16 Aug - 2023
Lec - 2

WAVE EQN IN THE STRING :

GOAL : To derive the 1D wave eqⁿ in a string.

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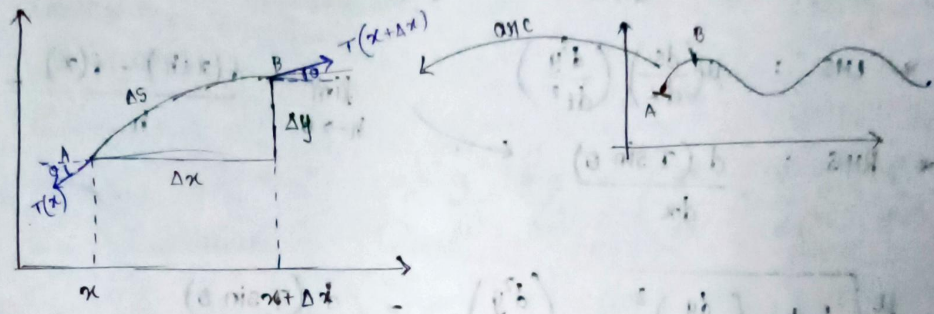


$$\therefore \Delta S^2 = \Delta x^2 + \Delta y^2 \longleftarrow \text{Pythogorus Theorem}$$

for small value ΔS is st. line.

$$\Rightarrow \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

* The magnitude of the tension point x & $x + \Delta x$ are resp. $T(x)$ & $T(x+\Delta x)$

* The net horizontal line is given as
$$[T(x+\Delta x) \cos\theta - T(x)\cos\theta]\,\hat{i}$$

* The net vertical line is given as :
$$[T(x+\Delta x)\sin\theta - T(x)\sin\theta]\,\hat{j}$$

* Let $\mu \longrightarrow$ mass per unit length.

* Because we are interested in the vertical ~~ortho~~ dispⁿ so, we are going to neglect horizontal motion.

* The notation of the transverse dispⁿ is denoted by y as a funcⁿ of x & t $[y(x,t)]$

* Using Newton's second law $[F = ma]$
$$\therefore [T(x+\Delta x)\sin\theta - T(x)\sin\theta] = (\mu \Delta S)\frac{d^2y}{dt^2}$$

now ~~multip~~ divide Δx both sides :

$$\left(\mu \frac{\Delta s}{\Delta x}\right) \frac{d^2y}{dt^2} = \frac{\left[T(x+\Delta x)\sin\theta - T(x)\sin\theta\right]}{\Delta x}$$

Take the limit $\Delta x \to 0$:

* LHS : $\mu\left(\frac{ds}{dx}\right)\left(\frac{d^2y}{dt^2}\right)$

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

* RHS : $\frac{d\,(T\sin\theta)}{dx}$

* $$\mu\sqrt{1+\left(\frac{dy}{dx}\right)^2}\left(\frac{d^2y}{dt^2}\right) = \frac{d\,(T\sin\theta)}{dx}$$

from pre page $\frac{\Delta s}{\Delta x} = (\ )$

we know, $\sin\theta = \frac{\cancel{T\sin}\ \tan\theta}{\sqrt{1+\tan^2\theta}}$

and also, $\tan\theta = \frac{dy}{dx} \to$ slope

$$\Rightarrow \mu\sqrt{1+\left(\frac{dy}{dx}\right)^2}\left(\frac{d^2y}{dt^2}\right) = \frac{d}{dx}\left(\frac{T\left(\frac{dy}{dx}\right)}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right)$$

* Let's assume, the tension in string T is continuous

we also assume $\left|\frac{dy}{dx}\right| << 1$

* $$\mu \frac{d^2y}{dt^2} = T\frac{d^2y}{dx^2} \longrightarrow \frac{T}{\mu} = \left[\frac{\text{force}}{\text{mass / length}}\right] = (\text{velocity})^2$$

$\Rightarrow \cancel{\frac{T}{\mu} = d}$

$$\Downarrow$$

$$\frac{1}{u^2}\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2}$$

velocity.

$$v^2 = \frac{T}{\mu}$$

$$\therefore v = \sqrt{\frac{T}{\mu}}$$

$$\frac{1}{u^2}\frac{d^2 \#}{dt^2} = \frac{d^2 \#}{dx^2} \xleftrightarrow[sp]{2D} \frac{1}{u^2}\left(\frac{d^2}{dt^2}\right)\# = \left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\#$$

Cartesian (1D space), $\# \to$ anyvalue (2D space)

$$\frac{1}{u^2}\frac{d^2 \#}{dt^2} = \left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)\#$$

(or)

$$\frac{1}{u^2}\frac{d^2 \#}{dt^2} = \nabla^2 \#$$

(3D space)

$\nabla^2 \to$ Laplacian

$$\nabla^2 = \left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)$$

* $$\frac{1}{u^2}\frac{d^2 f(x,y,z,t)}{dt^2} = \nabla^2 f(x,y,z,t)$$

(or)

$$\underbrace{\left(\frac{1}{u^2}\frac{d^2}{dt^2} - \nabla^2\right)}_{} f = 0$$

Box $\to \Box f = 0$ → This is a scalar wave eq^n.

