CS 230 : Discrete Computational Structures
Spring Semester, 2021
ASSIGNMENT #7
Due Date: Monday, March 29

Suggested Reading: Rosen Section 5.1 - 5.2; Lehman et al. Chapter 5.1 - 5.3

These are the problems that you need to turn in. For more practice, you are encouraged to work on the other problems. **Always explain your answers and show your reasoning.**

For Problems 1-4 and 6, prove the statements by mathematical induction. Clearly state your basis step and prove it. What is your inductive hypothesis? Prove the inductive step and show clearly where you used the inductive hypothesis.

1. **[5 Pts]** $n + 3 < 5n^2$, for all positive integers n.
   (a) Base case: $(1) + 3 < 5(1)^2$, $4 < 5$
   (b) Induction Hypothesis: Assume $k + 3 < 5k^2$
   (c) Prove $(k + 1) + 3 < 5(k + 1)^2$
   (d) $(k + 3) + 1 < 5k^2 + 10k + 5$
   (e) by IH, $(k + 3 < 5k^2)$. It now follows that we should prove $1 < 10k + 5$
   (f) $1 < 10k + 5$ because 1 is always less than 5, regardless of positive k.
   (g) Therefore, $(k + 3) + 1 < 5k^2 + 10k + 5$; QED.

2. **[5 Pts]** $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n + 1)! - 1$, for all positive integers n.
   (a) Base case: $(1) * (1)! = ((1) + 1)! - 1 = 1$
   (b) Induction Hypothesis: Assume $1 * 1! + 2 * 2! + ... + k * k! = (k + 1)! - 1$, $1 * 1! + 2 * 2! + ... + k * k! + 1 = (k + 1)!$
   (c) Prove $1 * 1! + 2 * 2! + ... + (k + 1) * (k + 1)! = (k + 1) + 1)! - 1$
   (d) $1 * 1! + 2 * 2! + ... + k * k! + (k + 1) * (k + 1)! = (k + 2)! - 1$
   (e) $1 * 1! + 2 * 2! + ... + k * k! + (k + 1) * (k + 1)! = (k + 1)!(k + 2) - 1$
   (f) $1 * 1! + 2 * 2! + ... + k * k! + 1 + (k + 1) * (k + 1)! = (k + 1)!(k + 2)$
   (g) By IH, $(k + 1)! + (k + 1) * (k + 1)! = (k + 1)!(k + 2)$
   (h) Dividing both sides by $(k + 1)!$: $1 + (k + 1) * 1 = 1 * (k + 2)$
   (i) Collecting terms, $k + 2 = k + 2$; QED

3. **[5 Pts]** $\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\cdots+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$, for all positive integers n.

   (a) Base case: $\frac{1}{(1)\cdot(1+1)}=\frac{(1)}{(1)+1}=\frac{1}{2}$

   (b) Induction Hypothesis: Assume $\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\cdots+\frac{1}{k\cdot(k+1)}=\frac{k}{k+1}$

   (c) Prove $\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\cdots+\frac{1}{k\cdot(k+1)}+\frac{1}{(k+1)\cdot((k+1)+1)}=\frac{(k+1)}{(k+1)+1}$

   (d) $\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\cdots+\frac{1}{k\cdot(k+1)}+\frac{1}{(k+1)\cdot(k+2)}=\frac{k+1}{k+2}$

   (e) By IH, $\frac{k}{k+1}+\frac{1}{(k+1)\cdot(k+2)}=\frac{k+1}{k+2}$

   (f) $\frac{k(k+2)}{(k+1)(k+2)}+\frac{1}{(k+1)\cdot(k+2)}=\frac{k+1}{k+2}$

   (g) $\frac{k^2+2k+1}{(k+1)(k+2)}=\frac{k+1}{k+2}$

   (h) $\frac{(k+1)(k+1)}{(k+1)(k+2)}=\frac{k+1}{k+2}$

   (i) $\frac{k+1}{k+2}=\frac{k+1}{k+2}$; QED

4. **[5 Pts]** 15 divides $4^{2n}-1$, for all natural numbers n.

