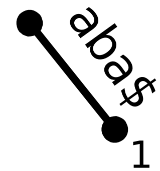
Ukkonen's suffix tree algorithm

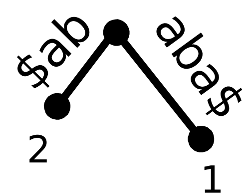
- Recall McCreight's approach:
  - For i = 1 .. n+1, build compressed trie of {x[j..n]$ | j ≤ i}
- Ukkonen's approach:
  - For i = 1 .. n+1, build compressed trie of {x[j..i]$ | j ≤ i}
  - Compressed trie of all suffixes of prefix x[1..i]$ of x$
  - A suffix tree except for "leaf" property

McCreight's algorithm, x=aba

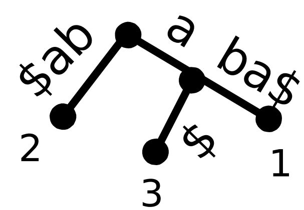
T_1 : {x$[j..n] | j ≤ 1 }



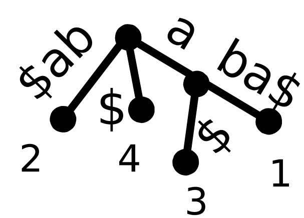
T_2 : {x$[j..n] | j ≤ 2 }



T_3 : {x$[j..n] | j ≤ 3 }

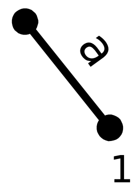


T_4 : {x$[j..n] | j ≤ 4 }

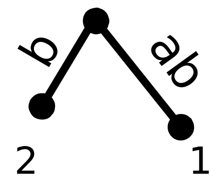


Ukkonen’s algorithm, x=aba

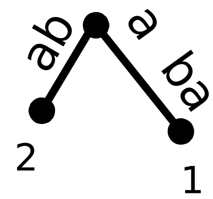
T_1 : {x$[j..1]}



T_2 : {x$[j..2]}

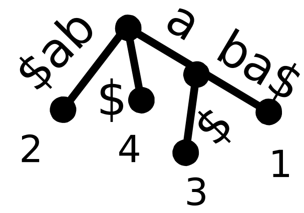


T_3 : {x$[j..3]}



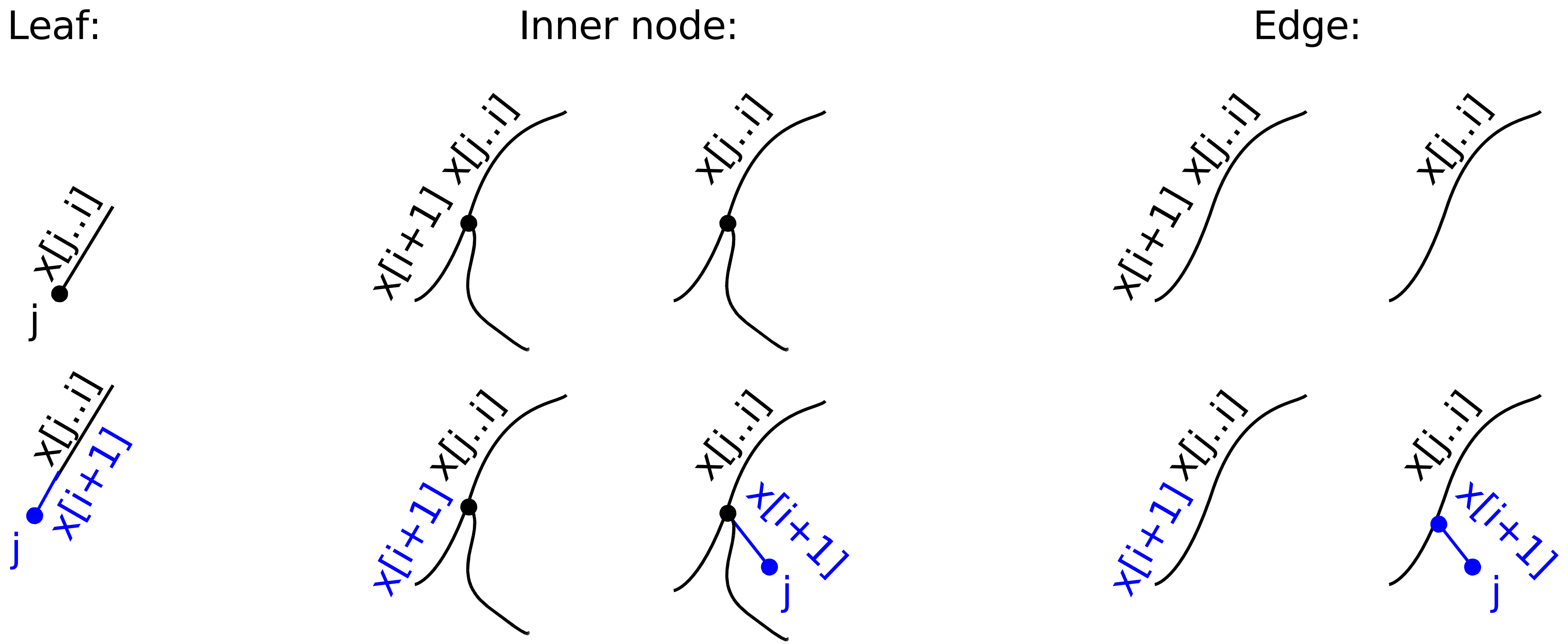
Note: no node for x[3..3] = “a”

