14-8-23

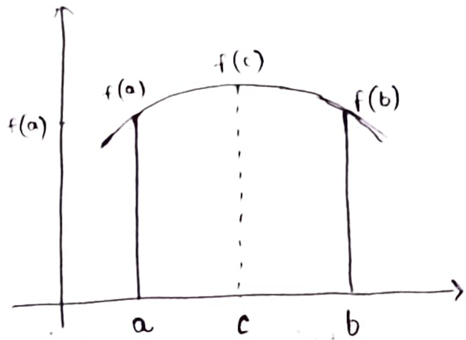
SINGLE VARIABLE CALCULAS (M-1)

Rolle's THEOREM:

* $f(x)$ is continuoes $[a,b]$
* $f(x)$ is diff^n on (a,b)

assumption → * $\boxed{c = \frac{a+b}{2}}$, then, $f(a) = f(b)$

* $f'(c) = 0$



⇒ $f(x) = \frac{x^3}{3} - 3x$, $[-3, 3]$

State and verify Rolle's Th^m & find 'c' ?

Sol^n) $\left.\begin{array}{l} f(-3) = 0 \\ f(3) = 0 \end{array}\right\}$ $f(-3) = f(3)$, Hence verified

$\cancel{\therefore c = \frac{-3+3}{2} = 0}$ ~~(Ans^n)~~

$f'(x) = x^2 - 3$

$\therefore x^2 - 3 = 0$

$x = \pm\sqrt{3} = c$

⇒ $f(x) = (x-a)^m (x-b)^n$, $[a,b]$.

Sol^n) $f(a) = 0 = f(b)$

$f'(x) = m(x-a)^{m-1}(x-b)^n + (x-a)^m n(x-b)^{n-1}$

$\therefore f'(x) = 0$

$\Rightarrow m(x-a)^{m-1}(x-b)^n = -n(x-a)^m \cdot (x-b)^{n-1}$

$\Rightarrow \frac{m}{(x-a)} = \frac{-n}{(x-b)}$

$\Rightarrow mx - mb = -nx + na$

$\Rightarrow x = \frac{na+mb}{m+n}$

$\Rightarrow f(x) = (x+\cancel{3}2)^3 (x-3)^4$, $[-2,3]$

Soln from pre question :

$m = 3$, $n = 4$, $a = -2$, $b = +3$.

$\therefore \quad x = \frac{na + mb}{m+n} = \frac{-8+9}{7} = \underline{\underline{\frac{1}{7}}}$.

LAAGRANGE'S MEAN VALUE THM :

* $f(x)$ is continuous on $[a,b]$
* $f(x)$ is derivable on (a,b)
* There is a point 'c' such that, $f'(c) = \frac{f(b)-f(a)}{b-a}$

$\Rightarrow f(x) = x(x-1)(x-2)$, determine 'c' lies b/w '0' & '½' by LMVT :

Soln $f(0) = 0$, $f(\frac{1}{2}) = \frac{3}{8}$

$\cancel{f'(x) = \frac{3/8 - 0}{\cancel{3/8}\,1/2} = \frac{3}{4}}$

$\Rightarrow f'(x) = x(x-1) + x(x-2) + (x-1)(x-2)$

$\Rightarrow f'(x) = x^2 - x + x^2 - 2x + x^2 - 2x - x + 2$

$\Rightarrow f'(x) = 3x^2 - 6x + 2$

$\therefore f'(c) = 3c^2 - 6c + 2 = \frac{3/8 - 0}{1/2 - 0}$

$\Rightarrow \quad c = \underline{\underline{\frac{1 \pm \sqrt{21}}{6}}} = 1.76, \; 0.24$

$\therefore$ 0.24 b/w 0 & ½

$\Rightarrow$ P.T. (If $0 < a < b < 1$), $\dfrac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \dfrac{b-a}{1+a^2}$,

Hence show that $\dfrac{\pi}{4} + \dfrac{3}{25} < \tan^{-1} 4/3 < \dfrac{\pi}{4} + \dfrac{1}{6}$

