CS230-HW10Sol

1. **Ying 5 pts**
If we choose 2 people from a group of m men and n women, that is, $m+n$ total people, we can count that as $\binom{m+n}{2}$. This is the lhs.
If we choose 2 people from a group of m men and n women, either we pick 2 women, or 2 men, or 1 man and 1 woman. There are $\binom{n}{2}$ ways to pick 2 women, $\binom{m}{2}$ ways to pick 2 men, and mn ways to choose 1 of each, for $\binom{m}{2}+\binom{n}{2}+mn$ ways to pick 2 people. This is the rhs.
Therefore, $\binom{m+n}{2}=\binom{m}{2}+\binom{n}{2}+mn$.

2. **Ying 8 pts**
(a) For this problem, we have n people, from whom we want to choose a k-person committee with a president, a secretary and a treasurer. So the left-hand-side of the equals sign, $P(n,3)C(n-3,k-3)$, can be interpreted as: count the number of possibilities if we first select the president, secretary and treasurer, and then choose $k-3$ committee members from the $n-3$ people remaining. The right-hand-side of the equals sign, $C(n,k)P(k,3)$ can be interpreted as: count the number of possibilities if we first choose k committee members, and then choose the president, secretary and treasurer from those k committee members. Since the end result is the same, a count of the possible k-person committees with a president, a secretary and a treasurer, the two sides of the equals sign are, indeed, equal.
(b)
$P(n,3)C(n-3,k-3) = \frac{n!}{(n-3)!}\frac{(n-3)!}{(k-3)!(n-3-(k-3))!} = \frac{n!}{(k-3)!(n-k)!} = \frac{n!}{(k-3)!(n-k)!}\frac{k!}{k!} =$
$\frac{n!}{k!(n-k)!}\frac{k!}{(k-3)!} = C(n,k)P(k,3)$

3. **Modeste 6 pts**
(a) We must pick at least 2 of each cookie. So let us pick 2 of each cookie first. Then there are 6 cookies left with 5 possible choices for each. Total: $C(6+5-1,6) = C(10,6) = 210$ ways.

(b) First, count the number of ways to pick at least 4 oatmeal cookies. Then, subtract the ones with more than 4 chocolate chip cookies.
(i) At least 4 oatmeal cookies: pick 4 oatmeal cookies, and pick remaining 12 cookies with 5 options. Total: $C(12+5-1,12) = C(16,12) = 1820$ ways.
(ii) At least 4 oatmeal cookies and at least 4+1 chocolate chip cookies (more than 4 chocolate chip cookies): 9 cookies are already picked, so pick remaining 7 cookies with 5 options.
Total: $C(7+5-1,7) = C(11,7) = 330$ ways.
Answer: $1820-330 = 1490$ ways.

4. **Modeste 9 pts**
(a) This problem is the same as the problem of placing 24 indistinguishable balls (the points) into 4 distinguishable bins (the variables). Thus, the solution is $\binom{24+4-1}{24}=\binom{27}{24}$.

(b) Again, we have 24 indistinguishable balls and 4 distinguishable bins. But now, before making any choices, $2+3+4+5 = 14$ of the balls are already assigned to bins, since at least two must end up in x_1, at least 3 must end up in x_2 and so forth. So we are only at liberty to place 10 balls in the 4 bins, and the number of possible choices is $\binom{10+4-1}{10} = \binom{13}{10}$.

(c) Let's split this problem into two parts. First, the number of choices where $x_1 > 4$ is the same as counting how to place $24-5 = 19$ indistinguishable balls into 4 distinguishable bins, which evaluates to $\binom{19+4-1}{19} = \binom{22}{19}$. Second, the number of choices available where $x_3 < 5$ is the same as the total number of choices minus the number of choices where $x_3 \geq 5$. In this specific situation, we care about counting the choices where $x_1 > 4$ and $x_3 \geq 5$. This means that $5+5 = 10$ balls (points) are already placed in bins, so the choices remaining count to $\binom{14+4-1}{14} = \binom{17}{14}$. Therefore, the answer to the question is the total number of choices when $x_1 > 4$ minus the number of choices when $x_1 > 4$ and $x_3 \geq 5$. In other words, $\binom{22}{15} - \binom{17}{14}$.

