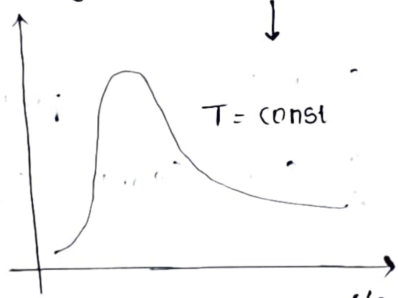
Lec- 14

QUANTUM MECHANICS :

Energy density (Hot body)



Planch's distribution :

(1900)

Planch $\longrightarrow$ $E_n = n(h\nu)$

$$\boxed{u(\nu)\,d\nu = \frac{8\pi h}{c^3} \cdot \frac{\nu^3\,d\nu}{e^{h\nu/KT} - 1}}$$

* Energy density of obj b/w ν to $\nu + d\nu$.

$x \rightarrow$ freq (or) wavelen

$y \rightarrow$ Energy density.

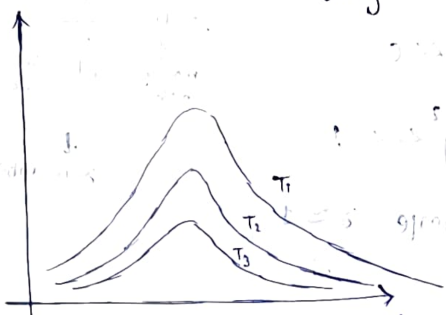
* $K_B \longrightarrow$ Boltzman const.

* $u(\nu)$ (or) $u(\lambda)$ (or) $u(\omega)$ (or) $u(E)$

omega

* $\hbar = \frac{h}{2\pi}$

* $u(\nu)$

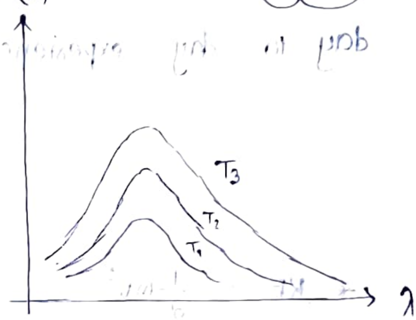


$T_1 \longrightarrow 8000\ K$

$T_2 \longrightarrow 5000\ K$

$T_3 \longrightarrow 2000\ K$.

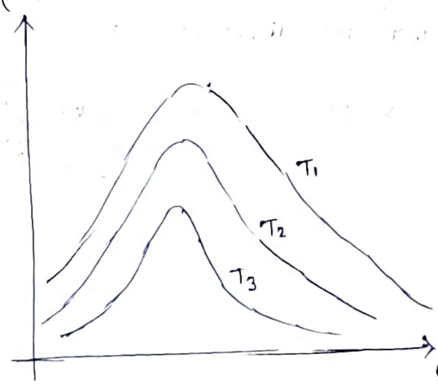
$u(\lambda)$



$\nu \propto T$

$\lambda \propto \frac{1}{T}$

* $u(\omega)$

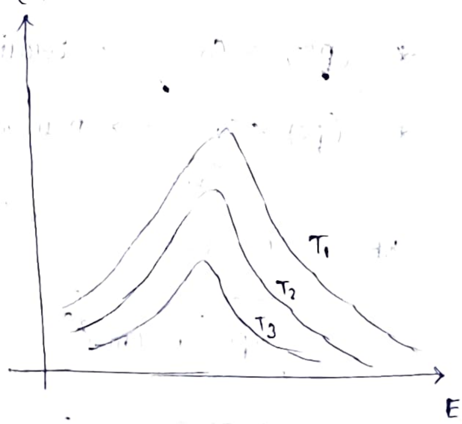


$\omega = 2\pi f$

$\therefore\ \nu \propto T$

$\omega \propto T$

* $u(E)$



$E = h\nu$

$\therefore\ \nu \propto T$

$E \propto T$

Topics :

* De Broglii's relation $\left(\lambda = \frac{h}{p}\right)$, $p \to mv$
* Compton's effect (particle nature of wave)
* Davisson Germer Experiment (wave nature of ~~par~~ wave particle)

if eV is given, $1eV = 1.6 \times 10^{-19} J$

De Broglii's Relation : $\boxed{\lambda = \frac{h}{mv}}$ $= \frac{h}{\sqrt{2m\,KE}}$

$$\boxed{\lambda = \frac{h}{\gamma m_0 u}} = \frac{h}{mu}, \quad \gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}, \quad E = mc^2$$

rest mass (m_0), moving mass (m)