T_4 : {x$[j..n]}



Tasks in iteration i

- In iteration i we must
  - Update each x[j..i] to x[j..i+1]
  - Add string x[i+1]   (special case of above)



Simple algorithm

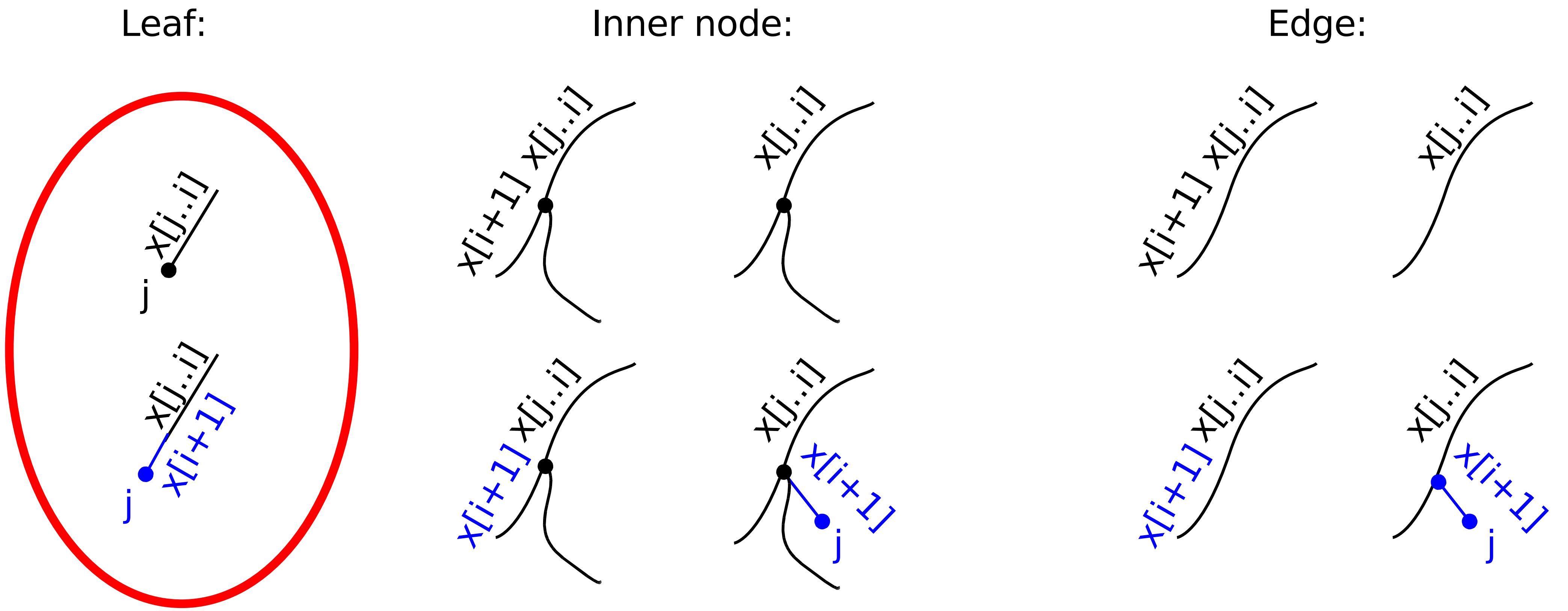
- "Obvious" algorithm:

```
For i=1,...,n+1:
    for j=1,...,i:
        find x[j..i]
        append x[i+1]
```

  - Running time $O(n^3)$
  - Need lots of tricks to get $O(n)$!

