MODULE - 7

→ Line integral : → Single integral.

→ Green's Thm (for a plane) } → Double integral

→ Stokes Thm

→ Gauss. Div. Thm → Triple integral.

LINE INTEGRAL :

$$W = \int_A^B \vec{F} \cdot \vec{dr}$$

⇒ if force $\vec{F} = 2x^2 y \hat{i} + 3xy \hat{j}$ displace a particle in xy-plane from (0,0) to (1,4) along curve $y = 4x^2$. Find work done.

Sol^n

$$W = \int_A^B \vec{F} \cdot \vec{dr}.$$

$$W = \int_0^1 2x^2 y \, dx + \int_0^4 3xy \, dy.$$

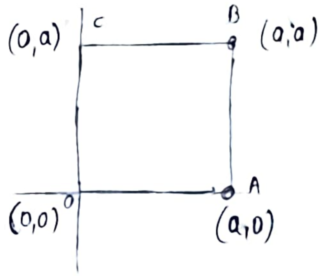
$$W = \int_0^1 2x^2 (4x^2) dx + \int_0^4 3\left(\frac{y}{4}\right)^{1/2} y \, dy$$

$$W = 8\left(\frac{x^5}{5}\right)_0^1 + \frac{3}{2} \cdot \frac{2}{5} \left(y^{5/2}\right)_0^4$$

$$W = \frac{8}{5} + \frac{3}{5}\left(2^5\right) = \underline{\frac{104}{5}} \text{ J (Ans)}$$

⇒ Evaluate $\int_C \vec{F} \cdot \vec{dr}$, where $\vec{F} = x^2 \hat{i} + xy \hat{j}$ & c is boundary of square in plane $z=0$ and bounded by lines $x=0$, $y=0$, $x=a$, $y=a$.

Sol^n) $\int_C \vec{F}\cdot d\vec{r} = \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} + \int_{BC} + \int_{CO}$



$d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$, $\vec{F} = x^2\hat{i} + xy\hat{j}$

$$\therefore \int_0^a x^2\,dx + \int_0^0 xy\,dy + \int_a^a x^2\,dx + \int_0^a xy\,dy + \int_a^0 x^2\,dx + \int_a^a$$

OA $\rightarrow$ $\therefore \int_0^a x^2\,dx = \frac{a^3}{3}$ $(\because y = 0)$

AB $\rightarrow$ $\int_0^a xy\,dy = \int_0^a a y\,dy = \frac{a^3}{2}$

BC $\rightarrow$ $\int_a^0 x^2\,dx = -\frac{a^3}{3}$.

CO $\rightarrow$ $\int_a^0 xy\,dy = \underline{\underline{0}}$.

$\therefore \int_C \vec{F}\cdot d\vec{r} = \underline{\underline{\frac{a^3}{2}}}$ (Ans)

$\Rightarrow$ Show that the vector field $\vec{F} = 2x(y^2+z^3)\hat{i} + 2x^2y\hat{j} + 3x^2z^2\hat{k}$ is conservative. Find its scalar potential and W in moving particle from $(-1,2,1)$ to $(2,3,4)$

Sol^n) conservative $\longrightarrow$ $\vec{\nabla}\times\vec{F} = 0$.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^2+2xz^3 & 2x^2y & 3x^2z^2 \end{vmatrix} = 0$$

$$\Rightarrow \hat{i}\left(\frac{\partial}{\partial y}(3x^2z^2) - \frac{\partial}{\partial z}(2x^2y)\right) - \hat{j}\left(\frac{\partial}{\partial z}(3x^2z^2) - \frac{\partial}{\partial z}(2xy^2+2xz^3)\right)$$

$$+\hat{k}\left(\frac{\partial}{\partial x}(2x^2y) - \frac{\partial}{\partial y}(2xy^2+2xz^3)\right)$$

$\Rightarrow \hat{i}(0-0) - (6xz^2 - 6xz^2)\hat{j} + (4xy - 4xy)\hat{k} = 0$

$\therefore$ it is irrotational / vector field is irrotational.

$\therefore \quad \vec{F} = \vec{\nabla}\phi$

$\phi = \int \vec{F} \cdot d\vec{r}$

$\phi = \int 2xy^2\, dx + 2xz^3\, dx + 2x^2y\, dy + 3x^2z^2\, dz$

$\phi = \int d(x^2y^2) + \int d(x^2z^3)$

$\phi = x^2y^2 + x^2z^3 + c$

$W = \int_{(-1,2,1)}^{(2,3,4)} \vec{F} \cdot d\vec{r} = (\phi)_{(-1,2,1)}^{(2,3,4)}$

$= [(4)(9) + (4)(64) - (1)(4) - (1)(1)]$

$= 36 + 256 - 4 - 1 = 256 - 5$

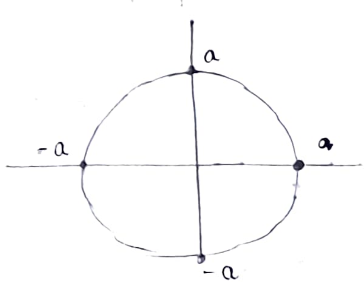
$= 289$

imp

$\Rightarrow \vec{F} = (\sin y)\hat{i} + x(1+\cos y)\hat{j}$. Evaluate line integral over circular path $x^2 + y^2 = a^2$, $z = 0$

Sol^n

$W = \int_c \vec{F} \cdot d\vec{r}$



$W = \int_c \sin y\, dx + x\, dy + x\cos y\, dy.$

$W = \int_c d(x\sin y) + \int_c x\, dy$

Let $x = a\cos\theta$, $y = a\sin\theta$ $\longrightarrow$ for circle, take parametric

$dx = -a\sin\theta\, d\theta$, $dy = a\cos\theta\, d\theta$

$dx = -y\, d\theta$, $dy = x\, d\theta$

$$W = \int_0^{2\pi} d\left[a\cos\theta \cdot \sin(a\sin\theta)\right] + \int_0^{2\pi} a\cos\theta \cdot a\cos\theta \, d\theta$$

$\cos 2\theta = 2\cos^2\theta - 1$

$$W = \left[a\cos\theta \cdot \sin(a\sin\theta)\right]_0^{2\pi} + \int a^2\cos^2\theta \, d\theta$$

$$W = \left[a(1) \cdot \sin(2a(0)) - 1\right] + \int_0^{2\pi} a^2\left[\frac{\cos 2\theta + 1}{2}\right] d\theta$$

$$W = [0] + \frac{a^2}{2}\left[\frac{\sin 2\theta}{2} + \theta\right]_0^{2\pi}$$

$$W = \frac{a^2}{2}[2\pi] = \underline{\pi a^2}.$$

$\Rightarrow \int (2xyz^2)\,dx + \oint(x^2z^2 + z\cos yz)\,dy + (2x^2yz + y\cos yz)\,dz$ is independent of path of integration? If so, evaluate it from $(1,0,1)$ to $(0, \frac{\pi}{2}, 1)$.