$E_0 = m_0 c^2$ — rest mass

$$\boxed{\therefore m = \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma m_0}$$

moving mass — remember

in day to day experience : $u << c$

$$\left(\frac{u}{c}\right)^2 <<< 1$$

here $\gamma \simeq 1$

* $KE = \frac{1}{2} mv^2$ — E when there is velocity
* $KE = E - E_0 \to (1)$ — where there is no velocity (i.e. rest mass)
* $E^2 = p^2c^2 + m_0^2 c^4$ — E for particle $= mv^2 = m_0 c^2$; E for wave $= \frac{hc}{\lambda} = pc$

Total energy $\Rightarrow E^2 = (pc)^2 + (E_0)^2 \to (2)$

* $(pc) > E_0 \to$ relativistic $\to$ then use above formula
* $(pc) < E_0 \to$ non-relativistic $\to$ then use formula : $\lambda = \frac{h}{\sqrt{2m\,KE}}$

E_0 = 511 keV

Derivation of KE.

* $KE = E - E_0$

$$= \sqrt{(pc)^2 + (m_0c^2)^2} - m_0c^2$$

$$= \sqrt{(m_0^2c^4)\left(1 + \frac{p^2}{m_0^2c^2}\right)} - m_0c^2$$

$$= m_0c^2\left[1 + \frac{p^2}{m_0^2c^2}\right]^{1/2} - m_0c^2$$

$$= m_0c^2\left[1 + \frac{1}{2}\frac{p^2}{m_0^2c^2}\right] - m_0c^2$$ [Binomial]

$$= \frac{1}{2}\frac{m_0^2u^2 m_0c^2}{m_0^2c^2} = \frac{1}{2}m_0u^2$$

⇒ Find de-Broglie λ of :

a) ball with velocity 30 m/s.

b) an e^- with velocity 10^7 m/s.

Sol^n a) $\frac{u}{c} = \frac{3\times10^1}{3\times10^8} = 10^{-7} <<< 1$

$\gamma \simeq 1 \Rightarrow \lambda = \frac{h}{m_0u}$.

b) $\frac{u}{c} = \frac{10\times10^6}{3\times10^8} = 3.3\times10^{-2}$

$\gamma \simeq 1$, $\cancel{\lambda =}$, $\lambda = \frac{h}{m_0u}$.

⇒ Find the KE of the proton where de-Broglie λ is 1 fm.

~~Sol^n~~ find pc, $E_0 \rightarrow m_0c^2$

$p = \frac{h}{\lambda}$, $pc = \frac{hc}{\lambda}$

Here we don't apply $\lambda = \frac{h}{\sqrt{2mKE}}$ because, proton has both wave (λ) & particle (mv^2)

~~Sol^n~~ $\boxed{\cancel{\lambda = \frac{h}{\sqrt{mv}}}, \text{ ~~but here~~}}$

$\cancel{\lambda = \frac{h}{mv} = \frac{h}{\sqrt{2mKE}} \Rightarrow KE = \frac{h^2}{2m\lambda^2} = \frac{(6.6\times10^{-34})^2}{2(1.6\times10^{-27})(10^{-15})^2}}$

$\cancel{\Rightarrow KE =}$

$pc = \frac{hc}{\lambda} \Rightarrow pc = \frac{(6.6\times10^{-34})(3\times10^8)}{(10^{-15})} =$

$\boxed{\cancel{p = \frac{h}{\lambda} \Rightarrow m_0u = \frac{h}{\lambda}}}$

Sol^n) ~~$KE = \sqrt{(pc)^2 + (E_0)^2}$~~

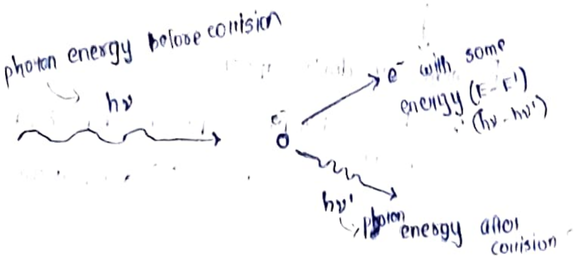
$$pc = \frac{hc}{\lambda} = \frac{6.6\times10^{-34}\ Js \times 3\times10^{8}\ m/s}{10^{-15}\ m} = 19.875\times10^{-11}\ J = 1.2410\ GeV$$

$$E_0 = m_0c^2 = 1.67\times10^{-27} \times (3\times10^{8})^2 = \underline{0.938\ GeV}.$$

$$E = \sqrt{(pc)^2 + (m_0c^2)^2} = \sqrt{\quad} = 1.515$$

$$\therefore KE = E - E_0 = 1.515 - 0.938 = \underline{\underline{577\ MeV}}$$

COMPTON EFFECT : (Particle / Wave nature of wave)



Treat this process as collision and derive the correlation b/w λ & λ'.