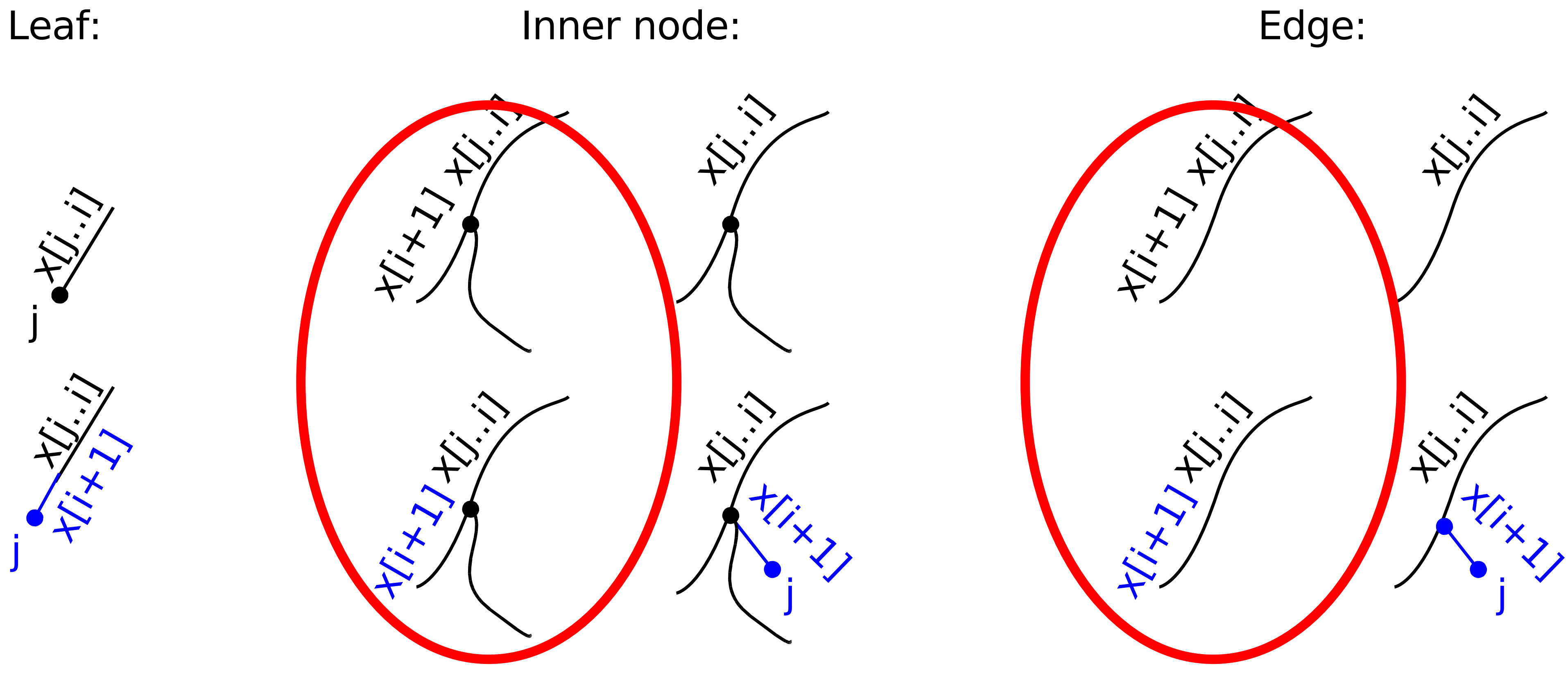
“Free” operations – leaves

- If we label leaves with (k,∞) – denoting “k to the current i”, updating a leaf is automatic



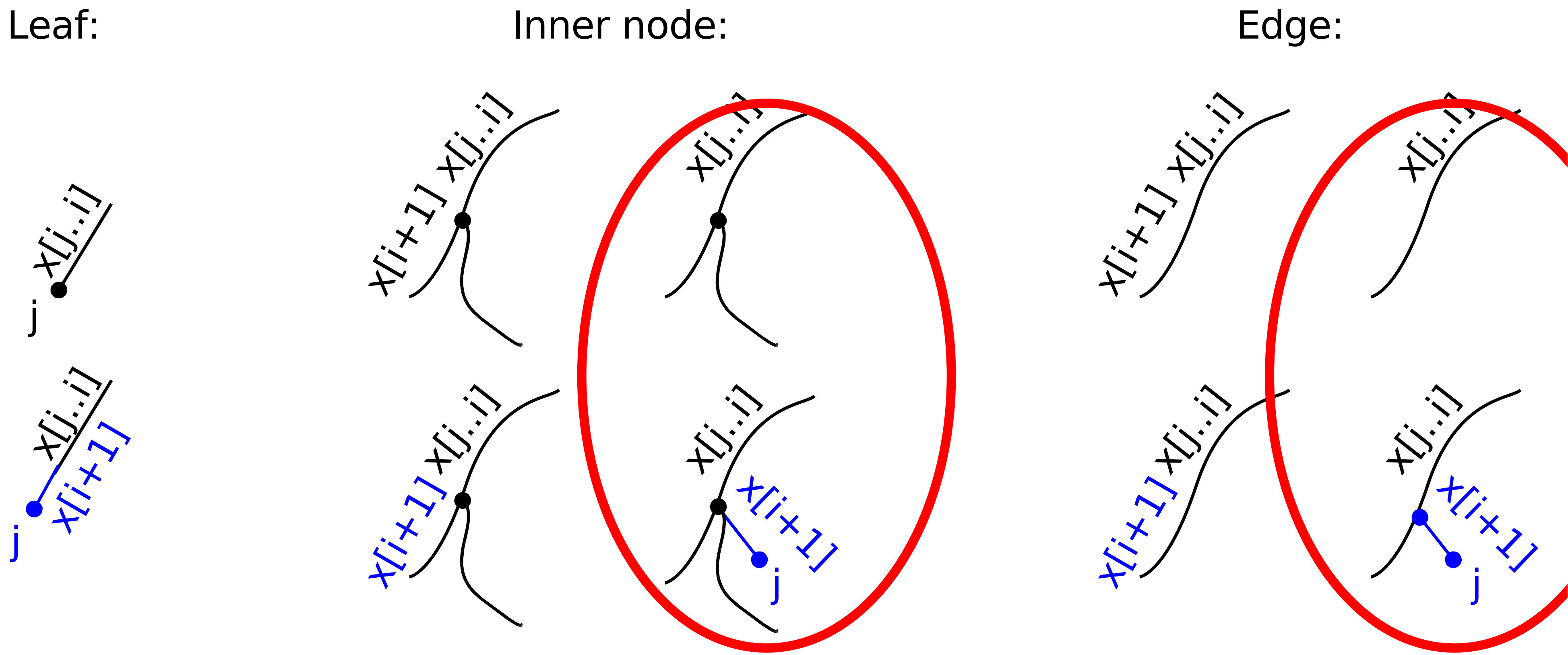
“Free” operations – existing strings

- If x[j..i+1] is already in the tree, the update is automatic



"Real" operations

- If we can recognize the free operations, we need only deal with the remaining



Lemma 5.2.4

- Let j denote suffix x[j..i] of x[1..i]

