_1 & O \\ O & P_2 \end{bmatrix}$ , where

$$P_1 = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1/4 & 1/3 & 1/2 \\ 1/4 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 \end{bmatrix} \end{matrix} \quad \text{and} \quad P_2 = \begin{matrix} & \begin{matrix} 4 & 5 \end{matrix} \\ \begin{matrix} 4 \\ 5 \end{matrix} & \begin{bmatrix} 1/3 & 3/4 \\ 2/3 & 1/4 \end{bmatrix} \end{matrix}$$

and  $O$  is an appropriately sized zero matrix. Using block multiplication,

$$P^k = \begin{bmatrix} P_1^k & O \\ O & P_2^k \end{bmatrix}$$

for all  $k > 0$ . Thus if state  $j$  is in one class and state  $i$  is in the other class, the  $(i, j)$ - and  $(j, i)$ -entries of  $P^k$  are zero for all  $k > 0$ . Thus both classes of this Markov chain must be recurrent. ■

**EXAMPLE 4** Consider altering the previous example slightly to get a Markov chain with transition matrix

$$P = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 1/4 & 1/3 & 1/2 & 0 & 0 \\ 1/4 & 1/3 & 1/4 & 0 & 0 \\ 1/2 & 1/3 & 1/4 & 0 & 1/4 \\ 0 & 0 & 0 & 1/3 & 1/2 \\ 0 & 0 & 0 & 2/3 & 1/4 \end{bmatrix} \end{matrix}$$

and the transition diagram given in Figure 2. Determine whether each of the communication classes is transient or recurrent.

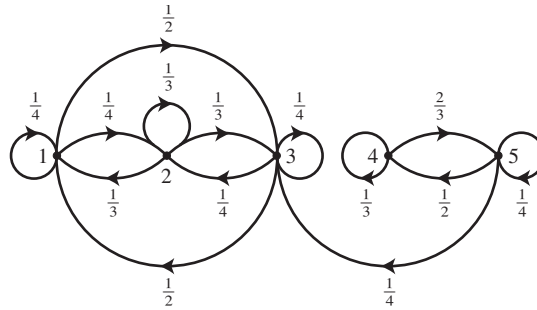


FIGURE 2 Transition diagram for Example 4.

**SOLUTION** The communication classes are still  $\{1, 2, 3\}$  and  $\{4, 5\}$ . Now the  $(3, 5)$ -entry is not zero, so  $\{4, 5\}$  is a transient class. By the remark above, the chain must have at least one recurrent class, so  $\{1, 2, 3\}$  must be that recurrent class. This result may also be proven using partitioned matrices. Let  $P = \begin{bmatrix} P_1 & S \\ O & Q \end{bmatrix}$ , where  $P_1$  is as in the previous example,

$$Q = \begin{matrix} & \begin{matrix} 4 & 5 \end{matrix} \\ \begin{matrix} 4 \\ 5 \end{matrix} & \begin{bmatrix} 1/3 & 1/2 \\ 2/3 & 1/4 \end{bmatrix} \end{matrix} \quad \text{and} \quad S = \begin{matrix} & \begin{matrix} 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1/4 \end{bmatrix} \end{matrix}$$



The submatrix  $P_1$  is a transition matrix in its own right: it describes transitions within the recurrent class  $\{1, 2, 3\}$ . Matrix  $S$  records the probabilities of transitions from the transient class  $\{4, 5\}$  into the recurrent class  $\{1, 2, 3\}$ . Matrix  $Q$  records the probabilities of transitions within the transient class  $\{4, 5\}$ . Block multiplication (see Section 2.4) now gives

$$P^k = \begin{bmatrix} P_1^k & S_k \\ O & Q^k \end{bmatrix}$$

for some nonzero matrix  $S_k$ . Since the lower left block is  $O$  for all matrices  $P^k$ , it is impossible to leave class  $\{1, 2, 3\}$  after entering it, and class  $\{1, 2, 3\}$  is a recurrent class. ■

In Examples 3 and 4, the states were ordered so that the members of each class were grouped together. In Example 4, the recurrent classes were listed first followed by the transient classes. This ordering was convenient, as it allowed for the use of partitioned matrices to determine the recurrent and transient classes. It is also possible to use block multiplication to compute powers of the transition matrix  $P$  if the states are reordered in the manner done in Examples 3 and 4: the states in each communication class are consecutive, and if there are any transient classes, the recurrent classes are listed first, followed by the transient classes. A transition matrix with its states thus reordered is said to be in **canonical form**. To see how this reordering works, consider the following example.

**EXAMPLE 5** The Markov chain in Example 1 has transition matrix

$$P = \begin{array}{ccccc} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} .8 & 1 & 0 & 0 & 0 \\ .2 & 0 & .3 & 0 & 0 \\ 0 & 0 & .6 & 0 & 0 \\ 0 & 0 & .1 & 0 & .4 \\ 0 & 0 & 0 & 1 & .6 \end{bmatrix} \end{array}$$

and its communication classes are  $\{1, 2\}$ ,  $\{3\}$ , and  $\{4, 5\}$ . To place the matrix in canonical form, reorder the classes  $\{1, 2\}$ ,  $\{4, 5\}$ , and  $\{3\}$ ; that is, rearrange the states in the order 1, 2, 4, 5, 3. To perform this rearrangement, first rearrange the columns, which produces the matrix

$$\begin{array}{ccccc} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} .8 & 1 & 0 & 0 & 0 \\ .2 & 0 & .3 & 0 & 0 \\ 0 & 0 & .6 & 0 & 0 \\ 0 & 0 & .1 & 0 & .4 \\ 0 & 0 & 0 & 1 & .6 \end{bmatrix} \end{array} \xrightarrow[\text{columns}]{\text{rearrange}} \begin{array}{ccccc} & \begin{matrix} 1 & 2 & 4 & 5 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} .8 & 1 & 0 & 0 & 0 \\ .2 & 0 & 0 & 0 & .3 \\ 0 & 0 & 0 & 0 & .6 \\ 0 & 0 & 0 & .4 & .1 \\ 0 & 0 & 1 & .6 & 0 \end{bmatrix} \end{array}$$

Now rearranging the rows produces the transition matrix in canonical form:

$$\begin{array}{ccccc} & \begin{matrix} 1 & 2 & 4 & 5 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} .8 & 1 & 0 & 0 & 0 \\ .2 & 0 & 0 & 0 & .3 \\ 0 & 0 & 0 & 0 & .6 \\ 0 & 0 & 0 & .4 & .1 \\ 0 & 0 & 1 & .6 & 0 \end{bmatrix} \end{array} \xrightarrow[\text{rows}]{\text{rearrange}} \begin{array}{ccccc} & \begin{matrix} 1 & 2 & 4 & 5 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} .8 & 1 & 0 & 0 & 0 \\ .2 & 0 & 0 & 0 & .3 \\ 0 & 0 & 0 & .4 & .1 \\ 0 & 0 & 1 & .6 & 0 \\ 0 & 0 & 0 & 0 & .6 \end{bmatrix} \end{array}$$

The transition matrix may be divided as follows:

$$P = \begin{array}{c} \begin{array}{cc|cc} 1 & 2 & 4 & 5 & 3 \\ \hline .8 & 1 & 0 & 0 & 0 \\ .2 & 0 & 0 & 0 & .3 \\ 0 & 0 & 0 & .4 & .1 \\ 0 & 0 & 1 & .6 & 0 \\ 0 & 0 & 0 & 0 & .6 \end{array} \begin{array}{l} 1 \\ 2 \\ 4 \\ 5 \\ 3 \end{array} \end{array} = \left[ \begin{array}{c|c} P_1 & S \\ \hline O & Q \end{array} \right]$$

In general, suppose that  $P$  is the transition matrix for a reducible Markov chain with  $r$  recurrent classes and one or more transient classes. A canonical form of  $P$  is

$$P = \left[ \begin{array}{ccc|c} P_1 & \cdots & O & S \\ \vdots & \ddots & \vdots & \vdots \\ O & \cdots & P_r & S \\ \hline O & \cdots & O & Q \end{array} \right]$$

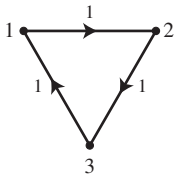
Here  $P_i$  is the transition matrix for the  $i^{\text{th}}$  recurrent class,  $O$  is an appropriately sized zero matrix,  $Q$  records transitions within the transient classes, and  $S$  contains the probabilities of transitions from the transient classes to the recurrent classes. Since  $P$  is a partitioned matrix, it is relatively easy to take powers of it using block multiplication:

$$P^k = \left[ \begin{array}{ccc|c} P_1^k & \cdots & O & S_k \\ \vdots & \ddots & \vdots & \vdots \\ O & \cdots & P_r^k & S_k \\ \hline O & \cdots & O & Q^k \end{array} \right]$$

for some matrix  $S_k$ . The matrices  $Q$ ,  $S$ , and  $S_k$  help to answer questions about the long-term behavior of the Markov chain which are addressed in Section 10.5.

## Periodicity

A final way of classifying states is to examine at what times it is possible for the system to return to the state in which it begins. Consider the following simple example.



**FIGURE 3**

Transition diagram for Example 6.

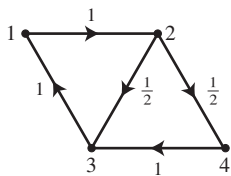
**EXAMPLE 6** A Markov chain on  $\{1, 2, 3\}$  has transition matrix

$$P = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \\ \hline 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \end{array}$$

The transition diagram, which is shown in Figure 3, is quite straightforward. The system must return to its starting point in three steps and every time the number of steps is a multiple of three.

**EXAMPLE 7** A Markov chain on  $\{1, 2, 3, 4\}$  has transition matrix

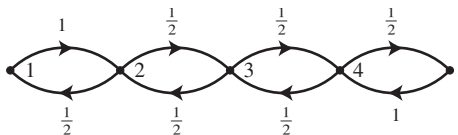
$$P = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \\ \hline 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1 \\ 0 & 1/2 & 0 & 0 \end{array} \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$



**FIGURE 4**  
Transition diagram for Example 7.

and the transition diagram shown in Figure 4. If the system starts in state 1, 2, or 3, the system may return to its starting point in three steps or in four steps, and may return every time the number of steps is  $3a + 4b$  for some non-negative integers  $a$  and  $b$ . It can be shown that every positive integer greater than 5 may be written in that form, so if the system starts in state 1, 2, or 3, it may also return to its starting point at any number of steps greater than 5. If the system starts in state 4, the system may return to its starting point in four steps or in seven steps, and a similar argument shows that the system may also return to its starting point at any number of steps greater than 17. ■

**EXAMPLE 8** The unbiased random walk on  $\{1, 2, 3, 4, 5\}$  with reflecting boundaries has the transition diagram shown in Figure 5. From this diagram, one can see that it will always take an even number of steps for the system to return to the state in which it started.



**FIGURE 5** Unbiased random walk with reflecting boundaries.

In Examples 6 and 8, the time steps at which the system may return to its initial site are multiples of a number  $d$ :  $d = 3$  for Example 6,  $d = 2$  for Example 8. This number  $d$  is called the *period* of the state, and is defined as follows.

### DEFINITION

The **period**  $d$  of a state  $i$  of a Markov chain is the greatest common divisor of all time steps  $n$  such that the probability that the Markov chain that started at  $i$  visits  $i$  at time step  $n$  is strictly positive.

Using a careful analysis of the set of states visited by the Markov chain, it may be shown that the period of each state in a given communication class is the same, so the period is a property of communication classes. See Appendix 2 for a proof of this fact, which leads to the following definition.

### DEFINITION

The **period** of a communication class  $C$  is the period of each state in  $C$ . If a Markov chain is irreducible, then the period of the chain is the period of its single communication class. If the period of every communication class (and thus every state) is  $d = 1$ , then the Markov chain is **aperiodic**.

The reason that the greatest common divisor appears in the definition is to allow a period to be assigned to all states of all Markov chains. In Example 7, the system may return to its starting state after any sufficiently large number of steps, so the period of each state is  $d = 1$ . That is, the Markov chain in Example 7 is aperiodic. Notice that this chain does not exhibit periodic behavior, so the term aperiodic is quite apt. Using the definition confirms that the period of the Markov chain in Example 6 is  $d = 3$ , while the period of the Markov chain in Example 8 is  $d = 2$ . The next theorem describes the transition matrix of an irreducible and aperiodic Markov chain.

**THEOREM 4**

Let  $P$  be the transition matrix for an irreducible, aperiodic Markov chain. Then  $P$  is a regular matrix.

**PROOF** Let  $P$  be an  $n \times n$  transition matrix for an irreducible, aperiodic Markov chain. To show that  $P$  is regular, a number  $k$  must be found for which every entry in  $P^k$  is strictly positive. Let  $1 \leq i, j \leq n$ . Since the Markov chain is irreducible, there must be a number  $a$  which depends on  $i$  and  $j$  such that the  $(i, j)$ -element in  $P^a$  is strictly positive. Since the Markov chain is aperiodic, there is a number  $b$  which depends on  $j$  such that the  $(j, j)$ -element in  $P^m$  is strictly positive for all  $m \geq b$ . Now note that since  $P^{a+m} = P^a P^m$ , the  $(i, j)$ -element in  $P^{a+m}$  must be greater than or equal to the product of the  $(i, j)$ -element in  $P^a$  and the  $(j, j)$ -element in  $P^m$ . Thus the  $(i, j)$ -element in  $P^{a+m}$  must be strictly positive for all  $m \geq b$ . Now let  $k$  be the maximum over all pairs  $(i, j)$  of the quantity  $a + b$ . This maximum exists because the state space is finite, and the  $(i, j)$ -element of  $P^k$  must be strictly positive for all pairs  $(i, j)$ . Thus every entry of  $P^k$  is strictly positive, and  $P$  is a regular matrix. ■

So, if  $P$  is the transition matrix for an irreducible, aperiodic Markov chain, then  $P$  must be regular and Theorem 1 must apply to  $P$ . Thus there is a steady-state vector  $\mathbf{q}$  for which

$$\lim_{n \rightarrow \infty} P^n \mathbf{x}_0 = \mathbf{q}$$

for any choice of initial probability vector  $\mathbf{x}_0$ . What can be said about the steady-state vector  $\mathbf{q}$  if an irreducible Markov chain has period  $d > 1$ ? The following result is proven in more advanced texts in probability theory.

**THEOREM 5**

Let  $P$  be the transition matrix for an irreducible Markov chain with period  $d > 1$ , and let  $\mathbf{q}$  be the steady-state vector for the Markov chain. Then, for any initial probability vector  $\mathbf{x}_0$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{d} (P^{n+1} + \cdots + P^{n+d}) \mathbf{x}_0 = \mathbf{q}$$

Theorem 5 says that in the case of an irreducible Markov chain with period  $d > 1$ , the vector  $\mathbf{q}$  is the limit of the average of the probability vectors  $P^{n+1} \mathbf{x}_0, P^{n+2} \mathbf{x}_0, \dots, P^{n+d} \mathbf{x}_0$ . When a Markov chain is irreducible with period  $d > 1$ , the vector  $\mathbf{q}$  may still be interpreted as a vector of occupation times.