Sol^n)

$$\int 2xyz^2\,dx + x^2z^2\,dy + 2x^2yz\,dz + z\cos yz\,dy + y\cos yz\,dz$$

$$\int d(x^2yz^2) + \int d(\sin yz)$$

$$x^2yz^2 + \sin yz + c$$

To check, whether it is independent of path of integration.

$\therefore \quad \vec{\nabla} \times \vec{F} = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^2 & (x^2z^2 + z\cos yz) & (2x^2yz + y\cos yz) \end{vmatrix}$$

$\Rightarrow \hat{i}(\longrightarrow) - \hat{j}(\longrightarrow) + \hat{k}(\longrightarrow) = \underline{0}$ (Hence independent)

$$\int_{(1,0,1)}^{(0,\pi/2,1)} x^2yz^2 + \int_{(1,0,1)}^{(0,\pi/2,1)} \sin yz.$$

$$0 - 0 + \sin \pi/2 - \sin 0 = \underline{\underline{1}} \quad (\text{Ans}).$$

$\Rightarrow$ $\vec{F} = (2y+3)\hat{i} + xz\hat{j} + (yz - x)\hat{k}$, Evaluate : $\int_C \vec{F}\cdot d\vec{r}$ along path c

is $x = 2t$, $y = t$, $z = t^3$ from $t = 0$, to $t = 1$

Sol^n) $\cancel{\int 2y\,dx + 3\,dx + xz\,dy + yz\,dz - x\,dz}$

$x = 2t$	$y = t$	$z = t^3$
$dx = 2\,dt$	$dy = dt$	$dz = 3t^2\,dt.$

$$\therefore \int (2t+3)(2\,dt) + \int (2t)(t^3)(dt) + \int ((t)(t^3) - (2t))(3t^2\,dt)$$

$$\int 4t\,dt + \int 6\,dt + \int 2t^4\,dt + \int 3t^6\,dt - 6t^3\,dt$$

$$\left(2t^2 + 6t + \frac{2t^5}{5} + \frac{3t^7}{7} - \frac{6t^4}{4}\right)_0^1$$

$$2 + 6 + \frac{2}{5} + \frac{3}{7} - \frac{3}{2} = \frac{140 + 420 + 28 + 30 + 105}{70} = \underline{7.32857}$$

GREEN'S THM : (for a plane)

* If $\phi(x,y)$ and $\psi(x,y)$ be continuous func^n over a region R bounded by simple closed curve C in xy plane, then

$$\oint_C \phi(x,y)\,dx + \psi(x,y)\,dy = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dx\,dy .$$

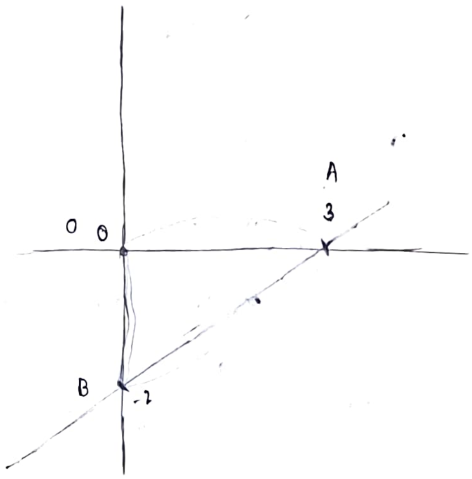
⇒ State and verify Green's Thm in plane for $\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\cdot dy$, where C is the boundary of the region bounded by $x \geq 0$, $y \leq 0$, $2x - 3y = 6$.

Soln/

$2x - 3y = 6$

$3y = 2x - 6$

$y = \frac{2}{3}x - 2$



$\therefore \phi = 3x^2 - 8y^2$

$\psi = 4y - 6xy$.

$\therefore \oint_C \phi\,dx + \psi\,dy$

$\Rightarrow \iint \left(\frac{d\psi}{dx} - \frac{d\phi}{dy} \right) dx\,dy$

$\Rightarrow \iint \left(-6y - (6x - 16y) \right) dx\,dy$ → RHS

$\Rightarrow \iint -6y + 16y \; dx\,dy = \int_0^3 \int_{(\frac{2}{3}x - 2)}^{0} 10y\,dx\,dy$

$\therefore 5 \int_0^3 (y^2)^{0}_{(\frac{2}{3}x-2)} = -5 \int_0^3 \frac{4}{9}x^2 + 4 - \frac{8}{3}x \; dx$

$= -5 \left[\frac{4}{9}\left(\frac{x^3}{3}\right)_0^3 + 4(x)_0^3 - \frac{8}{3}\left(\frac{x^2}{2}\right)_0^3 \right]$

$= -5 \left[4 + 12 - 12 \right] = -20$ → ①

LHS : $\oint_C \phi\, dx + \psi\, dy$

$\phi = 3x^2 - 8y^2$

$\psi = 4y - 6xy$

$y = \frac{2}{3}x - 2$

$\Rightarrow x = \frac{3}{2}y + 3$

$\boxed{dx = \frac{3}{2}dy}$

$\Rightarrow \int_{OB} \phi\, dx + \psi\, dy + \int_{BA} \phi\, dx + \psi\, dy + \int_{AO} \phi\, dx + \psi\, dy$

