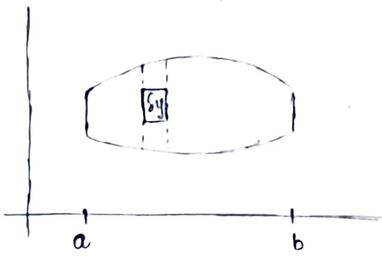
# MODULE - 4

## DOUBLE INTEGRAL

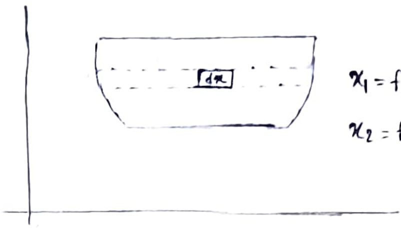
$y_1 = f_1(x)$ , $y_2 = f_2(x)$



* ~~$\iint_A f(x,y)\, dx\, dy$~~

* $$\iint_A f(x,y)\, dA = \int_a^b \int_{y_1}^{y_2} f(x,y)\, dy\, dx$$

* $$\iint_A f(x,y)\, dA = \int_c^d \int_{x_1}^{x_2} f(x,y)\, dx\, dy$$



$x_1 = f_1(y)$

$x_2 = f_2(y)$

* It is used to find area having two variables.

⇒ Evaluate : $\int_0^1 \int_0^x (x^2+y^2)\, dA$ , $dA \longrightarrow$ small area in x-y plane

Sol^n

$$I = \int_0^1 \int_0^x (x^2+y^2)\, dy\, dx = \int_0^1 \left[ x^2 y + \frac{y^3}{3} \right]_0^x dx$$

we can integrate first by dy or else dx depending upon limits.

$$= \int_0^1 \left[ x^2(x-0) + \frac{1}{3}(x^3-0) \right] dx$$

$$= \int_0^1 \left[ x^3 + \frac{x^3}{3} \right] dx = \int_0^1 \frac{4}{3} x^3\, dx = \frac{4}{3}\left[\frac{x^4}{4}\right]_0^1 = \frac{1}{3} \text{ sq. units}$$

⇒ Evaluate $\int_{-1}^{1} \int_0^{1-x} x^{1/3}\, y^{-1/2}\, (1-x-y)^{1/2}\, dy\, dx$.

Sol^n

$1 - x = c$ (let)

$$I = \int_{-1}^{1} x^{1/3}\, dx \int_0^c y^{-1/2} (c-y)^{1/2}\, dy$$

let $y = ct \Rightarrow dy = c\, dt$

$$I = \int_{-1}^{1} x^{1/3}\, dx \int_{0}^{c} c^{-1/2}\, t^{-1/2}\, (c - ct)^{1/2}\, c\, dt$$

$$I = \int_{-1}^{1} x^{1/3}\, dx \int_{0}^{c} c^{-1/2}\, t^{-1/2}\, c^{1/2}\, (1-t)^{1/2}\, c\, dt.$$

Beta & Gamma Funcn.

$$\left[ \int_{0}^{1} x^{l-1} (1-x)^{m-1}\, dx = \beta(l, m) \right]$$

$$I = \int_{-1}^{1} c\, x^{1/3}\, dx \int_{0}^{c} t^{-1/2} (1-t)^{1/2}\, dt$$

$$I = \int_{-1}^{1} c\, x^{1/3}\, dx \int_{0}^{c} t^{1/2 - 1} (1-t)^{1/2}\, dt = \int_{-1}^{1} c\, x^{1/3}\, dx\; \beta\left(\tfrac{1}{2}, \tfrac{3}{2}\right)$$

$$I = \int_{-1}^{1} c\, x^{1/3}\, dx\; \frac{\Gamma\tfrac{1}{2}\, \Gamma\tfrac{3}{2}}{\Gamma\left(\tfrac{1}{2} + \tfrac{3}{2}\right)} = \int_{-1}^{1} c\, x^{1/3}\, dx\; \frac{\Gamma\tfrac{1}{2} \cdot \tfrac{1}{2}\, \Gamma\tfrac{1}{2}}{\Gamma 2}$$

$$= \int_{-1}^{1} c\, x^{1/3}\, dx\; \frac{\sqrt{\pi}\, (\tfrac{1}{2})\, (\sqrt{\pi})}{1}$$

$$= \int_{-1}^{1} c\, x^{1/3}\, \frac{\pi}{2}\, dx = \frac{\pi}{2} \int_{-1}^{1} x^{1/3} \cdot c\, dx$$

put $c = 1 - x$

$$\frac{\pi}{2} \int_{-1}^{1} x^{1/3} (1-x)\, dx = \frac{\pi}{2} \int_{-1}^{1} \left(x^{1/3} - x^{4/3}\right) dx = \frac{9\pi}{28} \quad \text{(Ans)}$$

⇒ Evaluate $\iint_R (x+y)\, dy\, dx$, R is the region bound by $x = 0$, $x = 2$, $y = x$, $y = x+2$.