**EXAMPLE 9** The period of the irreducible Markov chain in Example 8 is  $d = 2$ , so the Markov chain has period  $d > 1$ . Let  $n$  be an even integer. Taking high powers of the transition matrix  $P$  shows that

$$P^n \longrightarrow \begin{array}{ccccc} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{bmatrix} 1/4 & 0 & 1/4 & 0 & 1/4 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 1/4 & 0 & 1/4 & 0 & 1/4 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \end{array}$$

and

$$P^{n+1} \longrightarrow \begin{array}{ccccc} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 1/4 & 0 & 1/4 & 0 \\ 1/2 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1/4 & 0 & 1/4 & 0 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix}$$

So for any initial probability vector  $\mathbf{x}_0$ ,

$$\lim_{n \rightarrow \infty} \frac{1}{2} (P^n + P^{n+1}) \mathbf{x}_0 = \begin{bmatrix} 1/8 & 1/8 & 1/8 & 1/8 & 1/8 \\ 1/4 & 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 & 1/4 \\ 1/8 & 1/8 & 1/8 & 1/8 & 1/8 \end{bmatrix} \mathbf{x}_0 = \begin{bmatrix} 1/8 \\ 1/4 \\ 1/4 \\ 1/4 \\ 1/8 \end{bmatrix}$$

But this vector was the steady-state vector for this Markov chain calculated in Exercise 32 in Section 10.2. Theorem 5 is thus confirmed in this case. ■

The steady-state vector for a *reducible* Markov chain will be discussed in detail in the next section.

### PRACTICE PROBLEM

1. Consider the Markov chain on  $\{1, 2, 3, 4\}$  with transition matrix

$$P = \begin{bmatrix} 1/4 & 1/3 & 1/2 & 0 \\ 0 & 1/3 & 0 & 1/3 \\ 3/4 & 0 & 1/2 & 1/3 \\ 0 & 1/3 & 0 & 1/3 \end{bmatrix}$$

Identify the communication classes of the chain as either recurrent or transient, and reorder the states to produce a matrix in canonical form.

## 10.4 EXERCISES

In Exercises 1–6, consider a Markov chain with state space with  $\{1, 2, \dots, n\}$  and the given transition matrix. Identify the communication classes for each Markov chain as recurrent or transient, and find the period of each communication class.

1.  $\begin{bmatrix} 1/4 & 0 & 1/3 \\ 1/2 & 1 & 1/3 \\ 1/4 & 0 & 1/3 \end{bmatrix}$

2.  $\begin{bmatrix} 1/4 & 1/2 & 1/3 \\ 1/2 & 1/2 & 1/3 \\ 1/4 & 0 & 1/3 \end{bmatrix}$

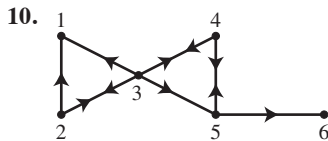
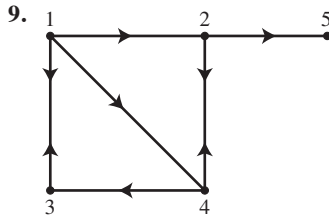
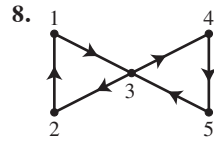
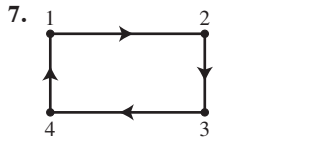
3.  $\begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$

4.  $\begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1/3 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 0 & 0 & 0 \\ 0 & 1/4 & 2/3 & 0 & 0 \\ 0 & 3/4 & 1/3 & 0 & 0 \end{bmatrix}$

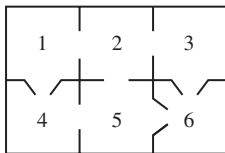
5.  $\begin{bmatrix} 0 & 0 & .4 & 0 & .8 & 0 \\ 0 & 0 & 0 & .7 & 0 & .5 \\ .3 & 0 & 0 & 0 & .2 & 0 \\ 0 & .1 & 0 & 0 & 0 & .5 \\ .7 & 0 & .6 & 0 & 0 & 0 \\ 0 & .9 & 0 & .3 & 0 & 0 \end{bmatrix}$

6.  $\begin{bmatrix} 0 & 1/3 & 0 & 2/3 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 1/3 & 0 \\ 0 & 2/3 & 0 & 1/3 & 0 & 0 & 2/5 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3/5 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2/3 & 0 \end{bmatrix}$

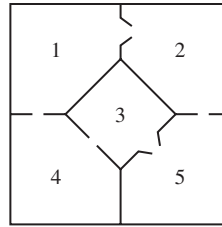
In Exercises 7–10, consider a simple random walk on the given directed graph. Identify the communication classes of this Markov chain as recurrent or transient, and find the period of each communication class.



11. Reorder the states in the Markov chain in Exercise 1 to produce a transition matrix in canonical form.
12. Reorder the states in the Markov chain in Exercise 2 to produce a transition matrix in canonical form.
13. Reorder the states in the Markov chain in Exercise 3 to produce a transition matrix in canonical form.
14. Reorder the states in the Markov chain in Exercise 4 to produce a transition matrix in canonical form.
15. Reorder the states in the Markov chain in Exercise 5 to produce a transition matrix in canonical form.
16. Reorder the states in the Markov chain in Exercise 6 to produce a transition matrix in canonical form.
17. Find the transition matrix for the Markov chain in Exercise 9 and reorder the states to produce a transition matrix in canonical form.
18. Find the transition matrix for the Markov chain in Exercise 10 and reorder the states to produce a transition matrix in canonical form.
19. Consider the mouse in the following maze from Section 10.1, Exercise 19.



- a. Identify the communication classes of this Markov chain as recurrent or transient.
  - b. Find the period of each communication class.
  - c. Find the transition matrix for the Markov chain and reorder the states to produce a transition matrix in canonical form.
20. Consider the mouse in the following maze from Section 10.1, Exercise 20.



- a. Identify the communication classes of this Markov chain as recurrent or transient.
- b. Find the period of each communication class.
- c. Find the transition matrix for the Markov chain and reorder the states to produce a transition matrix in canonical form.

In Exercises 21–22, mark each statement True or False. Justify each answer.

21. a. If two states  $i$  and  $j$  are both recurrent, then they must belong to the same communication class.  
b. All of the states in an irreducible Markov chain are recurrent.  
c. Every Markov chain must have at least one transient class.
22. a. If state  $i$  is recurrent and state  $i$  communicates with state  $j$ , then state  $j$  is also recurrent.  
b. If two states of a Markov chain have different periods, then the Markov chain is reducible.  
c. Every Markov chain must have exactly one recurrent class.
23. Confirm Theorem 5 for the Markov chain in Exercise 7 by taking powers of the transition matrix (see Example 9).
24. Confirm Theorem 5 for the Markov chain in Exercise 8 by taking high powers of the transition matrix (see Example 9).
25. Consider the Markov chain on  $\{1, 2, 3\}$  with transition matrix

$$P = \begin{bmatrix} 0 & 1/2 & 0 \\ 1 & 0 & 1 \\ 0 & 1/2 & 0 \end{bmatrix}$$

- a. Explain why this Markov chain is irreducible and has period 2.
- b. Find a steady-state vector  $\mathbf{q}$  for this Markov chain.
- c. Find an invertible matrix  $A$  and a diagonal matrix  $D$  such that  $P = ADA^{-1}$ . (See Section 5.3.)
- d. Use the result from part (c) to show that  $P^n$  may be written as

$$\begin{bmatrix} 1/4 & 1/4 & 1/4 \\ 1/2 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/4 \end{bmatrix} + (-1)^n \begin{bmatrix} 1/4 & -1/4 & 1/4 \\ -1/2 & 1/2 & -1/2 \\ 1/4 & -1/4 & 1/4 \end{bmatrix}$$

- e. Use the result from part (d) to confirm Theorem 5 for  $P$ .

26. Follow the plan of Exercise 25 to confirm Theorem 5 for the Markov chain with transition matrix

$$P = \begin{bmatrix} 0 & p & 0 \\ 1 & 0 & 1 \\ 0 & 1-p & 0 \end{bmatrix}$$

where  $0 < p < 1$ .

27. Confirm Theorem 5 for the Markov chain in Example 6.

28. Matrix multiplication can be used to find the canonical form of a transition matrix. Consider the matrix  $P$  in Example 5 and the matrix

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Notice that the rows of  $E$  are the rows of the identity matrix in the order 1, 2, 4, 5, 3.

- Compute  $EP$  and explain what has happened to the matrix  $P$ .
- Compute  $PE^T$  and explain what has happened to the matrix  $P$ .

- Compute  $EPE^T$  and explain what has happened to the matrix  $P$ .

29. Let  $A$  be an  $n \times n$  matrix and let  $E$  be an  $n \times n$  matrix resulting from permuting the rows of  $I_n$ , the  $n \times n$  identity matrix. The matrix  $E$  is called a **permutation matrix**.

- Show that  $EA$  is the matrix  $A$  with its rows permuted in exactly the same order that the rows of  $I_n$  were permuted to form  $E$ . *Hint:* Any permutation of rows can be written as a sequence of swaps of pairs of rows.
- Apply the result of part (a) to  $A^T$  to show that  $AE^T$  is the matrix  $A$  with its columns permuted in exactly the same order that the rows of  $I_n$  were permuted to form  $E$ .
- Explain why  $EAE^T$  is the matrix  $A$  with its rows and columns permuted in exactly the same order that the rows of  $I_n$  were permuted to form  $E$ .
- In the process of finding the canonical form of a transition matrix, does it matter whether the rows of the matrix or the columns of the matrix are permuted first? Why or why not?

### SOLUTION TO PRACTICE PROBLEM

- First note that states 1 and 3 communicate with each other, as do states 2 and 4. However, there is no way to proceed from either state 1 or state 3 to either state 2 or state 4, so the communication classes are  $\{1, 3\}$  and  $\{2, 4\}$ . Since the chain stays in class  $\{1, 3\}$  after it enters this class, class  $\{1, 3\}$  is recurrent. Likewise, there is a positive probability of leaving class  $\{2, 4\}$  at any time, so class  $\{2, 4\}$  is transient. One ordering of the states that produces a canonical form is 1, 3, 2, 4: the corresponding transition matrix is

$$P \xrightarrow[\text{columns}]{\text{rearrange}} \begin{bmatrix} 1 & 3 & 2 & 4 \\ 1/4 & 1/2 & 1/3 & 0 \\ 0 & 0 & 1/3 & 1/3 \\ 3/4 & 1/2 & 0 & 1/3 \\ 0 & 0 & 1/3 & 1/3 \end{bmatrix} \xrightarrow[\text{rows}]{\text{rearrange}} \begin{bmatrix} 1 & 3 & 2 & 4 \\ 1/4 & 1/2 & 1/3 & 0 \\ 3/4 & 1/2 & 0 & 1/3 \\ 0 & 0 & 1/3 & 1/3 \\ 0 & 0 & 1/3 & 1/3 \end{bmatrix}$$

## 10.5 THE FUNDAMENTAL MATRIX

The return time for a state in an irreducible Markov chain was defined in Section 10.3 to be the expected number of steps needed for the system to return to its starting state. This section studies the expected number of steps needed for a system to pass from one state to another state, which is called a transit time. Another quantity of interest is the probability that the system visits one state before it visits another. It is perhaps surprising that discussing these issues for irreducible Markov chains begins by working with reducible Markov chains, particularly those with transient states.

## The Fundamental Matrix and Transient States

The first goal is to compute the expected number of visits the system makes to a state  $i$  given that the system starts in state  $j$ , where  $j$  is a transient state. Suppose that a Markov chain has at least one transient state. Its transition matrix may be written in canonical form as

$$P = \left[ \begin{array}{c|c} R & S \\ \hline O & Q \end{array} \right]$$

Since at least one state is transient,  $S$  has at least one nonzero entry. In order for  $P$  to be a stochastic matrix, at least one of the columns of  $Q$  must sum to less than 1. The matrix  $Q$  is called a **substochastic matrix**. It can be shown that

$$\lim_{k \rightarrow \infty} Q^k = O$$

for any substochastic matrix  $Q$ . This fact implies that if the system is started in a transient class, it must eventually make a transition to a recurrent class and never visit any state outside that recurrent class again. The system is thus eventually **absorbed** by some recurrent class.

Now let  $j$  and  $i$  be transient states, and suppose that the Markov chain starts at state  $j$ . Let  $v_{ij}$  be the number of visits the system makes to state  $i$  before the absorption into a recurrent class. The goal is to calculate  $E[v_{ij}]$ , which is the expected value of  $v_{ij}$ . To do so, a special kind of random variable called an **indicator random variable** is useful. An **indicator random variable**  $I$  is a random variable which is 1 if an event happens and is 0 if the event does not happen. Symbolically,

$$I = \begin{cases} 0 & \text{if the event does not happen} \\ 1 & \text{if the event happens} \end{cases}$$

The expected value of an indicator random variable may be easily calculated:

$$E[I] = 0 \cdot P(I = 0) + 1 \cdot P(I = 1) = P(I = 1) = P(\text{event happens}) \quad (1)$$

Returning to the discussion of the number of visits to state  $i$  starting at state  $j$ , let  $I_k$  be the indicator random variable for the event “the system visits state  $i$  at step  $k$ .” Then

$$v_{ij} = I_0 + I_1 + I_2 + \dots = \sum_{k=0}^{\infty} I_k$$

since a visit to state  $i$  at a particular time will cause 1 to be added to the running total of visits kept in  $v_{ij}$ . Using Equation (1), the expected value of  $v_{ij}$  is

$$E[v_{ij}] = E\left[\sum_{k=0}^{\infty} I_k\right] = \sum_{k=0}^{\infty} E[I_k] = \sum_{k=0}^{\infty} P(I_k = 1) = \sum_{k=0}^{\infty} P(\text{visit to } i \text{ at step } k)$$

But  $P(\text{visit to } i \text{ at step } k)$  is just the  $(i, j)$ -entry in the matrix  $Q^k$ , so

$$E[v_{ij}] = \sum_{k=0}^{\infty} (Q^k)_{ij}$$

Thus the expected number of times that the system visits state  $i$  starting at state  $j$  is the  $(i, j)$ -entry in the matrix

$$I + Q + Q^2 + Q^3 + \dots = \sum_{k=0}^{\infty} Q^k$$



Using the argument given in Section 2.6,

$$I + Q + Q^2 + Q^3 + \dots = (I - Q)^{-1}$$

The matrix  $(I - Q)^{-1}$  is called the **fundamental matrix** of the Markov chain and is denoted by  $M$ . The interpretation of the entries in  $M$  is given in the following theorem.

### THEOREM 6

Let  $j$  and  $i$  be transient states of a Markov chain, and let  $Q$  be that portion of the transition matrix which governs movement between transient states.

- When the chain starts at a transient state  $j$ , the  $(i, j)$ -entry of  $M = (I - Q)^{-1}$  is the expected number of visits to the transient state  $i$  before absorption into a recurrent class.
- When the chain starts at a transient state  $j$ , the sum of the entries in column  $j$  of  $M = (I - Q)^{-1}$  is the expected number of time steps until absorption.

An alternative proof of Theorem 6 is given in Appendix 2.

**EXAMPLE 1** Consider an unbiased random walk on  $\{1, 2, 3, 4, 5\}$  with absorbing boundaries. If the system starts in state 3, find the expected number of visits to state 2 before absorption. Also find the expected number of steps until absorption starting at states 2, 3, and 4.

