MODULE - 2

LIMIT & PARTIAL DERIVATIVE :

* Step-1 : $f(x,y)$ is given

* Step-2 : Find $\lim\limits_{\substack{x\to a \\ y\to b}} f(x,y) = f_1$ (say)

$$\lim_{\substack{y\to b \\ x\to a}} f(x,y) = f_2 \text{ (say)}$$

* Step-3 : $f_1 = f_2$ (limit exist).

* Step-4 : let $y = mx$, $f(x,mx) = \lim\limits_{x\to 0} [\quad] \neq f_1$ or f_2 ~~1 (or)~~ $\cancel{\frac{m}{1+m^2}}$

$\underbrace{\qquad\qquad}_{\text{does not exist}}$

* Step-5 : $y = mx^2$

$$f(x, mx^2) = \lim_{x\to 0} [\quad] = 0$$

only do step 4 & 5 if $x\to 0$, $y\to 0$

$$\Rightarrow \lim_{\substack{x\to 0 \\ y\to 0}} f(x,y) = \frac{y^5}{x^{10}+y^5}$$

Sol^n)

$$\lim_{x\to 0} \left[\lim_{y\to 0} \frac{y^5}{x^{10}+y^5}\right] = 0 = f_1$$

$$\lim_{y\to 0} \left[\lim_{x\to 0} \frac{y^5}{x^{10}+y^5}\right] = 1 = f_2$$

$\therefore f_1 \neq f_2$ (does not exist)

$$\Rightarrow \lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = \frac{x^2y}{x^4+y^2}$$

Soln

$$\left.\begin{aligned} &\lim_{x\to 0}\left[\lim_{y\to 0} \frac{x^2y}{x^4+y^2}\right] = 0 \\ &\lim_{y\to 0}\left[\lim_{x\to 0} \frac{x^2y}{x^4+y^2}\right] = 0 \end{aligned}\right\} \quad f_1 = f_2 \quad (\text{exists})$$

$$f_3 = \lim_{x,y\to 0}\left[\frac{x^2y}{x^4+y^2}\right]$$

taking, $y = mx$

$$\lim_{x\to 0}\left[\frac{x^2 \cdot mx}{x^4+m^2x^2}\right] = \lim_{x\to 0}\left[\frac{mx}{x^2+m^2}\right] = \overset{= f_1}{\underline{\underline{0}}}$$ ~~it is not~~

taking, $y = mx^2$

$$\lim_{x\to 0}\left[\frac{x^2 \cdot mx^2}{x^4+m^2x^4}\right] = \frac{m}{1+m^2} \neq f_1 \longrightarrow \text{doesn't exist.}$$

———— x ————

we have to check that $f_1 = f_2 = f_3 = f_4 = \cdots\cdots$ equal.

for 5 mark que : put directly $y = mx^n$

where n is the diff b/w max^m powers of x in numerator & denominator.

ex : if $f = \dfrac{x^a y}{x^b + y^c}$, $n = a-b$.

for 10 mark que : we have to put, step by step.

$y = mx$, $y = mx^2$, $y = mx^3$, $\cdots$ $y = mx^n$.

CONTINUITY

- $f(a,b)$ should be well defined
- $\lim\limits_{x,y\to(a,b)} f(x,y) \longrightarrow$ should exist
- $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

} Rule.

$\Rightarrow f(x,y) = \begin{cases} \dfrac{x}{\sqrt{x^2+y^2}}, & x,y \neq 0 \\ 0, & x,y = 0. \end{cases}$

Sol^n i) it is well defined at $(0,0)$

ii) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x}{\sqrt{x^2+y^2}} = \lim\limits_{x\to 0}\left(\lim\limits_{y\to mx} \dfrac{x}{\sqrt{x^2+m^2x^2}}\right)$

$= \lim\limits_{x\to 0} \dfrac{1}{\sqrt{1+m^2}}$ → not exist So not continuous

~~Sol^n if $y = mx$. $\Rightarrow \dfrac{x}{\sqrt{x^2+m^2x^2}} = \dfrac{1}{\sqrt{1+m^2}}$, not continuous.~~

$\Rightarrow f(x,y) = \begin{cases} \dfrac{x^2+2y}{x+y^2}, & \cancel{x=1,\ y=2} \\ 1, & x=1,\ y=2 \end{cases}$