Sol^n^

$$\frac{1}{1+b^2} < \frac{\tan^{-1} b - \tan^{-1} a}{\cancel{1+b^2}\, b-a} < \frac{1}{1+a^2}$$

$$\frac{1}{1+b^2} < \frac{f(b) - f(a)}{b-a} < \frac{1}{1+a^2} \longrightarrow f(x) = \tan^{-1} x$$

$$f'(x) = \frac{1}{1+x^2}$$

$$f'(c) = \frac{1}{1+c^2}$$

$\therefore \quad a < c < b$

$$a^2 < c^2 < b^2$$

$$1+a^2 < 1+c^2 < 1+b^2$$

$$\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$$

$$\frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2} \Rightarrow \frac{1}{1+b^2} < f'(c) < \frac{1}{1+a^2}$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$\Rightarrow \frac{1}{1+c^2} = \frac{\tan^{-1}(b) - \tan^{-1}(a)}{b-a}.$$

$$\Rightarrow \frac{1}{1+b^2} < \frac{\tan^{-1} b - \tan^{-1} a}{b-a} < \frac{1}{1+a^2}$$

$\therefore$ from above ques: $\tan^{-1}(1) = \dfrac{\pi}{4}$

$$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$$

$$\frac{b-a}{1+b^2} + \tan^{-1} a < \tan^{-1} b < \frac{b-a}{1+a^2} + \tan^{-1} a$$

Hence $a = 1$, $b = 4/3$.

$$\frac{\pi}{4} + \frac{3}{25} < \tan^{-1} 4/3 < \frac{\pi}{4} + \frac{1}{6}$$

(Proved)

⇒ Verify LMVT, $f(x) = \sin x$ in $[0, \pi]$

Sol^n) $f'(x) = \cos x$

$$\therefore f'(c) = \cos c = \frac{\sin \pi - \sin 0}{\pi - 0}$$

$$\Rightarrow \cos c = 0$$

$$\Rightarrow c = \pi/2 \qquad \therefore \frac{\pi}{2} \text{ is b/w } 0 \text{ to } \pi.$$

⇒ Verify LMVT, $f(x) = \log_e x$ in $[1, e]$

Sol^n) $f'(x) = \frac{1}{x} = \frac{\log_e e - \log_e 1}{e - 1}$

$$\Rightarrow \frac{1}{x} = \frac{1}{e-1} \qquad \therefore e-1 \text{ is b/w } 1 \text{ \& } e$$

$$\therefore x = e - 1$$

INCREASING AND DECREASING FUNCTION

⇒ Find the critical points of $f(x) = x^3 - 12x - 5$ & find the range for that the fun^c is (↑) or (↓) ?

Sol^n $f'(x) = 3x^2 - 12 = 0$

$x = \pm 2$

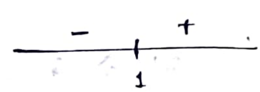
$f'(x) = (x-2)(x+2)$

∴ (↑) from $(-\infty, -2] \cup [2, \infty)$

(↓) from $[-2, 2]$

⇒ $f(x) = x^{1/3}(x-4)$

Sol^n $f'(x) = \frac{1}{3} x^{-2/3}(x-4) + x^{1/3} = 0$



$x^{1/3} = \frac{-1}{3x^{2/3}}(x-4)$ → but $x \neq 0$

∴ (↑) from $(1, \infty)$

(↓) from $(-\infty, 1) - \{0\}$

$3x = -x + 4$

$4x = 4$

$x = 1$

⇒ $f(x) = x^4 - 8x^2$

Sol^n $f'(x) = 4x^3 - 16x = 0$

$4x(x^2 - 4) = 0$

$4x = 0, \quad x = \pm 2$

∴ (↓) from $(-\infty, -2) \cup (0, 2)$

(↑) from $(-2, 0) \cup (2, \infty)$

$f'(x) = (x)(x-2)(x+2)$

⇒ $f(x) = x^2 e^{\frac{1}{1-x^2}}$

Sol^n $f'(x) = 2x e^{\frac{1}{1-x^2}} + x^2 \cancel{\frac{1}{(1-x^2)}} e^{\frac{1-1+x^2}{1-x^2}} \quad x^2 e^{\frac{1}{1-x^2}}$

