CS 230 : Discrete Computational Structures
Spring Semester, 2021
HOMEWORK ASSIGNMENT #8
Due Date: Wednesday, Apr 7

Suggested Reading: Rosen Sections 5.3; Lehman et al. Chapters 5, 6.1 - 6.2, 7

For the problems below, explain your answers and show your reasoning.

1. **[8 Pts]** Prove that $f_0 - f_1 + f_2 - \cdots - f_{2n-1} + f_{2n} = f_{2n-1} - 1$ where f_i are the Fibonacci numbers.

   (a) Base case: k = 1, $f_0 - f_1 + f_2 = f_1 - 1$ aka $0 - 1 + 1 = 1 - 1 = 0$

   (b) Induction Hypothesis: $f_0 - f_1 + f_2 - \cdots - f_{2k-1} + f_{2k} = f_{2k-1} - 1$

   (c) Prove: $f_0 - f_1 + f_2 - \cdots - f_{2k-1} + f_{2k} - f_{2(k+1)-1} + f_{2(k+1)} = f_{2(k+1)-1} - 1$

   (d) $(f_{2k-1} - 1) - f_{2(k+1)-1} + f_{2(k+1)} = f_{2(k+1)-1} - 1$ IH

   (e) $f_{2k-1} - f_{2k+1} + f_{2k+2} = f_{2k+1}$

   (f) $f_{2k-1} - f_{2k+1} + (f_{2k+1} + f_{2k}) = f_{2k+1}$ $f_{2k+2} = f_{2k+1} + f_{2k}$

   (g) $(f_{2k-1} + f_{2k}) = f_{2k+1}$

   (h) $f_{2k+1} = f_{2k+1}$ QED

2. **[12 Pts]** Let S defined recursively by (1) $4 \in S$ and (2) if $s \in S$ and $t \in S$, then $st \in S$. Let $A = \{4^i \mid i \in \mathcal{Z}^+\}$. Prove that

   (a) **[6 Pts]** $A \subseteq S$ by mathematical induction.

      i. Base case: k = 1, $4^1 = 4 \in S$

      ii. Induction Hypothesis: $s, t \in S$, $4^k = st \in S$

      iii. Prove: $x, y \in S$, $4^{k+1} = xy \in S$

      iv. $4 * 4^k = xy$

      v. $4 * st = xy$ IH

      vi. $4 \in S$ and $st \in S$, so $xy, 4^{k+1} \in S$ QED

   (b) **[6 Pts]** $S \subseteq A$ by structural induction.

      i. Base case: $4 = 4^1$, so $4 \in S$ and $4 \in A$

      ii. Induction Hypothesis: $x, y \in S$, assume $x, y \in A$. By (2), $xy \in S$

      iii. Prove: $xy \in A$

      iv. $x, y \in A$, so $x = 4^a$, $a \in Z^+$ and $y = 4^b$, $b \in Z^+$ IH

      v. $xy = 4^{a+b}$, $a + b \in Z^+$. Therefore, $xy \in A$ Def. of A; QED

3. **[5 Pts]** Define the set $S = \{2^k 3^m 5^p \mid k, m, p \in \mathcal{Z}\}$ inductively. You do not need to prove that your construction is correct.
   Basis: $k, m, p = 0$, $1 \in S$
   Inductive Step: if $a \in S$ and $b \in \{2, 3, 5\}$, then $ab \in S$

4. **[10 Pts]** Consider the following state machine with five states, labeled 0, 1, 2, 3 and 4. The start state is 0. The transitions are $0 \to 1$, $1 \to 2$, $2 \to 3$, $3 \to 4$, and $4 \to 0$.

   Prove that if we take n steps in the state machine we will end up in state 0 if and only if n is divisible by 5. Argue that to prove the statement above by induction, we first have to *strengthen the induction hypothesis.* State the strengthened hypothesis and prove it.

   (a) Predicate: P(n): After n steps, the state machine is in state 0 iff n is divisible by 5.

   (b) Basis: 5 | 0, so P(0) is true

   (c) Inductive Step: Assume P(k)

   (d) Prove: P(k + 1)

   (e) Suppose $5 \mid k$. By IH, statemachine is in 0 after k steps. However, $5 \not| \; k+1$ because the state machine is in state 1 after k + 1 steps. State machine is in state 0 after k + 1 steps iff $5 \mid k+1$. Both are false, therefore P(k + 1)

   (f) Should $5 \not| \; k$, By IH, state machine is in state 1, 2, 3, or 4. We cannot know if going 1 step after k steps will result in state 0. We need a stronger hypothesis to account for the other states.

   (g) Stronger Induction Hypothesis: P(n): After n steps, the state machine is in state p, where p = n % 5.

