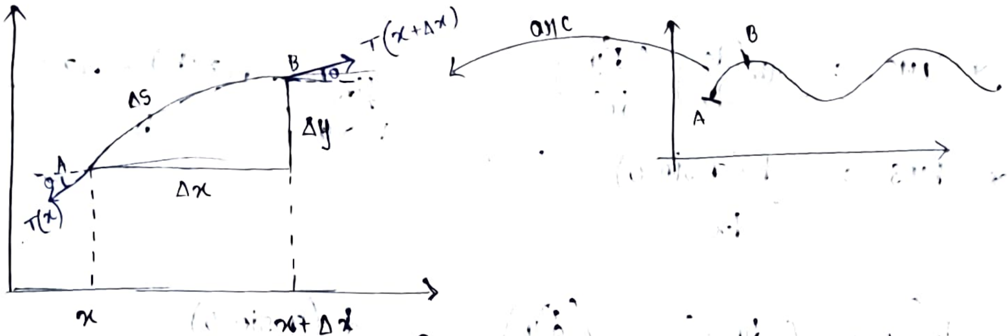
16 Aug - 2023
Lec- 2

WAVE EQN IN THE STRING :

\# GOAL: To derive the 1D wave eq^n in a string.

* 

$$\therefore \Delta S^2 = \Delta x^2 + \Delta y^2$$ ⟵ Pythogorus Theorem

$$\Rightarrow \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$ for small value ΔS is st. line.

* The magnitude of the tension point x & $x+\Delta x$ are resp. $T(x)$ & $T(x+\Delta x)$

* The net horizontal line is given as
$$\left[T(x+\Delta x)\cos\theta - T(x)\cos\theta\right]\hat{i}$$

* The net vertical line is given as :
$$\left[T(x+\Delta x)\sin\theta - T(x)\sin\theta\right]\hat{j}$$

* Let $\mu \longrightarrow$ mass per unit length.

* Because we are interested in the vertical ~~other~~ $disp^n$ so, we are going to neglect horizontal motion.

* The notation of the transverse $disp^n$ is denoted by y as a $func^n$ of x & t $[y(x,t)]$

* Using Newton's second law : $[F = ma]$
$$\therefore \left[T(x+\Delta x)\sin\theta - T(x)\sin\theta\right] = (\mu\,\Delta S)\frac{d^2y}{dt^2}$$

now ~~multip~~ divide Δx both sides :

$$\left(\mu \frac{\Delta s}{\Delta x}\right) \frac{d^2y}{dt^2} = \frac{\left[T(x+\Delta x)\sin\theta - T(x)\sin\theta\right]}{\Delta x}$$

Take the limit $\Delta x \to 0$:

* LHS : $\mu\left(\frac{ds}{dx}\right)\left(\frac{d^2y}{dt^2}\right)$

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

* RHS : $\frac{d(T\sin\theta)}{dx}$

* $$\mu\sqrt{1+\left(\frac{dy}{dx}\right)^2}\left(\frac{d^2y}{dt^2}\right) = \frac{d(T\sin\theta)}{dx}$$

from prev page $\frac{\Delta s}{\Delta x} = (\ )$

we know, $\sin\theta = \frac{\cancel{T\sin}\ \tan\theta}{\sqrt{1+\tan^2\theta}}$

and also, $\tan\theta = \frac{dy}{dx} \rightarrow$ slope

$$\Rightarrow \mu\sqrt{1+\left(\frac{dy}{dx}\right)^2}\left(\frac{d^2y}{dt^2}\right) = \frac{d}{dx}\left(\frac{T\left(\frac{dy}{dx}\right)}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right)$$