Soln $$I = \int_{0}^{2} dx \int_{x}^{x+2} (x+y)\, dy = \int_{0}^{2} \left[ xy + \frac{y^2}{2} \right]_{x}^{x+2} dx$$

$$I = \int_{0}^{2} \left[ x(x+2) + \frac{(x+2)^2}{2} - x^2 - \frac{x^2}{2} \right] dx$$

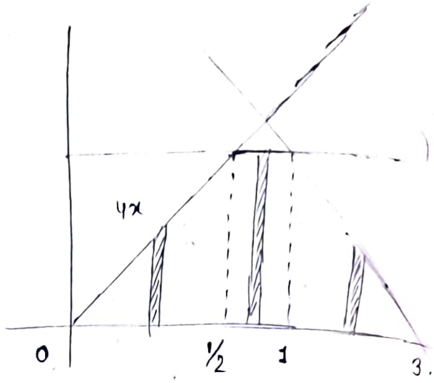
$$I = \int_{0}^{2} \left[ 2x + \cancel{\tfrac{4x}{2}} + 2 \right] dx = \underline{10}$$

⇒ Evaluate $\iint (x^2+y^2)\,dx\,dy$ throughout the area enclosed by the curves $y=4x$, $x+y=3$, $y=0$, $y=2$.

Sol^n)

$$A_1 = \int_0^{1/2} dx \int_0^{4x} (x^2+y^2)\,dy = \frac{19}{48}$$

$y=4x$




$$A_2 = \int_{1/2}^{1} dx \int_0^{2} (x^2+y^2)\,dy = \frac{23}{12}$$

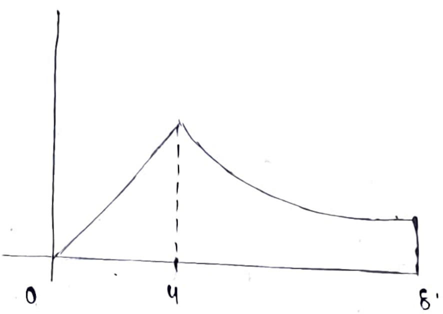
$$A_3 = \int_1^{3} dx \int_0^{-x+3} (x^2+y^2)\,dy = \frac{22}{3}$$

Add

⇒ Evaluate $\iint_A x^2\,dx\,dy$, $xy=16$, $y=x$, $y=0$, $x=8$

Sol^n)

$$\cancel{A_1 = \int_0^4 x^2\,dx \int_0^x dy + \int_4^8 x^2\,dx \int_0^{xy} dy}$$




$$A = \int_0^4 \int_0^x x^2\,dx\,dy + \int_4^8 \int_0^{16/x} x^2\,dx\,dy$$

$$= \int_0^4 x^2\,dx \int_0^x dy + \int_4^8 x^2\,dx \int_0^{16/x} dy$$

$$= 448 \text{ (Ans)}$$

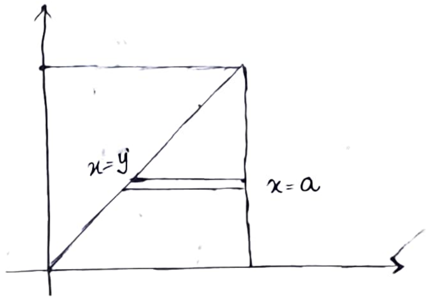
# CHANGE OF ORDER OF INTEGRATION

$\Rightarrow$ Evaluate : $\displaystyle\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$.

Sol$^n$) $\displaystyle I = \int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$.

$\longrightarrow$ here limits of $x$ are there

$x = a$, $x = y$, $y = 0$, $y = a$.




$\therefore$ the limit of $x$ is from left $\rightarrow$ right so convert into bottom $\rightarrow$ top.

bottom $\rightarrow$ top

to

left $\rightarrow$ right

$$I = \int_0^a \int_{y=0}^{y=x} \frac{x}{x^2+y^2}\, dx\, dy = \int_0^a x\, dx \int_0^x \frac{1}{x^2+y^2}\, dy$$

$$= \int_0^a x\, dx \left[\frac{1}{x}\tan^{-1}\frac{x}{y}\right]_0^x$$

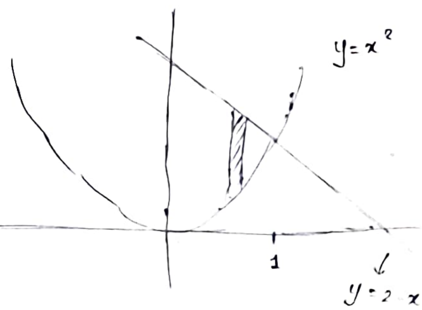
$$= \int_0^a \frac{x}{x}\, dx \left[\frac{\pi}{4}\right] = \frac{\pi}{4}\left[x\right]_0^a = \underline{\underline{\frac{a\pi}{4}}}$$

imp (must do)

$\Rightarrow$ Change order of integration : $\displaystyle I = \int_0^1 \int_{x^2}^{2-x} xy\, dx\, dy$.

Sol$^n$) It is from bottom to top.