⇒ $f(x) = 2x^3 + 3x^2 - 36x$

i) Find the intervals on which it is (↑) or (↓)

imp ii) Find the intervals on concavity & the inflection points

iii) Find the local max & min values of 'f'

Sol^n i) $f'(x) = 6x^2 + 6x - 36 = 0$

$$x^2 + x - 6 = 0$$

$$(x+3)(x-2) = 0$$

+   −   +

−3   +2

(↑) from $(-\infty, -3) \cup (2, \infty)$

(↓) from $(-3, +2)$

~~ii) $f''(x) = 12x + 6 = 0$~~

~~$6(2x+1) = 0$~~

~~$x = -1/2$ → point of inflection.~~

~~$f''(x) \neq 6$~~

ii) . $f''(x) = 12x + 6 = 0$

$$x = -1/2$$

Now, we have to find whether sign changes either sides of $-1/2$.

∴ $f''(0) = 6$, $f''(-1) = -6$.

Hence, $-1/2$ is inflection point

iii) $f'(x) = x^2 + x - 6 = 0$

$$x = -3, 2$$

$f(-3) = -54 + 27 + 108$

$= \underline{\underline{81}}$

↑

max

$f(2) = 16 + 12 - 72$

$= \underline{\underline{-46}}$

↑

min

∴

MAXIMA & MINIMA

* Step - 1 : $f(x) =$ ______
* Step - 2 : $f'(x) = 0$

  $x =$ —
* Step - 3 : $f''(x) =$ ______

  & check at each x

if $f''(x) > 0$ [min^m]

$f''(x) < 0$ [max^m]

Sometime $f''(x) = 0$

then check sign of $f'(x)$, if $f'(x)$ changes sign +ve to −ve

$max^m \leftarrow$ at $x =$ —

if $f'(x)$ changes sign −ve to +ve at $x =$ — $\leftarrow$ min^m

⇒ Find the max & min values of :

* $f(x) = 3x^4 - 2x^3 - 6x^2 + 6x + 1$

Sol^n $f'(x) = 12x^3 - 6x^2 - 12x + 6 = 0$

$= 6(2x^3 - x^2 - 2x + 1) = 0$

$= (x-1)(x+1)\left(x - \frac{1}{2}\right) = 0$

$x = 1, -1, \frac{1}{2}$

$\therefore f(1) = 3 - 2 - 6 + 6 + 1 = 2$

$f(-1) = 3 + 2 - 6 - 6 + 1 = -6$

$\cancel{f(1/2)}$

$f''(x) = 36x^2 - 12x - 12$

$= 12(3x^2 - x - 1)$

$f''(1) = 12(3\cancel{7}) = \cancel{36}\ 12 > 0$ (min^m)

$f''(-1) = 12(3) = 36 > 0$ (min^m)

$f''\left(\frac{1}{2}\right) = 12\left(\frac{3}{4} - \frac{1}{2} - 1\right) = 12\left(\frac{3-2-4}{4}\right) < 0$ (max^m)

* $f(x) = x^4 - 4x^3 + 10$

Soln $f'(x) = 4x^3 - 12x^2$

$= 4x^2(x-3) = 0$

$x = 0, \quad x = 3.$

$- \quad - \quad +$

$0 \quad 3.$

$f''(x) = 12x^2 - 24x$

$f''(0) = 0$ (Point of inflection)

$f''(3) = 144 - 72 = 72 > 0$ ($\min^m$)

* $f(x) = \sin x \, (1 + \cos x)$

Soln $f'(x) = \cos x \, (1 + \cos x) + \sin x \, (-\sin x)$

$= \cos x + \cos^2 x - \sin^2 x$

$= \cos x + \cos 2x = 0$

$= 2\cos^2 x + \cos x - 1 = 0$

$= (\cos x + 1)(2\cos x - 1) = 0$

$2t^2 + t - 1 = 0$

$2t^2 + 2t - t - 1 = 0$

$2t(t+1) - 1(t+1) = 0$

$\cos x = -1, \quad \cos x = 1/2$

$x = \pi, \quad x = \frac{\pi}{3}.$

$f''(x) = -\sin x + (-2\sin 2x)(2)$

$= -\sin x - 2\sin 2x.$

$\frac{60}{2(\cancel{180})}{\cancel{3}}$

$f''(\pi) = 0$

$f''\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} - 2\left[-\frac{\sqrt{3}}{2}\right]$