$\cancel{\Rightarrow \int_0^{-2} \phi\, dx + \int_0^{2} \psi\, dy + \int}$

$\Rightarrow \int_{OB} (3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy + \int_{BA} (3x^2 - 8y^2)\, dx + \oint (4y - 6xy)\, dy + \int_{AO} (3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy$

$\cancel{\Rightarrow \oint \left(3\left(\frac{9}{4}y^2 + 9 + 9y\right) - 8y^2\right)\frac{3}{2}dy + 4y - 6\left(\frac{3}{2}y + 3\right)y\, dy}$

$\cancel{\Rightarrow \frac{81}{8}y^2\, dy +}$

OB $\longrightarrow$ $y : 0 \to -2$, $x : 0 \to 0$

$\frac{21}{4}y^2 + 27 + 27y$

BA $\longrightarrow$ $y : -2 \to 0$, $x : 0 \to 3$

AO $\longrightarrow$ $y : 0 \to 0$, $x : 3 \to 0$

$\therefore \int_0^{-2} 4y\, dy + \int_{-2}^{0} \left(3\left(\frac{9}{4}y^2 + 9 + 9y\right) - 8y^2\right)\left(\frac{3}{2}dy\right) + \left(4y - 6\left(\frac{3}{2}y + 3\right)y\right) dy$

$+ \int_3^0 3x^2\, dx$

$\Rightarrow 2(y^2)_0^{-2} + \int_{-2}^{0} \left(\frac{81}{8}y^2 + \frac{81}{2} + \frac{81}{2}y\right) dy - \frac{24y^2}{2}dy + 4y\, dy - 9y^2\, dy - 18y\, dy$

$\Rightarrow 2[4] + \int_{-2}^{0} \left(\frac{9}{8}(6 + 3y)^2 - 21y^2 - 14y\right) dy + (-27)$

$\Rightarrow \underline{\underline{-20}}$

$\therefore$ LHS = RHS

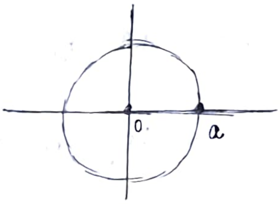
(Proved)

⇒ Apply Green's Th^m to evaluate $\int_c [(2x^2-y^2)dx + (x^2+y^2)dy]$, where c is boundary of area enclosed by x-axis and upper half of circle $x^2+y^2 = a^2$

Soln) $\phi = 2x^2-y^2$, $\psi = x^2+y^2$

$$\therefore \oint \phi\, dx + \psi\, dy = \iint \left(\frac{d\psi}{dx} - \frac{d\phi}{dy}\right) dx\, dy.$$

$$\iint (2x + 2y)\, dx\, dy.$$



Let $x = r\cos\theta$, $y = r\sin\theta$

$$2\iint r^2(\sin\theta + \cos\theta)\, dr\, d\theta$$

$\theta = 0 \to \pi$

$r = 0 \to a.$

$$2\int_0^{\pi} (\sin\theta + \cos\theta)\, d\theta \int_0^{a} r^2\, dr$$

$$2 \left[-\cos\theta + \sin\theta\right]_0^{\pi} \left[\frac{a^3}{3}\right] \Rightarrow \frac{2a^3}{3}\left[+1 + 0 - (-1) - 0\right]$$

$$= \frac{4a^3}{3} \text{ (Ans).}$$

Stoke's Thm : (Relation b/w line & surface integral).

$$\oint \vec{F}\cdot d\vec{r} = \iint_S \text{curl } \vec{F}\cdot \hat{n}\, ds$$

n

where : $\hat{n} = \dfrac{\text{grad } f}{|\text{grad } f|}$, $\cancel{\dfrac{dx\,dy}{\hat n\cdot\hat k}}$, $\boxed{ds = \dfrac{dx\,dy}{\hat{n}\cdot\hat{k}}}$

component of $\vec{F}$ along the normal $\Rightarrow \vec{F}\cdot\hat{n}$

⇒ Use Stoke's thm to evaluate $\int_C \vec{V}\cdot d\vec{r}$, where $\vec{V} = y^2\hat{i} + xy\hat{j} + xz\hat{k}$ and c is bounding curve of hemisphere $x^2+y^2+z^2=9$, $z>0$, oriented in +ve dirn.

Soln

$$\int_C \vec{V}\cdot d\vec{r} = \iint_S (\text{curl } \vec{V})\cdot\hat{n}\, ds = \iint_S (\nabla\times\vec{V})\cdot\hat{n}\, ds.$$

$$\therefore \nabla\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & xy & xz \end{vmatrix} = \hat{i}(0-0) - \hat{j}(z-0) + \hat{k}(y-2y)$$

$$= -z\hat{j} - y\hat{k}$$

$$\therefore \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-9)}{|\nabla\phi|}$$

$$\hat{n} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{3}$$

$$\therefore (\nabla\times\vec{V})\cdot\hat{n} = \frac{-2yz}{3} ,$$

$\therefore \cancel{\iint \frac{2yz}{3}}$

$$\hat{n}\cdot\hat{k}\, ds = dx\,dy.$$

$$\left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{3}\right)(\cdot\hat{k})\, ds = dx\,dy$$

$$\Rightarrow \frac{z}{3}\, ds = dx\,dy.$$

$$\Rightarrow ds = \frac{3}{z}\, dx\,dy.$$

$$\therefore \iint -\frac{2yz}{3}\left(\frac{3}{z}\,dx\,dy\right) \Rightarrow \iint -2y \;dx\,dy.$$

$$\Rightarrow \iint -2r\sin\theta \; r\,d\theta\,dr$$

$$\Rightarrow -2\int_0^{2\pi} \sin\theta\,d\theta \int_0^3 r^2\,dr$$

$$\Rightarrow -2\left(-\cos\theta\right)_0^{2\pi} \cdot \left[\frac{r^3}{3}\right]_0^3 = \underline{\underline{0}}$$

⇒ Verify Stoke's thm for func^n $\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$, where C is unit circle in xy-plane bounding hemisphere $z = \sqrt{1-x^2-y^2}$

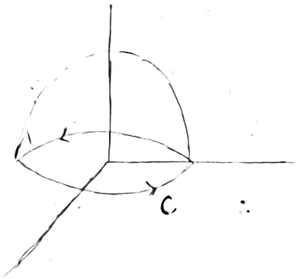
Sol^n $\quad \vec{F}\cdot d\vec{r} = z\,dx + x\,dy + y\,dz$