→ Sol^n i) well defined at $(1,2)$

ii) $\lim\limits_{(x,y)\to(1,2)} \dfrac{x^2+2y}{x+y^2} = \dfrac{1+4}{1+4} = 1 \to$ exist

iii) $\lim\limits_{(x,y)\to(a,b)} = f(a,b)$ → Hence continuous

~~Sol^n $y = mx$. $\dfrac{x^2+2mx}{x+m^2x^2} = \dfrac{x+2m}{1+m^2x}$.~~

~~$\dfrac{1+4}{1+4} \longrightarrow 1$~~

Exists

Continuous

$\Rightarrow f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & x,y \neq 0 \\ 0, & x,y = 0. \end{cases}$

Sol^n $y = mx$, $\dfrac{mx^2}{x^2+m^2x^2} = \dfrac{m}{1+m^2}$, does not exist

discontinuous

PARTIAL DERIVATIVE

$\Rightarrow$ Given, $U = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

P.T., $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

Sol^n) $\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(\frac{1}{y}\right) + \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right)$

$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{yx}{x^2+y^2}$.

$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-x^2/y^2}}\left(\frac{-x}{y^2}\right) + \frac{1}{1+y^2/x^2}\left(\frac{1}{x}\right)$.

$y\frac{\partial u}{\partial y} = \frac{-x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}$.

$\therefore$ (1) + (2) $= \underline{\underline{0}}$.

$\Rightarrow$ If, $V = (x^2+y^2+z^2)^{m/2}$, then find the value of 'm', $(m \neq 0)$.

which will make, $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$.

~~Sol^n~~) $\Rightarrow$ If, $V = (x^2-y^2)\, f(xy)$

P.T., $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = (x^4-y^4)\, f''(xy)$.

$\Rightarrow$ if, $z = x^y + y^x$, P.T. $\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}$.

$\uparrow$ $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$ $\qquad$ $\uparrow$ $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$.

Solⁿ 1) $V = (x^2+y^2+z^2)^{m/2}$

$$\frac{\partial v}{\partial x} = \frac{m}{n}(x^2+y^2+z^2)^{\frac{m-2}{2}}(2x)$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{2m}{n}\left[\frac{m-2}{2}(x^2+y^2+z^2)^{\frac{m-4}{2}}(x)(2x) + (x^2+y^2+z^2)^{\frac{m-2}{2}}\right]$$

$$\cancel{\frac{\partial^2 v}{\partial x^2} = \frac{2m}{n}\left[(x^2+y^2+z^2)^{\frac{m-4}{2}}\left[\frac{m-n}{n}(2x^2)(x^2+y^2+z^2)^{-1}+1\right]\right]}$$

$$\frac{\partial^2 v}{\partial x^2} = (mx^2)(m-2)(x^2+y^2+z^2)^{\frac{m-4}{2}} + m(x^2+y^2+z^2)^{\frac{m-2}{2}}$$

∴ similarly with $\frac{\partial^2 v}{\partial y^2}$ & $\frac{\partial^2 v}{\partial z^2}$

$$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$$

$$\underbrace{(m-2)(m)(x^2+y^2+z^2)^{\frac{m-4}{2}}}_{\text{common}}(x^2+y^2+z^2) + \underbrace{3m}_{\text{for 3 eq}^n.}(x^2+y^2+z^2)^{\frac{m-2}{2}} = 0$$

$$\Rightarrow m(x^2+y^2+z^2)^{\frac{m-4}{2}+1}(m-2) + 3m(x^2+y^2+z^2)^{\frac{m-2}{2}} = 0$$

$$\Rightarrow m(x^2+y^2+z^2)^{\frac{m-2}{2}}[m-2+3] = 0$$

$$\therefore m = -1 \quad \text{(Ans)}$$

Q.no 2) $v = (x^2 - y^2) f(xy)$

Soln) $\frac{\partial v}{\partial x} = (2x) f(xy) + (x^2-y^2) f'(xy)\, y.$

$\frac{\partial^2 v}{\partial x^2} = (2x)(f'(xy) \cdot y) + f(xy)(2) + y\left[(2x)f'(xy) + (x^2-y^2) f''(xy) y\right]$

$\frac{\partial^2 v}{\partial x^2} = 2xyf'(xy) + 2f(xy) + 2xyf'(xy) + y^2(x^2-y^2) f''(xy)$

$\frac{\partial v}{\partial y} = -2xyf'(xy) - 2f(xy) + (x^2-y^2) f''(xy)\, x^2 - 2xy\, f'(xy)$

$\therefore$ 1 + 2 $= \underline{\underline{(x^4 - y^4) f''(xy)}}$

Q.no 3) ~~Q. $z = x^y + y^x$~~

TOTAL DIFFERENTIAL CO EFFICIENT.

one variable

* let $z = f(x,y)$, $x = \phi(t)$, $y = \psi(t)$

$$\therefore \frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

Hence, $\frac{dz}{dt}$ is called total differential co-efficient of z.