* Let's assume, the tension in string T is continuous

we also assume $\left|\frac{dy}{dx}\right| << 1$

* $$\mu\frac{d^2y}{dt^2} = T\frac{d^2y}{dx^2} \longrightarrow \frac{T}{\mu} = \left[\frac{\text{force}}{\text{mass / length}}\right] = (\text{velocity})^2$$

$$\Rightarrow \cancel{\frac{T}{\mu} = \frac{d}{}}$$

$$\Downarrow$$

$$\boxed{\frac{1}{u^2}\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2}}$$

velocity

$$v^2 = \frac{T}{u}$$

$$\therefore \boxed{v = \sqrt{\frac{T}{\mu}}}$$

$$\frac{1}{u^2}\frac{d^2\#}{dt^2} = \frac{d^2\#}{dx^2} \quad \xleftrightarrow[\text{SP}]{\text{2D}} \quad \frac{1}{u^2}\left(\frac{d^2}{dt^2}\right)\# = \left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)\#$$

Cartesian, (1D space) , $\# \to$ anyvalue

(2D space)

$$\frac{1}{u^2}\frac{d^2\#}{dt^2} = \left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)\#$$

(or)

$$\frac{1}{u^2}\frac{d^2\#}{dt^2} = \nabla^2\#$$

(3D space)

$\nabla^2 \to$ Laplacian

$$\nabla^2 = \left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)$$

* $$\frac{1}{u^2}\frac{d^2 f(x,y,z,t)}{dt^2} = \nabla^2 f(x,y,z,t)$$

(or)

$$\underbrace{\left(\frac{1}{u^2}\frac{d^2}{dt^2} - \nabla^2\right)}_{\text{Box } \to\ \Box f = 0} f = 0 \quad \to \text{This is a scalar wave eq}^n$$

* Solution of $\Box f = 0$ (or) $\frac{1}{u^2}\frac{d^2f}{dt^2} = \frac{d^2f}{dx^2}$ (not imp)

we will see that the sol^n of above eq^n is of the format $f(x,t) = f(x \pm ut) \leftarrow$ wave is periodic, so $(x \pm ut)$

$$\boxed{x \pm ut = \xi \ (xi)}$$

* $\frac{df}{dx} = \frac{df}{d\xi} \Rightarrow \frac{d^2f}{dx^2} = \frac{d^2f}{d\xi^2}$, * $\frac{df}{dt} = (\pm u)\frac{d}{d\xi}$

$$\frac{d^2f}{dt^2} = u^2\frac{d^2}{d\xi^2}$$

LHS : $\frac{1}{\cancel{u^2}} \cdot \cancel{u^2} \frac{d^2 f}{d\xi^2} = \frac{d^2 f}{d\xi^2}$

RHS : $\frac{d^2 f}{dx^2} = \frac{d^2 f}{d\xi^2}$ (Proved)

$x - ut = \xi$

$\frac{d}{dx} = \frac{d}{d\xi} \frac{d\xi}{dx} = \frac{d}{d\xi}$

$\therefore f(x-ut)$ is 'a' soln of the above eqn

$\therefore$ Also $f = f(x+ut)$ is another soln also.

* $f = f(x-ut) + f(x+ut)$ (not imp)

$\hookrightarrow$ similar to $\sin(Kx - ut)$

17-8-23 (Lec-3)

* We will see EM wave eqn of the same nature when $\boxed{u = c}$

$\Box \vec{E} = 0$ (or) $\Box \vec{B} = 0$

* $f = f(x-ut) + f(x+ut)$ is called travelling wave eqn a soln of above partial diff. eqn (PDE)

ex: $\sin(Kx - ut)$ (or) $\cos(Kx - ut)$.

* $f = A \sin(Kx - \omega t - \phi)$

$\Rightarrow f = A \sin(Kx - \phi - \omega t)$

$\Rightarrow f = A \sin(Kx - (\omega t + \phi))$

H.W

$\Rightarrow$ * Take K, ω some value, put $t = t_0$ (const ~~cos ωt~~)

Compare graphs of f with & without ϕ. Plot it (Matplotlib)

Animate it.