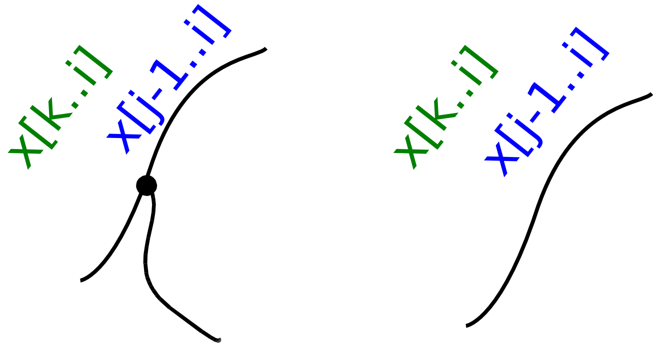
  a) If j>1 is a leaf node in T_i, then so is j-1

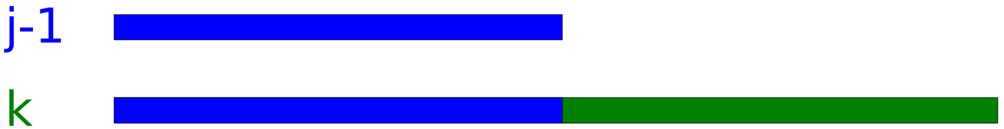
  b) If, from j<i, there is a path in T_i that begins with a, then there is a path in T_i from j+1 beginning with a

Proof of lemma 5.2.4 (a)

- If j>1 is a leaf node in T_i, then so is j-1

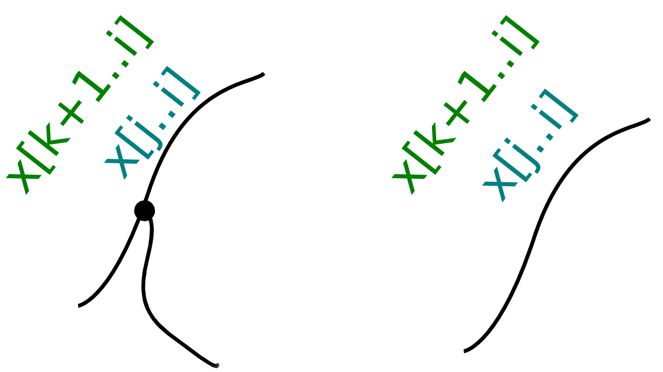
Assume j-1 is not a leaf. Then there exists k<j-1 such that:





Then

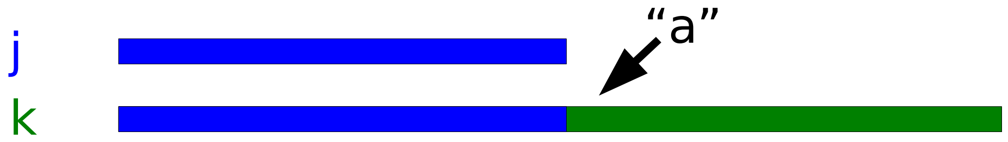
thus:



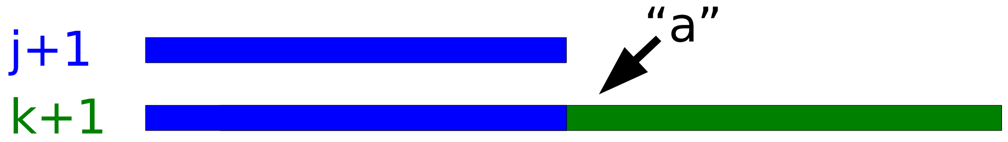
Proof of lemma 5.2.4 (b)

- If, from j<i, there is a path in T_i that begins with a, then there is a path in T_i from j+1 beginning with a

Assume j is followed by “a” there exists k<j such that:



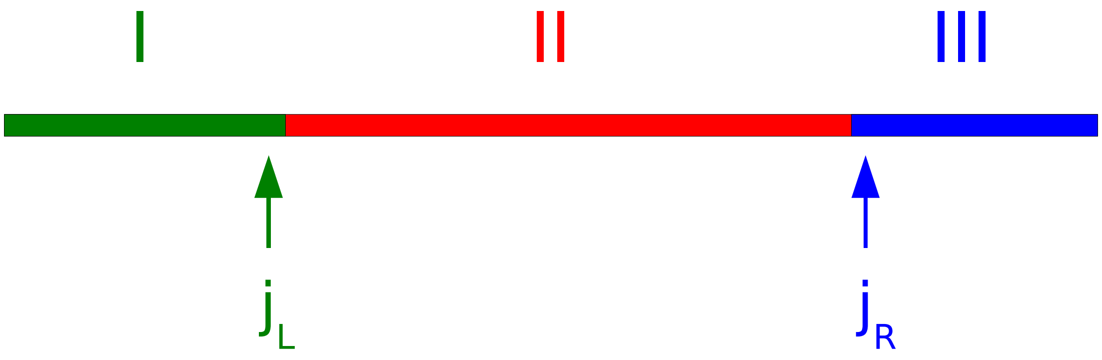
thus:



Hence j+1 is followed by “a”.