$= -\frac{\sqrt{3}}{2} + \frac{2\sqrt{3}}{2} = \frac{\sqrt{3}}{2} > 0$ ($\min^m$)

* $f(x) = \frac{a}{x} + bx$, $f(2) = 1$, has extreme value at $x = 2$

find a & b.

Sol^n $f'(x) = -\frac{a}{x^2} + b = 0$ $-ax^{-2}$

$-a(-2x^{-3})$

$\frac{2a}{x}$

$b = \frac{a}{x^2}$

$x^2 = \frac{a}{b} \Rightarrow x = \pm\sqrt{\frac{a}{b}}$

$\cancel{f''(x) = \frac{2a}{x^3}} = \frac{2a}{\left(\frac{a}{b}\right)^{3/2}}$ $\longrightarrow$ $\underline{a = 4b} \leftarrow x = 2$.

$f(2) = 1$

$\frac{a}{2} + 2b = 1$

$a + 4b - 2 = 0$.

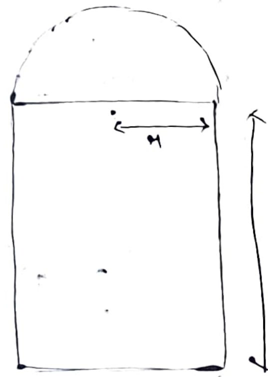
$8b = 2$

$b = 1/4$, $a = 1$

$\Rightarrow$ A window has the form of rectangle surmounted by a semi-circle. If the perimeter is 40ft, find its dimensions so that the greatest amount of light may be admitted.

Sol^n. Perimeter $= \pi r + (2l + 2r) = 40$

$\pi r + 2l + 2r = 40$.



Area $= \frac{\pi r^2}{2} + l(2r) + \frac{\pi r^2}{2} - \frac{\pi r^2}{2}$

$= \pi r^2 + 2rl - \frac{\pi r^2}{2}$

$= r(\pi r + 2l) - \frac{\pi r^2}{2}$

$= r(40 - 2r) - \frac{\pi r^2}{2}$

$\frac{dA}{dr}$ $= 40r - 2r^2 - \frac{\pi r^2}{2}$ diff & save

$\frac{dA}{dM}$

AREA BET^{W} THE CURVES :

→ Area bounded by the curve $y = f(x)$, $x = a, x = b$.

$$\int_a^b f(x)\, dx$$

→ Area bounded ⇒ $x = f(y)$, $y = a, y = b$.

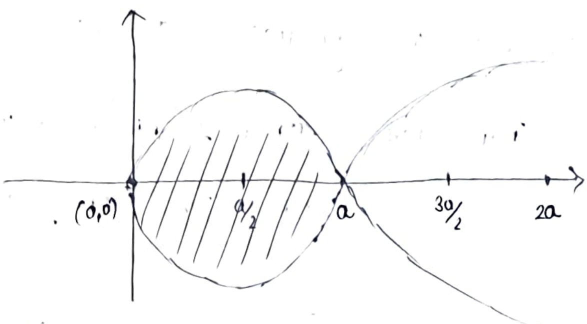
$$\int_a^b f(y)\, dy$$

⇒ $y^2 = \frac{x}{3a}(x-a)^2$

Sol^n $y = \pm \frac{\sqrt{x}(x-a)}{\sqrt{3a}} = f(x)$

x	0	$a/2$	a	$3a/2$	$2a$
y	0	$\frac{-a/2 \sqrt{a}}{\sqrt{3a}}$	0	$\frac{a/2 \frac{\sqrt{3a}}{2}}{\sqrt{3a}}$	$\frac{\sqrt{2a}\, a}{\sqrt{3a}}$