H.W

$\Rightarrow$ Take K, ω some value, put $x = x_0$ (const). Compare graphs of f with & without ϕ, Plot it & Animate it.

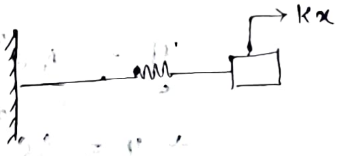
⇒ A transverse sinosuidal wave is generated at one end and down through a distⁿ 1·30 cm. The motion is continuous & repeated regularly 125 times/sec. If the distⁿ b/w the wave crest is observed, to be 15·6 cm. find the following:

i) Amplitude ii) Wavelength iii) Wave velocity

Soln * $f(x,t) = A \sin(Kx - \omega t)$

$\updownarrow$

* $f(x,t) = A e^{i(\omega t - Kx)}$ (or) $A e^{i(Kx - \omega t)}$



$m\ddot{x} + Kx = 0$

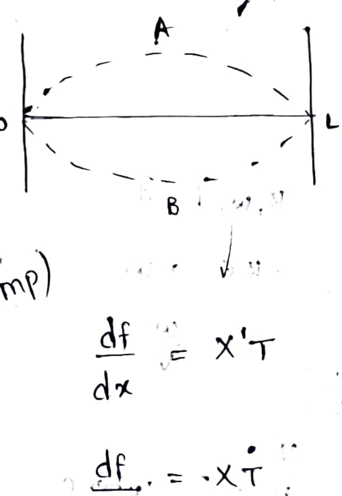
$\ddot{x} + \omega^2 x = 0$

$x(t) = \sin(\omega t)$
(or)
$\cos(\omega t)$

$\ddot{}$ means double diff hence accⁿ

$x(t) = a \sin(\omega t) + b \cos \omega t$
$x(t) = e^{i\omega t}$
$x(t) = e^{-i\omega t}$

Standing wave



$\Rightarrow \frac{1}{u^2} \frac{d^2 f}{dt^2} = \frac{d^2 f}{dx^2}$ from 1D wave eqⁿ

$\Rightarrow f(x,t) = X(x)\,T(t)$ seperation of variable

$\int d^2 f\, dx^2 = X(x)$
$\int d^2 f\, dt^2 = T(t)$

(imp)

$\frac{df}{dx} = X'T$ $\frac{d^2 f}{dx^2} = X''T$

$\frac{df}{dt} = X\dot{T}$, $\frac{d^2 f}{dt^2} = X\ddot{T}$

where $' \equiv \frac{d}{dx}$

$\dot{} \equiv \frac{d}{dt}$

$\therefore \frac{1}{u^2} X\ddot{T} = X''T$

$\Rightarrow \frac{1}{u^2} \frac{\ddot{T}}{T} = \frac{X''}{X} = \text{const}$

as LHS & RHS, both variables are same, means they are const

$\therefore \frac{X''}{X} = -K^2$, $\frac{\ddot{T}}{T} = -K^2 u^2$

just const (not any value)

$y=0$ at $x=0$ & $x=L$ ← (Pre diagram)

$$\therefore y = \underbrace{A e^{i(\omega t - Kx)}}_{\text{for upper wave}} + \underbrace{B e^{i(\omega t + Kx)}}_{\text{for lower wave}}$$

$$\therefore y = A e^{i(\omega t - Kx)} - A e^{i(\omega t + Kx)}$$ ($\because A = -B$, as B is reflected wave of A)

$$\Rightarrow y = A e^{i\omega t} (e^{-iKx} - e^{iKx})$$ ← $\boxed{e^{iKx} = \cos Kx + i \sin Kx}$

