* Let $x^n = y$:

$$n x^{n-1} dx = dy .$$

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$$

$$\Gamma n = \int_0^\infty e^{-y^{1/n}} \left(\frac{dy}{n}\right) = \frac{1}{n} \int_0^\infty e^{-y^{1/n}} dy$$

$$\therefore \Gamma \tfrac{1}{2} = \frac{1}{1/2} \int_0^\infty e^{-y^2} dy = 2 \int_0^\infty e^{-y^2} dy$$

$$= 2\left[\frac{\sqrt{\pi}}{2}\right] = \sqrt{\pi}$$

$$\therefore \Gamma \tfrac{1}{2} = \sqrt{\pi}$$

BETA FUNC^N :

$$\beta(l,m) = \int_0^1 x^{l-1} (1-x)^{m-1} dx$$

* $\beta(l,m) = \dfrac{\Gamma l \, \Gamma m}{\Gamma l+m}$

* Property : $\beta(l,m) = \beta(m,l)$.

$\Rightarrow \displaystyle\int_0^1 x^4 (1-\sqrt{x})^5 dx$

Sol^n/ $\sqrt{x} = t$

$x = t^2$

$dx = 2t\, dt$

$$\int_0^1 t^8 (1-t)^5 \, 2t\, dt$$

$$2\int_0^1 t^9 (1-t)^5 dt$$

$$2\beta(10,6) = 2\frac{\Gamma 10 \, \Gamma 6}{\Gamma 16} = \frac{2(9!)(5!)}{15!} = \frac{1}{15015}$$

Transformation of Beta Func^n :

$$\beta(l,m) = \int_0^1 x^{l-1}(1-x)^{m-1}\,dx. \qquad \alpha$$

$$\therefore \beta(l,m) = \int_0^\infty \frac{y^{l-1}}{(1+y)^{m+l}}\,dy = \int_0^\infty \frac{x^{l-1}}{(1+x)^{m+l}}\,dx$$

$$\Rightarrow \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx \qquad \alpha.$$

Sol^n)

Relation B/w Beta And Gamma :

$$* \ \beta(l,m) = \frac{\Gamma l\,\Gamma m}{\Gamma l+m}.$$

$$* \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{\Gamma\left[\frac{p+1}{2}\right]\Gamma\left[\frac{q+1}{2}\right]}{2\Gamma\left[\frac{p+q+2}{2}\right]}.$$

$$\Rightarrow \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Let $x = \sin^2\theta$, $\int_0^{\pi/2} (\sin\theta)^{2m-2}(\cos\theta)^{2n-2}\, 2\sin\theta\cos\theta\, d\theta$

$dx = \sin 2\theta\, d\theta$

$$\Rightarrow 2\int_0^{\pi/2} (\sin\theta)^{2m-1}(\cos\theta)^{2n-1}\,d\theta$$

Let $2m-1 = p$, $2n-1 = q$.

$$\Rightarrow \int_0^{\pi/2} \sin^p\theta\cos^q\theta\, d\theta = \frac{\beta(m,n)}{2} = \frac{\Gamma m\,\Gamma n}{2\Gamma m+n}$$

$$= \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\Gamma\frac{p+q+2}{2}}$$

(Proved)

$\Rightarrow \Gamma\frac{1}{2} = \sqrt{\pi}$ (Prove)

Sol^n $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \dfrac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}$.

Let $p = 0$, $q = 0$.

$\int_0^{\pi/2} d\theta = \dfrac{\left(\Gamma\frac{1}{2}\right)^2}{2} \Rightarrow \left(\frac{\pi}{2}\right)2 = \left(\Gamma\frac{1}{2}\right)^2$

$\Rightarrow \Gamma\frac{1}{2} = \sqrt{\pi}$.

$\Rightarrow$ Show that : $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \frac{1}{2}\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}$

Sol^n $\cot^{1/2}\theta = \dfrac{\cos^{1/2}\theta}{\sin^{1/2}\theta}$

$\int_0^{\pi/2} \cos^{1/2}\theta \sin^{-1/2}\theta \, d\theta = \dfrac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}} = \dfrac{\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}}{2\,\Gamma 1} = \frac{1}{2}\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}$

$\Rightarrow \int_{-1}^{1} (1+x)^{p-1} (1-x)^{q-1} \, dx$.

Sol^n $x = \cos 2\theta$.

$dx = -\sin 2\theta\,(2)\, d\theta$

$dx = -2\sin 2\theta\, d\theta$.

$\int_{\pi/2}^{0} (2\cos^2\theta)^{p-1} (2\sin^2\theta)^{q-1} (-2\sin 2\theta)\, d\theta$

$2^{p-1+q-1+1+1} \int_0^{\pi/2} \cos^{2p-1}\theta \sin^{2q-1}\theta \, d\theta$

$\dfrac{\Gamma\frac{2p-1+1}{2}\,\Gamma\frac{2q-1+1}{2}}{2\,\Gamma\frac{2p-1+2q-1+2}{2}} = \dfrac{\Gamma p\,\Gamma q}{2\,\Gamma p+q}$ (Ans)