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C z\,dx + x\,dy + y\,dz$$

on circle $x^2+y^2 = 1$, $z=0$ in xy plane

$$\therefore \oint_C \vec{F}\cdot d\vec{r} = \oint x\,dy,$$

let $x = 1\cos\phi$; $y = 1\sin\phi$.



$$\therefore \int_0^{2\pi} \cos\phi\,\cos\phi\,d\phi = \underline{\underline{\pi}}.$$

← LHS

RHS : $\iint (\text{curl } \vec{F})\cdot\hat{n}\,ds$ $\qquad \hat{n} = \dfrac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}} = x\hat{i}+y\hat{j}+z\hat{k}$

$\therefore \nabla\times\vec{F} = \hat{i}+\hat{j}+\hat{k}$, $\quad ds \to \sin\theta\,d\theta\,d\phi$

normal to $x^2+y^2+z^2=1$ is $x\hat{i}+y\hat{j}+z\hat{k}$ by find grad of $x^2+y^2+z^2=1$

$(\nabla\times\vec{F})\cdot\hat{n} = \sin\theta\cos\phi + \sin\theta\sin\phi + \cos\theta$

put $\theta \to 0$ to $\pi/2$, $\phi \to 0$ to 2π

$\therefore \underline{\underline{\pi}}$ $\qquad \therefore$ LHS = RHS

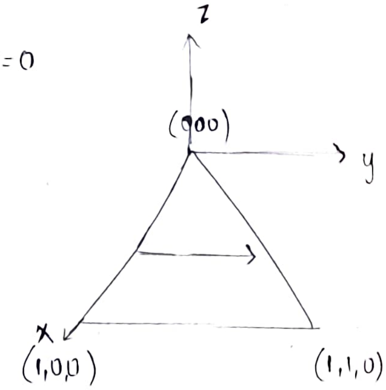
⇒ Evaluate $\oint_c \vec{F}\cdot d\vec{r}$ by Stoke's Th^m, where $\vec{F} = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}$

and c is boundary of Δ with vertices at $(0,0,0), (1,0,0), (1,1,0)$

Sol^n) curl $(\vec{F})$ = $\hat{j} + 2(x-y)\hat{k}$

$\therefore$ ~~curl $(\vec{F})\cdot\hat{n}$~~ $\hat{n} \rightarrow$ all points has $z=0$

so, $\hat{n} = \hat{k}$



$\therefore$ curl $(\vec{F})\cdot\hat{n} = 2(x-y)$

$\cancel{\int_{y=0}^{y=1}\int_{x=0}^{1} 2(x-y)\,dx\,dy}$

$$\int_0^1 \int_{y=0}^{y=x} 2(x-y)\,dx\,dy = \underline{\underline{1/3}}$$

v.imp

⇒ Verify Stoke's th^m for vector field $\vec{F} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$ over upper half of surface $x^2+y^2+z^2=1$, bounded by its projection on xy - plane.

Sol^n) $\int_c \vec{F}\cdot d\vec{r} = \int_c (2x-y)\,dx - \int yz^2\,dy - \int y^2z\,dz$

$x^2+y^2+z^2 = 1$ has projection on xy plane i.e, $x^2+y^2 = 1 \rightarrow c$

$\therefore$ Let $x = 1\cos\theta$, $y = 1\sin\theta$,

$dx = -\sin\theta\,d\theta$, $dy = \cos\theta\,d\theta$.

$\therefore \int 2\cos\theta - \sin\theta\,(-\sin\theta\,d\theta) - \int 0 - \int 0 \longrightarrow$ in xy plane.

$\therefore -\int 2\sin\theta\cos\theta\,d\theta + \int \sin^2\theta\,d\theta$

$$-\int_0^{2\pi} \sin 2\theta\,d\theta + \int_0^{2\pi} \frac{1-\cos 2\theta}{2}\,d\theta = \underline{\underline{\pi}} \longrightarrow \text{LHS}.$$

$\cancel{\text{curl}(\vec{F}) = \hat{k}}$

$\therefore$ ~~normal to $x^2+y^2 = 1$ is $\hat{k}$~~

$\therefore$ ~~curl $(\vec{F})\cdot\hat{n} =$~~

normal to $x^2+y^2+z^2 = 1$ is $x\hat{i}+y\hat{j}+z\hat{k}$

$$\therefore \hat{n} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}} = x\hat{i}+y\hat{j}+z\hat{k}$$

$x = \sin\phi\cdot\cos\theta$

$y = \sin\phi\,\sin\theta$

$z = \cos\phi$

$\therefore$ curl $(\vec{F})\cdot\hat{n} = z = \cos\phi$

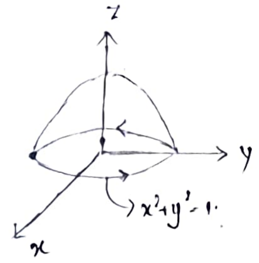
$$\therefore \iint \cos\phi\,|J|\,d\theta\,d\phi$$

RHS : curl $(\vec{F}) = \hat{k}$

$\therefore$ normal $= \hat{n}$

$$\therefore \iint \text{curl}(\vec{F})\cdot\hat{n}\,ds = \iint \hat{k}\cdot\hat{n}\,ds = \iint \hat{k}\cdot\hat{n}\,\frac{dx\,dy}{\hat{n}\cdot\hat{k}} = \iint dx\,dy$$

$$\therefore \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx\,dy = \pi$$

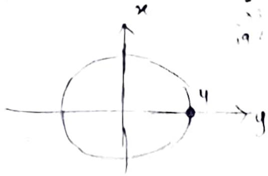


$\therefore$ LHS = RHS (verified).

$\Rightarrow$ verify Stoke's th^m for $\vec{F} = (x^2+y-4)\hat{i} + 3xy\hat{j} + (2xz+z^2)\hat{k}$ over the surface of hemisphere $x^2+y^2+z^2 = 16$ above xy plane.