   (h) Prove $\forall n, P(n)$

      i. Basis: n = 0, state machine is in state 0 and 0 = 0%5

      ii. Induction Steps:

         A. Case 1: k % 5 = 0, By IH, in state 0 after k steps.
         B. Case 2: k % 5 = 1, By IH, in state 1 after k steps.
         C. Case 3: k % 5 = 2, By IH, in state 2 after k steps.
         D. Case 4: k % 5 = 3, By IH, in state 3 after k steps.
         E. Case 5: k % 5 = 4, By IH, in state 4 after k steps.

      iii. Proof by Case:

         A. Case 1: k + 1 % 5 = 1, By IH, in state 1 after k + 1 steps.
         B. Case 2: k + 1 % 5 = 2, By IH, in state 2 after k + 1 steps.
         C. Case 3: k + 1 % 5 = 3, By IH, in state 3 after k + 1 steps.
         D. Case 4: k + 1 % 5 = 4, By IH, in state 4 after k + 1 steps.
         E. Case 5: k + 1 % 5 = 0, By IH, in state 5 after k + 1 steps.

5. **[10 Pts]** A robot wanders around a 2-dimensional grid. He starts out at (0,0) and can take the following steps: (-1,+3), (+2,-2) and (+4,0). Define a state machine for this problem. Then, define a Preserved Invariant and prove that the robot will never get to (2,0).

$$\text{States: } \{(a,b) \mid a,b \in Z\}, \text{ Transitions:}$$
$$\{(a,b) \to (a-1,b+3), (a,b) \to (a+2,b-2), (a,b) \to (a+4,b)\}$$

(a) Preserved Invariant Theorem: For every reachable state $s = (a,b)$, $(a-b)\ \%\ 4 = 0$

(b) Basis: $(0,0)$ is reachable in 0 steps.

(c) Induction Hypothesis: Assume $(r_1 - r_2)\ \%\ 4 = 0$ for any state $r = (r_1, r_2)$ that is reachable in k steps.

(d) Prove: $(s_1 - s_2)\ \%\ 4 = 0$ for any state $s = (s_1, s_2)$ that is reachable in k + 1 steps.

(e) $r_1 - r_2 = 4n$, $n \in Z$ — Definition of $\%\ 4 = 0$

(f) 3 CASES:

i. CASE 1: reach s in k + 1 steps

ii. $(s_1, s_2) = (r_1 - 1, r_2 + 3)$

iii. $r_1 - 1 - (r_2 + 3) = 4n - 4$ — by IH

iv. $4(n-1)$, so $(s_1 - s_2)\ \%\ 4 = 0$

i. CASE 2: reach s in k + 1 steps

ii. $(s_1, s_2) = (r_1 + 2, r_2 - 2)$

iii. $r_1 + 2 - (r_2 - 2) = 4n + 4$ — by IH

iv. $4(n+1)$, , so $(s_1 - s_2)\ \%\ 4 = 0$

i. CASE 3: reach s in k + 1 steps

ii. $(s_1, s_2) = (r_1 + 4, r_2)$

iii. $r_1 + 4 - r_2 = 4n + 4$ — by IH

iv. $4(n+1)$, , so $(s_1 - s_2)\ \%\ 4 = 0$

QED

(g) Therefore, (2, 0) is not reachable because 2 - 0 % 4 = 2, not 0.

6. **[15 Pts]** Let $L = \{(a,b) \mid a,b \in \mathcal{Z}, (a-b) \bmod 4 = 0\}$. We want to program a robot that can get to each point $(x,y) \in L$ starting at $(0,0)$.

(a) **[5 Pts]** Give an inductive definition of L. This will describe the steps you want the robot to take to get to points in L starting at $(0,0)$. Let L' be the set obtained by your inductive definition.
L' is the set of all points, starting at (0, 0), that is reachable taking the following steps: (-1,+3), (+2,-2), and (+4,0)

(b) **Extra Credit [5 Pts]** Prove inductively that $L' \subseteq L$, i.e., every point that the robot can get to is in L.

i. Base case: (0, 0) is a reachable in 0 steps defined by L' and in L because $(0-0)\ \%\ 4 = 0$

ii. Inductive Hypothesis: Suppose $(x,y) \in L'$, therefore reachable by the defined steps.