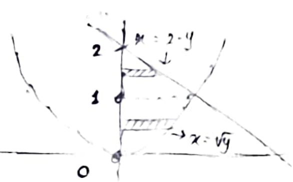
Convert form left to right.




$$I = \int_0^1 \int_0^{\sqrt{y}} xy\, dx\, dy + \int_1^2 \int_0^{2-y} xy\, dx\, dy$$

$$I = \int_0^1 y\, dy \int_0^{\sqrt{y}} x\, dx + \int_1^2 y\, dy \int_0^{2-y} x\, dx$$

$\Downarrow$



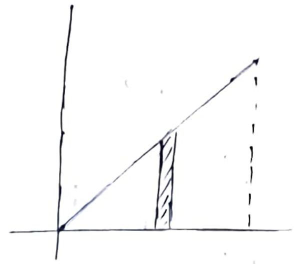

If we do partition then we have to change outer limit also

$$\underline{I = \frac{3}{8}} \quad \text{(Ans)}$$

$\Rightarrow \int_0^\infty \int_0^x x \exp\left(-\frac{x^2}{y}\right) dx\, dy \rightarrow$ By changing order of integration.

Sol$^n$

$$I = \int_0^\infty \int_0^x x e^{-\frac{x^2}{y}} dx\, dy$$



$$I = \int_0^\infty \int_y^\infty x e^{-\frac{x^2}{y}} dx\, dy.$$

$$I = \int_0^\infty dy \int_y^\infty x e^{-\frac{x^2}{y}} dx$$

$$I = \frac{1}{2}$$

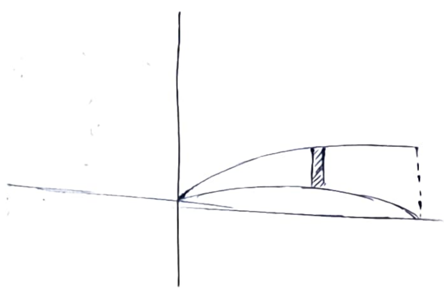
Generally

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} \longrightarrow \int_{y_3}^{y_4} \int_{x_3}^{x_4}$$

$\Rightarrow \int_0^{2a} \int_{\sqrt{2ax - x^2}}^{\sqrt{2ax}} 2\, dx\, dy \longrightarrow$ change order.

Sol$^n$

$$I = \int_0^{2a} dx \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} 2\, dy$$



$$\Rightarrow \int_a^{2a} \int_{y^2/2a}^{2a} 2\, dx\, dy + \int_0^a \int_{y^2/2a}^{a+\sqrt{a^2+y^2}} 2\, dx\, dy$$

$$+ \int_0^a \int_{a+\sqrt{a^2-y^2}}^{2a} 2\, dx\, dy.$$

Limits are $x = 0, x = 2a$

$y = \sqrt{2ax}$, $y = \sqrt{2ax - x^2}$

$y^2 = 2ax$ $\quad$ $x^2 = 0$

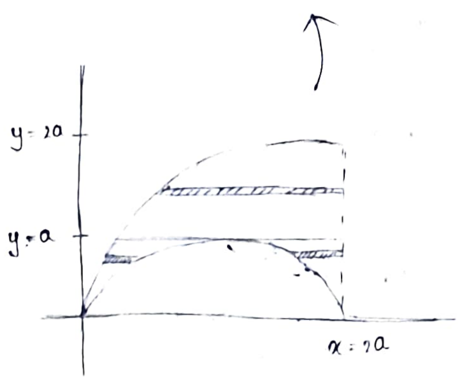
$\therefore (x-a)^2 + y^2 = a^2$

$x^2 + a^2 - 2ax + y^2 = a^2$

$x^2 = 0$ ← satisfied

$\therefore (x-a)^2 + y^2 = a^2$

⇒ Solve



# CHANGE OF VARIABLES

* $xy$ plane to $uv$ - plane. $\longrightarrow$ double integral can be solved easily.

* $\iint_R f(x,y)\,dx\,dy$ changed by the new variables $u,v$.

$$x = \phi(u,v) \quad , \quad y = \psi(u,v)$$

where $dx\,dy = |J|\,du\,dv$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} du\,dv.$$

after substituting :

$$\iint_{R_1} f(\phi(u,v), \psi(u,v))\ |J|\ du\,dv \}$$

$\Rightarrow$ Evaluate $\iint_R (x+y)^2\,dx\,dy$, where $R$ is the llgm in the $xy$-plane with vertices $(1,0)$, $(3,1)$, $(2,2)$, $(0,1)$. Using transformation

$$u = x+y \quad , \quad v = x-2y.$$

Sol$^n$ $\because$ $x = \frac{2u+v}{3}$ , $y = \frac{u-v}{3}$ $= \psi(u,v)$.