* Solution of $\Box f = 0$ (or) $\frac{1}{u^2}\frac{d^2 f}{dt^2} = \frac{d^2 f}{dx^2}$ (not imp)

we will see that the sol^n of above eq^n is of the format $f(x,t) = f(x \pm ut)$ ← wave is periodic, so $(x \pm ut)$

$$\boxed{x \pm ut = \xi \ (xi)}$$

* $\frac{df}{dx} = \frac{df}{d\xi} \Rightarrow \frac{d^2 f}{dx^2} = \frac{d^2 f}{d\xi^2}$, * $\frac{df}{dt} = (\pm u)\frac{d}{d\xi}$

$$\frac{d^2 f}{dt^2} = u^2 \frac{d^2}{d\xi^2}$$

LHS : $\frac{1}{\cancel{u^2}} \cdot \cancel{u^2} \frac{d^2 f}{d\xi^2} = \frac{d^2 f}{d\xi^2}$

RHS : $\frac{d^2 f}{dx^2} = \frac{d^2 f}{d\xi^2}$ (Proved) $\quad x - ut = \xi$

$$\frac{d}{dx} = \frac{d}{d\xi}\frac{d\xi}{dx} = \frac{d}{d\xi}$$

$\therefore f(x-ut)$ is 'a' soln of the above eqn

$\therefore$ Also $f = f(x+ut)$ is another soln also

* $f = f(x-ut) + f(x+ut)$ (not imp)

$\hookrightarrow$ similar to $\sin(Kx - ut)$

17-8-23 (Lec 3)

* We will see EM wave eqn of the same nature when $\boxed{u = c}$

$\Box \vec{E} = 0$ (or) $\Box \vec{B} = 0$

* $f = f(x-ut) + f(x+ut)$ is called travelling wave eqn a soln of above partial diff. eqn (PDE)

ex: $\sin(Kx - ut)$ (or) $\cos(Kx - ut)$.

* $f = A\sin(Kx - \omega t - \phi)$

$\Rightarrow f = A\sin(Kx - \phi - \omega t)$

$\Rightarrow f = A\sin(Kx - (\omega t + \phi))$

H.W

$\Rightarrow$ * Take K, ω some value, put $t = t_0$ (const ~~cos wt~~)

Compare graphs of f with & without ϕ. Plot it (Matplotlib)

H.W Animate it.

& $\Rightarrow$ Take K, ω some value, put $x = x_0$ (const). Compare graphs of f with & without ϕ, Plot it & Animate it.

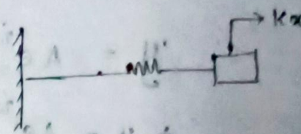
⇒ A transverse sinosuidal wave is generated at one end and down through a distⁿ 1·30 cm. The motion is continuous & repeated regularly 125 times/sec. If the distⁿ b/w the wave crest is observed to be 15·6 cm. find the following:

i) Amplitude ii) wavelength iii) Wave velocity.

Sol^n * $f(x,t) = A \sin(kx - \omega t)$

$\updownarrow$

* $f(x,t) = A e^{i(\omega t - kx)}$ (or) $A e^{i(kx - \omega t)}$



$m\ddot{x} + kx = 0$

means double diff hence accⁿ

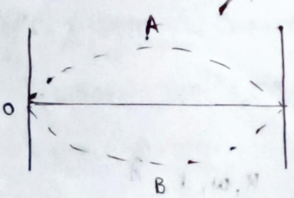
$\ddot{x} + \omega^2 x = 0$

$x(t) = \sin(\omega t)$
(or)
$\cos(\omega t)$

$x(t) = a\sin(\omega t) + b\cos\omega t$

$x(t) = e^{i\omega t}$

$x(t) = e^{-i\omega t}$



Standing wave

*

$\Rightarrow \dfrac{1}{u^2}\dfrac{d^2f}{dt^2} = \dfrac{d^2f}{dx^2}$ ← from 1D wave eqⁿ

