Modern Robotics by Lynch and Park

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Abstract

These are my study notes on Robotics to prepare myself for PhD applications. The notes follow the structure of the Northwestern Coursera course and the book by Lynch and Park.

1 Chapter 2 -Topology and Representation, Module 3 - Space Topology and Constraints

1.1 Topology

Until now we have been focusing on one important aspect of a robot's C-space– its dimension, or the number of degrees of freedom. However, the shape of the space is also important.

The idea that the two-dimensional surfaces of a small sphere, a large sphere, and a football all have the same kind of shape, which is different from the shape of a plane, is expressed by the topology of the surfaces. We do not attempt a rigorous treatment in this book,3 but we say that two spaces are **topologically equivalent if one can be continuously deformed into the other without cutting or gluing.**

The circle is mathematically written as S or S^1, a one-dimensional sphere. The line can be written as E or E^1, indicating a one dimensional Euclidean space. A point on E is usually represented by a real number and is often written as R or R^1. A closed interval of the line, which contains its endpoints, can be written $[a, b] \subset R^1$. An open interval (a, b) does not include the endpoints a and b and is topologically equivalent to a line, since the open interval can be stretched to a line.

In higher dimensions, R^n is the n-dimensional Euclidean space and S n-dimensional surface of a sphere in (n + 1)-dimensional space. For example, S^2 is the two-dimensional surface of a sphere in three-dimensional space. Some C-spaces can be expressed as the Cartesian product of two or more spaces of lower dimension; that is, points in such a C-space can be represented as the union of the representations of points in the lower-dimensional spaces.

While it is natural to choose a reference frame and length scale and to use a vector to represent points in a Euclidean space, representing a point on a curved space, such as a sphere, is less obvious. One solution for a sphere is to use latitude and longitude coordinates. A choice of n coordinates, or parameters, to represent an n-dimensional space is called an **explicit parametrization** of the space. A disadvantage can be presence of singularities.

Implicit Representation views the n-dimensional space as embedded in a Euclidean space of more than n dimensions, just as a two-dimensional unit sphere can be viewed as a surface embedded in a three-dimensional Euclidean space. A disadvantage of this approach is that the representation has more numbers than the number of degrees of freedom. An advantage is that there are no singularities in the representation.

We will use implicit representations throughout the book, beginning in the next chapter. In particular, we use nine numbers, subject to six constraints, to represent the three orientation freedoms of a rigid body in space.

- The C-space of a rigid body in the plane can be written as $\mathbb{R}^2 \times S^1$, since the configuration can be represented as the concatenation of the coordinates (x, y) representing $\mathbb{R}^2$ and an angle θ representing S^1.
- The C-space of a PR robot arm can be written $\mathbb{R}^1 \times S^1$. (We will occasionally ignore joint limits, i.e., bounds on the travel of the joints, when expressing the topology of the C-space; with joint limits, the C-space is the Cartesian product of two closed intervals of the line.)
- The C-space of a 2R robot arm can be written $S^1 \times S^1 = T^2$, where T^n is the n-dimensional surface of a torus in an $(n+1)$-dimensional space. (See Table 2.2.) Note that $S^1 \times S^1 \times \cdots \times S^1$ (n copies of S^1) is equal to T^n, not S^n; for example, a sphere S^2 is not topologically equivalent to a torus T^2.
- The C-space of a planar rigid body (e.g., the chassis of a mobile robot) with a 2R robot arm can be written as $\mathbb{R}^2 \times S^1 \times T^2 = \mathbb{R}^2 \times T^3$.
- As we saw in Section 2.1 when we counted the degrees of freedom of a rigid body in three dimensions, the configuration of a rigid body can be described by a point in $\mathbb{R}^3$, plus a point on a two-dimensional sphere S^2, plus a point on a one-dimensional circle S^1, giving a total C-space of $\mathbb{R}^3 \times S^2 \times S^1$.