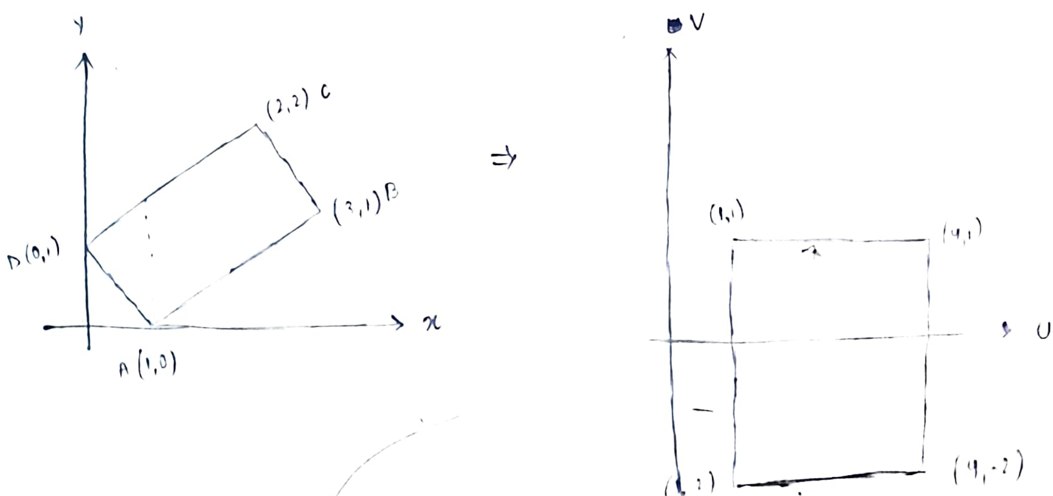
$(1,0) \Rightarrow u = 1, v = 1 \Rightarrow (1,1)$

$(3,1) \Rightarrow u = 4, v = 1 \Rightarrow (4,1)$

$(2,2) \Rightarrow u = 4, v = -2 \Rightarrow (4,-2)$

$(0,1) \Rightarrow u = 1, v = -2 \Rightarrow (1,-2)$

$$\int_{v=-2}^{1}\int_{u=1}^{4} \left[\frac{2u+v}{3} + \frac{u-v}{3}\right]^2 \frac{1}{3}\,du\,dv = \underline{\underline{21}} \text{ (Ans)}$$



$\Rightarrow$ $x+y = u$, $y = uv$, $x=0, y=0, x+y=1$.

PT. $\iint \left[ xy(1-x-y) \right]^{1/2} dx\,dy = \frac{2\pi}{105}$

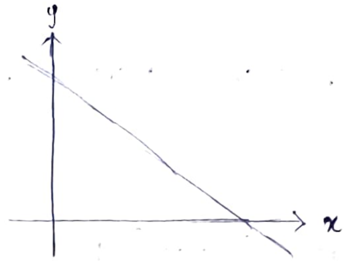
Sol$^n$) $x+y = u$, $\Rightarrow x = u - y$, $\Rightarrow x = u - uv$.

$\therefore dx\,dy = (J)\,du\,dv = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} du\,dv = u\,du\,dv$.

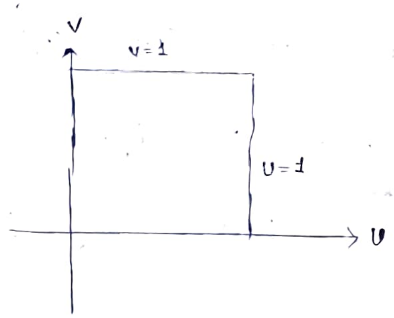
$x = 0 \Rightarrow u = 0, v = 1$

$y = 0 \Rightarrow u = 0, v = 0$

$x+y = 1 \Rightarrow u = 1$.




$\Downarrow$




$\therefore \iint \left[ (u-uv)(uv)(1-u) \right]^{1/2} u\,du\,dv$

$\int_0^1 \int_0^1 (u-uv)^{1/2} (uv)^{1/2} (1-u)^{1/2}\, u\,du\,dv$

$= \frac{2\pi}{105}$

$\Rightarrow \int_0^\infty \int_0^\infty e^{-(x+y)} \sin\left(\frac{\pi y}{x+y}\right) dx\,dy$ by means of transformation

$u = x+y$, $v = y$

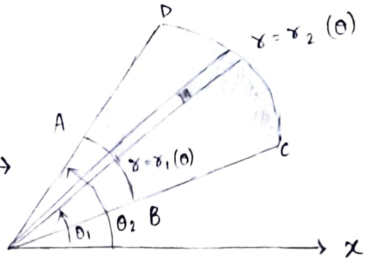
imp
(will come in exam)

$\Rightarrow x+y = u$, $y = uv$, PT: $\int_0^1 \int_0^{1-x} \frac{y}{e^{x+y}}\, dy\,dx = \frac{1}{2}(e-1)$. imp

must do

# EVALUATE DOUBLE INTEGRAL IN POLAR COORDINATES :

$$\int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_1(\theta)} f(\underset{\text{angle}}{r},\overset{\text{radius}}{\theta})\, dr\, d\theta$$

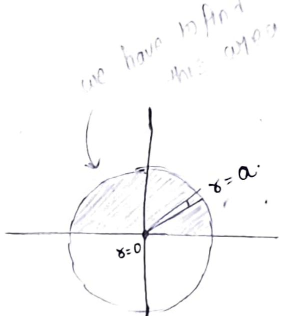



This represent part of circle.