**SOLUTION** Placing the states in the order 1, 5, 2, 3, 4 produces a transition matrix in canonical form:

$$\begin{array}{ccccc} \begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \quad 5 \\ \left[ \begin{array}{ccccc} 1 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1 \end{array} \right] \end{array} & \xrightarrow[\text{columns}]{\text{rearrange}} & \begin{array}{c} 1 \quad 5 \quad 2 \quad 3 \quad 4 \\ \left[ \begin{array}{ccccc} 1 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 & 1/2 \end{array} \right] \end{array} \\ & & \xrightarrow[\text{rows}]{\text{rearrange}} & \begin{array}{c} 1 \quad 5 \quad 2 \quad 3 \quad 4 \\ \left[ \begin{array}{cc|cc} 1 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 \\ \hline 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 0 \end{array} \right] \end{array} \end{array}$$

The matrix  $Q$  and the fundamental matrix  $M = (I - Q)^{-1}$  are

$$Q = \begin{array}{c} \begin{array}{ccc} 2 & 3 & 4 \\ \left[ \begin{array}{ccc} 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{array} \right] \end{array} \begin{array}{c} 2 \\ 3 \\ 4 \end{array} \quad \text{and} \quad M = \begin{array}{c} \begin{array}{ccc} 2 & 3 & 4 \\ \left[ \begin{array}{ccc} 3/2 & 1 & 1/2 \\ 1 & 2 & 1 \\ 1/2 & 1 & 3/2 \end{array} \right] \end{array} \begin{array}{c} 2 \\ 3 \\ 4 \end{array} \end{array}$$

Starting at state 3, the expected number of visits to state 2 until absorption is the entry of  $M$  whose row corresponds to state 2 and whose column corresponds to state 3. This value is 1, so the chain will visit state 2 once on the average before being absorbed. The sum of the column of  $M$  corresponding to state 2 (or state 4) is 3, so the expected number of steps until absorption is three if starting at state 2 (or state 4). Likewise, the expected number of steps until absorption starting at state 3 is four. ■

## Transit Times

Consider the problem of calculating the expected number of steps  $t_{ji}$  needed to travel from state  $j$  to state  $i$  in an irreducible Markov chain. If the states  $i$  and  $j$  are the same state, the value  $t_{jj}$  is the expected return time to state  $j$  found in Section 10.4. The value  $t_{ji}$  will be called the **transit time** (or **mean first passage time**) from state  $j$  to state  $i$ . Surprisingly, the insight into transient states provided by Theorem 6 is exactly what is needed to calculate  $t_{ji}$ .

Finding the transit time of a Markov chain from state  $j$  to state  $i$  begins by changing the transition matrix  $P$  for the chain. First reorder the states so that state  $i$  comes first. The new matrix has the form

$$\left[ \begin{array}{c|c} p_{ii} & S \\ \hline X & Q \end{array} \right]$$

for some matrices  $S$ ,  $X$ , and  $Q$ . Next change the first column of the matrix from  $\begin{bmatrix} p_{ii} \\ X \end{bmatrix}$  to  $\begin{bmatrix} 1 \\ O \end{bmatrix}$ , where  $O$  is a zero vector of appropriate size. In terms of probabilities, it is now impossible to leave state  $i$  after entering it. State  $i$  is now an absorbing state for the Markov chain, and the transition matrix now has the form

$$\left[ \begin{array}{c|c} 1 & S \\ \hline O & Q \end{array} \right]$$

The expected number of steps  $t_{ji}$  that it takes to reach state  $i$  after starting at state  $j$  may be calculated using Theorem 6(b): it will be the sum of the column of  $M$  corresponding to state  $j$ .

**EXAMPLE 2** Consider an unbiased random walk on  $\{1, 2, 3, 4, 5\}$  with reflecting boundaries. Find the expected number of steps  $t_{j4}$  required to get to state 4 starting at any state  $j \neq 4$  of the chain.

**SOLUTION** The transition matrix for this Markov chain is

$$P = \begin{bmatrix} 0 & 1/2 & 0 & 0 & 0 \\ 1 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1 \\ 0 & 0 & 0 & 1/2 & 0 \end{bmatrix}$$

First reorder the states to list state 4 first, then convert state 4 to an absorbing state.

$$\begin{array}{ccccc} \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ \begin{bmatrix} 0 & 1/2 & 0 & 0 & 0 \\ 1 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1 \\ 0 & 0 & 0 & 1/2 & 0 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \end{matrix} & \xrightarrow[\text{columns}]{\text{rearrange}} & \begin{matrix} 4 & 1 & 2 & 3 & 5 \\ \begin{bmatrix} 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1 \\ 1/2 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \end{matrix} \\ & & \xrightarrow[\text{rows}]{\text{rearrange}} & \begin{matrix} 4 & 1 & 2 & 3 & 5 \\ \begin{bmatrix} 0 & 0 & 0 & 1/2 & 1 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{matrix} 4 \\ 1 \\ 2 \\ 3 \\ 5 \end{matrix} \end{matrix} \end{array}$$

$$\xrightarrow[\text{state 4}]{\text{convert}} \begin{array}{c|ccccc} & 4 & 1 & 2 & 3 & 5 \\ \hline 4 & 1 & 0 & 0 & 1/2 & 1 \\ 1 & 0 & 0 & 1/2 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1/2 & 0 \\ 3 & 0 & 0 & 1/2 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 \end{array}$$

The matrix  $Q$  and the fundamental matrix  $M = (I - Q)^{-1}$  are

$$Q = \begin{array}{c|cccc} & 1 & 2 & 3 & 5 \\ \hline 1 & 0 & 1/2 & 0 & 0 \\ 2 & 1 & 0 & 1/2 & 0 \\ 3 & 0 & 1/2 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 \end{array} \quad \text{and} \quad M = \begin{array}{c|cccc} & 1 & 2 & 3 & 5 \\ \hline 1 & 3 & 2 & 1 & 0 \\ 2 & 4 & 4 & 2 & 0 \\ 3 & 2 & 2 & 2 & 0 \\ 5 & 0 & 0 & 0 & 1 \end{array}$$

Summing the columns of  $M$  gives  $t_{14} = 9$ ,  $t_{24} = 8$ ,  $t_{34} = 5$ , and  $t_{54} = 1$ . ■

## Absorption Probabilities

Suppose that a Markov chain has more than one recurrent class and at least one transient state  $j$ . If the chain starts at state  $j$ , then the chain will eventually be absorbed into one of the recurrent classes; the probability that the chain is absorbed into a particular recurrent class is called the **absorption probability** for that recurrent class. The fundamental matrix is used in calculating the absorption probabilities.

To calculate the absorption probabilities, begin by changing the transition matrix for the Markov chain. First write all recurrent classes as single states  $i$  with  $p_{ii} = 1$ ; that is, each recurrent class coalesces into an absorbing state. (Exercises 37 and 38 explore the information that the absorption probabilities give for recurrent classes with more than one state.) A canonical form for this altered transition matrix is

$$P = \left[ \begin{array}{c|c} I & S \\ \hline O & Q \end{array} \right]$$

where the identity matrix describes the lack of movement between the absorbing states.

Let  $j$  be a transient state and let  $i$  be an absorbing state for the changed Markov chain; to find the probability that the chain starting at  $j$  is eventually absorbed by  $i$ , consider the  $(i, j)$ -entry in the matrix  $P^k$ . This entry is the probability that a system which starts at state  $j$  is at state  $i$  after  $k$  steps. Since  $i$  is an absorbing state, in order for the system to be at state  $i$ , the system must have been absorbed by state  $i$  at some step at or before the  $k^{\text{th}}$  step. Thus the probability that the system has been absorbed by state  $i$  at or before the  $k^{\text{th}}$  step is just the  $(i, j)$ -entry in the matrix  $P^k$ , and the probability that the chain starting at  $j$  is eventually absorbed by  $i$  is the  $(i, j)$ -entry in  $\lim_{k \rightarrow \infty} P^k$ .

Computing  $P^k$  using rules for multiplying partitioned matrices (see Section 2.4) gives

$$P^2 = \left[ \begin{array}{c|c} I & S + SQ \\ \hline O & Q^2 \end{array} \right], \quad P^3 = \left[ \begin{array}{c|c} I & S + SQ + SQ^2 \\ \hline O & Q^3 \end{array} \right]$$

and it may be proved by induction (Exercise 39) that

$$P^k = \left[ \begin{array}{c|c} I & S_k \\ \hline O & Q^k \end{array} \right]$$

where

$$\begin{aligned} S_k &= S + SQ + SQ^2 + \dots + SQ^{k-1} \\ &= S(I + Q + Q^2 + \dots + Q^{k-1}) \end{aligned}$$

Since  $j$  is a transient state and  $i$  is an absorbing state, only the entries in  $S_k$  need be considered. The probability that the chain starting at  $j$  is eventually absorbed by  $i$  may thus be found by investigating the matrix

$$\begin{aligned} A &= \lim_{k \rightarrow \infty} S_k = \lim_{k \rightarrow \infty} S(I + Q + Q^2 + \dots + Q^{k-1}) \\ &= S(I + Q + Q^2 + \dots) = SM \end{aligned}$$

where  $M$  is the fundamental matrix for the Markov chain with coalesced recurrent classes. The  $(i, j)$ -entry in  $A$  is the probability that the chain starting at  $j$  is eventually absorbed by  $i$ . The following theorem summarizes these ideas; an alternative proof is given in Appendix 2.

### THEOREM 7

Suppose that the recurrent classes of a Markov chain are all absorbing states. Let  $j$  be a transient state and let  $i$  be an absorbing state of this chain. Then the probability that the Markov chain starting at state  $j$  is eventually absorbed by state  $i$  is the  $(i, j)$ -element of the matrix  $A = SM$ , where  $M$  is the fundamental matrix of the Markov chain and  $S$  is that portion of the transition matrix that governs movement from transient states to absorbing states.

**EXAMPLE 3** Consider the unbiased random walk on  $\{1, 2, 3, 4, 5\}$  with absorbing boundaries studied in Example 1. Find the probability that the chain is absorbed into state 1 given that the chain starts at state 4.

**SOLUTION** Placing the states in the order  $\{1, 5, 2, 3, 4\}$  gives the canonical form of the transition matrix:

$$\begin{array}{c} \begin{array}{cc|ccc} & \begin{array}{c} 1 \quad 5 \end{array} & \begin{array}{c} 2 \quad 3 \quad 4 \end{array} \\ \begin{array}{c} 1 \\ 5 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 0 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 5 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

The matrix  $Q$  and the fundamental matrix  $M = (I - Q)^{-1}$  are

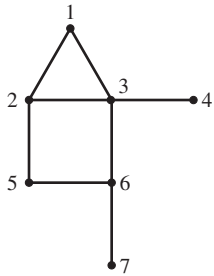
$$Q = \begin{array}{c} \begin{array}{ccc} & \begin{array}{c} 2 \quad 3 \quad 4 \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{bmatrix} \end{array} \quad \text{and} \quad M = \begin{array}{c} \begin{array}{ccc} & \begin{array}{c} 2 \quad 3 \quad 4 \end{array} \\ \begin{array}{c} 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 3/2 & 1 & 1/2 \\ 1 & 2 & 1 \\ 1/2 & 1 & 3/2 \end{bmatrix} \end{array} \end{array}$$

so

$$A = SM = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 3/2 & 1 & 1/2 \\ 1 & 2 & 1 \\ 1/2 & 1 & 3/2 \end{bmatrix} = \begin{bmatrix} 3/4 & 1/2 & 1/4 \\ 1/4 & 1/2 & 3/4 \end{bmatrix} \begin{array}{c} 1 \\ 5 \end{array}$$

The columns of  $A$  correspond to the transient states 2, 3, and 4 in that order, while the rows correspond to the absorbing states 1 and 5. The probability that the chain which started at state 4 is absorbed at state 1 is  $1/4$ . ■

Absorption probabilities may be used to compute the probability that a system modeled by an irreducible Markov chain visits one state before another.



**FIGURE 1**  
The graph for Example 4.

**EXAMPLE 4** Consider a simple random walk on the graph in Figure 1. What is the probability that a walker starting at state 1 visits state 4 before visiting state 7?

**SOLUTION** Changing state 4 and state 7 to absorbing states and then computing the absorption probabilities starting at state 1 will answer this question. Begin by reordering the states as 4, 7, 1, 2, 3, 5, 6 and rewrite states 4 and 7 as absorbing states:

$$\begin{array}{c}
 \begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \\ \left[ \begin{array}{ccccccc} 0 & 1/3 & 1/4 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/4 & 0 & 1/2 & 0 & 0 \\ 1/2 & 1/3 & 0 & 1 & 0 & 1/3 & 0 \\ 0 & 0 & 1/4 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 1/4 & 0 & 1/2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1/3 & 0 \end{array} \right] \end{array} \\
 \xrightarrow[\text{columns}]{\text{rearrange}} \\
 \begin{array}{c} 4 \quad 7 \quad 1 \quad 2 \quad 3 \quad 5 \quad 6 \\ \left[ \begin{array}{ccccccc} 0 & 0 & 0 & 1/3 & 1/4 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/4 & 1/2 & 0 \\ 1 & 0 & 1/2 & 1/3 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 0 & 1/4 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 \end{array} \right] \end{array} \\
 \xrightarrow[\text{rows}]{\text{rearrange}} \\
 \begin{array}{c} 4 \quad 7 \quad 1 \quad 2 \quad 3 \quad 5 \quad 6 \\ \left[ \begin{array}{ccccccc} 0 & 0 & 0 & 0 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1/3 & 1/4 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/4 & 1/2 & 0 \\ 1 & 0 & 1/2 & 1/3 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1/3 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 0 & 1/4 & 1/2 & 0 \end{array} \right] \end{array} \\
 \xrightarrow[\text{states 4 and 7}]{\text{convert}} \\
 \begin{array}{c} 4 \quad 7 \quad 1 \quad 2 \quad 3 \quad 5 \quad 6 \\ \left[ \begin{array}{ccccccc} 1 & 0 & 0 & 0 & 1/4 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1/3 & 1/4 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/4 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/3 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1/3 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 1/4 & 1/2 & 0 \end{array} \right] \end{array}
 \end{array}$$

The resulting transition matrix is  $\left[ \begin{array}{c|c} I & S \\ \hline O & Q \end{array} \right]$ , with

$$S = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 5 & 6 \\ \left[ \begin{array}{ccccc} 0 & 0 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/3 \end{array} \right] \end{array} \begin{array}{c} 4 \\ 7 \end{array}$$

and

$$Q = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} 0 & 1/3 & 1/4 & 0 & 0 \\ 1/2 & 0 & 1/4 & 1/2 & 0 \\ 1/2 & 1/3 & 0 & 0 & 1/3 \\ 0 & 1/3 & 0 & 0 & 1/3 \\ 0 & 0 & 1/4 & 1/2 & 0 \end{bmatrix} \end{matrix}$$

so

$$M = (I - Q)^{-1} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} 12/5 & 8/5 & 6/5 & 6/5 & 4/5 \\ 12/5 & 31/10 & 17/10 & 11/5 & 13/10 \\ 12/5 & 34/15 & 38/15 & 28/15 & 22/15 \\ 6/5 & 22/15 & 14/15 & 34/15 & 16/15 \\ 6/5 & 13/10 & 11/10 & 8/5 & 19/10 \end{bmatrix} \end{matrix}$$

and

$$A = SM = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 5 & 6 \end{matrix} \\ \begin{matrix} 4 \\ 7 \end{matrix} & \begin{bmatrix} 3/5 & 17/30 & 19/30 & 7/15 & 11/30 \\ 2/5 & 13/30 & 11/30 & 8/15 & 19/30 \end{bmatrix} \end{matrix}$$

Since the first column of  $A$  corresponds to state 1 and the rows correspond to states 4 and 7, respectively, the probability of visiting 4 before visiting 7 is  $3/5$ . ■

A mathematical model that uses Theorems 6 and 7 appears in Section 10.6.

### PRACTICE PROBLEMS

1. Consider a Markov chain on  $\{1, 2, 3, 4\}$  with transition matrix

$$P = \begin{bmatrix} 1 & 1/2 & 0 & 0 \\ 0 & 1/6 & 1/2 & 0 \\ 0 & 1/3 & 1/6 & 0 \\ 0 & 0 & 1/3 & 1 \end{bmatrix}$$

- If the Markov chain starts at state 2, find the expected number of steps before the chain is absorbed.
- If the Markov chain starts at state 2, find the probability that the chain is absorbed at state 1.