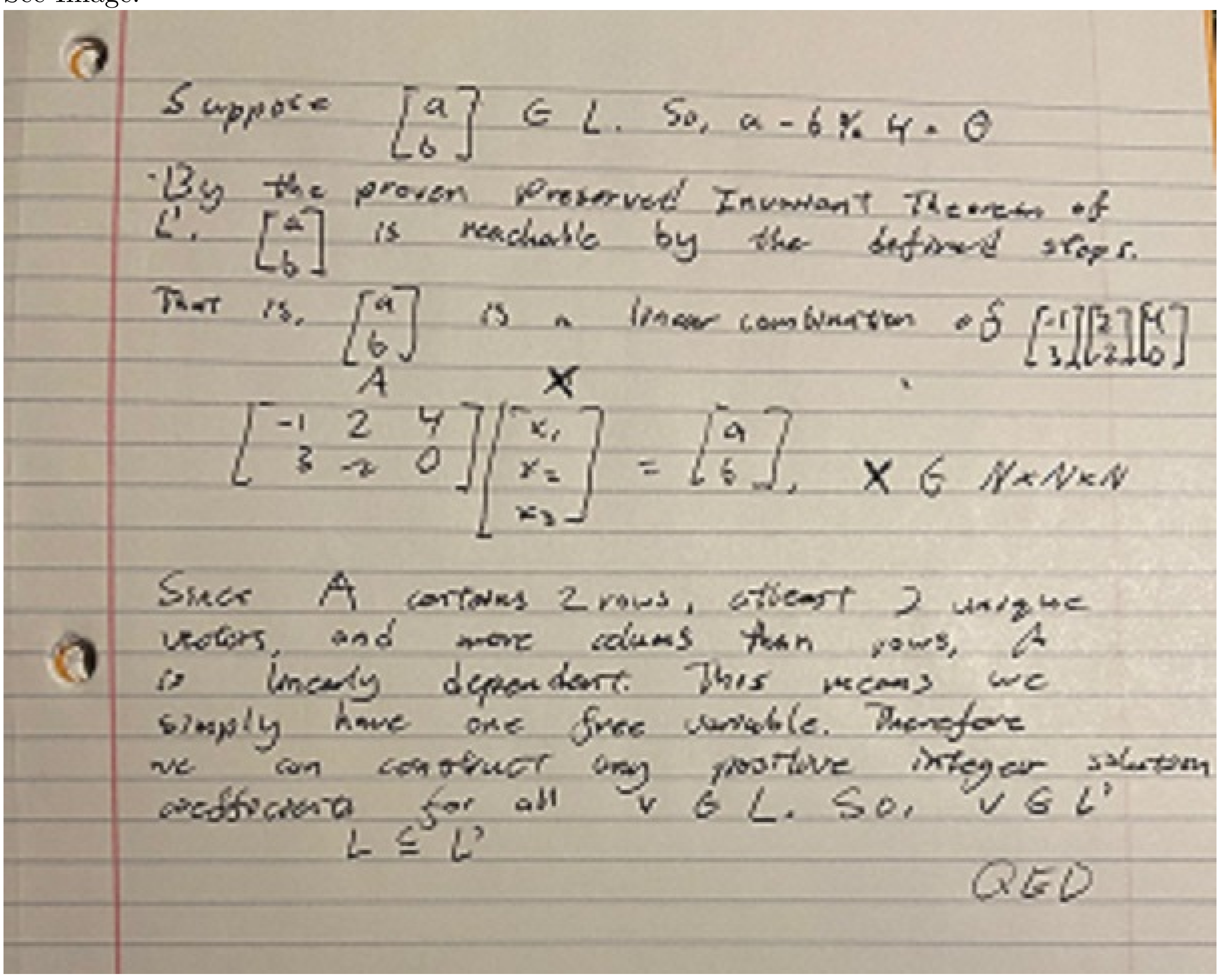
iii. Prove: $(x, y) \in L$.

iv. By the Preserved Invariant Theorem proved in question 5, if $(x, y) \in L'$, $(x-y)$ % $4 = 0$

v. By definition of L, $(x, y) \in L$ QED

(c) **Extra Credit [5 Pts]** Prove that $L \subseteq L'$, i.e., the robot can get to every point in L. To prove this, you need to give the path the robot would take to get to every point in L from $(0, 0)$, following the steps defined by your inductive rules.

See Image:



Suppose $\begin{bmatrix} a \\ b \end{bmatrix} \in L$. So, $a - b$ % $4 = 0$

By the proven Preserved Invariant Theorem of L', $\begin{bmatrix} a \\ b \end{bmatrix}$ is reachable by the defined steps.

That is, $\begin{bmatrix} a \\ b \end{bmatrix}$ is a linear combination of $\begin{bmatrix} -1 \\ 3 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} -1 & 2 & 4 \\ 3 & -2 & 0 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad X \in N \times N \times N$$

Since A contains 2 rows, atleast 2 unique vectors, and more colums than rows, A is linearly dependent. This means we simply have one free variable. Therefore we can construct any positive integer solution coefficients for all $v \in L$. So, $v \in L'$

$L \subseteq L'$

QED

For more practice, work on the problems from Rosen Sections 5.3; Lehman et al. Chapters 5, 6.1 - 6.2, 7