$$\Rightarrow y = (-2i) A e^{i\omega t} \sin Kx$$

Hence it matches with, $f(x,t) = X(x)\, T(t)$

Reference book
Modern Physics → Beiser

19-08-23

Lec-4

$$f(x,t) = (-2i) A e^{i\omega t} \sin Kx$$

$f(x,t) = 0$ at $x=0$ & $x=L$

$\sin Kx = 0$

$\Rightarrow K_n L = n\pi$

$\Rightarrow \omega_n L = n\pi v \Rightarrow \omega_n = \frac{v n\pi}{L}$

$\Rightarrow 2\pi f_n L = n\pi v \Rightarrow f_n = \frac{nv}{2L}$

$\Rightarrow 2\pi \cdot \frac{v}{\lambda_m} L = n\pi v \Rightarrow \lambda_n = \frac{2L}{n}$ → where n → no. of reso.

K, ω, f, λ

$K \leftrightarrow \omega$

$K = \frac{\omega}{v}$

$f = 2\pi\omega$
$= 2\pi \left(\frac{vn\pi}{L}\right)$

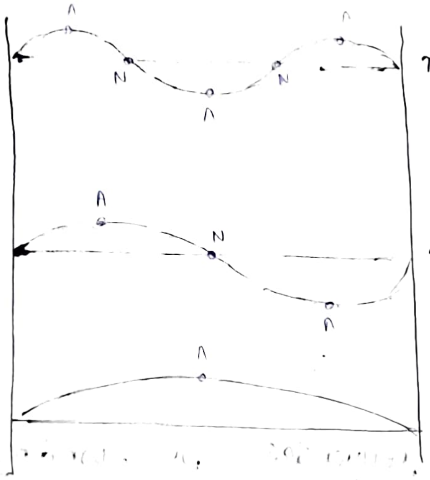
$\omega = 2\pi \left(\frac{nv}{2L}\right)$
$\frac{nv}{L}$

$v = \lambda f$

$f = \frac{v}{\lambda} = \frac{nv}{2L}$

Standard eqn of standing wave when superimposed :

$$y = \underbrace{A \sin Kx}_{\text{variable amplitude}} \cos \omega t$$



$n=3$ — 3^{rd} harmonic, 2^{nd} overtone, $f_3 = \frac{3v}{2L}$, $\lambda_3 = \frac{2L}{3}$

$n=2$ — 2^{nd} harmonic, 1^{st} overtone, $f_2 = \frac{v}{L}$, $\lambda_2 = L$

$n=1$ — fundamental, $f_1 = \frac{v}{2L}$, $\lambda_1 = 2L$

$Kx = \phi = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \leftarrow$ maxima $\rightarrow$ $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, \ldots$ gap $= \lambda/2$

$Kx = \phi = \pi, 2\pi, 3\pi, \ldots \leftarrow$ minima $\rightarrow$ $x = \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}, \ldots$ gap $= \lambda/2$

Travelling wave $\underset{}{\overset{\text{composed of}}{\rightleftharpoons}}$ Standing wave.

⇒ A violin string tuned to concert 440 Hz has a length of 0.34 m, Complete the following,

Q i) what are the 3 longest wavelength of resonance of the string.

ii) what are the corresponding λ that reach to the end of listeners.

Sol^n i) $L = 0.34$ m

$\lambda_1 = \frac{2L}{1} = 0.68$ m

$\lambda_2 = \frac{2L}{2} = 0.34$ m

$\lambda_3 = \frac{2L}{3} = 0.23$ m

$\Big\}$ using $\lambda_n = \frac{2L}{n}$

~~put 332 in Vair, Calculate Vstring~~

ii) $f = 440$ Hz

$\frac{v}{\lambda} = \text{const} = f$

$\frac{V_{string}}{\lambda_{string}} = \frac{V_{air}}{\lambda_{air}}$

$\therefore \lambda_{air} \propto \lambda_{string}$

$(\lambda_{air})_1, (\lambda_{air})_2, (\lambda_{air})_3$

⇒ What are the three lowest frequencies for standing wave on a wire 9.88 m long having mass of 0.107 kg which is stretched under a tension 236 N.