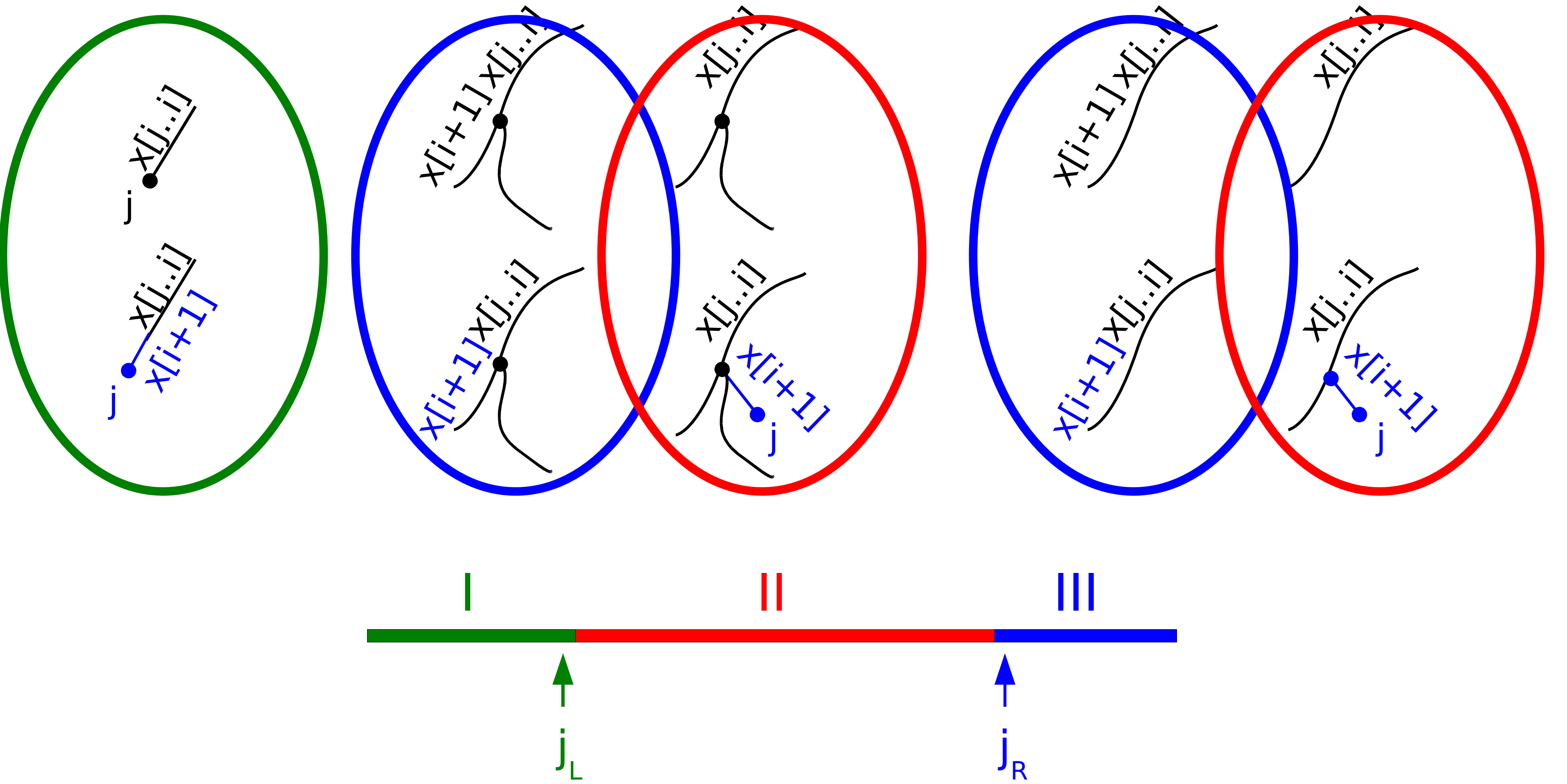
Corollary of lemma 5.2.4

- In iteration i, there exist indices j_L and j_R such that:
  - All suffixes $j \leq j_L$ are leaves
  - All suffixes $j \geq j_R$ are already in the trie