Soln
$$\int_C \vec{F}\cdot d\vec{r} = \int_C (x^2+y-4)\,dx + 3xy\,dy + (2xz+z^2)\,dz$$

let $x = 4\cos\theta$, $y = 4\sin\theta$, in xy plane
$dx = -4\sin\theta\,d\theta$, $dy = 4\cos\theta\,d\theta$, $z = 0$



$$\therefore \int 16\cos^2\theta + 4\sin\theta - 4\,(-4\sin\theta\,d\theta) + \int 3(16)\sin\theta\cos\theta\,(4\cos\theta\,d\theta)$$

$$\Rightarrow \int_0^{2\pi} -64\sin\theta\cos^2\theta - 16\sin^2\theta + 16\sin\theta\,d\theta + 192\sin\theta\cos^2\theta\,d\theta$$

$= -16\pi$

RHS : curl $(\vec{F}) = -2z\,\hat{j} + (3y-1)\,\hat{k}$

$$\hat{n} = \frac{x\,\hat{i} + y\,\hat{j} + z\,\hat{k}}{4}$$

$$\therefore \quad ds = \frac{dx\,dy}{\hat{n}\cdot\hat{k}} \Rightarrow ds = \frac{dx\,dy}{z/4} \Rightarrow ds = \frac{4}{z}\,dx\,dy$$

$$\therefore \iint \text{curl}\,(\vec{F})\cdot\hat{n}\,ds = \iint \frac{-2yz}{4} + \frac{3yz - z}{4}\left(\frac{4}{z}\,dx\,dy\right)$$

$$= \iint \frac{yz - z}{4}\left(\frac{4}{z}\,dx\,dy\right).$$

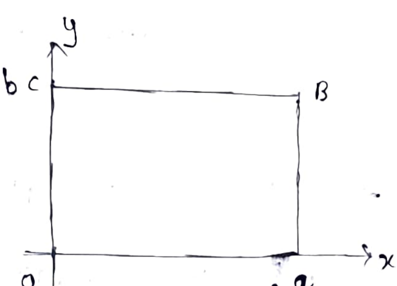
$$= \int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} y - 1\;dx\,dy.$$

$$= \underline{-16\pi}$$

$\therefore$ LHS = RHS (verified).

$\Rightarrow$ Verify Stoke's thm for vector field defined by $\vec{F} = (x^2 - y^2)\hat{i} + 2xy\hat{j}$ in rectangular in xy-plane bounded by lines $x = 0,\ y = 0,\ x = a,\ y = b$.

Soln



$$\therefore \oint_{OC} \vec{F}\cdot d\vec{r} = \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} + \int_{CO}$$

$$\int_{OA} \vec{F}\cdot d\vec{r} = \left(\frac{x^3}{3}\right)_0^a = \frac{a^3}{3}.$$

$$\int_{AB} \vec{F}\cdot d\vec{r} = 2a\left[\frac{y^2}{2}\right]_0^b = ab^2$$

$$\int_{BC} \vec{F}\cdot d\vec{r} = \int_a^0 (x^2 - b^2)\,dx = -\frac{a^3}{3} + b^2 a.$$

$$\int_{CO} \vec{F}\cdot d\vec{r} = \underline{0}$$

$\therefore$ LHS $= \dfrac{a^3}{3} + ab^2 - \dfrac{a^3}{3} + ab^2 + 0 = \underline{\underline{2ab^2}}$

RHS : $\text{curl}(\vec{F}) = 4y\hat{k}$

$\therefore \hat{n} = \hat{k}$ ($\perp$ to xy plane)

$\therefore ds = \dfrac{dx\,dy}{\hat{n}\cdot\hat{k}} = dx\,dy$

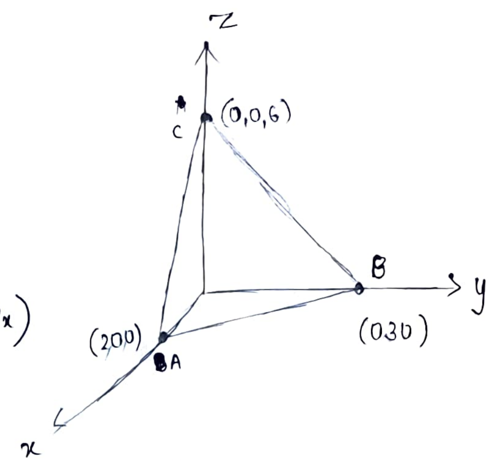
$\therefore \iint \text{curl}(\vec{F})\cdot\hat{n}\,ds = \int_0^a\int_0^b 4y\,dx\,dy = \underline{\underline{2ab^2}}$

$\therefore$ verified

v. imp

$\Rightarrow$ Verify stoke's thm for $\vec{F} = (x+y)\hat{i} + (2x-z)\hat{j} + (y+z)\hat{k}$ for the surface of Δ lamina with vertices $(2,0,0)$, $(0,3,0)$, $(0,0,6)$.

Soln $\oint_C \vec{F}\cdot d\vec{r} = \int_{AB} + \int_{BC} + \int_{CA}$



$\therefore \int_{AB} \vec{F}\cdot d\vec{r} = \int (x+y)\,dx + (2x-z)\,dy$

$= \int\left(x + 3 - \dfrac{3x}{2}\right)dx + \int 2x\cdot\left(\dfrac{-3}{2}\,dx\right)$

$= \underline{1}$

$\int_{BC} = 36$, $\int_{CA} = -16$.

eqn of AB

$\Rightarrow \dfrac{x}{2} + \dfrac{y}{3} = 1.$

eqn of BC

$\Rightarrow \dfrac{y}{3} + \dfrac{z}{6} = 1.$

eqn of CA

$\Rightarrow \dfrac{z}{6} + \dfrac{x}{2} = 1.$

$\therefore \oint \vec{F}\cdot d\vec{r} = 21$

$\therefore \text{curl}(\vec{F}) = 2\hat{i} + \hat{k}$

eqn of plane ABC $\Rightarrow \dfrac{x}{2} + \dfrac{y}{3} + \dfrac{z}{6} = 1$

$\therefore$ normal to plane $= \nabla\phi = \dfrac{\hat{i}}{2} + \dfrac{\hat{j}}{3} + \dfrac{\hat{k}}{6}$