∴ Area $= \int_0^a \frac{\sqrt{x}(x-a)}{\sqrt{3a}}\, dx$



$A = \frac{1}{\sqrt{3a}} \int_0^a x^{3/2} - a x^{1/2}\, dx$

$A = \frac{1}{\sqrt{3a}} \left[\frac{2x^{5/2}}{5} - \frac{2a x^{3/2}}{3} \right]_0^a$

$A = \frac{1}{\sqrt{3a}} \left[\frac{2}{5}(a)^{5/2} - \frac{2a}{3}(a^{3/2}) \right]$

$A = \frac{1}{\sqrt{3a}} \left[\frac{2}{5} a^{5/2} - \frac{2}{3} a^{5/2} \right]$

$A = \frac{a^{5/2}}{\sqrt{3}\, a^{1/2}} \left[\frac{6-10}{15} \right]$ ⇒ $A = \frac{+4a^2}{15\sqrt{3}}$

for both sides of x

∴ Ans $= 2\left(\frac{4a^2}{15\sqrt{3}}\right)$

⇒ Find area b/w $a^2x^2 = y^3(2a-y)$

Sol^n) $\underline{a^2x^2} = 2ay^3 - y^4$

$$x = \pm\sqrt{\frac{2ay^3 - y^4}{a^2}} = \frac{(y^{3/2})(\sqrt{2a-y})}{a}$$

Let $y = 2a\sin^2\theta$.

$$x = \frac{(2a\sin^2\theta)^{3/2}\sqrt{2a - 2a\sin^2\theta}}{a}$$

$$x = 2a^{1/2}\sin^2\theta\sqrt{2a(1-\sin^2\theta)}$$

$$x = 2a^{1/2}\sin^2\theta(\sqrt{2a})(\cos\theta)$$

$4\sin^2\theta\cos^2\theta$

$\sin^2 2\theta$

$$x = (2a^{1/2})(2^{1/2}\cdot a^{1/2})\sin^2\theta\cos^2\theta$$

$$x = (2^{3/2}a)\frac{\sin^2 2\theta}{4}$$

$$x = 2^{-1/2}a\sin^2 2\theta$$

$$x = 2^{-1/2}a$$

⇒ $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but there is no number c in $(-1,1)$ such that $f'(x) = 0$. Why does this not contradict Rolle's theorem?

Sol^n)

$$f(-1) = 1 - (-1)^{2/3} = 0$$

$$f(1) = 1 - (1)^{2/3} = 0$$

$$f'(x) = \frac{-2}{3} x^{-1/3} = 0$$, but $x \neq 0$ (otherwise not defined).

for rolle's theorem, $f(x)$ is continuous on $[a,b]$

$f(x)$ is diff on (a,b)

$f(a) = f(b)$.

here only $f(a) = f(b)$ is satisfied.

But $f(x)$ is not diff. in interval (a,b)

so, it does not contradict rolle's theorem.

(Proved).

⇒ If $f(x) = 2x^3 + 3x^2 - 36x$, find intervals on which it is (↑) or (↓), the local max^m & local min^m values of f, the intervals of concavity & inflection points.

Sol^n)

$$f'(x) = 6x^2 + 6x - 36 = 0$$

$$= x^2 + x - 6 = 0$$

$$\Rightarrow (x+3)(x-2) = 0$$

$$x = -3, 2$$

+		−		+
	-3		$+2$	

(↑) from $(-\infty, -3) \cup (2, \infty)$

(↓) from $(-3, 2)$

local max^m point at $x = -3$

local min^m point at $x = +2$

$\min^m$ value $= 2(2)^3 + 3(2)^2 - 36(2)$

$= 16 + 12 - 72 = \underline{\underline{-44}}$

$\max^m$ value $= 2(-3)^3 + 3(-3)^2 - 36(-3)$

$= -54 + 27 + 108$

$= \underline{\underline{81}}$.

$f''(x) = 2x + 1$

for concave upward : $f''(x) > 0$.

