$$\frac{\sin\theta\cos\theta}{r^2}\ \frac{\partial u}{\partial \theta}$$

$$\therefore \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial \phi}{\partial s}\cdot\frac{\partial s}{\partial x} = 0$$

$$= \frac{\partial \phi}{\partial r}\left(c - a\frac{\partial z}{\partial x}\right) + \frac{\partial \phi}{\partial s}\left(-b\frac{\partial z}{\partial x}\right) = 0$$

$$\Rightarrow a\frac{\partial z}{\partial x} = \frac{ac\,\dfrac{\partial \phi}{\partial r}}{a\dfrac{\partial \phi}{\partial r} + b\dfrac{\partial \phi}{\partial s}} \longrightarrow (1)$$

$$\frac{\partial \phi}{\partial y} = 0$$

$$\Rightarrow b\frac{\partial z}{\partial y} = \frac{bc\,\dfrac{\partial \phi}{\partial s}}{a\dfrac{\partial \phi}{\partial r} + b\dfrac{\partial \phi}{\partial s}} \longrightarrow (2)$$

$$\therefore (1) + (2) = \underline{\underline{c}}$$ (Proved)

If variable changes :

i.e., $u = f(x,y)$

in pre topic $\longrightarrow u = u(x,y)$

$\Rightarrow$ If $\omega = f(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$, ST:

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$$

Soln for this first diff x & y.

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\therefore \frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}(\cos\theta) + \frac{\partial f}{\partial y}(\sin\theta)$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y}(r\cos\theta)$$

$$\Rightarrow \frac{1}{r}\frac{\partial w}{\partial \theta} = -\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta$$

$$\therefore \quad (1)^2 + (2)^2 \Rightarrow \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \qquad \text{(Proved)}$$

imp

$\Rightarrow$ Transform the eqⁿ : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ into polar co-ordinates.

Solⁿ

Polar coordinates : $x = r\cos\theta$, $y = r\sin\theta$.

$$\therefore x^2 + y^2 = r^2.$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$r = \sqrt{x^2+y^2}$$

$$\frac{\partial r}{\partial x} = \frac{\frac{1}{2}(2x)}{\sqrt{x^2+y^2}} = \frac{x}{r}.$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$= \frac{\partial u}{\partial r}\left(\frac{x}{r}\right) + \frac{\partial u}{\partial \theta}\left(\frac{-y}{x^2+y^2}\right) = \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}.$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

use U.V

$$= \cos\theta\left[\cos\theta\frac{\partial^2 u}{\partial r^2} - \left(\frac{\sin\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \left(-\frac{\sin\theta}{r^2}\frac{\partial u}{\partial \theta}\right)\right)\right]$$

$$-\frac{\sin\theta}{r}\left[\cos\theta\frac{\partial^2 u}{\partial\theta\partial r} + (-\sin\theta)\frac{\partial u}{\partial r} - \left(\frac{\sin\theta}{r}\frac{\partial^2 u}{\partial\theta^2} + \left(\frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}\right)\right)\right]$$

$$\Rightarrow \cos^2\theta\frac{\partial^2 u}{\partial r^2} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial \theta}$$

$$- \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial\theta\,\partial r} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta}$$

$$\Rightarrow \cos^2\theta\,\frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\,\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} \longrightarrow (1)$$

Similarly for : $\dfrac{\partial^2 u}{\partial y^2}$

$$\Rightarrow \sin^2\theta\,\frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\,\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} \longrightarrow (2).$$

$\therefore$ (1) + (2) $\Rightarrow \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial\theta^2}$ (Proved).