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Stat HW11

8.4a. If $B(\hat{\theta}) = 0$, then $MSE(\hat{\theta}) = V(\hat{\theta})$

b. $MSE(\hat{\theta}) > V(\hat{\theta})$ because $MSE = V + B$

8.6a $\hat{\theta}_3 = a\hat{\theta}_1 + (1-a)\hat{\theta}_2$

$$E(\hat{\theta}_3) = a\,E(\hat{\theta}_1) + (1-a)\,E(\hat{\theta}_2)$$

$$= a\theta + (1-a)\theta$$
$$= a\theta + \theta - a\theta$$
$$= \theta$$

So, $\hat{\theta}_3$ is unbiased estimator of θ

8.8a $\hat{\theta}_1 = Y_1$, $E(\hat{\theta}_1) = E(Y_1) = \theta \rightarrow$ unbiased

$\hat{\theta}_2 = \frac{y_1 + y_2}{2}$, $E(\hat{\theta}_2) = 2\theta/2 = \theta \rightarrow$ unbiased

$E(\hat{\theta}_3) = 3\theta/3 \rightarrow$ unbiased

$E(\hat{\theta}_4) = \min(\theta, \theta, \theta) = \theta \rightarrow$ unbiased

$E(\hat{\theta}_5) = E((Y_1 + Y_2 + Y_3)/3) = 3\theta/3 = \theta \rightarrow$ unbiased

b. $V(\text{exp}) = \theta^2$

$V(\hat{\theta}_1) = V(Y_1) = \theta^2$

$\ldots = V\left(\frac{y_1 + y_2}{2}\right) = \frac{1}{4}\left(V(y_1) + V(y_2)\right) = \frac{\theta^2}{2}$

$\ldots = V\left(\frac{y_1}{3} + \frac{2y_2}{3}\right) = \frac{1}{9}\cancel{V(y_1)}\,\theta^2 + \frac{4}{9}\theta^2 = \frac{5\theta^2}{9}$

$\ldots = \min(\theta^2, \theta^2, \theta^2) = \theta^2$

$= \frac{1}{9}V(y_1) + \frac{1}{9}\theta^2 + \frac{1}{9}\theta^2 = \boxed{\frac{1}{3}\theta^2}$

8.12 a $\quad f(y) = 1 \qquad \overset{a}{\theta} \le y \le \overset{b}{\theta+1}$

$E(Y) = \theta + \frac{1}{2} \qquad E(\bar{Y}) = E\left(\frac{Y_1 + Y_2 \dots + Y_n}{n}\right)$

$E(\bar{Y}) \mathrel{!=} \theta$, so biased. $\qquad = \frac{1}{n}E(Y_1) + \dots$

$b(\theta) = \theta + \frac{1}{2} - \theta = \boxed{\frac{1}{2}} \qquad = \sum_{i=1}^{n} \frac{1}{n}\left(\theta + \frac{1}{2}\right) = \theta + \frac{1}{2}$

b. $E\left(\bar{Y} - \frac{1}{2}\right) = E(\bar{Y}) - E\left(\frac{1}{2}\right) = \theta$

$\boxed{\bar{Y} - \frac{1}{2}}$

c. $V(\bar{Y}) = \sum_{i=1}^{n} \frac{1}{n^2} V(Y_i) = \frac{1}{n^2}V(Y_1) + \frac{1}{n^2}V(Y_2) \dots = \frac{1}{12n}$

$V(Y) = 1/12 \qquad MSE(\bar{Y}) = \frac{1}{12n} + \left(\frac{1}{2}\right)^2 = \boxed{\frac{1+3n}{12n}}$

8.18 $\quad f(y) = \frac{1}{\theta} \quad 0 \le y \le \theta. \quad F(y) = \frac{y}{\theta}$

$Y_{(1)}: f_{Y_{(1)}}(y) = \frac{n\left(1 - \frac{y}{\theta}\right)^{n-1}}{\theta}$

$\cancel{E(Y_{(1)}) = \int_0^\theta \frac{n}{\theta}\left(1 - \frac{y}{\theta}\right)^{n-1} dy = \frac{n}{\theta}\left[\frac{\left(1 - \frac{y}{\theta}\right)^n}{n} \cdot -\frac{1}{\theta}\right]\Big|_0^\theta}$

$\cancel{- \frac{n}{\theta}\left(0 - 1\right) \cdot -\frac{1}{\theta^2}}$

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8.18 continued

$$E(Y_{(1)}) = \frac{n}{\theta}\int_0^\theta y\left(1-\frac{y}{\theta}\right)^{n-1} dy \qquad \begin{array}{cc} u & dv \\ y & \left(1-\frac{y}{\theta}\right)^{n-1} \\ - & \\ 1 & \cancel{-\theta\left(1-\frac{y}{\theta}\right)^n/n} \\ 0 & \end{array}$$

$$= \left(\frac{-y\theta\left(1-\frac{y}{\theta}\right)^n}{n} - \frac{\theta^2\left(1-\frac{y}{\theta}\right)^{n+1}}{n(n+1)}\right)\frac{n}{\theta}\Bigg|_0^\theta \qquad -\theta\left(1-\frac{y}{\theta}\right)^n/n \qquad \frac{\theta^2\left(1-\frac{y}{\theta}\right)^{n+1}}{n(n+1)}$$

$$= -y\left(1-\frac{y}{\theta}\right)^n - \frac{\theta\left(1-\frac{y}{\theta}\right)^{n+1}}{n+1}\Bigg|_0^\theta$$

$$(0-0) - \left(0 - \frac{\theta}{n+1}\right) = \boxed{\frac{\theta}{n+1}}$$

unbiased $= \frac{\theta}{n+1} \cdot n+1$, $\boxed{\text{so } Y_{(1)}(n+1) \text{ is unbiased}}$