$2x+1 > 0$

$\underline{\underline{x > \frac{-1}{2}}}$.

for concave downward : $\underline{\underline{x < \frac{-1}{2}}}$

$f''(x) = 2x + 1 = 0$

$\underline{\underline{x = \frac{-1}{2}}}$ (inflection point).

↳ point where concavity changes.

3) Find the interval of concavity & inflection point. Also find the extreme values on what interval is, f (↑) or (↓)

a) $f(x) = \sin x + \cos x \quad , \quad x \in [0, 2\pi]$

Sol^n) $f'(x) = \cos x - \sin x = 0$

$\tan x = 1$

$x = \frac{\pi}{4}$

$-$ $\quad$ $+$

$\pi/4$

$\therefore$ (↑) from $\left(\frac{\pi}{4}, 2\pi\right]$

(↓) from $\left[0, \frac{\pi}{4}\right)$

$f''(x) = -\sin x - \cos x = 0$

$\tan x = -1$

$x = \frac{3\pi}{4}$ (Point of inflection) for concave upward : $f''(x) > 0$

$$-\sin x - \cos x > 0$$

$$-\sin x > \cos x$$

$$\frac{\sin x}{\cos x} < -1$$

$$\tan x < -1$$

$$x < \frac{3\pi}{4}$$

$\therefore$ for concave downward : $x > \frac{3\pi}{4}$

b) $f(x) = e^{2x} + e^{-x}$

Soln $f'(x) = 2e^{2x} - e^{-x} = 0$

$$2e^{2x} = \frac{1}{e^x}$$

$$2e^{2x} \cdot e^x = 1$$

$$e^{3x} = \frac{1}{2}$$

$$3x = \ln\left(\frac{1}{2}\right)$$

$$x = \frac{1}{3}\ln\left(\frac{1}{2}\right)$$

$-$ $+$

$\frac{1}{3}\ln\left(\frac{1}{2}\right)$

($\uparrow$) from $x \in \left(\frac{1}{3}\ln\left(\frac{1}{2}\right), \infty\right)$

($\downarrow$) from $x \in \left(-\infty, \frac{1}{3}\ln\left(\frac{1}{2}\right)\right)$

c) $f''(x) = 4e^{2x} + e^{x} = 0$

$$4e^{2x} = -e^{x}$$

$$e^{x} = -1/4$$

it must be > 0

$$x = \ln(-1/4)$$

(No Inflection point)

~~for concave upward~~ :

c) $f(x) = x + 2\sin x$, $x \in [0, 2\pi]$

$\frac{\pi}{3}$

Sol^n $f'(x) = 1 + 2\cos x = 0$

$\cos x = -\frac{1}{2}$

$x = \frac{2\pi}{3}$.

$-$ $+$

$2\pi/3$

$\therefore$ ($\uparrow$) from $\left(\frac{2\pi}{3}, \infty\right)$

($\downarrow$) from $\left(-\infty, \frac{2\pi}{3}\right)$

$f''(x) = -2\sin x = 0$

$x = 0$ (Inflection point).

concave upward ~~from~~ $(x < 0)$

concave downward $(x > 0)$.

4) $f(x, y) = xy(5x + y - 15)$ $\longrightarrow$ find all critical points.

Sol^n ~~for~~ two variable eq^n, we must partial diff.

$f(x, y) = 5x^2 y + xy^2 - 15xy$

$$\frac{\partial f}{\partial x} = 10xy + y^2 - 15y = 0$$

$$\frac{\partial f}{\partial y} = 5x^2 + 2xy - 15x = 0.$$

$y(10x + y - 15) = 0$. | $x(5x + 2y - 15) = 0$.

one critical point is $(x, y) = (0, 0)$.

$y = 15 - 10x \longrightarrow 5x + 2(15 - 10x) - 15 = 0$

$5x + 30 - 20x - 15 = 0$

$15 = 15x \Rightarrow x = 1$

$y = 5$

The critical points are $(0,0)$ & $(1,5)$ (Ans)

The critical points are (0,0), (0,5), (1,0), (1,5)

⇒ Find the area b/w the curves, $y = x+2$, $y = x^2+x-2$

Sol^n $y = x+2$, $y = x^2+x-2$

$x+2 = x^2+x-2$

$x^2 = 4$

$x = \pm 2$

↳ Both curves meet at $x = +2$, $x = -2$.