* let $z = f(x,y)$, $x = \phi(u,v)$, $y = \psi(u,v)$ then

two variables

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}$$

⇒ If $u = x^3 + y^3$, where $x = a\cos t$, $y = b\sin t$, $\frac{du}{dt} = ?$

Sol^n
$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$$= (3x^2)(-a\sin t) + (3y^2)(b\cos t)$$

$$\Rightarrow -3a^3\cos^2 t\,\sin t + 3b^3\sin^2 t\,\cos t$$

⇒ if $z = f(x,y)$, where $x = e^u\cos v$, $y = e^u\sin v$, show that

$$y\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} = e^{2u}\frac{\partial z}{\partial y}.$$

Sol^n
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}(e^u\cos v) + \frac{\partial z}{\partial y}(e^u\sin v)$$

$$= x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} = \frac{\partial z}{\partial x}\cdot(e^u \sin v) + \frac{\partial z}{\partial y}(e^u \cos v)$$

$$= -y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}.$$

$$\therefore\ y\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) + x\left(-y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y}\right)$$

$$\frac{\partial z}{\partial y}(x^2+y^2) = \frac{\partial z}{\partial y}e^{2u}$$

$\Rightarrow$ If $u = u(y-z,\ z-x,\ x-y)$, PT. $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$

Sol^n Let $y-z = r$, $z-x = s$, $x-y = t$

$\therefore\ u = u(r, s, t)$

$$\therefore\ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}(-1) + \frac{\partial u}{\partial t}(1) = -\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}$$

$$\therefore\ \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} \cdot \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}.$$

$$= \frac{\partial u}{\partial r}(1) + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}(-1) = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t}.$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}.$$

$$= \frac{\partial u}{\partial r}(-1) + \frac{\partial u}{\partial s}(1) + \frac{\partial u}{\partial t}(0) = -\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}.$$

$\therefore$ 1 + 2 + 3 = 0

$\Rightarrow$ If $u = u\left[\frac{y-x}{xy}, \frac{z-x}{xz}\right]$, ST: $x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0$

Sol^n $u = u\left[\frac{1}{x} - \frac{1}{y}, \frac{1}{x} - \frac{1}{z}\right]$

$\therefore \frac{1}{x} - \frac{1}{y} = r, \quad \frac{1}{x} - \frac{1}{z} = s.$

$\therefore u = u(r, s)$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} = \frac{\partial u}{\partial r}\left(\frac{-1}{x^2}\right) + \frac{\partial u}{\partial s}\left(\frac{-1}{x^2}\right)$

$= -\frac{1}{x^2}\left[\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}\right].$

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y}$

$= \frac{\partial u}{\partial r}\left(\frac{1}{y^2}\right) + \frac{\partial u}{\partial s}(0) = \frac{1}{y^2} \frac{\partial u}{\partial r}.$

$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial z} = \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}\left(\frac{1}{z^2}\right) = \frac{1}{z^2} \frac{\partial u}{\partial s}$

$\therefore$ Hence proved

$\Rightarrow$ if $\phi(cx - az, cy - bz) = 0$, show that $ap + bq = c$:

where $p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}$

Sol^n ~~$\phi(r, s) = 0$~~

Let $r = cx - az, \quad s = cy - bz \qquad \therefore \phi(r, s) = 0$

$\frac{\partial r}{\partial x} = c - a\frac{\partial z}{\partial x}, \quad \frac{\partial s}{\partial y} = c - b\frac{\partial z}{\partial y}$

