CS 230 : Discrete Computational Structures

Spring Semester, 2021

HOMEWORK ASSIGNMENT #4

Due Date: Wednesday, March 3

Suggested Reading: Rosen Sections 2.1 - 2.3; Lehman et al. Chapter 4.1, 4.3, 4.4.

For the problems below, explain your answers and show your reasoning.

1. **[6 Pts]** Let A and B be non-empty sets. Prove that if $A \neq B$, then $A \times B \neq B \times A$. The definition of the Cartesian Products described would be $\{(a, b) | a \in A \wedge b \in B\}$ and $\{(b, a) | b \in B \wedge a \in A\}$. If $A \neq B$, then there exists some (a, b) and (b, a) that are not equal. QED

2. **[4 Pts]** Prove that $(A \cup B) - C = (A - C) \cup (B - C)$ using iff arguments and logical equivalences.

$$(A \cup B) - C$$

iff $(x \in A \vee x \in B) \wedge x \notin C$	Definition of $\cup$, Difference
iff $((x \in A \wedge x \notin C) \vee (x \in B \wedge x \notin C))$	Distribution
iff $(A - C) \cup (B - C)$	Definition of $\cup$, Difference; QED

3. **[8 Pts]** Disprove the statements below.

   (a) If (1) $A \cup C \subseteq B \cup C$ then (2) $A \subseteq B$.
   Suppose A = {1}, B = {2}, and C - {1}
   $A \cup C = \{1\}$, $B \cup C = \{1, 2\}$ therefore $A \cup C \subseteq B \cup C$ (1) is satisfied.
   However, $A \not\subseteq B$; (1) does not imply (2). QED

   (b) If (1) $A \cap C \subseteq B \cap C$ then (2) $A \subseteq B$.
   Suppose A = {1, 2}, B = {2}, and C = {2}
   $A \cap C = \{2\}$, $B \cap C = \{2\}$ therefore $A \cap C \subseteq B \cap C$ (1) is satisfied.
   However, $A \not\subseteq B$; (1) does not imply (2). QED

4. **[8 Pts]** Prove by contradiction that if (U) $A \cup C \subseteq B \cup C$ and (I) $A \cap C \subseteq B \cap C$ then $A \subseteq B$.

   (1) Suppose $A \not\subseteq B$. By definition, $\forall x(x \in A \rightarrow x \notin B)$.

   (2) Assuming $x \in A$, this means that x is also in $A \cup C$, which is a subset of $B \cup C$.

   (3) (2) implies that $x \in B \vee x \in C$ To prove this, $x \in B \wedge x \in C$ must be true.

   (4) Unfortuantely, if $x \in A$, then $x \notin B$. (4) Contradicts (3), so A must be a subset of B. QED

5. **[8 Pts]** Prove that (H) $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$ using subset argument. You *may not* use logical equivalences in your proof. Use general proof techniques like 'proof by contradiction' and 'proof by cases'.

(1) LHS of (H) is simply removing all common elements of A and B from the combination of A and B.

(2) RHS of (H) can be read as the combination of removing common elements of A and B from A and from B.

(3) Combining everything in (2), we get that this combination will not contain common elements of A and B, but everything that is not common.

(4) (1) and (3) are equivalent in English; QED

6. [**4 Pts**] Prove that $f(n) = 5n + 9$ is one-to-one, where the domain and co-domain of f is $\mathcal{Z}^+$. Show that f is not onto.
f is definitely not onto because there exists a positive integer, for example 2, in the co-domain that can't be an output.

$$f \text{ is one-to-one if } \forall x \in Z^+, \forall y \in Z^+(f(x) = f(y) \to x = y)$$

$$\begin{aligned} 5x + 9 &= 5y + 9 \\ 5x &= 5y \\ x &= y \\ &QED \end{aligned}$$

7. [**4 Pts**] Prove that $f(m, n) = m + n + mn$ is onto, where the domain of f is $\mathcal{Z} \times \mathcal{Z}$ and the co-domain of f is $\mathcal{Z}$. Show that f is not one-to-one.
Let $y \in Z$. If $y = f(n, m)$, then $y = m + n + mn$. This implies that $n = \frac{y-m}{1+m}$ and $m = \frac{y-n}{1+n}$. so for every $y \in Z$, there exists some $m, n \in Z$ such that $f(m, n) = y$. QED
f is not one-to-one because inputs (1,0) and (0,1) have the same output 1.

8. [**8 Pts**] Let g be a total function from A to B and f be a total function from B to C.

   (a) If $f \circ g$ is one-to-one, then is g one-to-one? Prove or give a counter-example.

   (1) Suppose $g \circ f$ and g is one-to-one.

   (2) $g(x) = g(y)$ for arbitrary $x, y \in A$

   (3) Composing: $f(g(x)) = f(g(y))$

   (4) Taking the inverse of $f \circ g$ of both sides gives $x = y$. This is consistent with supposition (1). QED

   (b) If $f \circ g$ is onto, then is g onto? Prove or give a counter-example.
   Counter-example: If the output of f is just a one element set, then $f \circ g$ will always be onto, regardless of g. QED

For more practice, work on the problems from Sections 2.1 - 2.3; Lehman et al. Chapter 4.1, 4.3, 4.4.