$\Rightarrow f(x,t) = X(x)\,T(t)$ ← seperation of variable

$\int d^2f\,dx^2 = X(x)$

$\int d^2f\,dt^2 = T(t)$

(imp)

$\dfrac{df}{dx} = X'T$, $\dfrac{d^2f}{dx^2} = X''T$

$\dfrac{df}{dt} = X\dot{T}$, $\dfrac{d^2f}{dt^2} = X\ddot{T}$

where $' \equiv \dfrac{d}{dx}$, $\dot{} \equiv \dfrac{d}{dt}$

$\therefore \dfrac{1}{u^2} X\ddot{T} = X''T$

$\Rightarrow \dfrac{1}{u^2}\dfrac{\ddot{T}}{T} = \dfrac{X''}{X} = \text{const}$ ← as LHS & RHS, both variables are same, means they are const

$\therefore \dfrac{X''}{X} = -k^2$, $\dfrac{\ddot{T}}{T} = -k^2u^2$

just const (not any value)

$y = 0$ at $x = 0$ & $x = L$ ← (Pre diagram)

$$\therefore y = \underbrace{Ae^{i(\omega t - Kx)}}_{\text{for upper wave}} + \underbrace{Be^{i(\omega t + Kx)}}_{\text{for lower wave}}$$

$$\therefore y = Ae^{i(\omega t - Kx)} - Ae^{i(\omega t + Kx)} \quad (\because A = -B \text{, as B is reflected wave of A})$$

$$\Rightarrow y = Ae^{i\omega t}\left(e^{-iKx} - e^{iKx}\right) \leftarrow \boxed{e^{iKx} = \cos Kx + i \sin Kx}$$

$$\Rightarrow y = (-2i)Ae^{i\omega t} \sin Kx$$

Hence it matches with, $f(x,t) = X(x)\,T(t)$

Reference book
Modern Physics → Beiser

19-08-23

Lec-4

$$f(x,t) = (-2i)Ae^{i\omega t} \sin Kx$$

$f(x,t) = 0$ at $x = 0$ & $x = L$

$$\sin Kx = 0$$

$$\Rightarrow K_n L = n\pi$$

$$\Rightarrow \omega_n L = n\pi v \Rightarrow \omega_n = \frac{vn\pi}{L}$$

$$\Rightarrow 2\pi f_n L = n\pi v \Rightarrow f_n = \frac{nv}{2L}$$

$$\Rightarrow 2\pi \cdot \frac{v}{\lambda_m} L = n\pi v \Rightarrow \lambda_n = \frac{2L}{n}$$

→ where $n \rightarrow$ no of reso.

K, ω, f, λ

$K \longleftrightarrow \omega$

$K = \frac{\omega}{v}$

$f = 2\pi\omega$

$= 2\pi\left(\frac{vn\pi}{L}\right)$

$\omega = 2\pi\left(\frac{nv}{2L}\right)$

$\frac{n\pi v}{L}$

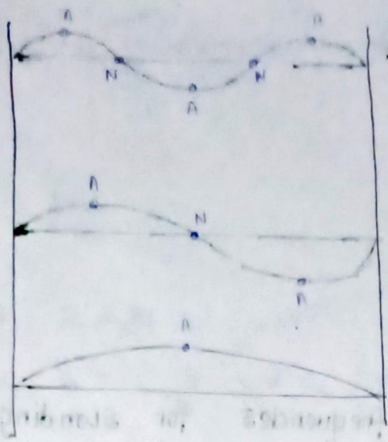
$f =$

$v = \lambda f$

$f = \frac{v}{\lambda} = \frac{nv}{2L}$

Standard eq^n of standing wave when superimposed:

$$y = \underbrace{A \sin Kx}_{\text{variable amplitude}} \cos \omega t$$



$n = 3$ — 3rd harmonic, 2nd overtone, $f_3 = \frac{3v}{2L}$, $\lambda_3 = \frac{2L}{3}$

$n = 2$ — 2nd harmonic, 1st overtone, $f_2 = \frac{v}{L}$, $\lambda_2 = L$

$n = 1$ — fundamental, $f_1 = \frac{v}{2L}$, $\lambda_1 = 2L$

$Kx = \phi = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \longleftarrow \text{maxima} \longrightarrow x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots \text{gap} = \lambda/2$

$Kx = \phi = \pi, 2\pi, 3\pi, \ldots \longleftarrow \text{minima} \longrightarrow x = \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}, \ldots \text{gap} = \lambda/2$

Travelling wave $\xrightarrow{\text{composed of}}$ Standing wave.