$\therefore$ $\boxed{\begin{array}{l} dx\,dy \longrightarrow r\,dr\,d\theta \\ f(x,y) \longrightarrow f(\theta, r) \end{array}}$

$\Rightarrow$ Transform integral to cartesian form & evaluate :

$$\int_0^{\pi}\int_0^{a} r^3 \sin\theta \cos\theta \, dr\, d\theta$$



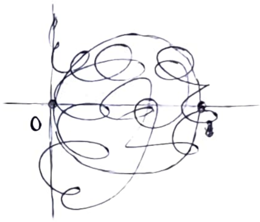

Soln

$$\int_0^{\pi}\int_0^{a} (r\sin\theta)(r\cos\theta)(r\,dr\,d\theta)$$

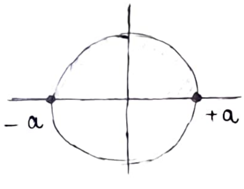
let $x = r\sin\theta$, $y = r\cos\theta$, $r\,dr\,d\theta = dx\,dy$.

$\Downarrow$

$$\int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} xy \, dy\, dx = \underline{0} \quad \text{(Ans)}.$$




when we get eq$^n$ of circle we mostly use polar to get product of values.



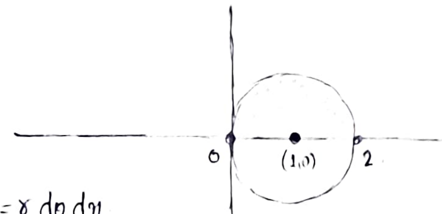

$\Rightarrow$ Evaluate $\displaystyle\int_0^{2}\int_0^{\sqrt{2x-x^2}} (x^2+y^2)\, dy\, dx$

Soln

$y = \sqrt{2x - x^2} \Rightarrow y^2 + x^2 - 2x = 0$

$\therefore$ centre $(1,0)$

$r = 1$

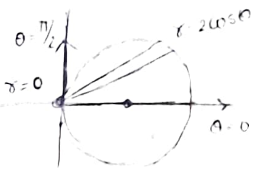



$\therefore$ let $x = r\cos\theta$, $y = r\sin\theta$, $dy\,dx = r\,d\theta\,dr$

$\therefore$ $r^2(\sin^2\theta + \cos^2\theta) - 2r\cos\theta = 0 \longleftarrow y^2 + x^2 - 2x = 0$

$r = 2\cos\theta$

$$\int_0^{\pi/2}\int_0^{2\cos\theta} r^3\, dr\, d\theta = \underline{\frac{3\pi}{4}} \quad \text{(Ans)}$$



# TRIPLE INTEGRATION :

$*$ $$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\,dx\,dy\,dz .$$

$$\Rightarrow \int_{x_1=a}^{x_2=b} \psi(x)\,dx \int_{y_1=\phi_1(x)}^{y_2=\phi_2(x)} \phi(x,y)\,dy \int_{z_1=f_1(x,y)}^{z_2=f_2(x,y)} f(x,y,z)\,dz .$$

$\Rightarrow$ $\iiint_R (x+y+z)\,dx\,dy\,dz$ , $R: 0 \le x \le 1,\ 1 \le y \le 2,\ 2 \le z \le 3.$

Sol$^n$ $$\int_0^1\int_1^2\int_2^3 (x+y+z)\,dx\,dy\,.\,dz .$$

$$\int_0^1 dx \int_1^2 dy \int_2^3 (x+y+z)\,dz .$$

$$\int_0^1 dx \int_1^2 \left[\frac{(x+y+z)^2}{2}\right]_2^3 dy = \frac{9}{2}$$

$$\Rightarrow \int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\,dx\,dy\,dz$$

Sol$^n$ $$\int_0^{\log 2} e^x dx \int_0^x e^y dy \int_0^{x+\log y} e^z dz = \int_0^{\log 2} e^x\,dx \int_0^x e^y\,dy \left(e^z\right)_0^{x+\log y}.$$

$$= \int_0^{\log 2} e^x\,dx \int_0^x e^y \left(e^{x+\log y} - 1\right) dy$$

$$= \int_0^{\log 2} e^x\,dx \int_0^x e^y \left(e^x \cdot y - 1\right) dy$$

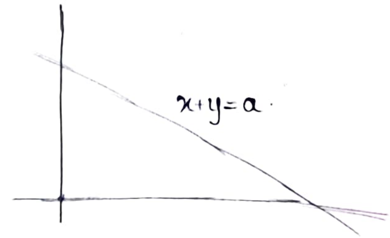
$$= \frac{8}{3}\log 2 - \frac{19}{9}$$

$\Rightarrow$ $\iiint_R (x^2+y^2+z^2)\,dx\,dy\,dz$ . where R denotes region bound by

$x=0,\ y=0,\ z=0$ & $x+y+z=a$.