Sol^n) $V = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{236}{0.107/9.88}} = \sqrt{\frac{236 \times 1000 \times 988}{107 \times 100}}$

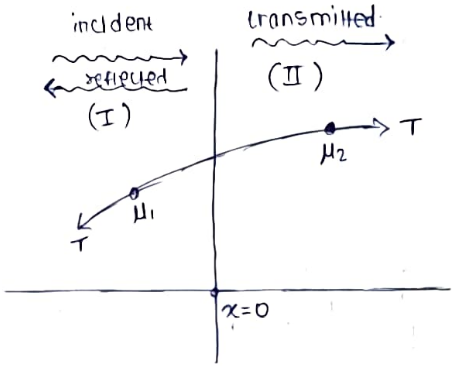
$\therefore \lambda = \frac{2L}{n}$, $\lambda_1 = 2L$, $\lambda_2 = 2L$, $\lambda_3 = \frac{2L}{3}$.

$v = \lambda f$

$\therefore f_1 = \frac{V}{\lambda_1}$, $f_2 = \frac{V}{\lambda_2}$, $f_3 = \frac{V}{\lambda_3}$

23-08-23

Transmission And Refletion Of Waves :



$$V = \sqrt{\frac{T}{\mu}}$$

as $\mu_1 \neq \mu_2$

so, $V_1 \neq V_2$ & $K_1 \neq K_2$ $\left(K = \frac{\omega}{V}\right)$

→ Goal :

To study the relation b/w the amplitude of incident wave, reflected wave & transmitted wave.

Region-1 $\begin{cases} f_I = Ae^{i(\omega t - K_1 x)} \\ f_R = Be^{i(\omega t + K_1 x)} \end{cases}$

Region-2 $\{ f_t = Ce^{i(\omega t - K_2 x)}$

* A geometric condition that the dispn is immediately same to the left & right of $x=0$?

$$(f_I + f_R)\Big|_{x=0} = f_T\Big|_{x=0}.$$

* f & its derivatives must be continuous across the boundary.

$$\frac{d}{dx}(f_I + f_R)\Big|_{x=0} = \frac{df_T}{dx}\Big|_{x=0} \longrightarrow (1)$$

* If the above condition is not true, then the accn would be infinity.

* $(f_I + f_R)|_{x=0} = f_T|_{x=0}$

$$e^{i\omega t}\left(Ae^{-K_1 x} + Be^{K_1 x}\right)\Big|_{x=0} = Ce^{i(\omega t - K_2 x)}\Big|_{x=0}$$

at $x = 0 \Rightarrow \boxed{A + B = C}$

* $\frac{d}{dx} f_I \Big|_{x=0} = -iK_1 f_I \Big|_{x=0}$

* $\frac{d}{dx} f_R \Big|_{x=0} = iK_1 f_R \Big|_{x=0}$

$\frac{d}{dx} f_T \Big|_{x=0} = -iK_2 f_T \Big|_{x=0}$

$$\therefore \boxed{\frac{B}{A} = \frac{(K_1 - K_2)}{(K_1 + K_2)}}, \quad \boxed{\frac{C}{A} = \frac{2K_1}{K_1 + K_2}}$$

$$\therefore \boxed{R = \frac{|B|^2}{|A|^2} = \frac{(K_1 - K_2)^2}{(K_1 + K_2)^2}},$$

reflection coefficient

$R + T = 1$

transmission coefficient $\rightarrow$ $T = 1 - R \Rightarrow \boxed{T = \frac{4K_1 K_2}{(K_1 + K_2)^2} = \frac{K_2}{K_1}\frac{|C|^2}{|A|^2}}$

from eqn (1)

* $\frac{d}{dx}\left(Ae^{i(\omega t - K_1 x)} + Be^{i(\omega t + K_1 x)}\right) = \frac{d}{dx}\left(Ce^{i(\omega t - K_2 x)}\right)$

