COM S 230 HW03

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1. Prove (p^3 is odd) $\iff$ (p is odd)

   Assume p^3 is odd

   $$\begin{aligned}&p^3 = 2n+1, n \in \mathbb{Z}\\&2n = p^3 - 1\\&2n = (p-1)(p^2+p+1)\end{aligned}$$

   $(p^2+p) = p(p+1)$ is even, because it is the product of an even and odd integer
   Therefore, p^2+p+1 is odd
   If p^2+p+1 is odd and $2n$ is even, then $p-1$ must be even for $2n = (p-1)(p^2+p+1)$ to remain consistent
   If $p-1$ is even, then p must be odd; QED

2. If x, y is rational and z is irrational, prove $x+yz$ is irrational
   We cannot use a diret proof because of the nature of irrational numbers
   For proof by contradiction, assume that $x+yz$ is rational, such that

   $$x = \tfrac{a}{b},\ y = \tfrac{c}{d}, \text{ and } \tfrac{a}{b} + \tfrac{c}{d}z = \tfrac{e}{f}$$
   $$z = (\tfrac{e}{f} - \tfrac{a}{b})\tfrac{d}{c}$$

   Left hand side is z, an irrational, and right hand side is sum/product of rational integers, which is rational. This is a contradiction. $x+yz$ cannot be rational; QED

3. Prove that if $mn > 35$, then $m \geq 6 \vee n \geq 8$

   (1) If $\neg(m \geq 6 \vee n \geq 8)$, then $mn \leq 35$ — Contrapositive of proposition
   (2) $\neg(m \geq 6 \vee n \geq 8)$ — Assumed
   (3) $(m < 6 \wedge n < 8)$ — De Morgan's (2)
   (4) $(m \leq 5 \wedge n \leq 7)$ — implied in (3) because m & n are integers
   (5) Because of (4), $mn \leq 35$
   (6) Contrapositive (1) of proposition is satisfied; QED

4. $fresh + soph + jun + sen = 32$. Prove $fresh \geq 5 \vee soph \geq 8 \vee jun \geq 10 \vee sen \geq 7$

   (1) $(fresh + soph + jun + sen = 32) \rightarrow (fresh \geq 5 \vee soph \geq 8 \vee jun \geq 10 \vee sen \geq 7)$ — Hyp. 1
   (2) $\neg(fresh \geq 5 \vee soph \geq 8 \vee jun \geq 10 \vee sen \geq 7) \rightarrow (fresh + soph + jun + sen \neq 32)$ — Contrapositive (1)

(3) $\neg(fresh \geq 5 \vee soph \geq 8 \vee jun \geq 10 \vee sen \geq 7)$ Assumed

(4) $fresh < 5 \wedge soph < 8 \wedge jun < 10 \wedge sen < 7)$ De Morgan's (3)

(5) $fresh \leq 4 \wedge soph \leq 7 \wedge jun \leq 9 \wedge sen \leq 6)$ Implied in (4) because all vars are ints

(6) $(fresh + soph + jun + sen \neq 32)$ implied by (5); all vars could be 0

(7) (2) is satisfied; QED

5. Prove $(p \geq 3 \vee p \leq -7) \rightarrow ((p+2)^2) \geq 25$

   (1) Case 1: $p \geq 3$, add 2: $p + 2 \geq 5$, square: $(p+2)^2 \geq 25$

   (2) Case 2: $p \leq -7$, add 2: $p + 2 \leq -5$, square: $(p+2)^2 \geq 25$

   (3) (1) & (2); QED

6. Prove $\sqrt{5}$ is irrational

   (1) Assume $\sqrt{5}$ is rational, that is $\sqrt{5} = \frac{a}{b}$ for some integers a, b.

   (2) Square it. $5 = \frac{a^2}{b^2}$

   (3) $5b^2 = a^2$

   (4) This implies 5 is a factor of a^2 or $a * a$

   (5) According to StackExchange, If a square is divisible by a prime, then its root is also divisible by the prime

   (6) (4) and (5) imply that 5 is a factor of a, and $a = 5x$ for some int x

   (7) $5b^2 = a^2$ and (6) implies that 5^2r^2 is a factor of b^2

   (8) (7) implies that 5 is also a factor of b.

   (9) (8) contradicts (1). a and b cannot have common factors; $\sqrt{5}$ is irrational; QED