$\therefore$ Area $= \int_{-2}^{+2} (x^2+x-2) - (x+2)\, dx$

Area $= \int_{-2}^{+2} x^2 - 4\, dx$

$A = \left(\frac{x^3}{3} - 4x\right)_{-2}^{+2} = \left(\frac{8}{3} - 8\right) - \left(\frac{-8}{3} + 8\right)$

$= \frac{8}{3} - 8 + \frac{8}{3} - 8 = \frac{16}{3} - 16$

$= \frac{16-48}{3} = \frac{32}{3}$ (Ans

⇒ $y = 3x^3 - x^2 - 10x$, $y = -x^2 + 2x$

Sol^n $3x^3 - x^2 - 10x = -x^2 + 2x$.

$3x^3 - 10x = 2x$

$3x^3 - 12x = 0$.

$x(3x^2 - 12) = 0$.

$x = 0$, $x = \pm 2$.

$A = \int_{-2}^{0} (3x^3 - x^2 - 10x + x^2 - 2x) + \int_{0}^{+2} (3x^3 - x^2 - 10x + x^2 - 2x)$

$A = \int_{-2}^{0} (3x^3 - 12x)\,dx + \int_{0}^{+2} (3x^3 - 12x)\, dx$

$A = \left(\frac{3}{4}x^4 - 6x^2\right)_{-2}^{0} + \left(\frac{3x^4}{4} - 6x^2\right)_{0}^{+2}$

$$A = \left| -\frac{3}{4}(-2)^4 + 6(-2)^2 \right| + \left| \frac{3}{4}(2)^4 - 6(2)^2 \right|$$

$$A = |-12 + 24| + |12 - 24| = \underline{\underline{24}}$$

$\Rightarrow$ $y = x^2$, $y = 2x - x^2$

Sol^n

$$x^2 = 2x - x^2 \qquad A = \int_0^1 x^2 - 2x + x^2 \, dx$$

$$2x^2 = 2x \qquad A = \int_0^1 2x^2 - 2x \, dx$$

$$x^2 - x = 0$$

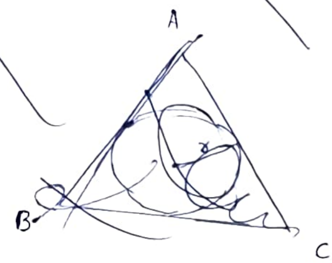
$$x(x-1) = 0 \qquad A = \left(\frac{2}{3}x^3 - x^2 \right)_0^1$$

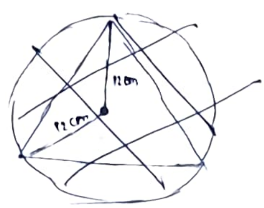
$$\underline{\underline{x = 0, 1}}$$

$$A = \frac{2}{3} - 1 = \underline{\underline{\frac{1}{3}}}$$

~~$\Rightarrow$ An isosceles Δ is inscribed in a circle of radius 12 cm Find max area of Δ.~~

~~Sol^n Area of Δ =~~

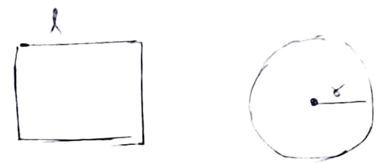




⇒ A piece of wire 100 cm long, which is cut into two pieces one piece is bent to form of a square & other is bent to form a circle. Determine how the wire should be cut so that the total area enclosed is max or min ?

Sol^n/

$L = 100$ cm.



$(area)_{squ} = \cancel{100}\ l^2$

$(area)_{circle} = \pi r^2$

$4l + 2\pi r = 100$

$2l + \pi r = 50$.

$l = \frac{50 - \pi r}{2}$.