Figure 1: C-space and Cartesian Product

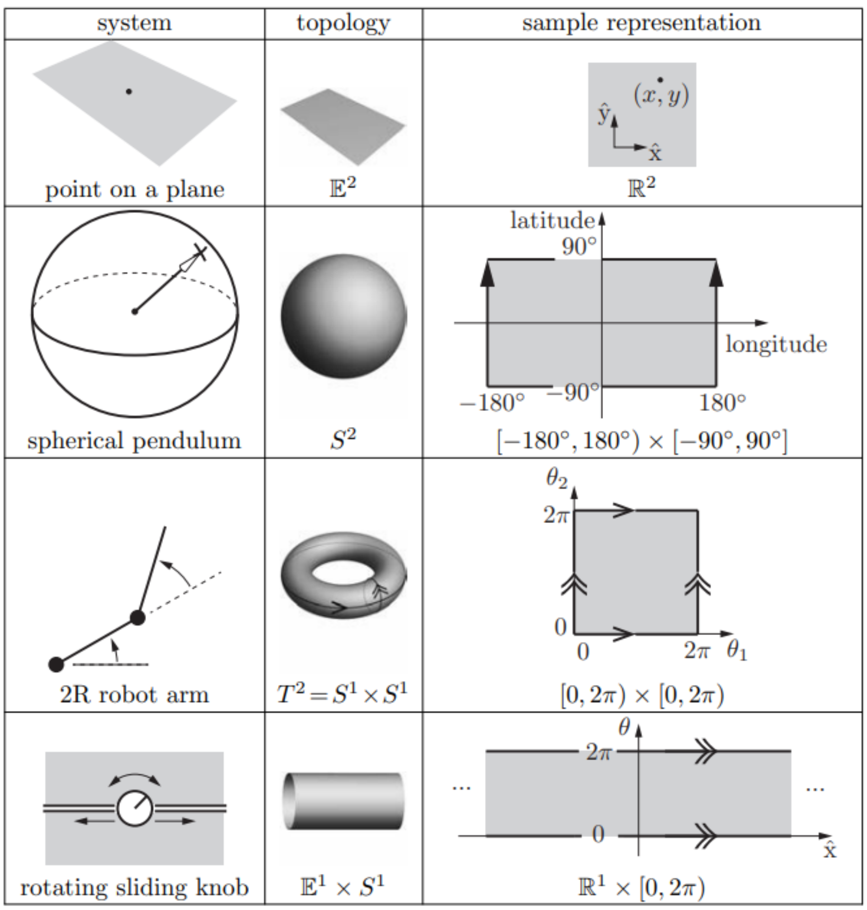

Figure 2: C-space: system, topology, representation

1.2 Configuration and Velocity Constraints

A holonomic constraint refers to a type of constraint that limits the motion of a system in such a way that it can be described using equations involving only **the system's position coordinates**. These constraints are integrable and directly reduce the number of degrees of freedom (DoF) of the system, i.e. the dimension of the C-space. Only limits the position! Holonomic constraints depend only on the system's position and can be expressed as an equation involving the system's configuration (e.g., coordinates like x,y,z). Example:

$x^2 + y^2 + z^2 = r^2$

Pfaffian constraints are the ones limiting velocities

$A(\theta)(\dot{\theta}) = 0$, One could regard $g(\theta)$ as being the integral of $A(\theta)$; for this reason, holonomic constraints of the form $g(\theta) = 0$ are also called integrable constraints - the velocity constraints that they imply can be integrated to give equivalent configuration constraints.

Consider a coin rolling on a plane without slipping.

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = r\dot{\theta} \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}$$

Collecting the four C-space coordinates into a single vector $q = [q_1, q_2, q_3, q_4]^T = [x, y, \phi, \theta]^T \in R^2 X T^2$

$$\begin{bmatrix} 1 & 0 & 0 & -r\cos q_3 \\ 0 & 1 & 0 & -r\sin q_3 \end{bmatrix} \dot{q} = 0$$

These constraints are not integrable. By inspection it should be clear that no such g1(q) exists etc. A Pfaffian constraint that is nonintegrable is called **a nonholonomic constraint.**

1.3 Task Space and Workspace

We now introduce two more concepts relating to the configuration of a robot: **the task space** and the workspace. Both relate to the configuration of the end-effector of a robot, not to the configuration of the entire robot. The task space is a space in which the robot's task can be naturally expressed.

The workspace is a specification of the configurations that the end-effector of the robot can reach. The definition of the workspace is primarily driven by the robot's structure, independently of the task.

1.4 Summary

1. A robot is mechanically constructed from links that are connected by various types of joint. The links are usually modeled as rigid bodies. An end-effector such as a gripper may be attached to some link of the robot. Actuators deliver forces and torques to the joints, thereby causing motion of the robot.