$\frac{\partial r}{\partial y} = -a\frac{\partial z}{\partial y}, \quad \frac{\partial s}{\partial x} = -b\frac{\partial z}{\partial x}$

$$\therefore \frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial \phi}{\partial s}\cdot\frac{\partial s}{\partial x} = 0$$

$$\frac{\sin\theta\cos\theta}{r^2}\quad\frac{\partial u}{\partial \theta}$$

$$= \frac{\partial \phi}{\partial r}\left(c - a\frac{\partial z}{\partial x}\right) + \frac{\partial \phi}{\partial s}\left(-b\frac{\partial z}{\partial x}\right) = 0$$

$$\Rightarrow a\frac{\partial z}{\partial x} = \frac{ac\frac{\partial \phi}{\partial r}}{a\frac{\partial \phi}{\partial r} + b\frac{\partial \phi}{\partial s}} \longrightarrow (1)$$

$$\frac{\partial \phi}{\partial y} = 0$$

$$\Rightarrow b\frac{\partial z}{\partial y} = \frac{bc\frac{\partial \phi}{\partial s}}{a\frac{\partial \phi}{\partial r} + b\frac{\partial \phi}{\partial s}} \longrightarrow (2)$$

$\therefore$ (1) + (2) $= \underline{\underline{c}}$ (Proved)

If variable changes :

i.e, $u = f(x,y)$

in pre topic $\longrightarrow$ $u = u(x,y)$

$\Rightarrow$ If $w = f(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$, ST:

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$$

Sol^n) for this first diff x & y.

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\therefore \frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}(\cos\theta) + \frac{\partial f}{\partial y}(\sin\theta)$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y}(r\cos\theta)$$

$$\Rightarrow \frac{1}{r}\frac{\partial w}{\partial \theta} = -\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta$$

$$\therefore \ (1)^2 + (2)^2 \Rightarrow \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \qquad \text{(Proved)}$$

imp

$\Rightarrow$ Transform the eq^n: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ into polar co-ordinates.

Sol^n

Polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$.

$$\therefore x^2 + y^2 = r^2.$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$r = \sqrt{x^2+y^2}$$

$$\frac{\partial r}{\partial x} = \frac{\frac{1}{2}(2x)}{\sqrt{x^2+y^2}} = \frac{x}{r}.$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$= \frac{\partial u}{\partial r}\left(\frac{x}{r}\right) + \frac{\partial u}{\partial \theta}\left(\frac{-y}{x^2+y^2}\right) = \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}.$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

use U.V

$$= \cos\theta\left[\cos\theta\frac{\partial^2 u}{\partial r^2} - \left(\frac{\sin\theta}{r}\frac{\partial^2 u}{\partial r\,\partial\theta} + \left(-\frac{\sin\theta}{r^2}\frac{\partial u}{\partial \theta}\right)\right)\right]$$

$$-\frac{\sin\theta}{r}\left[\cos\theta\frac{\partial^2 u}{\partial\theta\,\partial r} + (-\sin\theta)\frac{\partial u}{\partial r} - \left(\frac{\sin\theta}{r}\frac{\partial^2 u}{\partial\theta^2} + \left(\frac{\cos\theta}{r}\frac{\partial u}{\partial\theta}\right)\right)\right]$$

$$\Rightarrow \cos^2\theta\frac{\partial^2 u}{\partial r^2} - \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\,\partial\theta} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta}$$

$$- \frac{\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial\theta\,\partial r} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta}$$

$$\Rightarrow \cos^2\theta \frac{\partial^2 u}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\,\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} \longrightarrow (1)$$

Similarly for : $\dfrac{\partial^2 u}{\partial y^2}$

$$\Rightarrow \sin^2\theta \frac{\partial^2 u}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\,\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2} \longrightarrow (2).$$

$$\therefore \quad (1) + (2) \Rightarrow \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} \quad \text{(Proved)}.$$

⇒ If $u = f(r)$, $x = r\cos\theta$, $y = r\sin\theta$

P.T. $= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$.

Term ~~related~~ to x

~~so double diff~~

Sol^n $\frac{\partial u}{\partial x} = \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial x} = \frac{\partial f}{\partial r}\left(\frac{x}{r}\right)$

$x^2 + y^2 = r^2$

$r = \sqrt{x^2+y^2}$

imp

$\therefore \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}$.

$$\boxed{\frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r}\right) \cdot \frac{\partial r}{\partial x}}$$

$\therefore$ ~~$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 f}{\partial r^2} \cdot \frac{\partial r}{\partial x}$~~

diff wrt r ∵ f has r.