2. Consider a Markov chain on  $\{1, 2, 3, 4\}$  with transition matrix

$$P = \begin{bmatrix} 2/3 & 1/2 & 0 & 0 \\ 1/3 & 1/6 & 1/2 & 0 \\ 0 & 1/3 & 1/6 & 1/2 \\ 0 & 0 & 1/3 & 1/2 \end{bmatrix}$$

- If the Markov chain starts at state 2, find the expected number of steps required to reach state 4.
- If the Markov chain starts at state 2, find the probability that state 1 is reached before state 4.

## 10.5 EXERCISES

In Exercises 1–3, find the fundamental matrix of the Markov chain with the given transition matrix. Assume that the state space in each case is  $\{1, 2, \dots, n\}$ . If reordering of states is necessary, list the order in which the states have been reordered.

1. 
$$\begin{bmatrix} 1 & 0 & 1/6 & 0 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1/2 & 0 \end{bmatrix}$$

2. 
$$\begin{bmatrix} 1 & 0 & 0 & 1/4 & 1/5 \\ 0 & 1 & 0 & 1/8 & 1/10 \\ 0 & 0 & 1 & 1/8 & 1/5 \\ 0 & 0 & 0 & 1/4 & 3/10 \\ 0 & 0 & 0 & 1/4 & 1/5 \end{bmatrix}$$

3. 
$$\begin{bmatrix} 1/5 & 0 & 1/10 & 0 & 1/5 \\ 1/5 & 1 & 1/5 & 0 & 1/5 \\ 1/5 & 0 & 1/5 & 0 & 1/4 \\ 1/5 & 0 & 1/4 & 1 & 1/10 \\ 1/5 & 0 & 1/4 & 0 & 1/4 \end{bmatrix}$$

In Exercises 4–6, find the matrix  $A = \lim_{n \rightarrow \infty} S_n$  for the Markov chain with the given transition matrix. Assume that the state space in each case is  $\{1, 2, \dots, n\}$ . If reordering of states is necessary, list the order in which the states have been reordered.

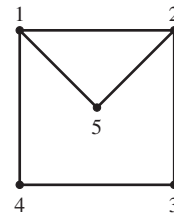
4. 
$$\begin{bmatrix} 1 & 0 & 1/6 & 0 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1/2 & 0 \end{bmatrix}$$

5. 
$$\begin{bmatrix} 1 & 0 & 0 & 1/4 & 1/5 \\ 0 & 1 & 0 & 1/8 & 1/10 \\ 0 & 0 & 1 & 1/8 & 1/5 \\ 0 & 0 & 0 & 1/4 & 3/10 \\ 0 & 0 & 0 & 1/4 & 1/5 \end{bmatrix}$$

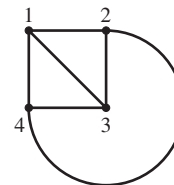
6. 
$$\begin{bmatrix} 1/5 & 0 & 1/10 & 0 & 1/5 \\ 1/5 & 1 & 1/5 & 0 & 1/5 \\ 1/5 & 0 & 1/5 & 0 & 1/4 \\ 1/5 & 0 & 1/4 & 1 & 1/10 \\ 1/5 & 0 & 1/4 & 0 & 1/4 \end{bmatrix}$$

7. Suppose that the Markov chain in Exercise 1 starts at state 3. How many visits will the chain make to state 4 on average before absorption?
8. Suppose that the Markov chain in Exercise 2 starts at state 4. How many steps will the chain take on average before absorption?

9. Suppose that the Markov chain in Exercise 3 starts at state 1. How many steps will the chain take on average before absorption?
10. Suppose that the Markov chain in Exercise 4 starts at state 3. What is the probability that the chain is absorbed at state 1?
11. Suppose that the Markov chain in Exercise 5 starts at state 4. Find the probabilities that the chain is absorbed at states 1, 2, and 3.
12. Suppose that the Markov chain in Exercise 6 starts at state 5. Find the probabilities that the chain is absorbed at states 2 and 4.
13. Consider a simple random walk on the following graph.

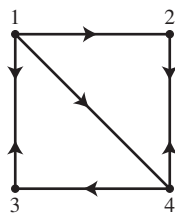


- a. Suppose that the walker begins in state 5. What is the expected number of visits to state 2 before the walker visits state 1?
- b. Suppose again that the walker begins in state 5. What is the expected number of steps until the walker reaches state 1?
- c. Now suppose that the walker starts in state 1. What is the probability that the walker reaches state 5 before reaching state 2?
14. Consider a simple random walk on the following graph.



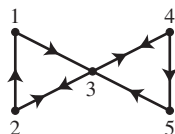
- a. Suppose that the walker begins in state 3. What is the expected number of visits to state 2 before the walker visits state 1?
- b. Suppose again that the walker begins in state 3. What is the expected number of steps until the walker reaches state 1?
- c. Now suppose that the walker starts in state 1. What is the probability that the walker reaches state 3 before reaching state 2?

15. Consider a simple random walk on the following directed graph. Suppose that the walker starts at state 1.



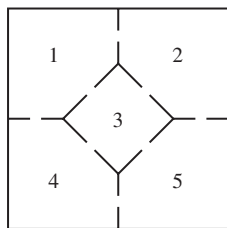
- How many visits to state 2 does the walker expect to make before visiting state 3?
- How many steps does the walker expect to take before visiting state 3?

16. Consider a simple random walk on the following directed graph. Suppose that the walker starts at state 4.



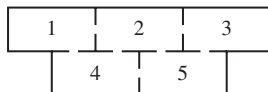
- How many visits to state 3 does the walker expect to make before visiting state 2?
- How many steps does the walker expect to take before visiting state 2?

17. Consider the mouse in the following maze from Section 10.1, Exercise 17.



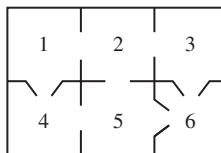
If the mouse starts in room 2, what is the probability that the mouse visits room 3 before visiting room 4?

18. Consider the mouse in the following maze from Section 10.1, Exercise 18.



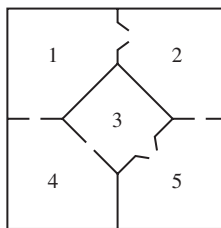
If the mouse starts in room 1, what is the probability that the mouse visits room 3 before visiting room 4?

19. Consider the mouse in the following maze from Section 10.1, Exercise 19.



If the mouse starts in room 1, how many steps on average will it take the mouse to get to room 6?

20. Consider the mouse in the following maze from Section 10.1, Exercise 20.



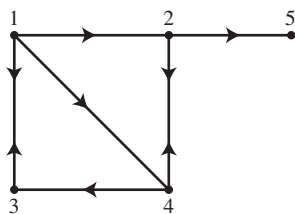
If the mouse starts in room 1, how many steps on average will it take the mouse to get to room 5?

In Exercises 21 and 22, mark each statement True or False. Justify each answer.

- The  $(i, j)$ -element in the fundamental matrix  $M$  is the expected number of visits to the transient state  $j$  prior to absorption, starting at the transient state  $i$ .
  - The  $(j, i)$ -element in the fundamental matrix gives the expected number of visits to state  $i$  prior to absorption, starting at state  $j$ .
  - The probability that the Markov chain starting at state  $i$  is eventually absorbed by state  $j$  is the  $(j, i)$ -element of the matrix  $A = SM$ , where  $M$  is the fundamental matrix of the Markov chain and  $S$  is that portion of the transition matrix that governs movement from transient states to absorbing states.
- The sum of the column  $j$  of the fundamental matrix  $M$  is the expected number of time steps until absorption.
  - Transit times may be computed directly from the entries in the transition matrix.
  - If  $A$  is an  $m \times m$  substochastic matrix, then the entries in  $A^n$  approach 0 as  $n$  increases.
- Suppose that the weather in Charlotte is modeled using the Markov chain in Section 10.1, Exercise 23. If it is sunny today, what is the probability that the weather will be cloudy before it is rainy?
- Suppose that the weather in Charlotte is modeled using the Markov chain in Section 10.1, Exercise 24. If it rained yesterday and today, how many days on average will it take before there are two consecutive days with no rain?

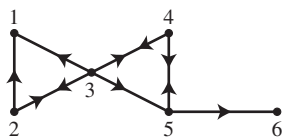


25. Consider a set of webpages hyperlinked by the given directed graph that was studied in Section 10.2, Exercise 25.



If a random surfer starts on page 1, how many mouse clicks on average will the surfer make before becoming stuck at a dangling node?

26. Consider a set of webpages hyperlinked by the given directed graph that was studied in Section 10.2, Exercise 26.



If a random surfer starts on page 3, what is the probability that the surfer will eventually become stuck on page 1, which is a dangling node?

Exercises 27–30 concern the Markov chain model for scoring a tennis match described in Section 10.1, Exercise 31. Suppose that players A and B are playing a tennis match, that the probability that player A wins any point is  $p = .6$ , and that the game is currently at “deuce.”

27. How many more points will the tennis game be expected to last?
28. Find the probability that player A wins the game.
29. Repeat Exercise 27 if the game is
- currently at “advantage A.”
  - currently at “advantage B.”
30. Repeat Exercise 28 if the game is
- currently at “advantage A.”
  - currently at “advantage B.”

Exercises 31–36 concern the two Markov chain models for scoring volleyball games described in Section 10.1, Exercise 32. Suppose that teams A and B are playing a 15-point volleyball game which is tied 15-15 with team A serving. Suppose that the probability  $p$  that team A wins any rally for which it serves is  $p = .7$ , and the probability  $q$  that team B wins any rally for which it serves is  $q = .6$ .

31. Suppose that rally point scoring is being used. How many more rallies will the volleyball game be expected to last?
32. Suppose that rally point scoring is being used. Find the probability that team A wins the game.
33. Suppose that side out scoring is being used. How many more rallies will the volleyball game be expected to last?

34. Suppose that side out scoring is being used. Find the probability that team A wins the game.

35. Rally point scoring was introduced to make volleyball matches take less time. Considering the results of Exercises 31 and 33, does using rally point scoring really lead to fewer rallies being played?

36. Since  $p = .7$  and  $q = .6$ , it seems that team A is the dominant team. Does it really matter which scoring system is chosen? Should the manager of each team have a preference?

37. Consider a Markov chain on  $\{1, 2, 3, 4, 5\}$  with transition matrix

$$P = \begin{bmatrix} 1/4 & 1/2 & 1/3 & 0 & 1/4 \\ 3/4 & 1/2 & 0 & 1/3 & 1/4 \\ 0 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/2 \end{bmatrix}$$

Find  $\lim_{n \rightarrow \infty} P^n$  by the following steps.

- What are the recurrent and transient classes for this chain?
  - Find the limiting matrix for each recurrent class.
  - Determine the long-range probabilities for the Markov chain starting from each transient state.
  - Use the results of parts (b) and (c) to find  $\lim_{n \rightarrow \infty} P^n$ .
  - Confirm your answer in part (d) by taking  $P$  to a high power.
38. Consider a Markov chain on  $\{1, 2, 3, 4, 5, 6\}$  with transition matrix

$$P = \begin{bmatrix} 1/3 & 1/2 & 0 & 0 & 1/2 & 0 \\ 2/3 & 1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/4 & 2/3 & 0 & 1/2 \\ 0 & 0 & 3/4 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/4 & 1/4 \\ 0 & 0 & 0 & 0 & 1/4 & 1/4 \end{bmatrix}$$

Find  $\lim_{n \rightarrow \infty} P^n$  by the following steps.

- What are the recurrent and transient classes for this chain?
- Find the limiting matrix for each recurrent class.
- Find the absorption probabilities from each transient state into each recurrent class.
- Use the results of parts (b) and (c) to find  $\lim_{n \rightarrow \infty} P^n$ .
- Confirm your answer in part (d) by taking  $P$  to a high power.

39. Show that if  $P = \begin{bmatrix} I & S \\ O & Q \end{bmatrix}$ , then  $P^k = \begin{bmatrix} I & S_k \\ O & Q^k \end{bmatrix}$ ,

where

$$\begin{aligned} S_k &= S + SQ + SQ^2 + \dots + SQ^{k-1} \\ &= S(I + Q + Q^2 + \dots + Q^{k-1}). \end{aligned}$$

## SOLUTIONS TO PRACTICE PROBLEMS

1. a. Since states 1 and 4 are absorbing states, reordering the states as  $\{1, 4, 2, 3\}$  produces the canonical form

$$P = \begin{array}{c} \begin{array}{cccc} & \begin{smallmatrix} 1 & 4 & 2 & 3 \end{smallmatrix} \\ \begin{smallmatrix} 1 \\ 4 \\ 2 \\ 3 \end{smallmatrix} & \begin{bmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1/6 & 1/2 \\ 0 & 0 & 1/3 & 1/6 \end{bmatrix} \end{array} \end{array}$$

So

$$Q = \begin{array}{c} \begin{array}{cc} & \begin{smallmatrix} 2 & 3 \end{smallmatrix} \\ \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} & \begin{bmatrix} 1/6 & 1/2 \\ 1/3 & 1/6 \end{bmatrix} \end{array} \quad \text{and} \quad M = \begin{array}{c} \begin{array}{cc} & \begin{smallmatrix} 2 & 3 \end{smallmatrix} \\ \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} & \begin{bmatrix} 30/19 & 18/19 \\ 12/19 & 30/19 \end{bmatrix} \end{array} \end{array}$$

The expected number of steps needed when starting at state 2 before the chain is absorbed is the sum of the entries in the column of  $M$  corresponding to state 2, which is

$$\frac{30}{19} + \frac{12}{19} = \frac{42}{19}$$

- b. Using the canonical form of the transition matrix, we see that

$$S = \begin{array}{c} \begin{array}{cc} & \begin{smallmatrix} 2 & 3 \end{smallmatrix} \\ \begin{smallmatrix} 1 \\ 4 \end{smallmatrix} & \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} \end{array} \quad \text{and} \quad A = SM = \begin{array}{c} \begin{array}{cc} & \begin{smallmatrix} 2 & 3 \end{smallmatrix} \\ \begin{smallmatrix} 1 \\ 4 \end{smallmatrix} & \begin{bmatrix} 15/19 & 9/19 \\ 4/19 & 10/19 \end{bmatrix} \end{array} \end{array}$$

The probability that the chain is absorbed at state 1 given that the Markov chain starts at state 2 is the entry in  $A$  whose row corresponds to state 1 and whose column corresponds to state 2; this entry is  $15/19$ .