   (a) Base case: $\frac{4^{2(1)}-1}{15}=1$

   (b) Induction Hypothesis: $\frac{4^{2k}-1}{15}=A$, $4^{2k}=15A+1$, $A\in N$

   (c) Prove $\frac{4^{2(k+1)}-1}{15}=B$, $B\in N$

   (d) $4^{2k+2}=15B+1$

   (e) $4^2*4^{2k}=15B+1$

   (f) By IH, $4^2*(15A+1)=15B+1$

   (g) $16*(15A+1)=15B+1$

   (h) $16*15A+16=15B+1$

   (i) $16*15A=15B-15$

   (j) $16A=B-1$

   (k) $B=16A+1$

   (l) Since $A\in N$, $B=16A+1\in N$; QED

5. **[9 Pts]** Let $P(n)$ be the statement that n-cent postage can be formed using just 4-cent and 7-cent stamps. Prove that $P(n)$ is true for all $n\geq18$, using the steps below.

   (a) First, prove $P(n)$ by regular induction. State your basis step and inductive step clearly and prove them.

      i. Base Case: $P(18)$, $18=4+(2)7$

      ii. Inductive Hypothesis: $k\geq18$ can be made by some linear combination of 4 and 7

      iii. Prove $k+1$ can be made by some linear combination of 4 and 7

      iv. CASE 1: Atleast 1 7 is used. Replace a 7 with 2 4's, $k-7+2(4)=k+1$

      v. CASE 2: No 7's, $k=4a$, $a\in N$. Since $k\geq18$, $4a\geq20$, $a\geq5$. Atleast 5 4's. Replace 5 4's with 3 7's. $k-(5)4+(3)7=k+1$

QED

(b) Now, prove $P(n)$ by strong induction. Again, state and prove your basis step and inductive step. Your basis step should have multiple cases.

i. Base Cases: $P(18)$, $18 = 4 + (2)7$; $P(19)$, $19 = (3)4 + 7$, $P(20)$, $20 = (5)4$, $P(21)$, $21 = (3)7$

ii. Inductive Hypothesis: Let $k \geq 21$. Assume all l where $18 \leq l \leq k$ can be formed using 4 and 7.

iii. Prove $k + 1$ can be formed.

iv. Since $k \geq 21$, $k - 3 \geq 18$, so $k - 3$ is possible by IH

v. $k - 3 + (1)4 = k + 1$.

QED

6. **[6 Pts]** Use mathematical induction to prove that DeMorgan's Law holds for the intersection of n sets, $n \in \mathcal{Z}^+$:

$$\overline{\left(\bigcap_{i=1}^{n} A_i\right)} = \bigcup_{i=1}^{n} \overline{A_i}$$

You may use DeMorgan's Law for two sets.

(a) Base Case: $\overline{A_1 \cap A_2 \cap A_3}$

i. $\overline{A_1 \cap (A_2 \cap A_3)}$ Intersection is Associative

ii. $\overline{A_1} \cup \overline{A_2 \cap A_3)}$ De Morgan's for Two Sets

iii. $\overline{A_1} \cup \overline{A_3} \cup \overline{A_3}$ De Morgan's for Two Sets

QED

(b) Induction Hypothesis: $\overline{A_1 \cap A_2 \cap ... \cap A_k} = \overline{A_1} \cup \overline{A_2} \cup ... \cup \overline{A_k}$

(c) Prove: $\overline{A_1 \cap A_2 \cap ... \cap A_k \cap A_{k+1}} = \overline{A_1} \cup \overline{A_2} \cup ... \cup \overline{A_k} \cup \overline{A_{k+1}}$

i. $\overline{(A_1 \cap A_2 \cap ... \cap A_k) \cap A_{k+1}}$ Intersection is Associative

ii. $\overline{A_1 \cap A_2 \cap ... \cap A_k} \cup \overline{A_{k+1}}$ De Morgan's for Two Sets

iii. $\overline{A_1} \cup \overline{A_2} \cup ... \cup \overline{A_k} \cup \overline{A_{k+1}}$ By IH

QED

For more practice, you are encouraged to work on other problems in Rosen Sections 5.1 and 5.2, like the ones below.

1. Rosen Section 5.1 Problem 4
2. Rosen Section 5.1 Problem 12
3. Rosen Section 5.1 Problem 31
4. Rosen Section 5.2 Problem 26