Corollary of lemma 5.2.4

- I and III are free operations



Updated algorithm

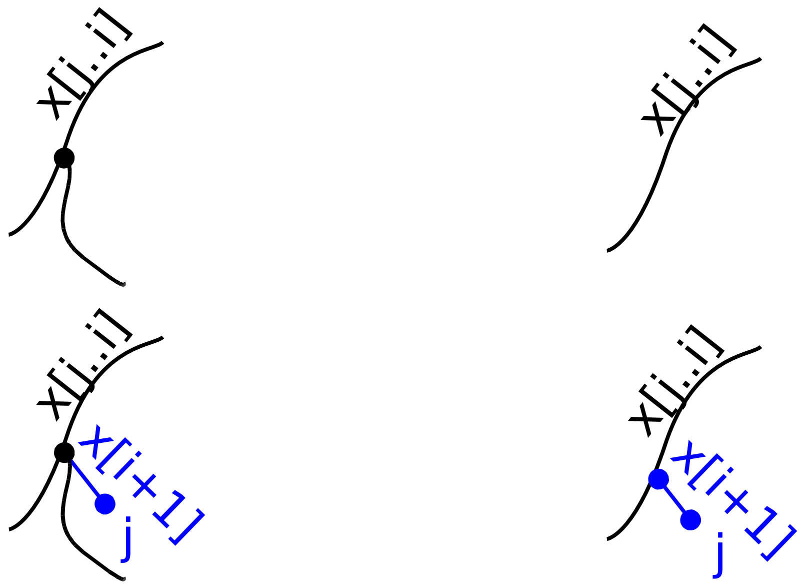
- Implicitly handling "free" operations:

```
For i=1,...,n+1:
    for j=j_L,...,j_R:
        find x[j..i]
        append x[i+1]
```

  - j_L in iteration i is the last leaf inserted in iteration i-1 ("once a leaf, always a leaf")
  - j_R in iteration i is the first index where x[j..i+1] is already in the trie

Handling index j in II

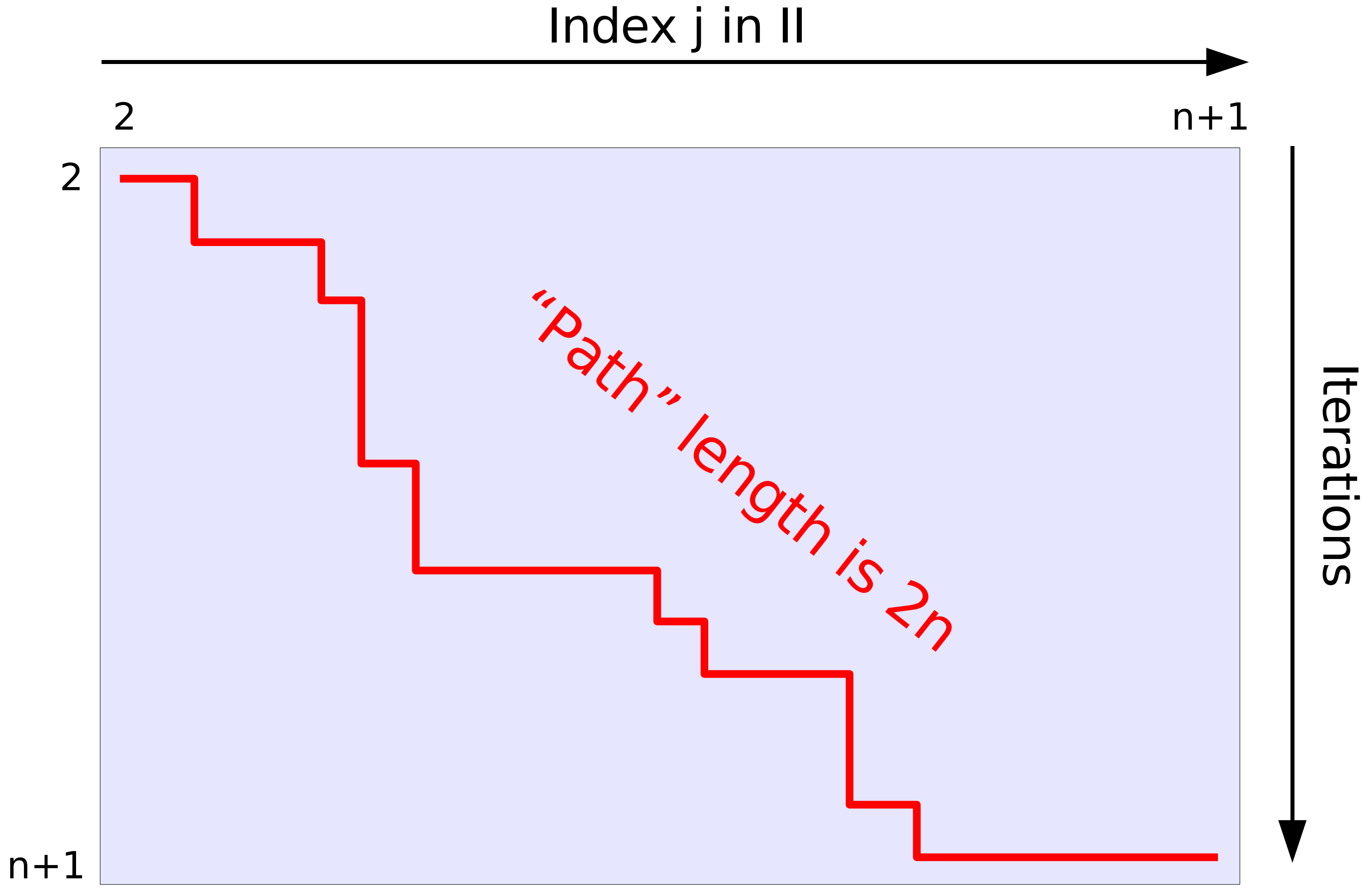
- Whenever $j_L < j < j_R$, j is made a leaf:



- Once j is a leaf, it will be in I and never in II again

Handling index j in II

- We handle j in II or implicitly in III 2n times:



Runtime

Only 2n of these:

```
For i=1,...,n+1:
    for j=j_L,...,j_R:
        find x[j..i]
        append x[i+1]
```

Constant time

- Running time is 2n*T(find x[j..i])
  - We just have to deal with T(find x[j..i]) in O(1)
  - No worries!