$\Rightarrow$ A violin string tuned to concert 440 Hz has a length of 0.34 m. Complete the following

i) What are the 3 longest wavelength of resonance of the string.

ii) What are the corresponding λ that reach to the end of listeners.

Sol^n i) $L = 0.34$ m

$$\left.\begin{aligned} \lambda_1 &= \frac{2L}{1} = 0.68\ \text{m} \\ \lambda_2 &= \frac{2L}{2} = 0.34\ \text{m} \\ \lambda_3 &= \frac{2L}{3} = 0.23\ \text{m} \end{aligned}\right\} \text{using } \lambda_n = \frac{2L}{n}$$

~~put 332 in Vair~~
~~Calculate Vstring~~

ii) $f = 440$ Hz

$\frac{v}{\lambda} = \text{const} = f$

$\frac{V_{string}}{\lambda_{string}} = \frac{V_{air}}{\lambda_{air}}$ $\therefore \lambda_{air}$ & λ_{string}

$(\lambda_{air})_1, (\lambda_{air})_2, (\lambda_{air})_3$

⇒ What are the three lowest frequencies for standing wave on a wire 9.88 m long having mass of 0.107 kg which is stretched under a tension 236 N.

Sol^n)

$$V = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{236}{\frac{0.107}{9.88}}} = \sqrt{\frac{236 \times 1000 \times 988}{107 \times 100}}$$

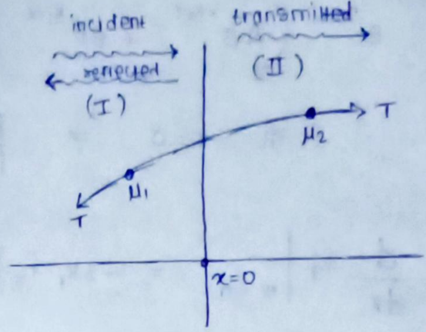
$$\therefore \lambda = \frac{2L}{n}, \quad \lambda_1 = 2L, \quad \lambda_2 = \frac{2L}{2}, \quad \lambda_3 = \frac{2L}{3}.$$

$v = \lambda f$

$$\therefore f_1 = \frac{V}{\lambda_1}, \quad f_2 = \frac{V}{\lambda_2}, \quad f_3 = \frac{V}{\lambda_3}$$

23-08-23

TRANSMISSION AND REFLECTION OF WAVES :



$V = \sqrt{\frac{T}{\mu}}$

as $\mu_1 \neq \mu_2$

so, $V_1 \neq V_2$ & $K_1 \neq K_2$ $\left(K = \frac{\omega}{V}\right)$

→ GOAL :

To study the relation b/w the amplitude of incident wave, reflected wave & transmitted wave.

Region-1 $\begin{cases} f_I = Ae^{i(\omega t - K_1 x)} \\ f_R = Be^{i(\omega t + K_1 x)} \end{cases}$

Region-2 $\{ f_t = Ce^{i(\omega t - K_2 x)}$

* A geometric condition that the dispⁿ is immediately same to the left & right of $x = 0$?

$$(f_I + f_R)\Big|_{x=0} = f_T\Big|_{x=0}$$

* f & its derivatives must be continuous across the boundary.

$$\frac{d}{dx}(f_I + f_R)\Big|_{x=0} = \frac{df_T}{dx}\Big|_{x=0} \longrightarrow ①$$

* If the above condition is not true, then the accⁿ would be infinity.

$$* \quad (f_I + f_R)\big|_{x=0} = f_T\big|_{x=0}$$

$$e^{i\omega t}\left(Ae^{-K_1 x} + Be^{K_1 x}\right)\Big|_{x=0} = Ce^{i(\omega t - K_2 x)}\Big|_{x=0}$$

at $x = 0 \Rightarrow \boxed{A + B = C}$

$$* \quad \frac{d}{dx} f_I\Big|_{x=0} = -iK_1 f_I\Big|_{x=0}$$

$$* \quad \frac{d}{dx} f_R\Big|_{x=0} = iK_1 f_R\Big|_{x=0}$$

$$\frac{d}{dx} f_T\Big|_{x=0} = -iK_2 f_T\Big|_{x=0}$$

$$\therefore \boxed{\frac{B}{A} = \frac{(K_1 - K_2)}{(K_1 + K_2)}}, \quad \boxed{\frac{C}{A} = \frac{2K_1}{K_1 + K_2}}$$

$$\therefore \boxed{R = \frac{|B|^2}{|A|^2} = \frac{(K_1 - K_2)^2}{(K_1 + K_2)^2}}, \quad \cancel{R+T=1}$$

