CS 230 : Discrete Computational Structures
Spring Semester, 2021
ASSIGNMENT #10
Due Date: Monday, April 26

Suggested Reading: Rosen Sections 6.4 - 6.5.

These are the problems that you need to turn in. Always explain your answers and show your reasoning. **Spend time giving a complete solution. You will be graded based on how well you explain your answers. Just correct answers will not be enough!**

1. **[5 Pts]** Prove, using a combinatorial argument, that $C(m+n,2) = C(m,2) + C(n,2) + mn$, where $m, n \geq 2$. To make your combinatorial argument, describe a problem that both the *lhs* and *rhs* expressions count.
   (a) I have two raffles: Raffle A with m participants and raffle B with n participants
   (b) How many ways are there to pick 2 tickets from raffle A and B pooled? Choose 2 from m+n
   (c) 3 possibilities: pick 2 only from Raffle A or Raffle B, or pick one from each bag
   (d) Choose 2 from m + Choose 2 from n + Choose 1 from m and 1 from n
   (e) $C(m+n,2) = C(m,2) + C(n,2) + mn$ QED

2. **[8 Pts]** Prove, (a) using a combinatorial argument, and (b) using an algebraic proof, that $P(n,3)C(n-3,k-3) = C(n,k)P(k,3)$.
   (a) i. Imagine the LHS, from n writers, is picking three writers for 1st-3rd place then selecting $k-3$ honorable mentions in no particular order from the remaining $n-3$ writers.
   ii. LHS gurantees 3 winners and $k-3$ honorable mentions. This results in k potential victors.
   iii. Imagine RHS is picking k potential winners from n writers and then selecting 1st-3rd place from that list of k writers, and the $k-3$ not chosen are honorable mentions.
   iv. This is removing 3 winners from a pool of 3 potential victors. It results in $k-3$ honorable mentions and 3 winners.
   v. LHS and RHS are the same QED
   (b) i. $\frac{n!}{(n-3)!} * \frac{(n-3)!}{(k-3)!(n-3-(k-3))!} = \frac{n!}{(k)!(n-k)!} * \frac{k!}{(k-3)!}$
   ii. $\frac{n!}{1} * \frac{1}{(k-3)!(n-k)!} = \frac{n!}{(n-k)!} * \frac{1}{(k-3)!}$
   iii. $\frac{n!}{(n-k)!(k-3)!} = \frac{n!}{(n-k)!(k-3)!}$ QED

3. **[6 Pts]** A cookie shop sells 5 different kinds of cookies. How many different ways are there to choose 16 cookies if (a) you pick at least two of each? (b) you pick at least 4 oatmeal cookies and at most 4 chocolate chip cookies?

   (a) i. 10 cookies already chosen. Choose 6 cookies to place in 5 bins
       ii. $\frac{(6+5-1)!}{6!4!}$
       iii. $\frac{10!}{6!4!}$

   (b) i. Count combos where 4 oatmeal already picked
       ii. Subtract where 4 oatmeal already picked AND 5 or more chocolate chips picked
       iii. $\frac{(12+5-1)!}{12!4!} - \frac{(7+5-1)!}{7!4!}$
       iv. $\frac{16!}{12!4!} - \frac{11!}{7!4!}$

4. **[9 Pts]** How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 24$, where x_i is a non-negative integer, for all i, if (a) there are no restrictions? (b) $x_1 > 1$, $x_2 > 2$, $x_3 > 3$, $x_4 > 4$? (c) $x_1 > 4$ and $x_3 < 5$?