Using **fastscan** and s(-)

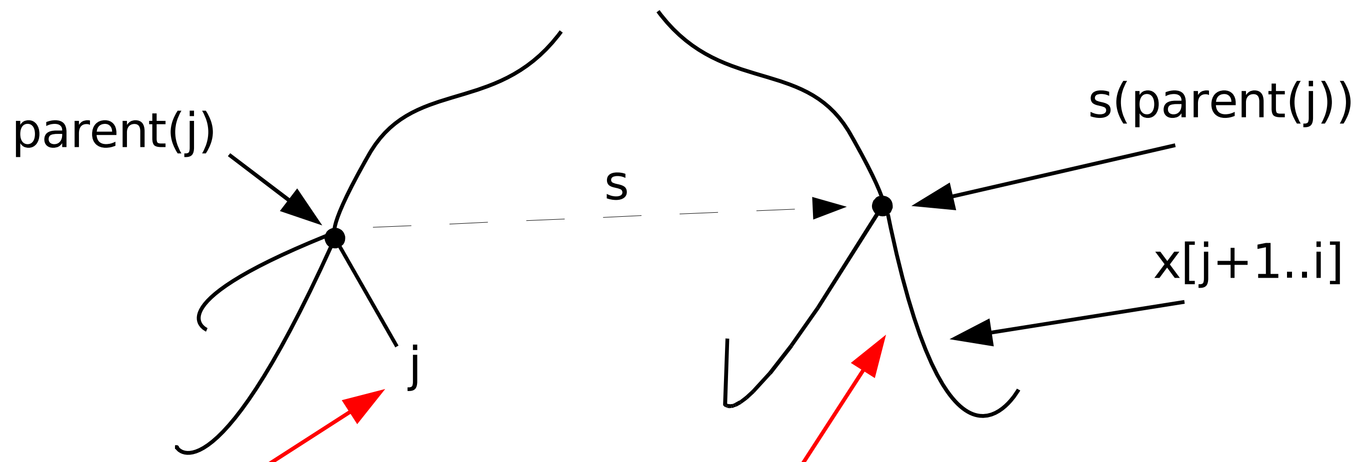
- When searching for x[j..i], it is already in the trie
  - We can use **fastscan** for the search
  - T(find x[j..i]) in O(d) where d is the (node-) depth of x[j..i]
- If we keep suffix links, s(-), in the tree we can use these as shortcuts

Suffix links

- **Invariant:** All inner nodes have suffix links

- Ensuring the invariant:
  - We only insert inner nodes x[j..i] when adding leaves j
  - Whenever we insert a new node, x[j..i] for some j<i, we also find or insert x[j+1..i], and can update s(x[j..i]) := x[j+1..i]
  - If we insert x[i..i], then s(x[i..i]) := ε

Finding x[j+1..i] from x[j..i]



Starting from here
(initial j is j_L and we can keep a pointer to that node between iterations)