Soln

$x+y+z=a$




in $x$-$y$ plane $\rightarrow$ $x+y=a$

$\Rightarrow y=a-x$

for $x$-axis $\longrightarrow$ $x=a$

$$\int_0^a dx \int_0^{a-x} dy \int_0^{a-x-y} (x^2+y^2+z^2)\,dz$$

$\hookrightarrow$ solve

imp (must).

Whenever one eqn comes then solve like this

# INTEGRATION BY CHANGING INTO ~~POLAR~~ SPERICAL COORDINATES :

$x = r \sin\theta \cos\phi$

$y = r\ \cancel{\cos\theta}\ \sin\theta \sin\phi$

$z = r\cos\theta$.

$dx\,dy\,dz = |J|\,dr\,d\theta\,d\phi = r^2 \sin\theta\,dr\,d\theta\,d\phi$

$\phi \rightarrow$ for $x$-$y$ plane

$\theta \rightarrow$ for $z$ plane


* It is used if the expression $x^2+y^2+z^2$ is involved in the problem

imp
* In a sphere $x^2+y^2+z^2=a^2$ the limits of $r$ are 0 and $a$ and limits of $\theta$ are $0, \pi$ and that of $\phi$ are 0 and $2\pi$.

$\Rightarrow$ $\iiint_R x^2+y^2+z^2\,dx\,dy\,dz$ , taken over the volume enclosed by sphere $x^2+y^2+z^2=1$

$\boxed{\phi \longrightarrow 0 \text{ to } \pi}$ $\rightarrow$ i.e from $+z$ to $-z$ in most que unless said in que like next que.

imp

Soln

$$\int_0^{2\pi}\int_0^{2\pi}\int_0^{1} r^2\,(r^2 \sin\theta\,dr\,d\theta\,d\phi)$$

$$\int_0^{2\pi} d\phi \int_0^{2\pi} \sin\theta\,d\theta \int_0^{1} r^4\,dr = \frac{4\pi}{5}$$

$\Rightarrow \iiint (x^2+y^2+z^2)\,dx\,dy\,dz$, over first octant of sphere $x^2+y^2+z^2 = a^2$.

Sol^n^ $\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a} r^2 (r^2 \sin\theta)\, dr\, d\theta\, d\phi$

first octant is
$\theta \to 0$ to $\pi/2$ &
$\phi \to 0$ to $\pi/2$

↳ solve

# SOLVING WHEN TWO OR MORE EQN ARE GIVEN :

$\Rightarrow$ Find the volume of cylinder $x^2+y^2=4$ & $y+z=4$ & $z=0$

Sol^n^ $z = 4-y$, $z=0$.

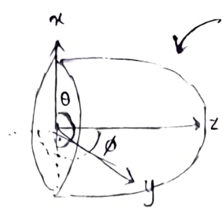
$$\int_{-2}^{+2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{0}^{4-y} (\underline{\quad\quad})$$

↳ any eq^n^ given.

# CONTINUING SPHERICAL COORDINATES :

$\Rightarrow \iiint_E (x^2+y^2)\,dv$, E is region portion of $x^2+y^2+z^2=4$, with $y \geq 0$

Sol^n^ from fig we can say :




Draw this in every que to understand que.

$\theta = 0 \to \pi$

$\phi = 0 \to \pi$.

$\therefore \int_0^{\pi}\int_0^{\pi}\int_0^{2} r^2 \sin^2\theta \; (r^2 \sin\theta \; d\theta\, d\phi\, dr)$

$\|$

$\frac{128\pi}{15}$

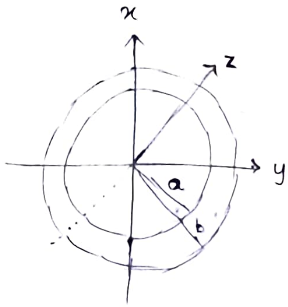
$\Rightarrow$ Find the volume of 2 concentric spheres $r_1 = a$ & $r_2 = b$

Sol^n here $r \Rightarrow a \to b$

around z-axis $\theta \Rightarrow 0 \to 2\pi$

$\phi \Rightarrow 0 \to \pi$

from +z to -z




$\therefore \int_0^{2\pi}\int_0^{\pi}\int_a^{b} r^2 \sin\theta \, d\theta \, d\phi \, dr$

↳ when nothing given in que, take $dv = r^2 \sin\theta \, d\theta \, d\phi \, dr$

(not in syllabus)

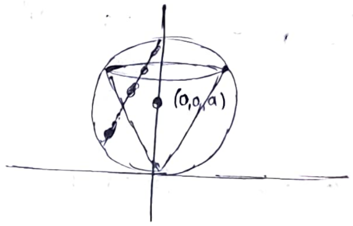
$\Rightarrow$ Find the volume of region above the cone $z^2 = x^2+y^2$ and inside the sphere $x^2+y^2+z^2 = 2az$ $(a>0)$.

Sol^n $x^2+y^2+z^2 - 2az = 0$

$x^2+y^2+(z-a)^2 = a^2$

$\therefore$ centre $(0,0,a)$.

$r = a$.