Re ~~(photon energy (↓) & e^- energy (↑))~~
~~so, photon λ (↑) & e^- λ (↓)~~

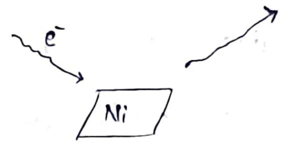
$$\Delta\lambda = (\lambda' - \lambda)$$ (after collision: λ'; before collision: λ)

$$\boxed{\Delta\lambda = \lambda_c\,(1-\cos\phi)}$$

where $\lambda_c = \dfrac{h}{m_0 c}$ (Compton wavelength).

$$\lambda_c = \frac{h}{(m_0)_e\, c} = \underline{2.463\ pm}$$ → imp

DAIRSSON GERMER : (WAVE NATURE OF PARTICLE).



Wave Nature	Particle Nature.
Braggs Theory	de-Broglie
λ_{Brags}	$\lambda_{de\text{-}brog}$.

$\boxed{2d \sin\theta = n\lambda}$

$$\lambda_{Brags} = \lambda_{debrog}$$

HEISENBERG UNCERTAINITY THEORY :

* $\Delta P_x \cdot \Delta x \geq \hbar/2 = \frac{h}{4\pi}$

  $\Delta P_y \cdot \Delta y \geq \hbar/2$

  $\Delta P_z \cdot \Delta z \geq \hbar/2$.

* $\Delta E \cdot \Delta t \geq \hbar/2$

* $\Delta J \cdot \Delta \phi \geq \hbar/2$.

$J \rightarrow$ angular momentum.

$\phi \rightarrow$ angle.

WAVE FUNCTION :

* Notation $\longrightarrow$ $\Psi = \Psi(\vec{r}, t)$ Psi

  $\psi = \psi(\vec{r})$ psi

* Ψ is in general complex.

* $|\Psi|^2$ has meaning. It represents the probability density of

$|\Psi|^2 \cdot dx$ is the probability of finding the object (e^-, proton, etc) b/w x & $x + dx$.

$$P(x) = \int |\Psi|^2 \, dx$$

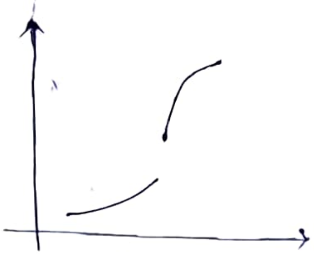
* Properties of Ψ :

i) Ψ is continuous, single values

ii) The derivative $\frac{\partial \Psi}{\partial x}$, $\frac{\partial \Psi}{\partial y}$, $\frac{\partial \Psi}{\partial z}$ are also continuous

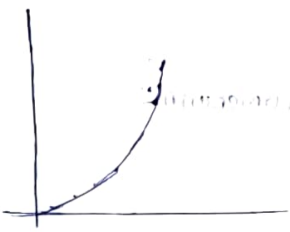
iii) Ψ must be normalisable.

iv) Ψ goes to zero at $x \to \pm\infty$.



Not continuous (x)

Not diff (x)



Not goes to zero at $x \to \pm\infty$ (x)

There are two values at one x (x).

$m_p = 1.67 \times 10^{-27}$ kg $= m_n$

$m_e = 9.1 \times 10^{-31}$ kg

* Zero point energy of e^- means $n = 1$.

SCHRODINGER'S TIME DEPENDENT EQN : (STDE)

* we know :

$$\frac{1}{u^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} \quad , \quad \text{where } f = f(x,t)$$

ODE : ordinary diff. eq^n

PDE : partial DE.

* GOAL : we have to find differential eq^n (ODE / PDE), which will describe the micro level or quantum level description of the particle.

