CS 230 : Discrete Computational Structures

**Spring Semester, 2021**

HOMEWORK ASSIGNMENT #6

**Due Date:** Monday, March 22

**Suggested Reading:** Rosen Section 2.5

For the problems below, explain your answers and show your reasoning.

1. **[14 Pts]** Show that the following sets are countably infinite, by defining a bijection between $\mathcal{N}$ (or $\mathcal{Z}^+$) and that set. You do not need to prove that your function is bijective.

   (a) **[4 Pts]** the set of non-negative integers divisible by 5
   $\{5n \mid n \in N\}$

   (b) **[5 Pts]** the set of integers divisible by 5
   $\{(-5n, 5n) \mid n \in N\}$

   (c) **[5 Pts]** $\{0, 1, 2, 3\} \times \mathcal{N}$
   $\{((0, n), (1, n), (2, n), (3, n)) \mid n \in N\}$

2. **[14 Pts]** Determine whether the following sets are countable or uncountable. Prove your answer. To prove countable, describe your enumeration precisely, There is no need to define a bijection.

   (a) **[7 Pts]** the set of real numbers with decimal representation consisting of all 5's (5.55 and 55.555... are such numbers).
   This can be enumerated by defining the set of all real numbers with n fives before and m fives after the decimal. $(n, m) \in N \times N$, where n and m count the number of fives before and after the decimal. $N \times N$ is countably infinite, so these numbers are also countably infinite.

   (b) **[7 Pts]** the set of real numbers with decimal representation consisting of 1's, 3's and 5's.
   Suppose $r \in R$, where r $= 0.d_{i1}d_{i2}d_{i3}...d_{ij}$, for $d_{ij} \in \{1, 3, 5\}$

| | | | | |
|---|---|---|---|---|
| $r_1$ . | $d_{11}$ | $d_{12}$ | $d_{13}$ | ... |
| $r_2$ . | $d_{21}$ | $d_{22}$ | $d_{23}$ | ... |
| $r_3$ . | $d_{31}$ | $d_{32}$ | $d_{33}$ | ... |
| $r_4$ . | $d_{41}$ | $d_{42}$ | $d_{43}$ | ... |
| ... | ... | ... | ... | ... |

   Now, $z = 0.z_1 z_2 z_2 z_3$, where

$$z_i = \begin{cases} 1 & d_{ii} \in \{3, 5\} \\ 5 & d_{ii} \in \{1, 3\} \end{cases}$$

   In general, for all k, $z_k \neq d_{kk}$ so $z \neq r_k$
   So, $z \notin \{r_1, r_2, r_3, ...\}$, but z is in all numbers with digits 1, 3, 5 and within [0, 1]
   We assume $[0, 1] \in \{r_1, r_2, r_3, ...\}$. Contradiction! This set of numbers is uncountable.

3. **[6 Pts]** Prove that the set of functions from $\mathcal{N}$ to $\mathcal{N}$ is uncountable, by using a diagonalization argument. If all functions from $\mathcal{N}$ to $\mathcal{N}$ is countable, then these functions are listable as $F = \{f_1, f_2, f_3, ...\}$
   I define a function $g(x) = 2f(x)$. However, this makes $g(x) \notin F$, but still from $\mathcal{N}$ to $\mathcal{N}$
   Contradiction! This set of functions is uncountable.

4. **[6 Pts]** Argue that a countably infinite union of countable infinite sets is countably infinite.
Suppose $S_1, S_2, S_3, ..., S_i$ are countably infinite sets. These sets are countable as $\{s_{i1}, s_{i2}, s_{i3}, ..., s_{ij}\}$.
Even if all these sets are disjoint, the union can be counted as $\{s_{11}, s_{12}, s_{13}, ..., s_{1j}, s_{21}, s_{22}, s_{23}, ..., s_{2j}, ...\}$.
If they aren't disjoint, then those unions will too be countable. They simply have less elements because they will have repeated values between sets.