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial^2 f}{\partial r^2} \cdot \frac{\partial r}{\partial x}\right) \cdot \frac{x}{r} + \frac{\partial f}{\partial r}\left[\frac{r - x\frac{\partial r}{\partial x}}{r^2}\right]$$

using chain rule, ~~∵ f has ∂r~~ when f has one variable ~~not ∂x~~

using $\frac{u}{v}$

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial^2 f}{\partial r^2}\right)\left(\frac{x}{r}\right)^2 + \frac{\partial f}{\partial r}\left[\frac{r^2 - x^2}{r^3}\right]$$

$$= \frac{\partial^2 f}{\partial r^2}\left(\frac{x^2}{r^2}\right) + \frac{\partial f}{\partial r}\left(\frac{y^2}{r^3}\right).$$

Similarly : $\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 f}{\partial r^2}\left(\frac{y^2}{r^2}\right) + \frac{\partial f}{\partial r}\left(\frac{x^2}{r^3}\right)$.

$\therefore$ (1) + (2) $= f''(r) + \frac{1}{r}f'(r)$ (Proved)

$\Rightarrow$ if $x+y = 2e^{\theta}\cos\phi$; $x-y = 2ie^{\theta}\sin\phi$.

ST : $\frac{\partial^2 v}{\partial\theta^2} + \frac{\partial^2 v}{\partial\phi^2} = 4xy\frac{\partial^2 v}{\partial x\,\partial y}$.

Sol^n ATQ : $v = f(x,y)$.

$\therefore \frac{\partial v}{\partial\theta} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial\theta} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial\theta}$

so, here, we need either x or y :

$\therefore 2x = 2e^{\theta}(\cos\phi + i\sin\phi) \Rightarrow x = e^{\theta+i\phi}$

$2y = 2e^{\theta}(\cos\phi - i\sin\phi) \Rightarrow y = e^{\theta-i\phi}$

$\therefore \frac{\partial x}{\partial\theta} = e^{\theta+i\phi} = x, \quad \frac{\partial y}{\partial\theta} = e^{\theta-i\phi} = y.$

imp $\frac{\partial x}{\partial\phi} = e^{\theta+i\phi}(i) = ix$, $\frac{\partial y}{\partial\phi} = e^{\theta-iy}(-i) = -iy$.

$\therefore \frac{\partial v}{\partial\theta} = \frac{\partial v}{\partial x}(x) + \frac{\partial v}{\partial y}(y)$ ← ~~has no term related to θ so diff~~

$\frac{\partial^2 v}{\partial\theta^2} = \frac{\partial}{\partial\theta}\left(\frac{\partial v}{\partial\theta}\right) = \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right)$

not using chain rule

$\because$ ~~here v has $d\theta$ & diff also by $d\theta$~~

$= x\frac{\partial}{\partial x}\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) + y\frac{\partial}{\partial y}\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right)$

$\because$ f has 2 variables. for two variables we have to multiply.

$= x\left(\frac{\partial v}{\partial x} + x\frac{\partial^2 v}{\partial x^2} + y\frac{\partial^2 v}{\partial x\,\partial y}\right) + y\left(x\frac{\partial^2 v}{\partial y\,\partial x} + \frac{\partial v}{\partial y} + y\frac{\partial^2 v}{\partial y^2}\right)$

$\frac{\partial^2 v}{\partial\theta^2} = x^2\frac{\partial^2 v}{\partial x^2} + y^2\frac{\partial^2 v}{\partial y^2} + 2xy\frac{\partial^2 v}{\partial x\,\partial y} + \left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right)$

Similarly : $\frac{\partial^2 v}{\partial\phi^2} = -\left[x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right] + 2xy\frac{\partial^2 v}{\partial x\,\partial y} - \left[x^2\frac{\partial^2 v}{\partial x^2} + y^2\frac{\partial^2 v}{\partial y^2}\right]$

Add both

(Hence Proved)

~~Double partial diff. has two types :~~

~~i) if $\frac{\partial u}{\partial x}$ has a term related to x~~

~~then $\frac{\partial^2 u}{\partial x^2} = 0$ ex : last 2nd question~~

~~ii) if $\frac{\partial u}{\partial x}$ has no term related to x.~~

~~ex : last question.~~

ERRORS : (APPLICATION).

we know, $\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$

hence : $\frac{\delta y}{\delta x} = \frac{dy}{dx}$ (approx)

$\Rightarrow \delta y = \left(\frac{dy}{dx}\right) \delta x$ (approx)

where : $\delta x \longrightarrow$ absolute error.