* wave eq^n : $f(x,t) = Ae^{i(\omega t - Kx)}$

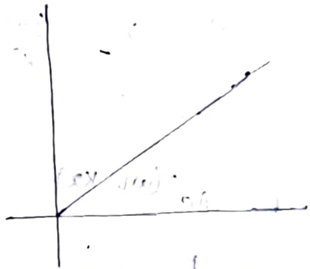
$$\frac{\partial f}{\partial t} = i\omega f \qquad \frac{\partial f}{\partial x} = -iKf$$

$$\frac{\partial^2 f}{\partial t^2} = (i\omega)^2 f \qquad \frac{\partial^2 f}{\partial x^2} = (-iK)^2 f$$

→ put in above eq^n ←

$$\therefore \frac{\omega^2}{u^2} = K^2 \quad (\text{or}) \quad \omega = Ku.$$

$\boxed{K = \frac{\omega}{c}}$ → module - 1



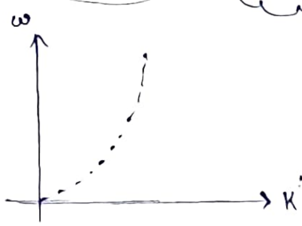
(dispersion relation).

* For Quantum level :

$$E = \frac{p^2}{2m} \quad , \quad \text{if } V(x) = 0 \quad , \quad p = \hbar K, \quad \hbar = \frac{h}{2\pi}.$$

$$\Rightarrow \hbar\omega = \frac{\hbar^2 K^2}{2m}$$

$$\Rightarrow \underline{\omega \propto K^2}$$



* If we have to satisfy this dispersion relation, what is the governing ODE / PDE ?

$$\boxed{E = \frac{hc}{\lambda} = h\nu = 2\pi\hbar\left(\frac{\omega}{2\pi}\right) = \hbar\omega}$$

~~p~~

$$\boxed{\lambda = \frac{h}{p} \Rightarrow p = \frac{h}{\lambda} \Rightarrow p = \frac{h\nu}{c} \Rightarrow p = \frac{h}{2\pi}\left(\frac{\omega}{c}\right) = \frac{hK}{2\pi} = \hbar K}$$

de-Broglie

* Assume the governing ODE /PDE sol^n of the following:

format $f(x,t) = Ae^{i(\omega t - Kx)}$

$\omega \rightarrow$ relation to time derivati.

$K \rightarrow$ relation to space derivati.

$\omega \,\&\, K^2$

$\Rightarrow \omega = \gamma K^2$ where γ = const.

LHS $\longrightarrow$ one time derivation

RHS $\rightarrow$ two space derivation

* $\frac{\partial f}{\partial t}$ & $\frac{\partial^2 f}{\partial x^2}$

$\Rightarrow i\omega f = \gamma(-iK)^2 f$

$\Rightarrow \gamma = \frac{i\omega}{K^2}$

* $\omega \,\&\, K^2$

$\frac{\partial f}{\partial t} = \gamma\left(\frac{\partial^2 f}{\partial x^2}\right)$

$(-i\omega)f = \gamma(iK)^2 f \longrightarrow ①$

* $f = Ae^{i(\omega t - Kx)}$

follow

$\frac{\partial f}{\partial t} = i\omega f$

$\frac{\partial^2 f}{\partial x^2} = (-iK)^2 f$

* $f = Ae^{i(Kx - \omega t)}$

follow

$\frac{\partial f}{\partial t} = (-i\omega) f$

$\frac{\partial^2 f}{\partial x^2} = (iK)^2 f$

* from eq ① :

$\gamma = \left(\frac{i\omega}{K^2}\right) = \frac{i\left(E/\hbar\right)}{\left(P/\hbar\right)^2} = \frac{iE}{\hbar}\frac{\hbar^2}{p^2} = \frac{i\hbar E}{p^2}$

$\Rightarrow \gamma = \frac{i\hbar}{2m}$

$$\boxed{\omega = 2\pi\nu = \frac{2\pi v}{\lambda} = \frac{2\pi v p}{h} = \frac{pv}{\hbar} = \frac{hc/\lambda}{\hbar} = \frac{E}{\hbar}}$$

$\lambda\nu = \lambda f$

$$\boxed{K^2 = \frac{\omega^2}{c^2} = \left(\frac{\omega}{c}\right)^2 = \left(\frac{2\pi f}{c}\right)^2 = \left(\frac{hf}{\hbar\lambda}\right)^2 = \left(\frac{P}{\hbar}\right)^2}$$