5. **Ling 10 pts**

   (a) Since each committee has a different task, we must pick people for the first, pick people from the remaining for the second, pick people from the remaining for the third, and so on until each committee is filled.

   Therefore, $C(30,5)C(25,5)C(20,5)C(15,3)C(12,3)C(9,3)C(6,3)C(3,3) = \frac{30!}{5!5!5!3!3!3!3!3!}$.

   (b) If they have the same task, then there are two sets of indistinguishable bins for the objects to be placed in. The three 5-member committees are indistinguishable and the five 3-member teams are indistinguishable, so we divide by 3! and 5!. Therefore the answer is $\frac{30!}{(5!5!5!3!3!3!3!3!)(5!3!)}$.

   (c) If some of the committees have been given the same task while some have not been given a task yet, we have a set of indistinguishable committees we can process like in (b), but taking special care to consider the remaining committees with no task. Since there are three 3-member committees without a task, two of the five 3-member teams are distinct from the other three. So, we divide by 3!, 2! and 3!. Therefore the answer is $\frac{30!}{(5!5!5!3!3!3!3!3!)(3!2!3!)}$.

6. **Ling 6 pts**

   We will solve the second problem first. If the books and boxes are identical, then the problem is "how many ways can 6 be written as a sum of positive integers?" The ways are:

   6, 5+1, 4+2, 4+1+1, 3+3, 3+2+1, 3+1+1+1, 2+2+2, 2+2+1+1, 2+1+1+1+1, 1+1+1+1+1+1
   for a total of 11 ways.

   If the books are different, then the problem breaks into the following cases, where each number represents a box containing that many books (empty boxes are left out for convenience):

   1,1,1,1,1,1: There is only one way to put one book in each box.
   2,1,1,1,1: There are $\binom{6}{2}$ ways to pick a pair of books to share a box.
   2,2,1,1: There are $\binom{6}{2}$ ways to pick the first pair, then $\binom{4}{2}$ ways to pick the second pair.

However, picking (A,B) then (C,D) is the same as picking (C,D) then (A,B), so this counts twice the number of solutions. We need to divide by 2 for an accurate count.
2,2,2: $\binom{6}{2}$ ways to pick the first pair, then $\binom{4}{2}$ ways to pick the second pair, then divide by 3! since the order among the three pairs doesn't matter.
3,1,1,1: $\binom{6}{3}$ ways to pick the triplet.
3,2,1: $\binom{6}{3}$ ways to pick the triplet, then $\binom{3}{2}$ ways to pick the pair.
3,3: $\binom{6}{3}$ ways to pick the triplet, then divide by 2 for the same reason as the 2,2,1,1 case.
4,1,1: $\binom{6}{4}$ ways to pick the quadruplet.
4,2: $\binom{6}{4}$ ways to pick the quadruplet.
5,1: $\binom{6}{5}$ ways to pick the quintuplet.
6: There is only one way to put all the books in the same box.
The sum is 1+15+15*6/2+15*6/6+20+20*3+20/2+15+15+6+1=203 ways to put 6 different books into 6 identical boxes.

7. **Jonathan 6 pt**

   (a) We use the "stars and bars" counting method, with 12 stars and 4 bars, because 4 bars are enough to divide the books into 5 separate areas. So the answer is $\binom{12+5-1}{12} = \binom{16}{12} = 1820$.

   (b) Order all the books first, then count the number of ways they can be put on the shelves in that order. The number of ways to order the books is 12!, while the number of ways a single ordering can be put on the shelves is the answer to (a). So the answer to (b) is $12!\binom{16}{12}$.