9.2 a. $E(\hat{\mu}_1) = \frac{1}{2}(\mu+\mu) = \mu$ ✓

$E(\hat{\mu}_2) = \frac{1}{4}\mu + \frac{\cancel{(n-2)}\mu}{\cancel{2}(\cancel{n-2})} + \frac{1}{4}\mu = \mu$ ✓

$E(\hat{\mu}_3) = \frac{n\mu}{n} = \mu$ ✓

b. $V(\hat{\mu}_1) = \frac{1}{4}(\sigma^2+\sigma^2) = \frac{\sigma^2}{2}$

$V(\hat{\mu}_2) = \frac{1}{16}\sigma^2 + \frac{\cancel{(n-2)}\sigma^2}{4(n-2)^{\cancel{2}}} + \frac{1}{16}\sigma^2 = \cancel{\frac{3\sigma^2}{8}} \quad \frac{n\cdot\sigma^2}{8(n-2)}$

$V(\hat{\mu}_3) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$

$$\boxed{\begin{aligned} &\mathrm{Eff}(\hat{\mu}_3, \hat{\mu}_1) = \frac{\sigma^2}{2} \Big/ \frac{\sigma^2}{n} = n/2 \\ &\mathrm{Eff}(\hat{\mu}_3, \hat{\mu}_2) = \frac{n\cdot\sigma^2}{8(n-2)} \Big/ \frac{\sigma^2}{n} = \frac{n^2}{8(n-2)} \end{aligned}}$$

9.3a from 8.12: $E(\bar{Y}) = \theta + \frac{1}{2}$

$$E(\hat{\theta}_1) = \theta + \tfrac{1}{2} - \tfrac{1}{2} = \theta \checkmark$$

$$f_{Y_n}(y) = n(y-\theta)^{n-1}$$

$$E(Y_{(n)}) = \int_\theta^{\theta+1} ny(y-\theta)^{n-1} \qquad u = y-\theta \quad y = u+\theta \quad du = dy$$

$$= n\int_0^1 (u+\theta)(u)^{n-1}\,du = n\int_0^1 u^n + \theta u^{n-1}\,du$$

$$= n\left(\frac{u^{n+1}}{n+1} + \frac{\theta u^n}{n}\Big|_0^1\right) = \frac{n}{n+1} + \theta$$

So,

$$E(\hat{\theta}_2) = E(Y_{(n)}) - \frac{n}{n+1} = \theta \checkmark$$

b. From 8.12: $V(\bar{Y}) = \frac{1}{12n}$

$$E(Y_{(n)}^2) = n\int_\theta^{\theta+1} y^2(y-\theta)^{n-1}\,dy = n\int_0^1 (u+\theta)^2 u^{n-1}\,du$$

$$= n\int_0^1 (u^2+2u\theta+\theta^2)u^{n-1}\,du = n\int_0^1 u^{n+1} + 2u^n\theta + \theta^2 u^{n-1}\,du$$

$$= n\left(\frac{u^{n+2}}{n+2} + \frac{2u^{n+1}\theta}{n+1} + \frac{\theta^2 u^n}{n}\Big|_0^1\right) = \frac{n}{n+2} + \theta^2 + \frac{2n\theta}{n+1}$$

$$V(Y_{(n)}) = \frac{n}{n+2} + \theta^2 + \frac{2n\theta}{n+1} - \left(\left(\frac{n}{n+1}\right)^2 + \theta^2 + \frac{2n\theta}{n+1}\right)$$

$$= \frac{n}{n+2} - \frac{n^2}{(n+1)^2} = \frac{n}{(n+2)(n+1)^2}$$

$$V(\hat{\theta}_2) = \frac{n}{(n+2)(n+1)^2} + 0 \qquad \mathrm{Eff}(\hat{\theta}_1, \hat{\theta}_2) = \boxed{\frac{12n^2}{(n+2)(n+1)^2}}$$

9.7 $f(y) = \frac{1}{\theta} e^{-y/\theta}$, $E(y) = \theta$, $V(Y) = \theta^2$

$\hat{\theta}_1 = nY_{(1)}$ $MSE(\hat{\theta}_1) = \theta^2$

$\hat{\theta}_2 = \bar{Y}$ $V(\hat{\theta}_2) = V\left(\frac{Y_1 + Y_2 \ldots + Y_n}{n}\right)$

$= \frac{1}{n^2} V(Y_1 + Y_2 + \ldots + Y_n) = \frac{1}{n^2} n\theta^2 = \frac{\theta^2}{n}$

Since $\hat{\theta}_1$ is unbiased, $b(\hat{\theta}_1) = 0$

$MSE(\hat{\theta}_1) = V(\hat{\theta}_1)$

$E(\bar{Y}) = E(Y_1 + Y_2 + \ldots + Y_n)\frac{1}{n} = \frac{n\theta}{n} = \theta$

So, $\bar{\theta}_2$ is unbiased, $b(\hat{\theta}_2) = 0$

oops, this is not necessary

$MSE(\hat{\theta}_2) = V(\hat{\theta}_2)$

$Eff(\hat{\theta}_1, \hat{\theta}_2) = \frac{V(\hat{\theta}_2)}{V(\hat{\theta}_1)} = \frac{\theta^2}{n} \cdot \frac{1}{MSE(\hat{\theta}_1)} = \frac{\theta^2}{n} \cdot \frac{1}{\theta^2} = \boxed{\frac{1}{n}}$