# CARTESIAN TO CYLINDRICAL COORDINATE :

$$\Rightarrow \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} x^2 \, dz \, dy \, dx$$

$x \to r\cos\theta$
$y \to r\sin\theta$
$z \to z$

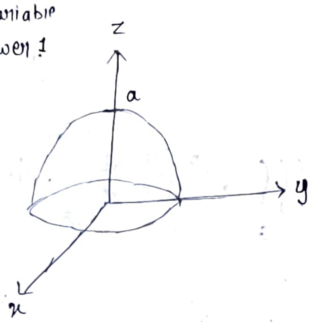
Sol^n)

* We use cylindrical coordinates in case of paraboloids, cones, etc.

* $x^2 + y^2 = a - z \longrightarrow$ eq^n of paraboloid.

two variable with power 2

one variable with power 1

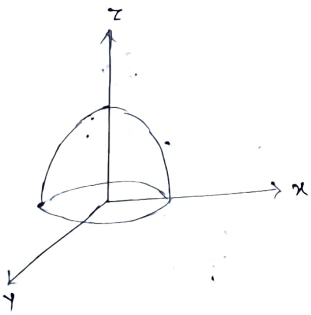



when $x = y = 0$
then $z = 0$
$x^2 + y^2 \to$ circle

volume que.
(not in syllabus)

$\Rightarrow$ Using cylindrical coordinates, find volume of solid region bound by paraboloid $x^2 + y^2 = 4 - z$ and plane $z = 0$

Sol^n) $x^2 + y^2 = 4 - z$




$z = 0$
means $xy$ plane

intersecting both eq^ns.

$\therefore \boxed{x^2 + y^2 = 4} \longrightarrow$ eq^n of circle.

for cylindrical, $x \to r\cos\theta$
$y \to r\sin\theta$
$z \to z$

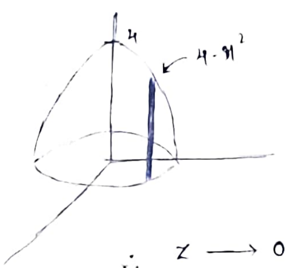
then $dx\,dy\,dz = |J|\,dz\,dr\,d\theta = r\,dz\,dr\,d\theta$

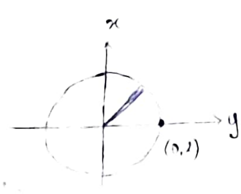
$\therefore$ we know.

$\therefore x^2 + y^2 = 4 - z$

$r^2 = 4 - z$

$z = 4 - r^2$




$\therefore z \longrightarrow 0$ to $4 - r^2$

$r \longrightarrow 0$ to $2$

$\theta \longrightarrow 0$ to $2\pi$

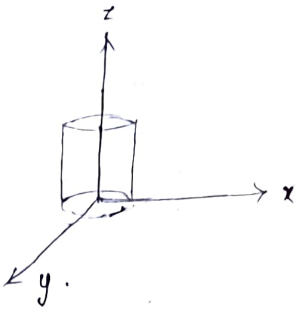
$$\therefore \int_0^{2\pi}\int_0^{2}\int_0^{4-r^2} r\, dz\, dr\, d\theta$$ ← when no eq$^n$ given in integration take $r\, dz\, dr\, d\theta = dv$

$$= \underline{\underline{8\pi}}$$

imp.

* Eq$^n$ of cylinder $\Rightarrow x^2 + y^2 = r^2$

means axis is z and radius is r.



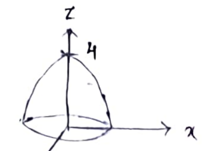

~~⇒ Using spherical, evaluate $\iiint_S e^{(x^2+y^2+z^2)^{3/2}} dv$~~

~~Sol$^n$~~

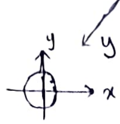
⇒ Using cylindrical ; $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{4-x^2-y^2} x^2\, dz\, dy\, dx$ imp

Sol$^n$ whenever, eq$^n$ of cylinder, paraboloid or cone comes solve using cylinderical.

$\therefore$ drawing eq$^n$ : $x^2+y^2 = 4-z$ →

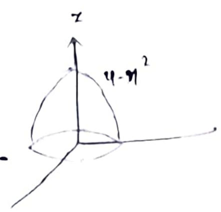



$x^2+y^2 = 4$ →




$x = \pm 2$.

$\therefore$ put $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.




| | | |
|---|---|---|
| $\therefore r^2 = 4 - z$ | $\therefore r^2 = 4$ | |
| $z = 4 - r^2$ | $r = 2$. | |


limits are : $z : 0 \longrightarrow 4-r^2$

$r : 0 \longrightarrow 2$

$\theta : 0 \longrightarrow 2\pi$

$$\therefore \int_0^{2\pi}\int_0^{2}\int_0^{4-r^2} r^2\cos^2\theta\,(r\,dz\,dr\,d\theta) = \underline{\underline{\frac{16\pi}{3}}} \text{ (Ans)}$$

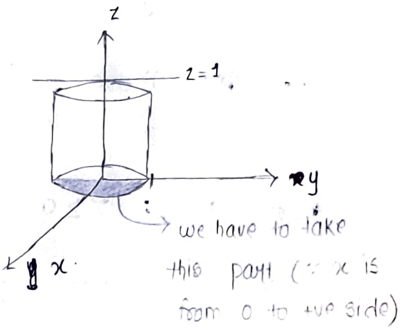
for cylindrical

$$\boxed{dv = |J|\,dz\,dr\,d\theta = r\,dz\,dr\,d\theta}$$

$\Rightarrow \displaystyle\int_{-1}^{1}\int_0^{\sqrt{1-y^2}}\int_0^{1} (x^2+y^2)\,dz\,dx\,dy$ , using cylinder.