2. The most widely used one-dof joints are the revolute joint, which allows rotation about the joint axis, and the prismatic joint, which allows translation in the direction of the joint axis. Some common two-dof joints include the cylindrical joint, which is constructed by serially connecting a revolute and prismatic joint, and the universal joint, which is constructed by orthogonally connecting two revolute joints. The spherical joint, also known as the ball-and-socket joint, is a three-dof joint whose function is similar to the human shoulder joint.

3. The configuration of a rigid body is a specification of the location of all its points. For a rigid body moving in the plane, three independent parameters are needed to specify the configuration. For a rigid body moving in three-dimensional space, six independent parameters are needed to specify the configuration.

4. The configuration of a robot is a specification of the configuration of all its links. The robot's configuration space is the set of all possible robot configurations. The dimension of the C-space is the number of degrees of freedom of a robot.

5. The number of degrees of freedom of a robot can be calculated using Grubler's formula.

6. A robot's C-space can be parametrized explicitly or represented implicitly. For a robot with n degrees of freedom, an explicit parametrization uses n coordinates, the minimum necessary. An implicit representation involves m coordinates with $m => n$, with the m coordinates subject to $m - n$ constraint equations. With an implicit parametrization, a robot's C-space can be viewed as a surface of dimension n embedded in a space of higher dimension m.

7. A robot's motion can also be subject to velocity constraints of the form $A(\theta)\dot{\theta} = 0$ where $A(\theta)$ is k by m. that cannot be expressed as the differential of some function $g(\theta)$. Such constraints are said to be nonholonomic constraints, or nonintegrable constraints. These constraints reduce the dimension of feasible velocities of the system but do not reduce the dimension of the reachable C-space. Nonholonomic constraints arise in robot systems subject to conservation of momentum or rolling without slipping.

8. A robot's task space is a space in which the robot's task can be naturally expressed. A robot's workspace is a specification of the configurations that the end-effector of the robot can reach.

1.5 Exercises from Module 3

1. To deform one n-dimensional space into another topologically equivalent space, which operations are you allowed to use?
   (a) stretching

2. True or false? An n-dimensional space can be topologically equivalent to an m-dimensional space, where $m! = n$.
   (a) False. Deforming a space by stretching cannot change its dimension.

3. True or false? An explicit parametrization uses fewer numbers to represent a configuration than an implicit representation.
   (a) True

4. A k-dimensional space is represented by 7 coordinates subject to 3 independent constraints. What is k?
   (a) 4

5. True or false? A nonholonomic constraint implies a configuration constraint.
   (a) False. A nonholonomic constraint is a velocity constraint that cannot be integrated to a configuration constraint.

6. True or false? A Pfaffian velocity constraint is necessarily nonholonomic.
   (a) False. A Pfaffian velocity constraint can be nonholonomic (nonintegrable), or it can be the derivative of a holonomic configuration constraint.

7. A wheel moving in free space has the six degrees of freedom of a rigid body. If we constrain it to be upright on a plane (no "leaning") and to roll without slipping, how many holonomic and nonholonomic constraints is the wheel subject to?
   (a) Two holonomic constraints and two nonholonomic constraints.

8. How many degrees of freedom does the upright wheel on the plane have?
   (a) 4. Two variables specify the contact point on the plane, a third specifies the point on the wheel in contact with the ground, and a fourth specifies the heading direction of the wheel.

9. If the task is to control the orientation of a spaceship simulator, but not its position, how many degrees of freedom does the task space have?

   (a) 3. The space of orientations has 3 degrees of freedom.

10. True or false? The workspace depends on the robot's joint limits but the task space does not.

    (a) The task space depends on the task, not the robot. The workspace of a particular robot is independent of the task.

11. The chassis of a mobile robot moving on a flat surface can be considered as a planar rigid body. Assume that the chassis is circular, and the mobile robot moves in a square room. Which of the following could be a mathematical description of the C-space of the chassis while it is confined to the room?

    (a) $[a, b]X[a, b]XS^1$

12. Which of the following is a possible mathematical description of the C-space of a rigid body in 3-dimensional space?

    (a) $R^3XS^2XS^1$

13. A spacecraft is a free-flying rigid body with a 7R arm mounted on it. The joints have no joint limits. Give a mathematical description of the C-space of this system. (See Chapter 2.3.1 for related discussion.)

    (a) $R^3XS^2XT^8$