$\Rightarrow Ae^{i(\omega t - K_1 x)}(-K_1) + Be^{i(\omega t + K_1 x)}(K_1) = Ce^{i(\omega t - K_2 x)}(-K_2)$

$\Rightarrow -K_1 A\cancel{e^{i\omega t}} + K_1 B\cancel{e^{i\omega t}} = -K_2 C\cancel{e^{i\omega t}}$

$\Rightarrow -K_1 A + K_1 B = -K_2 C$

$\Rightarrow K_1(A - B) = K_2 C$

$\Rightarrow K_1(A - B) = K_2(A + B)$

$$K_1\left(\frac{A}{B}-1\right) = K_2\left(\frac{A}{B}+1\right)$$

$$\frac{A}{B}(K_1-K_2) = K_1+K_2$$

$$\boxed{\frac{A}{B} = \frac{K_2+K_1}{K_1-K_2}}$$

24/08/2023

→ **Impedence :**

$$Z \doteq \frac{T}{V}, \quad V = \sqrt{\frac{T}{\mu}}$$

imp.

:= (or) $\doteq$ is called definition

$$\Rightarrow T = V^2\mu.$$

$$\therefore \boxed{Z = \mu V}$$

we have used notation 'T' for "tension" awa "transmission coeff"

$$T = \mu v^2 = \frac{\mu\omega^2}{K^2} \quad \left(\because v = \frac{\omega}{K}\right).$$

$$\Rightarrow K \propto \sqrt{\mu} \Rightarrow \lambda \propto \frac{1}{\sqrt{\mu}}$$

$$Z = \mu\sqrt{\frac{T}{\mu}} = \sqrt{\mu T}$$

$$\therefore \frac{B}{A} = \frac{K_1-K_2}{K_1+K_2} = \frac{\sqrt{\mu_1}-\sqrt{\mu_2}}{\sqrt{\mu_1}+\sqrt{\mu_2}}$$

$$\frac{C}{A} = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1}+\sqrt{\mu_2}}.$$

⇒ Express the ratio $\frac{B}{A}$ & $\frac{C}{A}$ in terms of Z_1 & Z_2 ?

Sol^n) $\therefore Z = \mu V$, where $Z = \frac{T}{V}$

here v & μ are variable, so making one variable

$$Z = \mu\sqrt{\frac{T}{\mu}} = \sqrt{\mu T} \quad \therefore Z \propto \sqrt{\mu}$$

$$\therefore Z \propto K.$$

$$\therefore \frac{B}{A} = \frac{Z_1-Z_2}{Z_1+Z_2}, \quad \frac{C}{A} = \frac{2Z_1}{Z_1+Z_2}$$

⇒ A transvers sin wave of amplitude 3.0 cm & wavelength 25 cm travels along a light string of linear mean density 1 gm cm^{-1}. which is joined with heavier string of 4.0 gm cm^{-1}. The joined strings are held under constant tension. Complete :

i) what is the wavelength a, & amplitude of the wave as it travels in the heavier string?

ii) what fraction of the wave power is reflected at the boundary of the two strings ?

Soln) i) $\lambda \propto \frac{1}{\sqrt{\mu}}$

$$\frac{C}{A} = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1}+\sqrt{\mu_2}} = \underline{\underline{2\text{ cm}}} \text{ (Amplitude)}.$$

$$\frac{\lambda_2}{\lambda_1} = \sqrt{\frac{\mu_1}{\mu_2}}$$

$$\underline{\underline{\lambda_2 = 12.5\text{ cm}}}$$

ii) $R = \frac{|B|^2}{|A|^2} = \left(\frac{B}{A}\right)^2 = \left(\frac{\sqrt{\mu_1}-\sqrt{\mu_2}}{\sqrt{\mu_1}+\sqrt{\mu_2}}\right)^2 = \underline{\underline{\frac{1}{9}}}$ (Ans)

reflection coeff
(or)
reflection power.