* $\frac{\partial f}{\partial t} = \frac{i\hbar}{2m} \frac{\partial^2 f}{\partial x^2}$

$$\longrightarrow \frac{\partial \Psi}{\partial t} = \frac{i\hbar}{2m} \frac{\partial^2 f}{\partial x^2}$$

$f \equiv \Psi(x,t)$

$$i\hbar \frac{\partial \Psi}{\partial t} = \frac{-\hbar^2}{2m} \cdot \frac{\partial^2 \Psi}{\partial x^2} \quad , \text{ if } V(x) = 0$$

* In the presence of the potential $V(x)$, the STDE :

$$i\hbar \frac{\partial \Psi}{\partial t} = \frac{-\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V(x)\, \Psi(x,t)$$

↳ This is not a derivation of STDE :

*⇒ What would be the STDE in presence or absence of $V(x)$?

~~Sol~~ ↳ May come in FAT

Ans ⟶ above the pg.

~~SCHRODINGER TIME INDEPENDENT Eq^N : (STIE)~~

* 1D (space) : Cartisian $i\hbar \dfrac{\partial \Psi(x,t)}{\partial t} = -\dfrac{\hbar^2}{2m} \dfrac{\partial^2 \Psi(x,t)}{\partial x^2} + V(x)\,\Psi(x,t)$

* 2D (space) : $i\hbar \dfrac{\partial \Psi(x,y,t)}{\partial t} = -\dfrac{\hbar^2}{2m}\left[\dfrac{\partial^2 \Psi(x,y,t)}{\partial x^2} + \dfrac{\partial^2 \Psi(x,y,t)}{\partial y^2}\right] + V(x,y)\,\Psi(x,y,t)$

* 3D (space) : $i\hbar \dfrac{\partial \Psi(x,y,z,t)}{\partial t} = -\dfrac{\hbar^2}{2m}\left[\dfrac{\partial^2 \Psi(x,y,z,t)}{\partial x^2} + \dfrac{\partial^2 \Psi(x,y,z,t)}{\partial y^2} + \dfrac{\partial^2 \Psi(x,y,z,t)}{\partial z^2}\right] + V(x,y,z)\,\Psi(x,y,z,t)$

SCHRODINGER TIME INDEPENDENT Eq^N : (STIE)

* STDE : $i\hbar \dfrac{\partial \Psi}{\partial t} = -\dfrac{\hbar^2}{2m} \dfrac{\partial^2 \Psi}{\partial x^2} + V(x)\,\Psi$

$\therefore \quad \Psi = \Psi(x,t)$

Assume : $\Psi(x,t) = f(x)\, g(t)$

$$\frac{\partial \Psi}{\partial t} = f(x)\, \frac{dg}{dt}$$

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{d^2 f}{dx^2} \cdot g$$

$$i\hbar\, f(x) \frac{dg(t)}{dt} = -\frac{\hbar^2}{2m} \frac{d^2 f(x)}{dx^2}\, g(t) + V(x)\, f(x)\, g(t).$$

* Divide LHS & RHS by $f(x)\,g(t)$

$$i\hbar \frac{1}{g(t)} \cdot \frac{d(g(t))}{dt} = -\frac{\hbar^2}{2m} \frac{1}{f(x)} \frac{d^2 f(x)}{dx^2} + V(x) = E \text{ (const)}$$

→ $\ln(g(t))$

i) $i\hbar \displaystyle\int \frac{d(g(t))}{dt} \cdot \frac{1}{g(t)} = \int E\, dt \Rightarrow g(t) = e^{-\frac{iEt}{\hbar}}$

ii) $-\frac{\hbar^2}{2m} \frac{1}{f(x)} \frac{d^2 f(x)}{dx^2} + V(x) = E$

$\therefore$ eqn (ii) can be written as :

$$\frac{-\hbar^2}{2m} \frac{1}{f(x)} \frac{d^2 f}{dx^2} + V(x) = E$$

$$\Rightarrow \frac{d^2 f}{dx^2} + \frac{2m}{\hbar^2} \left(E - V(x)\right) f = 0$$

$\Psi(x,t) = f(x)\, g(t)$ is called seperation of variables.

$\psi = \psi(x) \rightarrow \Rightarrow \left.\frac{d^2 \psi}{dx^2} + \frac{2m}{\hbar^2} \left(E - V(x)\right) \psi = 0 \right\} \rightarrow$ STIE