$\frac{\delta x}{x} \rightarrow$ relative error

$\frac{\delta x}{x} \times 100 \longrightarrow$ percentage error.

$\Rightarrow$ The power dissipated in a resistor is given by $P = \frac{E^2}{R}$.
Find by using calculus, the approx % change in P when E is (↑) by 3% & R is (↓) by 2%.

Soln) $P = \frac{E^2}{R} \Rightarrow \log P = 2 \log E - \log R$.

$\therefore \frac{\delta P}{P} = \frac{2}{E}\delta E - \frac{\delta R}{R}$.

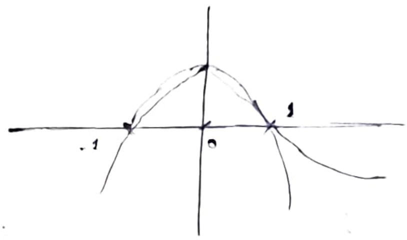
$\Rightarrow 100\left(\frac{\delta P}{P}\right) = 2 \times \left(\frac{\delta E}{E} \times 100\right) - \left(\frac{\delta R}{R} \times 100\right)$

$\Rightarrow \left(\frac{\delta P}{P}\right) 100 = 6 - (-2) = 8\%$ (Ans).

⇒ Find the area of the region bounded by the curves

$y = \cos\frac{\pi x}{2}$, $y = 1 - x^2$.

Sol^n^ $\cos\frac{\pi x}{2} = 1 - x^2$



$\therefore x = 0, 1$.

$$A = \int_0^1 \cos\frac{\pi x}{2} - (1 - x^2)\, dx$$

$$A = \left[\frac{\sin\frac{\pi x}{2}}{\pi/2} - x + \frac{x^3}{3}\right]_0^1 = \frac{1}{\pi/2} - 1 + \frac{1}{3} = 2\left|\frac{2}{\pi} - \frac{2}{3}\right|$$

for two regions

Module-2 : Limit / Continuity → 5/10M.

Partial → 5M

Chain rule → 5M

Application → 5/10 M.

↳ upto Errors.

⇒ The diameter & altitude of a can in shape of right circular cylinder are measured as 40 & 64cm resp. The possible error in each measurement is ±5%. Find the app. the max^m^ possible error in computed value for the volume & the lateral surface. Find % error ?

Sol^n^ $V = \pi r^2 h = \frac{\pi}{4} D^2 h$

$\log V = \log\frac{\pi}{4} + 2\log D + \log h$.

$$\frac{\delta V}{V} = 0 + 2\frac{\delta D}{D} + \frac{\delta h}{h} = 2(\pm 5) + (\pm 5) = \pm 15 \quad (\text{Max}^m)$$

$\therefore S = 2\pi r l = \pi D h$

$\log S = \log \pi + \log D + \log h$

$$\frac{\delta S}{S} = 0 + \frac{\delta D}{D} + \frac{\delta h}{h} = (\pm 5) + (\pm 5) = \pm 10 \quad (\text{Max}^m)$$

⇒ The period T of simple pendulum is $T = 2\pi\sqrt{\frac{l}{g}}$.

Find the max^m error in T due to possible errors upto 1% in l & 2% in g.

Sol^n $\log T = \log 2\pi + \frac{1}{2}\left[\log l + \log g\right]$.

$$\frac{\delta T}{T} = \frac{1}{2}\frac{\delta l}{l} + \frac{1}{2}\frac{\delta g}{g}$$

$$= \frac{1}{2}(1) + \frac{1}{2}(2) = \underline{\underline{3/2}}$$

⇒ A balloon is in the form of right circular cylinder of radius 1.5 m & length 4m & is surmounted by hemispherical ends. If the radius is (↑) by 0.01 m & the length by 0.05 m. Find % change in volume of balloon.