reflection coefficient

$$R + T = 1$$

transmission coefficient $\rightarrow T = 1 - R \Rightarrow \boxed{T = \frac{4K_1 K_2}{(K_1 + K_2)^2} = \frac{K_2}{K_1}\frac{|C|^2}{|A|^2}}$

$$* \quad \frac{d}{dx}\left(Ae^{i(\omega t - K_1 x)} + Be^{i(\omega t + K_1 x)}\right) = \frac{d}{dx}\left(Ce^{i(\omega t - K_2 x)}\right) \rightarrow \text{from eqn (1)}$$

$$\Rightarrow Ae^{i(\omega t - K_1 x)}(-K_1) + Be^{i(\omega t + K_1 x)}(K_1) = Ce^{i(\omega t - K_2 x)}(-K_2)$$

$$\Rightarrow -K_1 A\cancel{e^{i\omega t}} + K_1 B\cancel{e^{i\omega t}} = -K_2 C\cancel{e^{i\omega t}}$$

$$\Rightarrow -K_1 A + K_1 B = -K_2 C$$

$$\Rightarrow K_1(A - B) = K_2 C$$

$$\Rightarrow K_1(A - B) = K_2(A + B)$$

$$K_1\left(\frac{A}{B}-1\right) = K_2\left(\frac{A}{B}+1\right)$$

$$\frac{A}{B}(K_1-K_2) = K_1+K_2$$

$$\boxed{\frac{A}{B} = \frac{K_2+K_1}{K_1-K_2}}$$

24/08/2023

→ **Impedence :**

$Z \doteq \frac{T}{V}$, $V = \sqrt{\frac{T}{\mu}}$:= (or) $\doteq$ is called definition

imp.

$\Rightarrow T = V^2\mu$.

$$\therefore \boxed{Z = \mu V}$$

we have used notation 'T' for "tension" awa "transmission coeff"

$T = \mu V^2 = \mu\frac{\omega^2}{K^2}$ $\left(\because V = \frac{\omega}{K}\right)$.

$\Rightarrow K \propto \sqrt{\mu} \Rightarrow \lambda \propto \frac{1}{\sqrt{\mu}}$

$Z = \mu\sqrt{\frac{T}{\mu}} = \sqrt{\mu T}$

$$\therefore \frac{B}{A} = \frac{K_1-K_2}{K_1+K_2} = \frac{\sqrt{\mu_1}-\sqrt{\mu_2}}{\sqrt{\mu_1}+\sqrt{\mu_2}}$$

$$\frac{C}{A} = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1}+\sqrt{\mu_2}}.$$

⇒ Express the ratio $\frac{B}{A}$ & $\frac{C}{A}$ in terms of Z_1 & Z_2 ?

Sol^n) $\therefore Z = \mu V$, where $Z = \frac{T}{V}$

here v & μ are variable, so making one variable

$Z = \mu\sqrt{\frac{T}{\mu}} = \sqrt{\mu T}$ $\therefore Z \propto \sqrt{\mu}$

$\therefore Z \propto K$.

$$\therefore \frac{B}{A} = \frac{Z_1-Z_2}{Z_1+Z_2}, \quad \frac{C}{A} = \frac{2Z_1}{Z_1+Z_2}$$

⇒ A transvers sin wave of amplitude 3.0 cm & wavelength 25 cm travels along a light string of linear mean density 1 gm cm^{-1} which is joined with heavier string of 4.0 gm cm^{-1}. The joined strings are held under constant tension. Complete :

i) What is the wavelength a, & amplitude of the wave as it travels in the heavier string?

ii) What fraction of the wave power is reflected at the boundary of the two strings?

Sol^n) i) $\lambda \propto \frac{1}{\sqrt{\mu}}$ $\frac{C}{A} = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1}+\sqrt{\mu_2}} = \underline{\underline{2\text{ cm}}}$ (Amplitude).

$$\frac{\lambda_2}{\lambda_1} = \sqrt{\frac{\mu_1}{\mu_2}}$$

$$\underline{\underline{\lambda_2 = 12.5\text{ cm}}}$$

ii) $R = \frac{|B|^2}{|A|^2} \Rightarrow \left(\frac{B}{A}\right)^2 = \left(\frac{\sqrt{\mu_1}-\sqrt{\mu_2}}{\sqrt{\mu_1}+\sqrt{\mu_2}}\right)^2 = \underline{\underline{\frac{1}{9}}}$ (Ans)

reflection coeff
(or)
reflection power.