CS230-HW11Sol

1. **10 Pts** No, consider a graph where $V = \{1, 2, 3, 4, 5\}$ and $E = \{(1, 2), (1, 4), (2, 3), (2, 4), (3, 4)\}$. As we can see, there are 5 nodes and 5 edges, but the graph is not connected because there is no edge to node 5.

2. **8 Pts** The sum of the degrees of all the vertices in G is $5+4+3+3+2+2+1 = 20$. By the Handshaking Theorem, the sum of the degrees of the vertices is twice the number of edges. Therefore, the number of edges in graph G is $20/2 = 10$.

3. **12 Pts** Base Case: $h = 0$. A complete binary tree of height 0 is just a single node with no children, and therefore has 1 leaf. $1 = 2^0$, so the base case holds.
   Induction Hypothesis: Suppose that for some $k \geq 0$, a complete binary tree of height $h = k$ has 2^k leaves.
   Induction Step: Let T be a complete binary tree of height $h = k+1$. Then T's left and right subtrees are each complete binary trees of height k, and thus, by the I.H., have 2^k leaves. The number of leaves in T is equal to the sum of the number of leaves in T's subtrees, which must be equal to $2^k + 2^k = 2^(k+1)$. Hence the hypothesis holds for $k+1$, as required.

4. **Jonathan 20 Pts** First, suppose graph G is a tree. Then, G is connected and acyclic. We show that adding any edge $(v, w) \notin E$ will create a cycle in G. Since G is connected, there is a path $v, u_1, u_2, \ldots, u_k, w$ in G. So, adding the edge (v, w) will create the cycle $v, u_1, u_2, \ldots, u_k, w, v$. Therefore, G is acyclic but adding any edge will create a cycle.

   Now, suppose G is acyclic, but adding any edge to G will create a cycle. We prove that G is connected. Choose any two vertices x and y in G. Either $(x, y) \in E(G)$ or $(x, y) \notin E(G)$. (i) If $(x, y) \in E(G)$, then clearly there is a path from x to y in G. (ii) If $(x, y) \notin E(G)$, then, by the assumption, adding the edge (x, y) to G creates a cycle C which contains (x, y). Let $C = \langle x, y, v_1, \ldots, v_k, x \rangle$. The path $\langle y, v_1, \ldots, v_k, x \rangle$ is a path between y and x in G. So, in either case there is a path between x and y in G. Since this is true for any pair of vertices, G is connected. Since G is acyclic as well, therefore, G is a tree.