2. a. Reorder the states as  $\{4, 1, 2, 3\}$  and make state 4 into an absorbing state to produce the canonical form

$$P = \begin{array}{c} \begin{array}{cccc} & \begin{smallmatrix} 4 & 1 & 2 & 3 \end{smallmatrix} \\ \begin{smallmatrix} 4 \\ 1 \\ 2 \\ 3 \end{smallmatrix} & \begin{bmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 2/3 & 1/2 & 0 \\ 0 & 1/3 & 1/6 & 1/2 \\ 0 & 0 & 1/3 & 1/6 \end{bmatrix} \end{array} \end{array}$$

So

$$Q = \begin{array}{c} \begin{array}{ccc} & \begin{smallmatrix} 1 & 2 & 3 \end{smallmatrix} \\ \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} & \begin{bmatrix} 2/3 & 1/2 & 0 \\ 1/3 & 1/6 & 1/2 \\ 0 & 1/3 & 1/6 \end{bmatrix} \end{array} \quad \text{and} \quad M = \begin{array}{c} \begin{array}{ccc} & \begin{smallmatrix} 1 & 2 & 3 \end{smallmatrix} \\ \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} & \begin{bmatrix} 14.25 & 11.25 & 6.75 \\ 7.50 & 7.50 & 4.50 \\ 3.00 & 3.00 & 3.00 \end{bmatrix} \end{array} \end{array}$$

The expected number of steps required to reach state 4, starting at state 2, is the sum of the entries in the column of  $M$  corresponding to state 2, which is

$$11.25 + 7.50 + 3.00 = 21.75$$

- b. Make states 1 and 4 into absorbing states and reorder the states as  $\{1, 4, 2, 3\}$  to produce the canonical form

$$P = \begin{array}{c} \begin{array}{cccc} & \overset{1}{1} & \overset{4}{4} & \overset{2}{2} & \overset{3}{3} \\ \begin{array}{l} 1 \\ 4 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1/6 & 1/2 \\ 0 & 0 & 1/3 & 1/6 \end{bmatrix} \end{array} \end{array}$$

So

$$Q = \begin{array}{c} \begin{array}{cc} \overset{2}{2} & \overset{3}{3} \\ \begin{array}{l} 2 \\ 3 \end{array} & \begin{bmatrix} 1/6 & 1/2 \\ 1/3 & 1/6 \end{bmatrix} \end{array}, \quad M = \begin{array}{c} \begin{array}{cc} \overset{2}{2} & \overset{3}{3} \\ \begin{array}{l} 2 \\ 3 \end{array} & \begin{bmatrix} 30/19 & 18/19 \\ 12/19 & 30/19 \end{bmatrix} \end{array}, \\ S = \begin{array}{c} \begin{array}{cc} \overset{2}{2} & \overset{3}{3} \\ \begin{array}{l} 1 \\ 4 \end{array} & \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} \end{array}, \quad \text{and} \quad A = SM = \begin{array}{c} \begin{array}{cc} \overset{2}{2} & \overset{3}{3} \\ \begin{array}{l} 1 \\ 4 \end{array} & \begin{bmatrix} 15/19 & 9/19 \\ 4/19 & 10/19 \end{bmatrix} \end{array}$$

Thus the probability that, starting at state 2, state 1 is reached before state 4 is the entry in  $A$  whose row corresponds to state 1 and whose column corresponds to state 2; this entry is  $15/19$ .

## 10.6 MARKOV CHAINS AND BASEBALL STATISTICS

Markov chains are used to model a wide variety of systems. The examples and exercises in this chapter have shown how Markov chains may be used to model various situations. The final example to be explored is a model for how runners proceed around the bases in baseball. This model leads to useful measures of expected run production both for a team and for individual players.

### Baseball Modeled by a Markov Chain

Many baseball fans carefully study the statistics of their favorite teams. The teams themselves use baseball statistics for individual players to determine strategy during games, and to make hiring decisions.<sup>4</sup> This section shows how a Markov chain is used to predict the number of earned runs a team will score and to compare the offensive abilities of different players. Some exercises suggest how to use Markov chains to investigate matters of baseball strategy, such as deciding whether to attempt a sacrifice or a steal.

The Markov chain in this section provides a way to analyze how runs are scored during one half-inning of a baseball game. The states of the chain are the various configurations of runners on bases and the number of outs. See Table 1.

The first state in the left column of Table 1 (“no bases occupied, 0 outs”) is the initial state of the chain, when the baseball half-inning begins (that is, when one team becomes the team “at bat”). The four states in the far right column describe the various ways the half-inning can end (when the third out occurs and the teams trade places). Physically, the half-inning is completed when the third out occurs. Mathematically, the Markov

<sup>4</sup>The use of statistical analysis in baseball is called **sabermetrics** as a tribute to SABR, the Society for American Baseball Research. An overview of sabermetrics can be found at <http://en.wikipedia.org/wiki/Sabermetrics>.

**TABLE 1** The 28 States of a Baseball Markov Chain

Bases Occupied	Outs	State	Left on Base	Outs	State
None	0	0:0	0	3	0:3
First	0	1:0	1	3	1:3
Second	0	2:0	2	3	2:3
Third	0	3:0	3	3	3:3
First and Second	0	12:0			
First and Third	0	13:0			
Second and Third	0	23:0			
First, Second, and Third	0	123:0			
None	1	0:1			
First	1	1:1			
Second	1	2:1			
Third	1	3:1			
First and Second	1	12:1			
First and Third	1	13:1			
Second and Third	1	23:1			
First, Second, and Third	1	123:1			
None	2	0:2			
First	2	1:2			
Second	2	2:2			
Third	2	3:2			
First and Second	2	12:2			
First and Third	2	13:2			
Second and Third	2	23:2			
First, Second, and Third	2	123:2			

chain continues in one of the four “final” states. (The model only applies to a game in which each half-inning is completed.) So, each of these four states is an absorbing state of the chain. The other 24 states are transient states, because whenever an out is made, the states with fewer outs can never occur again.

The Markov chain moves from state to state because of the actions of the batters. The transition probabilities of the chain are the probabilities of possible outcomes of a batter’s action. For a Markov chain, the transition probabilities must remain the same from batter to batter, so the model does not allow for variations among batters. This assumption means that each batter for a team hits as an “average batter” for the team.<sup>5</sup>

The model also assumes that only the batter determines how the runners move around the bases. This means that stolen bases, wild pitches, and passed balls are not considered. Also, errors by the players in the field are not allowed, so the model only calculates earned runs—runs that are scored without the benefit of fielding errors. Finally, the model considers only seven possible outcomes at the plate: a single (arriving

<sup>5</sup>This unrealistic assumption can be overcome by using a more complicated model which uses different transition matrices for each batter. Nevertheless, the model presented here can lead to useful information about the team. Later in the section, the model will be used to evaluate individual players.

safely at first base and stopping there), a double (arriving safely at second base), a triple (arriving safely at third base), a home run, a walk (advancing to first base without hitting the ball), a hit batsman (a pitched ball hits the batter, and the batter advances to first base), and an “out.” Thus, the model allows no double or triple plays, no sacrifices, and no sacrifice flies. However, Markov chain models can be constructed that include some of these excluded events.<sup>6</sup>

## Constructing the Transition Matrix

The  $28 \times 28$  transition matrix for the Markov chain has the canonical form

$$P = \begin{bmatrix} I_4 & S \\ O & Q \end{bmatrix} \quad (1)$$

where  $I_4$  is the  $4 \times 4$  identity matrix (because the only recurrent states are the four absorbing states, one of which is entered when the third out occurs),  $S$  is a  $4 \times 24$  matrix, and  $Q$  is a  $24 \times 24$  substochastic matrix. The columns of  $S$  and  $Q$  correspond to the transient states, in the order shown in Table 1. The entries in  $S$  describe the transitions from the 24 transient states (with 0, 1, or 2 outs) to the absorbing states (with 3 outs). Note that the only way to enter an absorbing state is to come from a state with 2 outs. Let  $p_O$  denote the probability that the batter makes an out. Then  $S$  may be written in block form, with three  $4 \times 8$  blocks, as

$$S = \begin{bmatrix} O & O & X \end{bmatrix}$$

where  $X = \begin{bmatrix} \begin{matrix} 0:2 & 1:2 & 2:2 & 3:2 & 12:2 & 13:2 & 23:2 & 123:2 \\ p_O & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p_O & p_O & p_O & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_O & p_O & p_O & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_O \end{matrix} & \begin{matrix} 0:3 \\ 1:3 \\ 2:3 \\ 3:3 \end{matrix} \end{bmatrix} \quad (2)$

The matrix  $X$  describes the transitions from the transient states with 2 outs to the absorbing states with 3 outs. (For example, columns 2, 3, and 4 of  $X$  list the probabilities that the batter makes the third out when one runner is on one of the three bases.) The substochastic matrix  $Q$  has the following block form, with  $8 \times 8$  blocks,

$$Q = \begin{bmatrix} \begin{matrix} 0 & 1 & 2 \\ A & O & O \\ B & A & O \\ O & B & A \end{matrix} & \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} \end{bmatrix} \quad (3)$$

The labels on the rows and columns of  $Q$  represent the number of outs. The four zero blocks in  $Q$  reflect the facts that the number of outs cannot go from 1 to 0, from 2 to 0 or 1, or from 0 directly to 2 in one step. The matrix  $A$  describes how the various base configurations change when the number of outs does not change.

The entries in  $A$  and  $B$  depend on how the batter’s action at the plate affects any runners that may already be on base. The Markov chain model presented here makes the assumptions shown in Table 2. The exercises consider some alternative assumptions.

The entries in the  $8 \times 8$  matrices  $A$  and  $B$  are constructed from the probabilities of the six batting events in Table 2. Denote these probabilities by  $p_W$ ,  $p_1$ ,  $p_2$ ,  $p_3$ ,  $p_H$ , and  $p_O$ , respectively. The notation  $p_O$  was introduced earlier during the construction of the matrix  $S$ .

<sup>6</sup>Other models use “play-by-play” data. The numbers of transitions between states are counted and scaled to produce a transition matrix. For these models it does not matter *how* the runners advance, merely that they do.

**TABLE 2** Assumptions about Advancing Runners

Batting Event	Outcome
Walk or Hit Batsman	The batter advances to first base. A runner on first base advances to second base. A runner on second base advances to third base only if first base was also occupied. A runner on third base scores only if first base and second base were also occupied.
Single	The batter advances to first base. A runner on first base advances to second base. A runner on third base scores. A runner on second base advances to third base half of the time and scores half of the time.
Double	The batter advances to second base. A runner on first base advances to third base. A runner on second base scores. A runner on third base scores.
Triple	The batter advances to third base. A runner on first base scores. A runner on second base scores. A runner on third base scores.
Home Run	The batter scores. A runner on first base scores. A runner on second base scores. A runner on third base scores.
Out	No runners advance. The number of outs increases by one.

The  $8 \times 8$  matrix  $B$  involves the change of state when the number of outs increases. In this case, the configuration of runners on the bases does not change (see Table 2). So

$$B = p_O I$$

where  $I$  is the  $8 \times 8$  identity matrix.<sup>7</sup>

Matrix  $A$  concerns the situations in which the batter does not make an out and either succeeds in reaching one of the bases or hits a home run. The construction of  $A$  is discussed in Example 1 below and in the exercises. The labels on the rows and columns of  $A$  correspond to the states in Table 2. Here  $k$  is the fixed number of outs: either 0, 1, or 2.

$$A = \begin{array}{c} \begin{array}{cccccccc} & 0:k & 1:k & 2:k & 3:k & 12:k & 13:k & 23:k & 123:k \end{array} \\ \left[ \begin{array}{cccccccc} p_H & p_H & p_H & p_H & p_H & p_H & p_H & p_H \\ p_W + p_1 & 0 & .5p_1 & p_1 & 0 & 0 & .5p_1 & 0 \\ p_2 & 0 & p_2 & p_2 & 0 & 0 & p_2 & 0 \\ p_3 & p_3 & p_3 & p_3 & p_3 & p_3 & p_3 & p_3 \\ 0 & p_W + p_1 & p_W & 0 & .5p_1 & p_1 & 0 & .5p_1 \\ 0 & 0 & .5p_1 & p_W & 0 & 0 & .5p_1 & 0 \\ 0 & p_2 & 0 & 0 & p_2 & p_2 & 0 & p_2 \\ 0 & 0 & 0 & 0 & p_W + .5p_1 & p_W & p_W & p_W + .5p_1 \end{array} \right] \begin{array}{l} 0:k \\ 1:k \\ 2:k \\ 3:k \\ 12:k \\ 13:k \\ 23:k \\ 123:k \end{array} \end{array}$$

The analysis in Example 1 below requires two facts from probability theory. If an event can occur in two mutually exclusive ways, with probabilities  $p_1$  and  $p_2$ , then the probability of the event is  $p_1 + p_2$ . The probability that two independent events both occur is the product of the separate probabilities for each event.

<sup>7</sup> A batter can make an out in three ways—by striking out, by hitting a fly ball that is caught, or by hitting a ground ball that is thrown to first base before the batter arrives. When the second or third case occurs, a runner on a base sometimes can advance one base, but may also make an out and be removed from the bases. Table 2 excludes these possibilities.

**EXAMPLE 1**

- Justify the transition probabilities for the initial state “no bases occupied.”
- Justify the transition probabilities for the initial state “second base occupied.”

**SOLUTION**

- For the first column of  $A$ , the batter either advances to one of the bases or hits a home run. So the probability that the bases remain unoccupied is  $p_H$ . The batter advances to first base when the batter either walks (or is hit by a pitch) or hits a single. Since the desired outcome can be reached in two different ways, the probability of success is the sum of the two probabilities—namely,  $p_W + p_1$ . The probabilities of the batter advancing to second base or third base are, respectively,  $p_2$  and  $p_3$ . All other outcomes are impossible, because there can be at most one runner on base after one batter when the starting state has no runners on base.
- This concerns the third column of  $A$ . The initial state is  $2:k$  (a runner on second base,  $k$  outs). For entry  $(1, 3)$  of  $A$ , the probability of a transition “to state  $0:k$ ” is required. Suppose that only second base is occupied and the batter does not make an out. Only a home run will empty the bases, so the  $(1, 3)$ -entry is  $p_H$ .

Entry  $(2, 3)$ : (“to state  $1:k$ ”) To leave a player only on first base, the batter must get to first base and the player on second base must reach home plate successfully.<sup>8</sup> From Table 2, the probability of reaching home plate successfully from second base is  $.5$ . Now, assume that these two events are independent, because only the actions of the batter (and Table 2) influence the outcome. In this case, the probability of both events happening at the same time is the product of these two probabilities, so the  $(2, 3)$ -entry is  $.5p_1$ .

Entry  $(3, 3)$ : (“to state  $2:k$ ”) To leave a player only on second base, the batter must reach second base (a “double”) and the runner on second base must score. The second condition, however, is automatically satisfied because of the assumption in Table 2. So the probability of success in this case is  $p_2$ . This is the  $(3, 3)$ -entry.

Entry  $(4, 3)$ : (“to state  $3:k$ ”) By an argument similar to that for the  $(3, 3)$ -entry, the  $(4, 3)$ -entry is  $p_3$ .

Entry  $(5, 3)$ : (“to state  $12:k$ ”) To leave players on first base and second base, the batter must get to first base and the player on second base must remain there. However, from Table 2, if the batter hits a single, the runner on second base will at least get to third base. So, the only way for the desired outcome to occur is for the batter to get a walk or be hit by a pitch. The  $(5, 3)$ -entry is thus  $p_W$ .

Entry  $(6, 3)$ : (“to state  $13:k$ ”) This concerns the batter getting to first base and the runner on second base advancing to third base. This can happen only if the batter hits a single, with probability  $p_1$ , and the runner on second base stops at third base, which happens with probability  $.5$  (by Table 2). Since both events are required, the  $(6, 3)$ -entry is the product  $.5p_1$ .

Entry  $(7, 3)$ : (“to state  $23:k$ ”) To leave players on second base and third base, the batter must hit a double and the runner on second base must advance only to third base. Table 2 rules this out—when the batter hits a double, the runner on second base scores. Thus the  $(7, 3)$ -entry is zero.

Entry  $(8, 3)$ : The starting state has just one runner on base. The next state cannot have three runners on base, so the  $(8, 3)$ -entry is zero. ■

<sup>8</sup> The only other way to make the player on second base “disappear” would be for the player to be tagged “out,” but the model does not permit outs for runners on the bases.

**EXAMPLE 2** Batting statistics are often displayed as in Table 3. Use the data in Table 3 to obtain the transition probabilities for the 2002 Atlanta Braves.