$\therefore \hat{n} = \underline{\underline{\dfrac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{14}}}}$

$\iint \text{curl}(\vec{F}) \cdot \hat{n} \, ds$

$$ds = \frac{dx\,dy}{1/\sqrt{14}}$$

$$\therefore \iint \frac{7}{\sqrt{14}} \frac{dx\,dy}{1/\sqrt{14}} = 7\iint dx\,dy = 7\left[\frac{1}{2} \times 2 \times 3\right] = \underline{\underline{21}}$$

Area of ABC

verified

GAUSS - DIVERGENCE THM

(Relation b/w surface to volume integral)

* $\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \operatorname{div}\vec{F}\, dv$, $dv = dx\,dy\,dz$

⇒ State Gauss's Divergence Thm $\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint \operatorname{div}\vec{F}\, dv$

where S is surface of sphere $x^2+y^2+z^2=16$ and $\vec{F} = 3x\hat{i} + 4y\hat{j} + 5z\hat{k}$

Sol^n) $\nabla\cdot\vec{F} = 3+4+5 = 12$

$\iiint_V 14\, dv \Rightarrow 12V$

$= 12\left(\frac{4}{3}\pi(4)^3\right) = \underline{\underline{\frac{3584\pi}{3}}}$

⇒ Evaluate $\iint_S \vec{F}\cdot\hat{n}\, ds$; where $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ and S is the surface of cube bounded by $x=0, x=1, y=0, y=1, z=0, z=1$

Sol^n) $\vec{\nabla}\cdot\vec{F} = 4z - 2y + y = 4z - y$

$\therefore \int_0^1\int_0^1\int_0^1 (4z-y)\, dx\,dy\,dz = 3/2$.

xyz / y dv $dx\,dy/dz$

⇒ Evaluate $\iint_S (y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k})\cdot\hat{n}\, ds$, where S is the part of sphere $x^2+y^2+z^2=1$, above xy-plane and bounded by this plane

Sol^n) $\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \vec{\nabla}\cdot\vec{F}\, dv$.

$\therefore \vec{\nabla}\cdot\vec{F} = 2zy^2$

$\therefore \iiint 2zy^2\, dx\, dy\, dz$

$x = r \sin\phi \cos\theta$

$y = r \sin\phi \sin\theta$

$z = r \cos\phi$

$\therefore dx\,dy\,dz = r^2 \sin\theta \, dr\, d\theta\, d\phi$

$$\therefore \int_0^{\pi/2}\int_0^{2\pi}\int_0^{1} 2\,(r\cos\phi)\,(r^2\sin^2\phi\,\sin^2\theta)\,(r^2\sin\theta\,dr\,d\theta\,d\phi)$$

$\Rightarrow \pi/12$ (Ans).

More examples form this topic.

$\Rightarrow$ Find $\iint \vec{F}\cdot\hat{n}\,ds$, $\vec{F} = (2x+3z)\hat{i} - (xz+y)\hat{j} + (y^2+2z)\hat{k}$ and S is surface of sphere having center $(3,-1,2)$ radius 3.

Sol^n now eq^n of sphere : $(x-3)^2 + (y+1)^2 + (z-2)^2 = 9$.

But $\hat{n}$ is diff. to find.

So use ; $\iint \vec{F}\cdot\hat{n}\,ds = \iiint \operatorname{div}\vec{F}\,dv$.

$\therefore \operatorname{div}\vec{F} = 3$

$\therefore \iiint 3\,dv \Rightarrow 3\iiint dv$

$= 3\left[\frac{4}{3}\pi r^3\right] = 108\pi$.

⇒ use Div. Th^m to evaluate $\iint_S \vec{A}\cdot\vec{ds}$

where $\vec{A} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$, S is surface of sphere $x^2+y^2+z^2 = a^2$

Sol^n) $\iint_S \vec{A}\cdot\vec{ds} = \iiint \text{div}\,\vec{A}\cdot dv$

$$\boxed{\iint_S \vec{A}\cdot\vec{ds} = \iint_S \vec{A}\cdot\hat{n}\ \vec{ds}}$$

imp.

$\therefore$ div $\vec{A} = 3x^2 + 3y^2 + 3z^2$

$$\therefore 3\iiint (x^2+y^2+z^2)\,dv = 3\int_0^{\pi/2}\int_0^{2\pi}\int_0^{a} (r^2)(r^2\sin\theta)\,dr\,d\theta\,d\phi$$

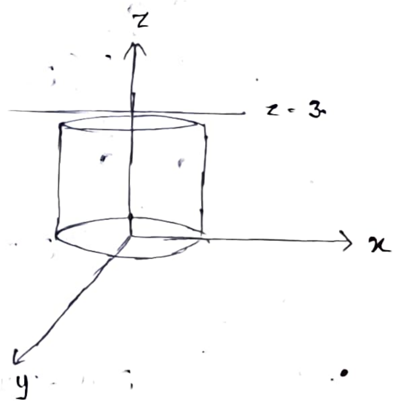
$$= \frac{12\pi a^5}{5}$$

⇒ Use div. th^m to evaluate $\iint_S \vec{F}\cdot\vec{ds}$ where $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$

and S is the surface of region; $x^2+y^2 = 4$, $z = 0$, & $z = 3$.

Sol^n)

$$\therefore \iint_S \vec{F}\cdot\hat{n}\ \vec{ds} = \iiint \text{div}\,\vec{F}\,dv$$



div $\vec{F} = 4 - 4y + 2z$

$\therefore$ Let $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

$dx\,dy\,dz = r\,dr\,d\theta\,dz$

$$\therefore \int_0^2\int_0^{2\pi}\int_0^3 (4 - 4r\sin\theta + 2z)\,r\,dr\,d\theta\,dz$$

$$\int_0^2\int_0^{2\pi}\left(4z - 4zr\sin\theta + z^2\right)_0^3 (r\,dr\,d\theta)$$

$$\int_0^2\int_0^{2\pi} (12 - 12r\sin\theta + 9)\,(r\,dr\,d\theta)$$

$$\int_0^2 (21\theta + 12r\cos\theta)_0^{2\pi}\,(r\,dr)$$

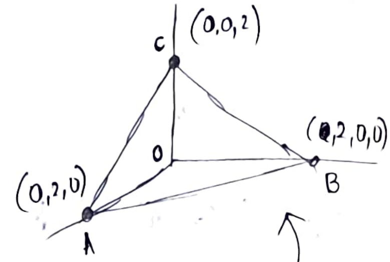
$$\int_0^2 (42\pi + 12r - 12r)\,(r\,dr)$$

$$\left(\frac{42r^2\pi}{2}\right)_0^2 = 21(4\pi) = 84\pi$$

⇒ Evaluate surface integral $\iint \vec{F}\cdot\hat{n}\, ds$, where $\vec{F} = (x^2+y^2+z^2)(\hat{i}+\hat{j}+\hat{k})$