Sol^n) if limits do not have variables then take according to derivatives order

$$\therefore \int_{y=-1}^{y=1}\int_{x=0}^{x=\sqrt{1-y^2}}\int_{z=0}^{z=1} (x^2+y^2)\,dz\,dx\,dy.$$




$\therefore$ $z = 0 \longrightarrow$ xy plane

$z = 1 \longrightarrow$ plane $\parallel$ to xy plane.

$x = \sqrt{1-y^2} \Rightarrow x^2+y^2 = 1 \longrightarrow$ eq^n of cylinder in 3D

$y = \pm 1.$

$\therefore$ $x = r\cos\theta,\ y = r\sin\theta,\ z = z.$

$\therefore$ $r^2 = 1 \Rightarrow r = \pm 1 = 1.$

$\therefore$ $z : 0 \longrightarrow 1$

$r : 0 \longrightarrow 1$

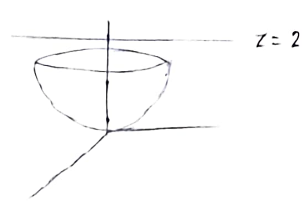
$\theta : 0 \longrightarrow 2\pi$

$$\therefore \int_0^{\pi}\int_0^{1}\int_0^{1} r^2\,(r\,dz\,dr\,d\theta) = \underline{\underline{\pi/4}}.$$

$\Rightarrow \displaystyle\iiint_S (x^2+y^2)\,dx\,dy\,dz$ , using cylinder, bound by $x^2+y^2 = 2z$ & $z = 2$ plane

Sol^n) $\therefore x^2+y^2 = 2z \longrightarrow$ paraboloid.

it is +ve, so open up.




$\therefore x^2+y^2 = 4 \longrightarrow$ cylinder in 3D.

$r^2 = 4$

$\boxed{r = 2}$

$x^2+y^2 = 2z$

$\boxed{\frac{r^2}{2} = z}$

$z : r^2/2 \longrightarrow 2$

$r : 0 \longrightarrow 2$

$\theta : 0 \longrightarrow 2\pi$

$$\int_0^{2\pi}\int_0^{2}\int_{r^2/2}^{2} r^2 \,(r\, dz\, dr\, d\theta) = \underline{\underline{\frac{16\pi}{3}}} \quad \text{(Ans)}$$

# TRIPLE INT OF SPHERICAL COORDINATES USING CONE :

⇒ Given $x^2+y^2+z^2=1$, $z=\sqrt{x^2+y^2}$, $\iiint 3z\, dx\, dy\, dz = ?$

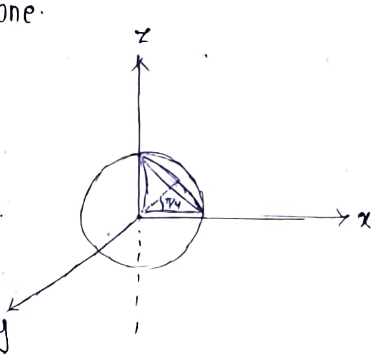
Soln) $z=\sqrt{x^2+y^2}$

$\boxed{r^2 z^2 = x^2+y^2 \longrightarrow \text{eq}^n \text{ of cone}}$

∴ from above . $z^2 = x^2+y^2 \longrightarrow$ so use cone.

from que : $z \geq 0$.

Hence, we need upper half of xy plane.




∴ put same values as spherical.

$$\int_0^{\pi/4}\int_0^{2\pi}\int_0^{1} 3r\cos\theta \; r^2 \sin\theta \; dr\, d\theta\, d\phi = \underline{\underline{\frac{3\pi}{8}}} \quad \text{(Ans)}.$$

← radius.

← for sphere

for cone

# PRACTISE PROBLEM :

⇒ If $u^3 - v^3 = x + y$ , $u^2 + v^2 = x^3 + y^3$ , PT. $\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{2}\frac{(y^2 - x^2)}{uv(u-v)}$.

⇒ Expand $x^2y + 3y - 2$ in power of $(x-1)$ & $(y+2)$ using Taylor upto 3rd term.

⇒ Find extreme values , $u = x^2y^2 - 5x^2 - 8xy - 5y^2$

⇒ Use the method of Lagrenge's multiplier method to find the volume of largest rect. llpiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

⇒ $\int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx$ , use change of order of int.

⇒ Using transformation , $x + y = u$ , $y = uv$

PT : $\iint [xy(1-x-y)]^{1/2} \, dx \, dy = \frac{2\pi}{105}$