   (a) i. 24 objects in 4 bins.
       ii. $\frac{(24+4-1)!}{24!3!}$
       iii. $\frac{27!}{24!3!}$

   (b) i. 14 items already placed. choose 10 to place into 4 bins
       ii. $\frac{(10+4-1)!}{10!3!}$
       iii. $\frac{13!}{10!3!}$

   (c) i. 5 objects already placed. Compute that total
       ii. Subtract the combinations where $x_1 > 4$ (5 objects placed) AND $x_3 \geq 5$ (5 objects placed)
       iii. $\frac{(19+4-1)!}{19!3!} - \frac{(14+4-1)!}{14!3!}$
       iv. $\frac{22!}{19!3!} - \frac{17!}{14!3!}$

5. **[10 Pts]** How many ways are there to split 30 people into three committees of 5 people each and five committees of 3 people each if (a) all eight committees have different tasks? (b) all eight committees have the same task? (c) the three 5-member committees and two of the 3-member committees are all given the same task while the remaining three 3-member committees are not given any task yet?

   (a) i. 30 choose 5 * 25 choose 5 * 20 choose 5 * 15 choose 3 * 12 choose 3 * 9 choose 3 * 6 choose 3 * 3 choose 3
       ii. $\frac{30!}{5!25!} * \frac{25!}{5!20!} * \frac{20!}{5!15!} * \frac{15!}{3!12!} * \frac{12!}{3!!} * \frac{9!}{3!6!} * \frac{6!}{3!3!} * \frac{3!}{3!0!}$
       iii. $\frac{30!}{5!5!5!3!3!3!3!3!}$

   (b) i. Each pair of committees of size 5 is interchangeable and each pair of committees of size 3 is interchangeable. Division Rule on part a answer:
       ii. $\frac{30!}{(5!5!5!*3!)(3!3!3!3!3!*5!)}$

   (c) i. Use division rule on committees that are indistinguishable (same/no task)
       ii. $\frac{30!}{(5!5!5!*3!)(3!3!*2!)(3!3!3!*3!)}$

6. **[6 Pts]** How many ways are there to pack 6 different books into 6 identical boxes with no restrictions placed on how many can go in a box (some boxes can be empty)? What if the books are identical?

   (a) i. 6 distinguishable objects in 6 identical bins: 11 Cases enumerated by brute force: 6 — 5,1 — 4,2 — 4,1,1 — 3,3 — 3,2,1 — 3,1,1,1

   ii. 6: 6 choose 6 = 1

   iii. 5,1: 6 choose 5 ways = 6

   iv. 4,2: 6 choose 4 $= \frac{6!}{4!2!} = 15$

   v. 4,1,1: 6 choose 4 $= \frac{6!}{4!2!} = 15$

   vi. 3,3: 6 choose 3, division rule to remove duplciates $= \frac{6!}{3!3!}/2 = 10$

   vii. 3,2,1: 6 choose 3 $= \frac{6!}{3!3!} = 20$

   viii. 3,1,1,1: 6 choose 3 $= \frac{6!}{3!3!} = 20$

   ix. 2,2,2: 6 choose 2, division rule to remove duplicates $= \frac{6!}{2!4!}/3 = 5$

   x. 2,2,1,1: 6 choose 2 $= \frac{6!}{2!4!} = 15$

   xi. 2,1,1,1,1: 6 choose 2 $= \frac{6!}{2!4!} = 15$

   xii. 1,1,1,1,1,1: 6 choose 1, division rule to remove duplicates $= \frac{6!}{1!5!}/6 = 1$

   xiii. $1 + 6 + 15 + 15 + 10 + 20 + 20 + 5 + 15 + 15 + 1 = 123$ ways

   b) 6 identical objects in 6 identical bins: Just the # of cases! 11 ways

7. **[6 Pts]** How many ways can we place 12 books on a bookcase with 5 shelves if the books are (a) indistinguishable copies (b) all distinct? Note that the position of the books on the shelves matter.
   a) 12 identical objects in 5 distinguishable bins: $\frac{(12+5-1)!}{12!4!} = \frac{16!}{12!4!}$
   b) 12! ways to order the 12 books. There are 13 spots to place 4 divideres $13 * 4$ (5 bins). Every permutation of books has $13 * 4$ ways to place the dividers. $12! * 52$

For more practice, work on the problems from Rosen Sections 6.4 - 6.5.