S is surface of tetrahedron $x=0, y=0, z=0$, $x+y+z=2$, n is unit normal in outward dr^n to closed surface S

Sol^n $$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint \operatorname{div}\vec{F}\, dv$$



$$\frac{x}{2}+\frac{y}{2}+\frac{z}{2} = 1$$

$$\operatorname{div}\vec{F} = 2(x+y+z)$$

$x=2 \quad y=2-x \quad z=2-x-y$

$$\therefore 2\int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{2-x-y} (x+y+z)\, dx\, dy\, dz$$

$$\therefore 2\int_0^2\int_0^{2-x} \left(xz+yz+\frac{z^2}{2}\right)_0^{2-x-y}$$

$$2\int_0^2\int_0^{2-x} \left[x(2-x-y)+y(2-x-y)+\frac{(2-x-y)^2}{2}\right] dx\, dy$$

$$2\int_0^2\int_0^{2-x} \left[2x-x^2-yx+2y-xy-y^2+\frac{(2-x-y)^2}{2}\right] dx\, dy$$

$$\cancel{2\int_0^2 \left[2(2-x)-(2-x)^2-y(2-x)+\right.}$$

$$\Rightarrow 2\int_0^2 \left[2xy-x^2y-\frac{y^2x}{2}+2y^2-\frac{xy^2}{2}-\frac{y^3}{3}+\frac{(2-x-y)^3}{6}\right]_0^{2-x} dx$$

$$\Rightarrow 2\int_0^2 \left[2x(2-x)-x^2(2-x)-\frac{x}{2}(2-x)^2+2(2-x)^2-\frac{x}{2}(2-x)^2-\frac{(2-x)^3}{3}+\frac{(2-x-y)^3}{6}\right] dx$$

$$\Rightarrow 2\int_0^2 \left[4x-2x^2-2x^2+x^3-x(2-x)^2+2(2-x)^2-\frac{(2-x)^3}{3}+\frac{(2-x-y)^3}{6}\right] dx$$

$$\Rightarrow 2\left[2x^2-\frac{4x^3}{3}+\frac{x^4}{4}-2x^2+\frac{4x^3}{3}-\frac{x^4}{4}-\frac{(2-x)^3}{3}+\frac{(2-x)^4}{12}-\frac{(2-x)^4}{24}\right]_0^2$$

$$= \underline{\underline{4}}$$

* if in pre ques :

$$\iiint \operatorname{div} \vec{F}\, dv = \iiint \overset{\text{const}}{a}\, dv$$

$$= a \iiint dv$$ volume of tetrahedron $\longrightarrow \frac{1}{3}(AOB)(OC)$

pre diagram

$\Rightarrow$ Use div. thm to evaluate : $\iint x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$

over the surface of sphere radius A.

Soln) $\iint (f_1\, dy\, dz + f_2\, dz\, dx + f_3\, dx\, dy)$

$$\Rightarrow \iiint \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \right) dx\, dy\, dz.$$

$$\therefore \iiint \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\, dx\, dy\, dz = 3 \iiint dv$$

$$= 3\left(\frac{4}{3}\pi (A)^3\right) = \underline{4\pi A^3}$$

$\Rightarrow \iint yz\, dy\,dz + zx\, dz\, dx + xy\, dy\, dx$, $S = x^2 + y^2 + z^2 = 4$

Same as pre question :

Soln) $\iiint \frac{\partial (yz)}{\partial x} + \frac{\partial (zx)}{\partial y} + \frac{\partial (xy)}{\partial z}\, dx\, dy\, dz.$

$$\iiint (0)\, dx\, dy\, dz = \underline{0}$$

$\Rightarrow$ ~~$\iint xz^2 dy$~~

$dx = 0$ so directly put $dy\,dz$ as it is on y-z axis)

⇒ $\iint xz^2\, dy\, dz + (x^2y - z^3)\, dz\, dx + (2xy + y^2z)\, dx\, dy$ where

S is surface of hemispherical regio bound by

$z = \sqrt{a^2 - x^2 - y^2}$, $z = 0$.

Sol^n $\iiint \frac{\partial (xz^2)}{\partial x} + \frac{\partial (x^2y - z^3)}{\partial y} + \frac{\partial (2xy + y^2z)}{\partial (z)}\, dx\, dy\, dz.$

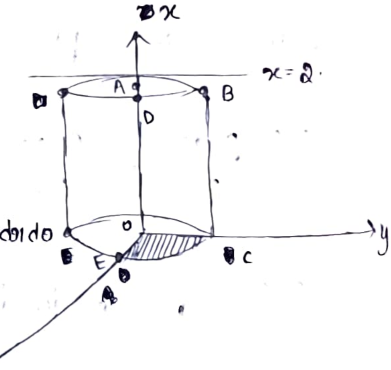
⇒ $\iiint z^2 + x^2 + y^2 \, dx\, dy\, dz.$

$y = r \sin\theta \sin\phi$, $x = r\sin\theta \cos\phi$, $z = r\cos\theta$.

$\therefore \int_0^{\theta=\pi/2}\int_0^{\phi=2\pi}\int_0^{a} r^2 (r^2 \sin\theta)\, dr\, d\theta\, d\phi$

⇒ verify div. th^m $\vec{F} = 2x^2y\,\hat{i} - y^2\,\hat{j} + 4xz^2\,\hat{k}$ taken over the region in first octant bounded by $y^2+z^2 = 9$, $x = 2$.