**TABLE 3** Atlanta Braves Batting Statistics—2002 Season

Walks	Hit Batsman	Singles	Doubles	Triples	Home Runs	Outs
558	54	959	280	25	164	4067

**SOLUTION** The sum of the entries in Table 3 is 6107. This is the total number of times that Atlanta Braves players came to bat during the 2002 baseball season. From the first two columns, there are 612 walks or hit batsmen. So,  $p_W = 612/6107 = .1002$ . Of the 6107 times a player came to bat, a player hit a single 959 times, so  $p_1 = 959/6107 = .1570$ . Similar calculations provide  $p_2 = .0458$ ,  $p_3 = .0041$ ,  $p_H = .0269$ , and  $p_O = .660$ . These values are placed in the matrices shown above to produce the transition matrix for the Markov chain. ■

## Applying the Model

Now that the data for the stochastic matrix is available, Theorems 6 and 7 from Section 10.5 can provide information about how many earned runs to expect from the Atlanta Braves during a typical game. The goal is to calculate how many earned runs the Braves will score on average in each half-inning. First, observe that since three batters must make an out to finish one half-inning, the number of runs scored in that half-inning is given by

$$[\# \text{ of runs}] = [\# \text{ of batters}] - [\# \text{ of runners left on base}] - 3 \quad (4)$$

If  $R$  is the number of runs scored in the half-inning,  $B$  is the number of batters, and  $L$  is the number of runners left on base, Equation (4) becomes

$$R = B - L - 3 \quad (5)$$

The quantity of interest is  $E[R]$ , the expected number of earned runs scored. Properties of expected value indicate that

$$E[R] = E[B] - E[L] - 3 \quad (6)$$

Each batter moves the Markov chain ahead one step. So, the expected number of batters in a half-inning  $E[B]$  is the expected number of steps to absorption (at the third out) when the chain begins at the initial state “0 bases occupied, 0 outs.” This initial state corresponds to the fifth column of the transition matrix

$$P = \begin{bmatrix} I_4 & S \\ O & Q \end{bmatrix}$$

In baseball terms, Theorem 6 shows that

The expected number of players that bat in one half-inning is the sum of the entries in column 1 of the fundamental matrix  $M = (I - Q)^{-1}$ .

Thus  $E[B]$  may be computed. The other quantity needed in Equation (6) above is  $E[L]$ , the expected number of runners left on base in a typical half-inning. This is given by the following sum:

$$E[L] = 0 \cdot P(L = 0) + 1 \cdot P(L = 1) + 2 \cdot P(L = 2) + 3 \cdot P(L = 3) \quad (7)$$



Theorem 7 can provide this information because the recurrent classes for the chain are just the four absorbing states (at the end of the half-inning). The probabilities needed in Equation (7) are the probabilities of absorption into the four final states of the half-inning given that the initial state of the system is “0 bases occupied, 0 outs.” So the desired probabilities are in column 1 of the matrix  $SM$ , where  $M$  is the fundamental matrix of the chain and  $S = \begin{bmatrix} O & O & X \end{bmatrix}$  as in Equation (2). The probabilities can be used to calculate  $E[L]$  using Equation (7), and thus to find  $E[R]$ .

**EXAMPLE 3** When the Atlanta Braves data from Example 2 is used to construct the transition matrix (not shown here), it turns out that the sum of the first column of the fundamental matrix  $M$  is 4.5048, and the first column of the matrix  $SM$  is

$$\begin{bmatrix} .3520 \\ .3309 \\ .2365 \\ .0805 \end{bmatrix}$$

Compute the number of earned runs the Braves can expect to score per inning based on their performance in 2002. How many earned runs does the model predict for the entire season, if the Braves play  $1443\frac{2}{3}$  innings, as they did in 2002?

**SOLUTION** The first column of  $SM$  shows that, for example, the probability that the Braves left no runners on base is .3520. The expected number of runners left on base is

$$E[L] = 0(.3520) + 1(.3309) + 2(.2365) + 3(.0805) = 1.0454$$

The expected number of batters is  $E[B] = 4.5048$ , the sum of the first column of  $M$ . From Equation (6), the expected number of earned runs  $E[R]$  is

$$E[R] = E[B] - E[L] - 3 = 4.5048 - 1.0454 - 3 = .4594$$

The Markov chain model predicts that the Braves should average .4594 earned run per inning. In  $1443\frac{2}{3}$  innings, the total number of earned runs expected is

$$.4594 \times 1443.67 = 663.22$$

The actual number of earned runs for the Braves in 2002 was 636, so the model’s error is 27.22 runs, or about 4.3%. ■

Mathematical models are used by some Major League teams to compare the offensive profiles of single players. To analyze a player using the Markov chain model, use the player’s batting statistics instead of a team’s statistics. Compute the expected number of earned runs that a team of such players would score in an inning. This number is generally multiplied by 9 to yield what has been termed an “offensive earned run average.”

**EXAMPLE 4** Table 4 shows the career batting statistics for Jose Oquendo, who played for the New York Mets and St. Louis Cardinals in the 1980’s and 1990’s. Compute his offensive earned run average.

**TABLE 4** Jose Oquendo Batting Statistics

Walks	Hit Batsman	Singles	Doubles	Triples	Home Runs	Outs
448	5	679	104	24	14	2381

**SOLUTION** Construct the transition matrix from this data as described in Example 2, and then compute  $M$  and  $SM$ . The sum of the first column of  $M$  is 4.6052, so a team entirely composed of Jose Oquendos would come to bat an average of 4.6052 times per inning. That is,  $E[B] = 4.6052$ . The first column of  $SM$  is

$$\begin{bmatrix} .2844 \\ .3161 \\ .2725 \\ .1270 \end{bmatrix}$$

so the expected number of runners left on base is

$$E[L] = 0(.2844) + 1(.3161) + 2(.2725) + 3(.1270) = 1.2421$$

From Equation (6), the expected number of earned runs is

$$E[R] = E[B] - E[L] - 3 = 4.6052 - 1.2421 - 3 = .3631$$

The offensive earned run average for Jose Oquendo is  $.3631 \times 9 = 3.2679$ . This compares with an offensive earned run average of about 10 for teams composed of the greatest hitters in baseball history. See the Exercises. ■

### PRACTICE PROBLEMS

1. Let  $A$  be the matrix just before Example 1. Explain why entry (3, 6) is zero.
2. Explain why entry (6, 3) of  $A$  is  $.5p_1$ .

## 10.6 EXERCISES

In Exercises 1–6, justify the transition probabilities for the given initial states. See Example 1.

1. First base occupied
2. Third base occupied
3. First and second bases occupied
4. First and third bases occupied
5. Second and third bases occupied
6. First, second, and third bases occupied
7. Major League batting statistics for the 2006 season are shown in Table 5. Compute the transition probabilities for this data as was done in Example 2, and find the matrix  $A$  for this data.
8. Find the complete transition matrix for the model using the Major League data in Table 5.

9. It can be shown that the sum of the first column of  $M$  for the 2006 Major League data is 4.53933, and that the first column of  $SM$  for the 2006 Major League data is

$$\begin{bmatrix} .34973 \\ .33414 \\ .23820 \\ .07793 \end{bmatrix}$$

Find the expected number of earned runs per inning in a Major League game in 2006.

10. The number of innings batted in the Major Leagues in the 2006 season was 43,257, and the number of earned runs scored was 21,722. What is the total number of earned runs scored for the season predicted by the model, and how does it compare with the actual number of earned runs scored?

**TABLE 5** Major League Batting Statistics—2006 Season

Walks	Hit Batsman	Singles	Doubles	Triples	Home Runs	Outs
15,847	1817	29,600	9135	952	5386	122,268

TABLE 6 Batting Statistics for Leading Batters

Name	Walks	Hit Batsman	Singles	Doubles	Triples	Home Runs	Outs
Barry Bonds	2558	106	1495	601	77	762	6912
Babe Ruth	2062	43	1517	506	136	714	5526
Ted Williams	2021	39	1537	525	71	521	5052

11. Batting statistics for three of the greatest batters in Major League history are shown in Table 6. Compute the transition probabilities for this data for each player.
12. The sums of the first columns of  $M$  for the player data in Table 6 and the first columns of  $SM$  for the player data in Table 6 given in Table 7. Find and compare the offensive earned run averages of these players. Which batter does the model say was the best of these three?

TABLE 7 Model Information for Batting Statistics

	Sum of First Columns of $M$	First Column of $SM$
Barry Bonds	5.43012	$\begin{bmatrix} .281776 \\ .292658 \\ .258525 \\ .167041 \end{bmatrix}$
Babe Ruth	5.70250	$\begin{bmatrix} .268150 \\ .295908 \\ .268120 \\ .167822 \end{bmatrix}$
Ted Williams	5.79929	$\begin{bmatrix} .233655 \\ .276714 \\ .290207 \\ .199425 \end{bmatrix}$

13. Consider the second columns of the matrices  $M$  and  $SM$ , which correspond to the “Runner on first, none out” state.
- a. What information does the sum of the second column of  $M$  give?
- b. What value can you calculate using the second column of  $SM$ ?
- c. What would the calculation of expected runs scored using the data from the second columns mean?

Exercises 14–18 show how the model for run production in the text can be used to determine baseball strategy. Suppose that you are managing a baseball team and have access to the matrices  $M$  and  $SM$  for your team.

14. The sum of the column of  $M$  corresponding to the “Runner on first, none out” state is 4.53933, and the column of  $SM$  corresponding to the “Runner on first, none out” state is

$$\begin{bmatrix} .06107 \\ .35881 \\ .41638 \\ .16374 \end{bmatrix}$$

Your team now has a runner on first and no outs. How many earned runs do you expect your team to score this inning?

15. The sum of the column of  $M$  corresponding to the “Runner on second, none out” state is 4.53933, and the column of  $SM$  corresponding to the “Runner on second, none out” state is

$$\begin{bmatrix} .06107 \\ .47084 \\ .34791 \\ .12018 \end{bmatrix}$$

How many earned runs do you expect your team to score if there is a runner on second and no outs?

16. The sum of the column of  $M$  corresponding to the “Bases empty, one out” state is 3.02622, and the column of  $SM$  corresponding to the “Bases empty, one out” state is

$$\begin{bmatrix} .48513 \\ .31279 \\ .16060 \\ .04148 \end{bmatrix}$$

How many earned runs do you expect your team to score if the bases are empty and there is one out?

17. Suppose that a runner for your team is on first base with no outs. You have to decide whether to tell the baserunner to attempt to steal second base. If the steal is successful, there will be a runner on second base and no outs. If the runner is caught stealing, the bases will be empty and there will be one out. Suppose further that the baserunner has a probability of  $p = .8$  of stealing successfully. Does attempting a steal in this circumstance increase or decrease the number of earned runs your team will score this inning?
18. In the previous exercise, let  $p$  be the probability that the baserunner steals second base successfully. For which values of  $p$  would you as manager call for an attempted steal?

**SOLUTIONS TO PRACTICE PROBLEMS**

1. For entry  $(3, 6)$  of  $A$ , the probability of a transition from state  $13:k$  to state  $2:k$  is required. Suppose that first and third bases are occupied and that the batter does not make an out. To leave a player on second base, the batter must hit a double and the players on first and third base must both reach home plate successfully. This cannot happen according to the model, so the  $(3, 6)$ -entry is 0.
2. For entry  $(6, 3)$  of  $A$ , the probability of a transition from state  $2:k$  to state  $13:k$  is required. Suppose that only second base is occupied and that the batter does not make an out. To leave players on first base and on third base, the batter must get to first base, and the player on second base must advance to third. The desired outcome occurs when the batter hits a single, but the runner from second base will then stop at third base with probability .5. The  $(6, 3)$ -entry is thus  $.5p_1$ .



# Answers to Odd-Numbered Exercises Chapter 10

## Chapter 10

### Section 10.1, page 9

1. a. Stochastic.

b. Not stochastic. Columns do not sum to 1.

$$3. \mathbf{x}_3 = \begin{bmatrix} .556 \\ .444 \end{bmatrix}$$

5. 109/216

7. 13/36

9. a. .53125

b. 0

11. a. 5/8

b. 1/8

$$13. P = \begin{bmatrix} 0 & 1/3 & 0 & 1/2 & 1/2 \\ 1/3 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1/3 & 0 & 1/2 & 0 \\ 1/3 & 0 & 1/2 & 0 & 0 \\ 1/3 & 1/3 & 0 & 0 & 0 \end{bmatrix}$$

$$15. P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1 & 0 & 0 \end{bmatrix}$$

$$17. a. P = \begin{bmatrix} 0 & 1/3 & 1/4 & 1/3 & 0 \\ 1/3 & 0 & 1/4 & 0 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 & 1/3 \\ 1/3 & 0 & 1/4 & 0 & 1/3 \\ 0 & 1/3 & 1/4 & 1/3 & 0 \end{bmatrix},$$

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$b. \mathbf{x}_3 = \begin{bmatrix} .25926 \\ .11111 \\ .25926 \\ .11111 \\ .25926 \end{bmatrix}$$

$$19. a. P = \begin{bmatrix} 0 & 1/3 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 1/3 & 0 \\ 0 & 1/3 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 1/3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/3 & 1 \end{bmatrix},$$

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$b. \mathbf{x}_4 = \begin{bmatrix} .12963 \\ 0 \\ .12963 \\ 0 \\ .43518 \\ .30556 \end{bmatrix}$$

23. Sunny with probability .406, cloudy with probability .145375, rainy with probability .448625.

$$25. \mathbf{x}_3 = \begin{bmatrix} 1/6 \\ 5/18 \\ 5/18 \\ 5/18 \end{bmatrix}$$

## A2 Answers to Odd-Numbered Exercises

$$27. \text{ a. } P = \begin{bmatrix} p^2 & p(1-p) & p(1-p) & (1-p)^2 \\ p(1-p) & p^2 & (1-p)^2 & p(1-p) \\ p(1-p) & (1-p)^2 & p^2 & p(1-p) \\ (1-p)^2 & p(1-p) & p(1-p) & p^2 \end{bmatrix} \quad \text{b. .94206}$$

$$29. \text{ a. } P = \begin{bmatrix} 1-p & p/6 & 0 & 0 & 0 & 0 & 0 \\ p & 1-p & p/3 & 0 & 0 & 0 & 0 \\ 0 & 5p/6 & 1-p & p/2 & 0 & 0 & 0 \\ 0 & 0 & 2p/3 & 1-p & 2p/3 & 0 & 0 \\ 0 & 0 & 0 & p/2 & 1-p & 5p/6 & 0 \\ 0 & 0 & 0 & 0 & p/3 & 1-p & p \\ 0 & 0 & 0 & 0 & 0 & p/6 & 1-p \end{bmatrix} \quad \text{b. .19290}$$

$$31. \text{ a. } P = \begin{bmatrix} 0 & 1-p & p & 0 & 0 \\ p & 0 & 0 & 0 & 0 \\ 1-p & 0 & 0 & 0 & 0 \\ 0 & p & 0 & 1 & 0 \\ 0 & 0 & 1-p & 0 & 1 \end{bmatrix} \quad \text{b. .192}$$

33. Suppose that  $P$  is an  $m \times m$  stochastic matrix all of whose entries are greater than or equal to  $p$ . The proof proceeds by induction. Notice that the statement to be proven is thus true for  $n = 1$ . Assume the statement is true for  $n$ , and let  $B = P^n$ . Then, since

$$P^{n+1} = BP, \text{ the } (i, j)\text{-entry in } P^{n+1} \text{ is } \sum_{k=1}^m b_{ik} p_{kj}. \text{ Since } b_{ik} \geq p \text{ by the induction hypothesis, } \sum_{k=1}^m b_{ik} p_{kj} \geq p \sum_{k=1}^m p_{kj}. \text{ But } \sum_{k=1}^m p_{kj} = 1 \text{ since } P \text{ is a stochastic matrix, so all of the entries in } P^{n+1} \text{ are greater than or equal to } p.$$

### Section 10.2, page 21

$$1. P^{10} = \begin{bmatrix} .33333 & .33333 \\ .66667 & .66667 \end{bmatrix}$$

$$\mathbf{q} = \begin{bmatrix} .33333 \\ .66667 \end{bmatrix} \text{ The probability is .33333.}$$

$$3. P^{20} = \begin{bmatrix} .21429 & .21429 & .21429 \\ .57143 & .57143 & .57143 \\ .21429 & .21429 & .21429 \end{bmatrix}$$

$$\mathbf{q} = \begin{bmatrix} 3/14 \\ 4/7 \\ 3/14 \end{bmatrix} \approx \begin{bmatrix} .21429 \\ .57143 \\ .21429 \end{bmatrix} \text{ The probability is .21429.}$$

$$5. \begin{bmatrix} 8/17 & 8/17 \\ 9/17 & 9/17 \end{bmatrix}$$

7.  $P$  is regular since all entries in  $P^2$  are positive.