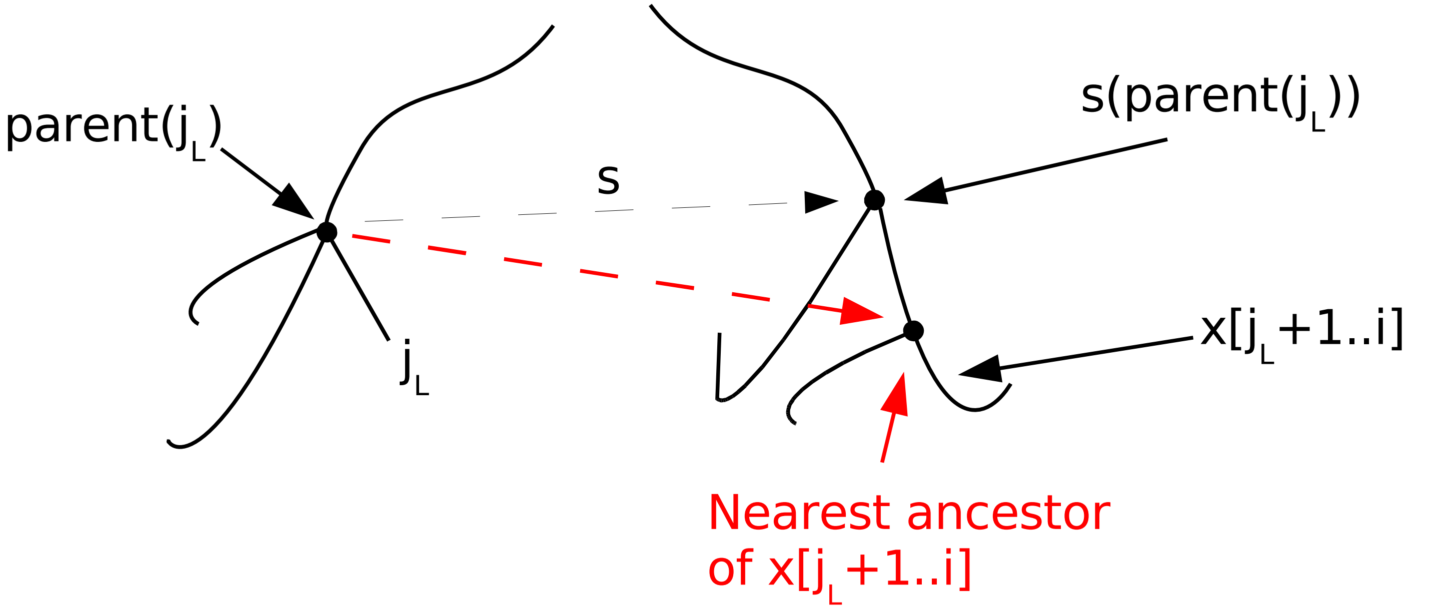
Using **fastscan** here

Bound on **fastscan**

- Time usage by **fastscan** is bounded by
  - n – for the maximal (node-)depth in the trie
  - + total decrease of (node-)depth
- Decrease in depth:
  - Moving to parent(j): 1
  - Moving to s(parent(j)): max 1
  - "Restarting" at j_L: **?**

Hacking the suffix links

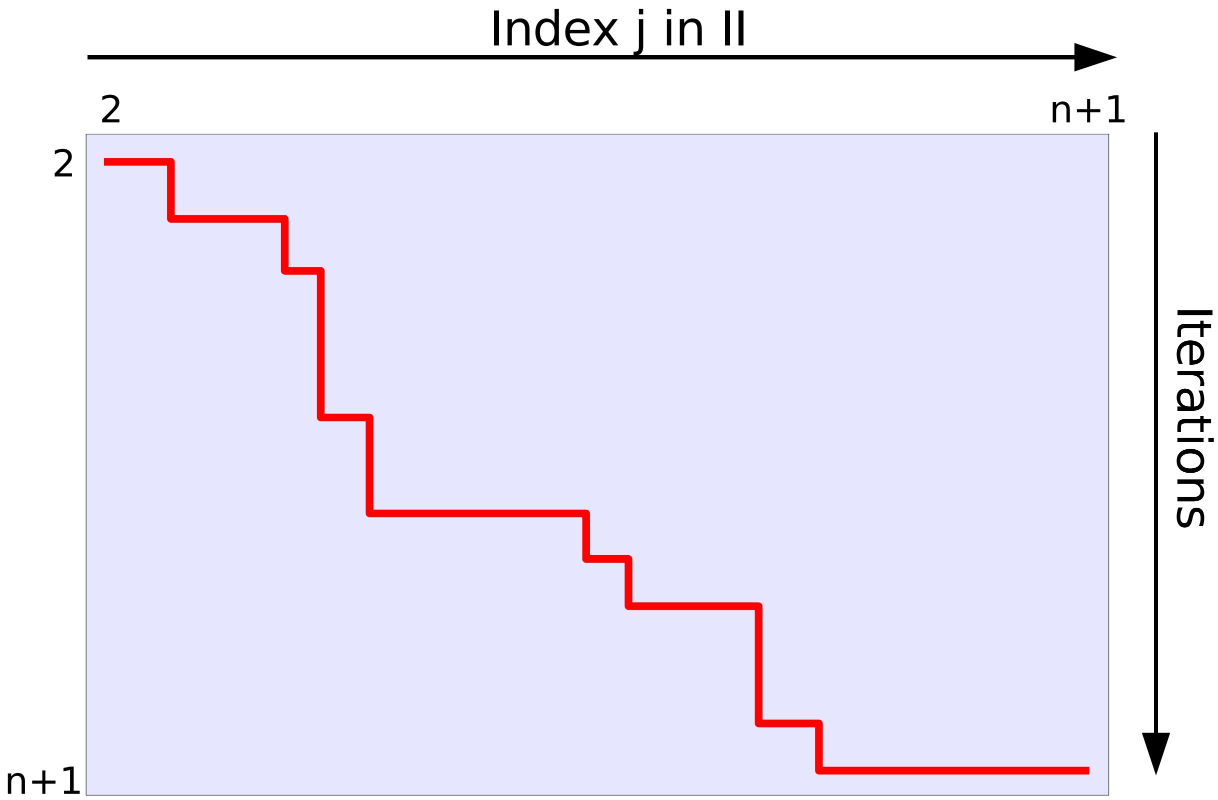
- When searching for $x[j_L+1..i]$, update $s(x[j_L..i])$ to point to the nearest ancestor of $x[j_L+1..i]$



- "Restarting" becomes essentially free

Running time

- Vertical steps are paid for by the previous horizontal step (free restarting)
- Horizontal steps are total **fastscan** bounded by O(n)
- **Runtime O(n)**



Why Ukkonen?

- Ukkonen’s algorithm is an “online” algorithm:
  - As long as no suffix is a prefix of another, the intermediate trees are suffix trees
  - Generalized suffix trees can be built one string at a time