Sol^n div $\vec{F} = 4xy - 2y + 8xz$



Let $x = x$, $y = r\sin\theta$, $z = r\cos\theta$

$\therefore \int_0^3\int_0^{\pi/2}\int_0^2 4x(r\sin\theta) - 2(r\sin\theta) + 8(x)(r\cos\theta)\, r\, dx\, dr\, d\theta$

$\therefore \int_0^3\int_0^{\pi/2} \left[2x^2 r\sin\theta - 2xr\sin\theta + 4x^2 r\cos\theta\right]_0^2 r\, dr\, d\theta$

$\int_0^3\int_0^{\pi/2} \left[8r\sin\theta - 4r\sin\theta + 16r\cos\theta\right] r\, dr\, d\theta$

$\int_0^3 \left[-4r\cos\theta + 16r\sin\theta\right]_0^{\pi/2} r\, dr$

$\int_0^3 \left[16r + 4r\right] r\, dr \Rightarrow \left(\frac{20r^3}{3}\right)_0^3 = \frac{20(27)}{3} = \underline{\underline{180}}$ RHS.

LHS : $\iint \vec{F}\cdot\hat{n}\, ds$

Here we don't have common normal to every surface

so, take parts.

$$\Rightarrow \iint \vec{F}\cdot\hat{n}\,ds = \underset{\substack{OABC\\ \uparrow \\ \text{rect}}}{\iint} + \underset{\substack{OCE\\ \uparrow \\ \text{sector}}}{\iint} + \underset{\substack{ABD\\ \uparrow \\ \text{sector}}}{\iint} + \underset{\substack{OADE\\ \uparrow \\ \text{rect}}}{\iint} + \underset{\substack{BDEC\\ \uparrow \\ \text{curve surface.}}}{\iint}$$

i) $\underset{BDEC}{\iint} \vec{F}\cdot\hat{n}\,ds$, $\hat{n} = \dfrac{\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(y^2+z^2-9)}{|\nabla\phi|} = \dfrac{y\hat{j}+z\hat{k}}{3}$.

$$\Rightarrow \underset{BDEC}{\iint} (2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k})\cdot\frac{y\hat{j}+z\hat{k}}{3}\,ds = \underset{BDEC}{\iint} \frac{(-y^3+4xz^3)\,ds}{3}.$$

$$\Rightarrow \frac{1}{3}\underset{BDEC}{\iint} (-y^3+4xz^3)\frac{dx\,dy}{z/3} = \int_0^2 dx \int_0^3 \left(\frac{-y^3}{z} + 4xz^2\right) dy$$

$$\Rightarrow \int_0^2 dx \int_0^{\pi/2} \frac{-27\sin^3\theta}{3\cos\theta} + 4x\,(9\cos^2\theta) = \underline{\underline{108}}$$

ii) $\underset{OABC}{\iint} \vec{F}\cdot\hat{n} = \iint (2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k})\cdot(-\hat{k})\,ds =$

$= \iint 4xz^2\,ds = 0$ ($\because$ in plane OABC , $z=0$).

iii) $\underset{OADE}{\iint} = \iint (2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k})\cdot(-\hat{j})\,ds = \underline{\underline{0}}$

take dir^n where the desired part is not there.

iv) $\underset{ABC}{\iint} = \iint (\text{———})\cdot(\hat{i})\,ds = \underset{ABD}{\iint} 2x^2y\, dy\, dz$

Here we didn't do $ds = \dfrac{dx\,dy}{\hat{n}\cdot\hat{k}}$ ($\because$ here $dx=0$ so directly put $dy\,dz$ as it is on y-z axis)

$$\Rightarrow \int_0^3 dz \int_0^{\sqrt{9-x^2}} 2(2)^2(y)\,dy = \underline{\underline{72}}$$

$\therefore \iint \vec{F}\cdot\hat{n}\,ds = 108 + 72 = \underline{\underline{180}}$

$\therefore$ LHS = RHS

(Proved)

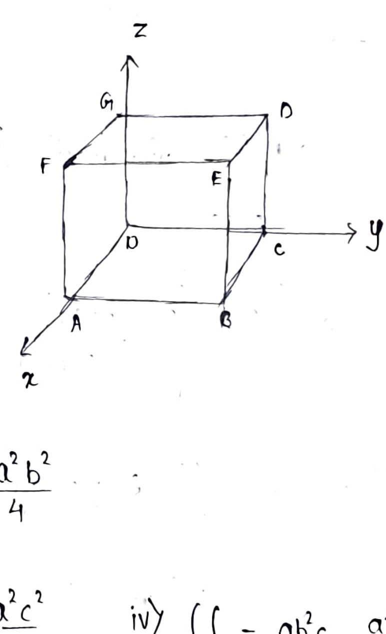
$\Rightarrow$ Verify Gauss div. Th^m :

$\vec{F} = (x^2-yz)\hat{i} + (y^2-zx)\hat{j} + (z^2-xy)\hat{k}$, taken over rect. llpiped

$x \in [0,a]$, $y \in [0,b]$, $z \in [0,c]$.

Soln $\iiint \text{div}\,\vec{F}\cdot dv = abc\,(a+b+c)$. $\longrightarrow$ RHS.

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{OABC} + \iint_{DEFG} + \iint_{OAFG} + \iint_{BCDE} + \iint_{ABEF} + \iint_{OCDG}.$$



$\Rightarrow$ i) $\iint_{OABC} (x^2-yz)\hat{i} + (y^2-zx)\hat{j} + (z^2-xy)\hat{k}\cdot(-\hat{k})\,ds$

$$\Rightarrow -\int_0^a\int_0^b (z^2-xy)\,dx\,dy = \frac{a^2b^2}{4}$$

ii) $\iint_{DEFG} = abc^2 - \frac{a^2b^2}{4}$ iii) $\iint_{OAFG} = \frac{a^2c^2}{4}$ iv) $\iint_{BCDE} = ab^2c - \frac{a^2c^2}{4}$

v) $\iint_{ABEF} = a^2bc - \frac{b^2c^2}{4}$ vi) $\iint_{OCDG} = \frac{b^2c^2}{4}$.

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = abc\,(a+b+c)$$

$\therefore$ Proved