9. a. The transition matrix is

$$P = \begin{bmatrix} 0 & 1/4 & 0 & 0 & 0 \\ 1 & 0 & 1/2 & 0 & 0 \\ 0 & 3/4 & 0 & 3/4 & 0 \\ 0 & 0 & 1/2 & 0 & 1 \\ 0 & 0 & 0 & 1/4 & 0 \end{bmatrix}$$

The  $(0, 0)$ -entry in  $P^k$  will be zero if  $k$  is odd while the  $(0, 1)$ -entry in  $P^k$  will be zero if  $k$  is even. Thus  $P$  is not regular.

b. Compute that

$$\mathbf{q} = \begin{bmatrix} 1/16 \\ 1/4 \\ 3/8 \\ 1/4 \\ 1/16 \end{bmatrix}$$

so the chain will spend the most steps in state 2, which corresponds to both urns containing 2 molecules.

11. a. The transition matrix is

$$P = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1 \\ 0 & 0 & 1/2 & 0 \end{bmatrix}$$

The  $(1, 1)$ -entry in  $P^k$  will be zero if  $k$  is odd while the  $(1, 2)$ -entry in  $P^k$  will be zero if  $k$  is even. Thus  $P$  is not regular.

b. Compute that

$$\mathbf{q} = \begin{bmatrix} 1/6 \\ 1/3 \\ 1/3 \\ 1/6 \end{bmatrix}$$

so the chain will spend the most steps in states 2 and 3.

$$13. \mathbf{q} = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/6 \\ 1/6 \\ 1/6 \end{bmatrix}$$

$$15. \mathbf{q} = \begin{bmatrix} 3/13 \\ 3/13 \\ 3/13 \\ 4/13 \end{bmatrix}$$

17. Since  $\mathbf{q} = \begin{bmatrix} .1875 \\ .1875 \\ .25 \\ .1875 \\ .1875 \end{bmatrix}$ , the probability is .25.

19. Since

$$P\mathbf{q} = \begin{bmatrix} 0 & 1/3 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 1/3 & 0 \\ 0 & 1/3 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 1/3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$\mathbf{q}$  is a steady-state vector for the Markov chain. Room 6 is an absorbing state for the chain—once the mouse moves into room 6, it will stay there forever.

23. To the nearest day, 152 are sunny, 52 are cloudy, and 161 are rainy.

25. The Google matrix and its steady-state vector are

$$G = \begin{bmatrix} .03 & .03 & .88 & .03 & .2 \\ .313333 & .03 & .03 & .455 & .2 \\ .313333 & .03 & .03 & .455 & .2 \\ .313333 & .455 & .03 & .03 & .2 \\ .03 & .455 & .03 & .03 & .2 \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} .231535 \\ .208517 \\ .208517 \\ .208517 \\ .142915 \end{bmatrix}$$

and the PageRanks are 1; 2, 3, and 4 (tied); 5.

27. a. If a dominant (AA) individual is mated with a hybrid (Aa), then the dominant individual will always contribute an A. One half of the time the hybrid will also contribute an A, leading to a dominant offspring. The other half of the time, the hybrid will contribute an a, yielding a hybrid offspring.
- b. If a recessive (aa) individual is mated with a hybrid (Aa), then the recessive individual will always contribute an a. One half of the time the hybrid will also contribute an a, leading to a recessive offspring. The other half of the time, the hybrid will contribute an A, yielding a hybrid offspring.
- c. If a hybrid (Aa) is mated with another hybrid (Aa), then a dominant offspring will result when both hybrids contribute an A, which happens  $(1/2)(1/2) = 1/4$  of the time. Likewise a recessive offspring will result when both hybrids contribute an a, which also happens  $(1/2)(1/2) = 1/4$  of the time. Finally, in all other cases, a hybrid offspring will be produced, which happens  $1 - 1/4 - 1/4 = 1/2$  of the time.
29. a. Confirm that all entries in  $P^6$  are strictly positive.

b. Compute that  $\mathbf{q} = \begin{bmatrix} 1/64 \\ 3/32 \\ 15/64 \\ 5/16 \\ 15/64 \\ 3/32 \\ 1/64 \end{bmatrix}$  so the chain spends the most steps in state 3, which corresponds to both urns containing 3 molecules. The fraction of steps the chain spends there is  $5/16$ .

c. Compute that

$$P\mathbf{q} = \begin{bmatrix} 1-p & p/6 & 0 & 0 & 0 & 0 & 0 \\ p & 1-p & p/3 & 0 & 0 & 0 & 0 \\ 0 & 5p/6 & 1-p & p/2 & 0 & 0 & 0 \\ 0 & 0 & 2p/3 & 1-p & 2p/3 & 0 & 0 \\ 0 & 0 & 0 & p/2 & 1-p & 5p/6 & 0 \\ 0 & 0 & 0 & 0 & p/3 & 1-p & p \\ 0 & 0 & 0 & 0 & 0 & p/6 & 1-p \end{bmatrix} \begin{bmatrix} 1/64 \\ 3/32 \\ 15/64 \\ 5/16 \\ 15/64 \\ 3/32 \\ 1/64 \end{bmatrix} = \begin{bmatrix} 1/64 \\ 3/32 \\ 15/64 \\ 5/16 \\ 15/64 \\ 3/32 \\ 1/64 \end{bmatrix} = \mathbf{q}$$

so the result of part (b) does not depend on the value of  $p$ .



**31.** Compute that

$$\begin{bmatrix} 1 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} q \\ 0 \\ 0 \\ 0 \\ 1-q \end{bmatrix} = \begin{bmatrix} q \\ 0 \\ 0 \\ 0 \\ 1-q \end{bmatrix}$$

**33.** Compute that

$$\begin{bmatrix} 1/4 & 1/3 & 1/2 & 0 & 0 \\ 1/4 & 1/3 & 1/4 & 0 & 0 \\ 1/2 & 1/3 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 3/4 \\ 0 & 0 & 0 & 2/3 & 1/4 \end{bmatrix} \begin{bmatrix} 4/11 \\ 3/11 \\ 4/11 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4/11 \\ 3/11 \\ 4/11 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} 1/4 & 1/3 & 1/2 & 0 & 0 \\ 1/4 & 1/3 & 1/4 & 0 & 0 \\ 1/2 & 1/3 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 3/4 \\ 0 & 0 & 0 & 2/3 & 1/4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 9/17 \\ 8/17 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 9/17 \\ 8/17 \end{bmatrix}$$

If the chain is equally likely to begin in each of the states, then it begins in state 1, 2, or 3 with probability 3/5, and in state 4 or 5 with probability 2/5. Since

$$\frac{3}{5} \begin{bmatrix} 4/11 \\ 3/11 \\ 4/11 \\ 0 \\ 0 \end{bmatrix} + \frac{2}{5} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 9/17 \\ 8/17 \end{bmatrix} = \begin{bmatrix} 12/55 \\ 9/55 \\ 12/55 \\ 18/85 \\ 16/85 \end{bmatrix}$$

the probability of the chain being in state 1 after many steps is 12/55.

**35. a.** The matrix  $P$  will be a stochastic matrix if  $p + q \leq 1$ . It will be a regular stochastic matrix if in addition  $p \neq 1$  and  $q \neq 1$ .

**b.** A steady-state vector for  $P$  is

$$\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

**37. a.** Let  $\mathbf{v}$  be an eigenvector of  $P$  associated with  $\lambda = 1$ . Let  $P\mathbf{v} = \mathbf{y}$ . Then, by Exercise 36,

$$|y_1| + \dots + |y_m| \leq |v_1| + \dots + |v_m|$$

But  $P\mathbf{v} = \mathbf{v}$ , so  $\mathbf{y} = \mathbf{v}$  and

$$|y_1| + \dots + |y_m| = |v_1| + \dots + |v_m|$$

Since equality holds, each nonzero entry in  $\mathbf{v}$  must have the same sign by Exercise 36.

**b.** By part (a), each nonzero entry in  $\mathbf{v}$  must have the same sign. Since  $\mathbf{v}$  is an eigenvector,  $\mathbf{v} \neq \mathbf{0}$  and so must have at least one nonzero entry. Thus the sum of the entries in  $\mathbf{v}$  will not be zero, so one may define

$$\frac{1}{v_1 + \dots + v_m} \mathbf{v}$$

This vector will also be an eigenvector of  $P$  associated with  $\lambda = 1$ , each entry in this vector will be

nonnegative, and the sum of the entries in this vector will be 1. It is thus a steady-state vector for  $P$ .

**39. a.** Since  $\mathbf{x}_0 = c_1\mathbf{q} + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n$  and  $\lambda_1 = 1$ , Equation (2) indicates that

$$\mathbf{x}_k = P^k\mathbf{x}_0 = c_1\mathbf{q} + c_2\lambda_2^k\mathbf{v}_2 + \dots + c_n\lambda_n^k\mathbf{v}_n$$

**b.** By part (a),

$$\mathbf{x}_k - c_1\mathbf{q} = c_2\lambda_2^k\mathbf{v}_2 + \dots + c_n\lambda_n^k\mathbf{v}_n$$

and  $\mathbf{x}_k \rightarrow c_1\mathbf{q}$  since  $|\lambda_i| < 1$ . Since  $|\lambda_2|$  is the largest magnitude eigenvalue remaining, the  $c_2\lambda_2^k\mathbf{v}_2$  will be the largest of the error terms and will thus govern how quickly  $\{\mathbf{x}_k\}$  converges to  $c_1\mathbf{q}$ .

### Section 10.3, page 31

1.  $\{1, 3\}, \{2\}$ ; reducible
3.  $\{1\}, \{2\}, \{3\}$ ; reducible
5.  $\{1, 3, 5\}, \{2, 4, 6\}$ ; reducible
7.  $\{1, 2, 3, 4, 5\}, \{6\}$ ; reducible
9.  $\{1, 2, 3, 4\}, \{5\}$
11.  $\{1, 2, 3, 4\}$ ; irreducible
13. Every state is reachable from every other state in two steps or fewer, so the Markov chain is irreducible. The return times are  
State 1: 4  
State 2: 4  
State 3: 6  
State 4: 6  
State 5: 6
15. Every state is reachable from every other state in three steps or fewer, so the Markov chain is irreducible. The return times are  
State 1: 13/3  
State 2: 13/3  
State 3: 13/3  
State 4: 13/4
17. 4 steps.
19. 15/2 steps.
23. 2.27368 days.
25. **a.** Since each entry in  $G$  is positive, the Markov chain is irreducible.  
**b.** 4.31901 mouse clicks.
27. 8/3 steps.
29.  $16/5 = 3.2$  draws.
31. {deuce, advantage A, advantage B}, {A wins the game}, {B wins the game}
33. Every state is reachable from every other state in three steps or fewer, so the Markov chain is irreducible.
35. Each dangling node forms a separate communication class for the Markov chain.

## Section 10.4, page 40

1. The communication classes are  $\{1, 3\}$  and  $\{2\}$ . Class  $\{1, 3\}$  is transient while class  $\{2\}$  is recurrent. All classes have period 1.
3. The communication classes are  $\{1\}$ ,  $\{2\}$ , and  $\{3\}$ . Class  $\{1\}$  is recurrent while classes  $\{2\}$  and  $\{3\}$  are transient. All classes have period 1.
5. The communication classes are  $\{1, 3, 5\}$  and  $\{2, 4, 6\}$ . Both classes are recurrent and have period 2.
7. The communication class is  $\{1, 2, 3, 4\}$ , which must be recurrent. The class has period 4.
9. The communication classes are  $\{1, 2, 3, 4\}$ , which is transient, and  $\{5\}$ , which is recurrent. Both classes have period 1.
11. Ordering the states 2, 1, 3 gives the matrix
13. The matrix is already in canonical form.
15. Ordering the states 1, 3, 5, 2, 4, 6 gives the matrix

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1/4 & 1/3 \\ 0 & 1/4 & 1/3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & .4 & .8 & 0 & 0 & 0 \\ .3 & 0 & .2 & 0 & 0 & 0 \\ .7 & .6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .7 & .5 \\ 0 & 0 & 0 & .1 & 0 & .5 \\ 0 & 0 & 0 & .9 & .3 & 0 \end{bmatrix}$$

17. The original transition matrix is

$$\begin{bmatrix} 1/3 & 0 & 1 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 & 0 \\ 1/3 & 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1 \end{bmatrix}$$

Ordering the states 5, 1, 2, 3, 4 gives the matrix

$$\begin{bmatrix} 1 & 0 & 1/2 & 0 & 0 \\ 0 & 1/3 & 0 & 1 & 0 \\ 0 & 1/3 & 0 & 0 & 1/2 \\ 0 & 1/3 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 & 0 \end{bmatrix}$$

19. a. The communication classes are  $\{1, 2, 3, 4, 5\}$  and  $\{6\}$ . Class  $\{1, 2, 3, 4, 5\}$  is transient while class  $\{6\}$  is recurrent.
- b. Both classes have period 1.
- c. The original transition matrix is

$$\begin{bmatrix} 0 & 1/3 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 1/3 & 0 \\ 0 & 1/3 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/3 & 0 \\ 0 & 1/3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 1/3 & 1 \end{bmatrix}$$

Ordering the states 6, 1, 2, 3, 4, 5 gives the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1/2 & 0 & 1/3 \\ 0 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 1/3 \\ 0 & 0 & 1/3 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 1/3 & 0 & 1 & 0 \end{bmatrix}$$

23. It is easy to compute that  $\mathbf{q} = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$  for the matrix  $P$ .

We further find that

$$P^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad P^3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\text{and } P^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

by direct computation. Thus  $P^5 = P$ ,  $P^6 = P^2$ ,  $P^7 = P^3$ ,  $P^8 = P^4 = I$ , and so on. So no matter the value of  $n$ , one of the four matrices  $P^{n+1}$ ,  $P^{n+2}$ ,  $P^{n+3}$ , and  $P^{n+4}$  will be  $P$ , one will be  $P^2$ , one will be  $P^3$ , and one will be  $P^4 = I$ . Therefore

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{4} (P^{n+1} + P^{n+2} + P^{n+3} + P^{n+4}) \\ &= \frac{1}{4} (P + P^2 + P^3 + P^4) \\ &= \frac{1}{4} \left( \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \right. \\ & \quad \left. + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix} \end{aligned}$$

as promised in Theorem 5.

## A6 Answers to Odd-Numbered Exercises

25. a. It is possible to go from any state to any other state in any even number of steps, so the Markov chain is irreducible with period 2.

b.  $\mathbf{q} = \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}$

- c. One choice is

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -2 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

- d. Compute that

$$\begin{aligned} P^n &= AD^nA^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -2 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/4 & 1/4 & 1/4 \\ 1/4 & -1/4 & 1/4 \\ -1/2 & 0 & 1/2 \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} 1 + (-1)^n & 1 - (-1)^n & 1 + (-1)^n \\ 2 - 2(-1)^n & 2 + 2(-1)^n & 2 - 2(-1)^n \\ 1 + (-1)^n & 1 - (-1)^n & 1 + (-1)^n \end{bmatrix} \\ &= \begin{bmatrix} 1/4 & 1/4 & 1/4 \\ 1/2 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/4 \end{bmatrix} + (-1)^n \begin{bmatrix} 1/4 & -1/4 & 1/4 \\ -1/2 & 1/2 & -1/2 \\ 1/4 & -1/4 & 1/4 \end{bmatrix} \end{aligned}$$

- e. The second terms in the expressions for  $P^n$  and  $P^{n+1}$  will cancel each other when added, so

$$(1/2)(P^n + P^{n+1}) = \begin{bmatrix} 1/4 & 1/4 & 1/4 \\ 1/2 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/4 \end{bmatrix}$$

as promised in Theorem 5.