Sol^n $V = \pi r^2 h + 2\left[\frac{2}{3}\pi r^3\right] = \pi r^2 h + \frac{4}{3}\pi r^3$.

$$\cancel{\frac{\delta v}{v}} \quad \delta V = \pi(2r)\,\delta r\, h + \pi r^2 \cdot \delta h + \frac{4}{3}\pi\, 3r^2\, \delta r.$$

imp

$$\therefore \frac{\delta v}{V} = \frac{\pi r\left[2\,\delta r\, h + r\,\delta h + 4r\,\delta r\right]}{\pi r^2 h + \frac{4}{3}\pi r^3}$$

Any value Function radius

$$\frac{\delta v}{V} = \frac{(2)(0.01)(4) + (1.5)(0.05) + (4)(1.5)(0.01)}{(1.5)(4) + \frac{4}{3}(1.5)^2}$$

$$100 \times \frac{\delta v}{V} = \frac{0.215}{9} \times 100$$

% change = $\underline{\underline{2.389\,\%}}$

We didn't use log here for finding error because here the values are added, i.e, $\log V = \log\left(\pi r^2 h + \frac{4}{3}\pi r^3\right)$ & we can't find $\underline{\log(A+B)}$, so in this que we found δv & V seperately.

JACOBIAN

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$\Rightarrow$ If $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial(x,y)}{\partial(r,\theta)}$

Sol^n $\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = \underline{r}$

$\Rightarrow$ find $\frac{\partial(r,\theta)}{\partial(x,y)}$

Sol^n $\begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \frac{1}{r}$

* $\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1 \longrightarrow$ Property.

$\Rightarrow$ $U = xyz$
$V = x^2+y^2+z^2$, $\frac{\partial(u,v,w)}{\partial(x,y,z)} = ?$
$w = x+y+z$

Sol^n $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} yz & xz & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$

$\Rightarrow 1(2xz^2 - 2xy^2) - 1(2yz^2 - 2x^2y) + 1(2y^2z - 2x^2z)$

$\Rightarrow 2xz^2 - 2xy^2 - 2yz^2 + 2x^2y + 2y^2z - 2x^2z$

* If u, v, w are independent and x, y, z are not, then :

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0 \quad , (u,v,w) \text{ are functionally dependent}$$

⇒ If, $u = xy + yz + xz$, $v = x^2 + y^2 + z^2$, $w = x + y + z$. Determine (u,v,w) are functionally dependent, if so then find the relation b/w them ?

Sol^n

→ Property - 1

⇒ If $x = uv$, $y = \frac{u+v}{u-v}$, $\frac{\partial(u,v)}{\partial(x,y)} = ?$

Sol^n Here $\frac{\partial u}{\partial x}$ is difficult to find, so, find $\frac{\partial x}{\partial u}$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{-2v}{(u-v)^2} & \frac{2u}{(u-v)^2} \end{vmatrix} = \frac{4uv}{(u-v)^2}$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$$

$$\Rightarrow \frac{\partial(u,v)}{\partial(x,y)} = \frac{(u-v)^2}{4uv}$$

→ Property - 2 :

⇒ $u = x^2 - y^2$, $v = 2xy$, $x = r\cos\theta$, $y = r\sin\theta$

$\frac{\partial(u,v)}{\partial(r,\theta)} = ?$

Sol^n $\dfrac{\partial(u,v)}{\partial(r,\theta)} = \dfrac{\partial(u,v)}{\partial(x,y)} \cdot \dfrac{\partial(x,y)}{\partial(r,\theta)}$

$\dfrac{\partial(u,v)}{\partial(x,y)} = 4r^2 \quad , \quad \dfrac{\partial(x,y)}{\partial(r,\theta)} = r.$

$\therefore \dfrac{\partial(u,v)}{\partial(r,\theta)} = \underline{\underline{4r^3}}.$

→ Property - 3 :

⇒ If $u = xy + yz + zx$, $v = x^2 + y^2 + z^2$, $w = x + y + z$, determine whether there is a functional relation b/w u, v, w & if so, find it.

Sol^n to find functional relation, $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} y+z & x+z & x+y \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

~~Hence~~

$$= 2\begin{vmatrix} x+y+z & x+y+z & x+y+z \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} \rightarrow R_1 \rightarrow R_1 + R_2$$

Hence, functional relation exists.

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = \underline{\underline{0}}$$

$\therefore$ The relation is :

$w^2 = v + 2u$

$\Rightarrow w^2 - v - 2u = 0$ (Ans)

Learn Jacobian Implicit → Pass → Syllabus.