27. It is easy to compute that  $\mathbf{q} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$  for the matrix  $P$ . We further find that

$$P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } P^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

by direct computation. Thus  $P^4 = P$ ,  $P^5 = P^2$ ,  $P^6 = P^3 = I$ , and so on. So no matter the value of  $n$ , one of the three matrices  $P^{n+1}$ ,  $P^{n+2}$ , and  $P^{n+3}$  will be  $P$ , one will be  $P^2$ , and one will be  $P^3 = I$ . Therefore

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{3} (P^{n+1} + P^{n+2} + P^{n+3}) &= \frac{1}{3} (P + P^2 + P^3) \\ &= \frac{1}{3} \left( \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \end{aligned}$$

as promised in Theorem 5.

29. a. Since any permutation of rows may be written as a sequence of row swaps, the permutation of rows may be performed by multiplying  $A$  on the left by a sequence of elementary matrices  $E_1, \dots, E_k$ . By the comment on page 106,  $E = E_k \cdots E_1$ , and  $EA$  will be the matrix  $A$  with its rows permuted in exactly the same order in which the rows of  $I_n$  were permuted to form  $E$ .
- b. By part (a),  $EA^T$  will be the matrix  $A^T$  with its rows permuted in exactly the same order in which the rows of  $I_n$  were permuted to form  $E$ . Thus  $(EA^T)^T$  will be the matrix  $A$  with its columns permuted in exactly the same order in which the rows of  $I_n$  were permuted to form  $E$ , and since  $(EA^T)^T = (A^T)^T E^T = AE^T$ , the result follows.
- c. The matrix  $EAE^T$  is  $(EA)E^T$ . By part (a),  $EA$  is the matrix  $A$  with its rows permuted in exactly the same order in which the rows of  $I_n$  were permuted to form  $E$ . Applying part (b) to  $EA$ ,  $(EA)E^T$  is the matrix  $EA$  with its columns permuted in exactly the same order in which the rows of  $I_n$  were permuted to form  $E$ . Thus  $EAE^T$  is the matrix  $A$  with its rows and columns permuted in exactly the same order in which the rows of  $I_n$  were permuted to form  $E$ .
- d. Since matrix multiplication is associative,  $(EA)E^T = E(AE^T)$  and it does not matter whether the rows of matrix  $A$  or the columns of matrix  $A$  are permuted first.

## Section 10.5, page 50

1.  $\begin{bmatrix} 3 & 2 \\ 3/2 & 2 \end{bmatrix}$
3. Using reordering 2, 4, 1, 3, 5:
- $$\begin{bmatrix} 1075/736 & 125/368 & 185/368 \\ 25/46 & 35/23 & 15/23 \\ 105/184 & 55/92 & 155/92 \end{bmatrix}$$
5.  $\begin{bmatrix} 10/21 & 3/7 \\ 5/21 & 3/14 \\ 2/7 & 5/14 \end{bmatrix}$
7.  $3/2$                       9.  $1895/736$
11. At state 1:  $10/21$ ; at state 2:  $5/21$ ; at state 3:  $2/7$
13. a.  $9/11$               b.  $29/11$               c.  $3/7$
15. a. 1                      b.  $10/3$
17.  $5/7$                       19.  $38/5$
23.  $2/7$                       25.  $19/2$                       27.  $3.84615$
29. Advantage A:  $2.53846$   
Advantage B:  $3.30769$
31.  $3.01105$                       33.  $3.92932$
35. From the results of Exercises 31 and 33, using rally point scoring led to  $3.92932 - 3.01105 = .91827$  fewer rallies being played.

37. a.  $\{1, 2\}$  is a recurrent class;  $\{3, 4, 5\}$  is a transient class.  
b. The limiting matrix for  $\{1, 2\}$  is

$$\begin{bmatrix} 2/5 & 2/5 \\ 3/5 & 3/5 \end{bmatrix}$$

- c. Since there is only one recurrent class, the probability that the chain is absorbed into  $\{1, 2\}$  is 1. Thus if the chain is started in any transient state, the probability of being at state 1 after many time steps is  $2/5$ , the probability of being at state 2 after many time steps is  $3/5$ , and the probability of being at state 3, 4, or 5 after many time steps is 0.

- d. Since the  $i^{\text{th}}$  column of  $\lim_{n \rightarrow \infty} P^n$  gives the long-range probabilities for the chain started at state  $i$ ,

$$\lim_{n \rightarrow \infty} P^n = \begin{bmatrix} 2/5 & 2/5 & 2/5 & 2/5 & 2/5 \\ 3/5 & 3/5 & 3/5 & 3/5 & 3/5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

e.  $P^{50} \approx \begin{bmatrix} .4 & .4 & .4 & .4 & .4 \\ .6 & .6 & .6 & .6 & .6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

39. The result is trivially true if  $n = 1$ . Assume that the result is true for  $n = k$ ; that is,  $P^k = \begin{bmatrix} I & S_k \\ O & Q^k \end{bmatrix}$ , where  $S_k = S(I + Q + Q^2 + \dots + Q^{k-1})$ . Then

$$\begin{aligned} P^{k+1} &= P^k \cdot P \\ &= \begin{bmatrix} I & S_k \\ O & Q^k \end{bmatrix} \cdot \begin{bmatrix} I & S \\ O & Q \end{bmatrix} \\ &= \begin{bmatrix} I & S + S_k Q \\ O & Q^{k+1} \end{bmatrix} \end{aligned}$$

But

$$\begin{aligned} S + S_k Q &= S + S(I + Q + Q^2 + \dots + Q^{k-1})Q \\ &= S + S(Q + Q^2 + \dots + Q^k) \\ &= S(I + Q + Q^2 + \dots + Q^k) \\ &= S_{k+1} \end{aligned}$$

and the result is proven by induction.

## Section 10.6, page 61

1. This concerns the second column of  $A$ . The initial state is  $1:k$  (a runner on first base,  $k$  outs). For entry  $(1, 2)$  of  $A$ , the probability of a transition “to state  $0:k$ ” is required. Suppose that only first base is occupied and the batter does not make an out. Only a home run will empty the bases, so the  $(1, 2)$ -entry is  $p_H$ .

Entry  $(2, 2)$ : (“to state  $1:k$ ”) To leave a player on first base, the batter must get to first base and the player on first base must reach home plate successfully. This cannot happen according to the model, so the  $(2, 2)$ -entry is 0.

Entry (3, 2): (“to state 2: $k$ ”) To leave a player on second base, the batter must get to second base and the player on first base must reach home plate successfully. This cannot happen according to the model, so the (3, 2)-entry is 0.

Entry (4, 2): (“to state 3: $k$ ”) To leave a player on third base, the batter must get to third base and the player on first base must reach home plate successfully. This can happen only if the batter hits a triple, so the (4, 2)-entry is  $p_3$ .

Entry (5, 2): (“to state 12: $k$ ”) To leave players on first base and second base, the batter must get to first base and the player on first base must advance to second. The desired outcome occurs when the batter either hits a single, gets a walk, or is hit by a pitch. The (5, 2)-entry is thus  $p_W + p_1$ .

Entry (6, 2): (“to state 13: $k$ ”) This concerns the batter getting to first base and the runner on first base advancing to third base. This cannot happen according to the model, so the (6, 2)-entry is 0.

Entry (7, 2): (“to state 23: $k$ ”) To leave players on second base and third base, the batter must hit a double and the runner on first base must advance only to third base. Thus the (7, 2)-entry is  $p_2$ .

Entry (8, 2): (“to state 123: $k$ ”) The starting state has just one runner on base. The next state cannot have three runners on base, so the (8, 2)-entry is 0.

3. This concerns the fifth column of  $A$ . The initial state is 12: $k$  (runners on first base and second base,  $k$  outs). For entry (1, 5) of  $A$ , the probability of a transition “to state 0: $k$ ” is required. Suppose that first and second bases are occupied and the batter does not make an out. Only a home run will empty the bases, so the (1, 5)-entry is  $p_H$ .

Entry (2, 5): (“to state 1: $k$ ”) To leave a player on first base, the batter must get to first base and both players on base must reach home plate successfully. This cannot happen according to the model, so the (2, 5)-entry is 0.

Entry (3, 5): (“to state 2: $k$ ”) To leave a player on second base, the batter must get to second base and both players on base must reach home plate successfully. This cannot happen according to the model, so the (3, 5)-entry is 0.

Entry (4, 5): (“to state 3: $k$ ”) To leave a player on third base, the batter must get to third base and the players on base must reach home plate successfully. This can happen only if the batter hits a triple, so the (4, 5)-entry is  $p_3$ .

Entry (5, 5): (“to state 12: $k$ ”) To leave players on first base and second base, the batter must get to first base, the player on first base must advance to second base, and the player on second base must reach home plate successfully. The desired outcome occurs when the batter hits a single, but the runner from second will then reach home with probability .5. The (5, 5)-entry is thus  $.5p_1$ .

Entry (6, 5): (“to state 13: $k$ ”) This concerns the batter getting to first base and the runner on first base advancing to

third base. This cannot happen according to the model, so the (6, 5)-entry is 0.

Entry (7, 5): (“to state 23: $k$ ”) To leave players on second base and third base, the batter must hit a double, in which case the runner on first base must advance to third base and the runner on second base must reach home. Thus the (7, 5)-entry is  $p_2$ .

Entry (8, 5): (“to state 123: $k$ ”) To leave runners on first, second, and third bases, the batter must reach first and the two runners must each advance one base. This happens when the batter is walked, is hit by a pitch, or hits a single but the runner on second base does not reach home. Thus the (8, 5)-entry is  $p_W + .5p_1$ .

5. This concerns the seventh column of  $A$ . The initial state is 23: $k$  (runners on second and third bases,  $k$  outs). For entry (1, 7) of  $A$ , the probability of a transition “to state 0: $k$ ” is required. Suppose that second and third bases are occupied and the batter does not make an out. Only a home run will empty the bases, so the (1, 7)-entry is  $p_H$ .

Entry (2, 7): (“to state 1: $k$ ”) To leave a player on first base, the batter must get to first base and the players on second base and third base must reach home plate successfully. The desired outcome occurs when the batter hits a single, but the runner from second will then reach home with probability .5. Thus the (2, 7)-entry is  $.5p_1$ .

Entry (3, 7): (“to state 2: $k$ ”) To leave a player on second base, the batter must reach second base (a “double”) and the runners on second and third bases must score. The second condition, however, is automatically satisfied because of the assumption in Table 2. So the probability of success in this case is  $p_2$ . This is the (3, 7)-entry.

Entry (4, 7): (“to state 3: $k$ ”) By an argument similar to that for the (3, 6)-entry, the (4, 7)-entry is  $p_3$ .

Entry (5, 7): (“to state 12: $k$ ”) To leave players on first base and second base, the batter must get to first base and the player on second base must remain there while the runner on third base reaches home. This is impossible, so the (5, 7)-entry is 0.

Entry (6, 7): (“to state 13: $k$ ”) This concerns the batter getting to first base and the runner on second base advancing to third base while the runner on third base reaches home. This can happen only if the batter hits a single, with probability  $p_1$ , and the runner on second base stops at third base, which happens with probability .5 (by Table 2). Since both events are required, the (6, 7)-entry is the product  $.5p_1$ .

Entry (7, 7): (“to state 23: $k$ ”) To leave players on second base and third base, the batter must hit a double and the runner on second base must advance only to third base. This cannot happen, so the (7, 7)-entry is 0.

Entry (8, 7): (“to state 123: $k$ ”) To leave runners on first, second, and third bases, the batter must reach first base and the two runners must each fail to advance one base. This happens when the batter is walked or is hit by a pitch. Thus the (8, 7)-entry is  $p_W$ .

7.  $p_W = .0954785$ ,  $p_1 = .159996$ ,  $p_2 = .049377$ ,  $p_3 = .00514581$ ,  $p_H = .0291127$ ,  $p_O = .66089$ . Thus

$$A = \begin{array}{c} \begin{array}{cccccccc} 0:k & 1:k & 2:k & 3:k & 12:k & 13:k & 23:k & 123:k \end{array} \\ \left[ \begin{array}{cccccccc} .0291127 & .0291127 & .0291127 & .0291127 & .0291127 & .0291127 & .0291127 & .0291127 \\ .255474 & 0 & .0799978 & .159996 & 0 & 0 & .0799978 & 0 \\ .049377 & 0 & .049377 & .049377 & 0 & 0 & .049377 & 0 \\ .00514581 & .00514581 & .00514581 & .00514581 & .00514581 & .00514581 & .00514581 & .00514581 \\ 0 & .255474 & .0954785 & 0 & .0799978 & .159996 & 0 & .0799978 \\ 0 & 0 & .0799978 & .0954785 & 0 & 0 & .0799978 & 0 \\ 0 & .049377 & 0 & 0 & .049377 & .049377 & 0 & .049377 \\ 0 & 0 & 0 & 0 & .175476 & .0954785 & .0954785 & .175476 \end{array} \right] \begin{array}{l} 0:k \\ 1:k \\ 2:k \\ 3:k \\ 12:k \\ 13:k \\ 23:k \\ 123:k \end{array} \end{array}$$

9. The sum of the first column of  $M$  shows that

$E[B] = 4.53933$ . The first column of  $SM$  allows  $E[L]$  to be computed:

$$E[L] = 0(.34973) + 1(.33414) + 2(.23820) + 3(.07793) = 1.04433$$

Thus

$$E[R] = E[B] - E[L] - 3 = 4.53933 - 1.04433 - 3 = .495$$

11. Bonds:  $p_W = .212933$ ,  $p_1 = .119495$ ,  $p_2 = .0480377$ ,  $p_3 = .00615458$ ,  $p_H = .0609064$ ,  $p_O = .552474$ .

Ruth:  $p_W = .2004$ ,  $p_1 = .144421$ ,  $p_2 = .0481721$ ,  $p_3 = .0129474$ ,  $p_H = .0679741$ ,  $p_O = .526085$ .

Williams:  $p_W = .210936$ ,  $p_1 = .157383$ ,  $p_2 = .0537579$ ,  $p_3 = .00727012$ ,  $p_H = .0533484$ ,  $p_O = .517305$ .

13. a. The sum of the second column of  $M$  will tell the expected number of batters that will come to the plate starting with a runner on first and none out.  
b. The second column of  $SM$  will give the probabilities of leaving 0, 1, 2, or 3 runners on base starting with a runner on first and none out. Thus the expected number

of runners left on base starting with a runner on first and none out could be calculated.

- c. The expected number of runs scored using the second column data will give the expected number of runs scored starting with a runner on first and none out.

15. The sum of the column of  $M$  is 4.53933. One batter has already reached base, so  $E[B] = 1 + 4.53933 = 5.53933$ . The column of  $SM$  allows  $E[L]$  to be computed:

$$E[L] = 0(.06107) + 1(.47084) + 2(.34791) + 3(.12108) = 1.52990$$

Thus

$$E[R] = E[B] - E[L] - 3 = 5.53933 - 1.52990 - 3 = 1.00943$$

17. If the baserunner does not attempt a steal, you expect to score .85654 runs by Exercise 14. If the runner attempts a steal and succeeds, you expect to score 1.00943 runs by Exercise 15. If the runner attempts a steal and does not succeed, you expect to score .26779 runs by Exercise 16. Thus the expected number of runs scored if a steal is attempted is  $1.00943(.8) + .26779(.2) = .861102$ . Attempting a steal thus increases the expected number of runs scored.