$Area = l^2 + \pi r^2$

$= \frac{(50 - \pi r)^2}{2} + \pi r^2$

$$\frac{dA}{dr} = \frac{1}{2}\left[2(50 - \pi r)(-\pi)\right] + 2\pi r = 0$$

$$(50 - \pi r)(-\pi) + 2\pi r = 0$$

$$-50\pi + \pi^2 r + 2\pi r = 0$$

$$-50 + \pi r + 2r = 0.$$

$$r = \frac{50}{2+\pi}$$

$$l = \frac{50 - \pi\left(\frac{50}{2+\pi}\right)}{2} = \frac{100 + 50\pi - 50\pi}{2(2+\pi)}$$

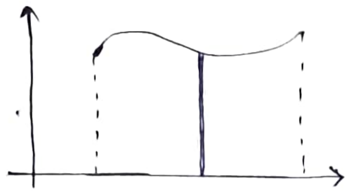
$$= \frac{50}{4 + 2\pi}$$

$$\therefore 4l = \frac{200}{4+2\pi} \quad \text{(Ans)}$$

$$2\pi r = \frac{2\pi(50)}{2+\pi} = \frac{100\pi}{2+\pi} \quad \text{(Ans)}$$

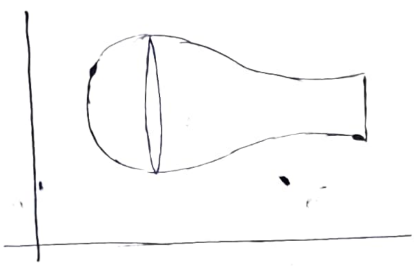
Solids Of Revolution (Disk Method)

* In 2-D $\longrightarrow$



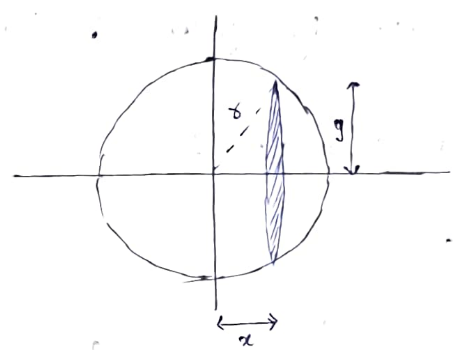
↳ we take a line & using int we find area

* But in 3-D :



we assume a circle or disk to find the whole volume

⇒ Find volume of sphere :



Sol^n We assume a disc whose radius is y, & at dist^n x from center of sphere.

$\therefore \quad x^2 + y^2 = r^2 \longrightarrow$ Pytho.

$$\text{Area of disc} = \pi y^2 = \pi(r^2 - x^2)$$

$$\text{Volume} = \int_{-r}^{r} \pi(r^2 - x^2)\,dx \Rightarrow 2\pi \int_{0}^{r} (r^2 - x^2)\,dx$$

$$\Rightarrow 2\pi \left[r^2 x - \frac{x^3}{3} \right]_0^r = \frac{4}{3}\pi r^3$$

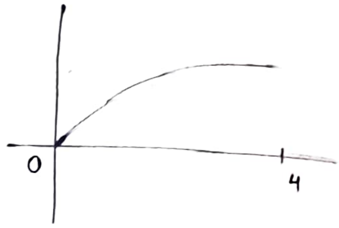
⇒ The region b/w curve $y = \sqrt{x}$, $x \in [0,4]$ & the x-axis is revolved about the x-axis to generate solid. Find volume.

Sol^n) $V = \int_0^4 \pi y^2 \, dx$

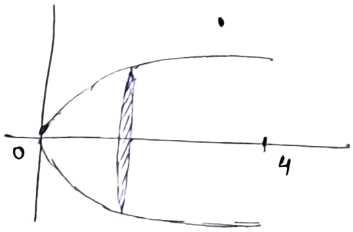
$V = \int_0^4 \pi (\sqrt{x})^2 \, dx$.

↳ the func^n is at $\sqrt{x}$ dist^n from x-axis.

$V = \int_0^4 \pi x \, dx = \left(\frac{\pi x^2}{2}\right)_0^4 = \frac{\pi}{2}[16] = \underline{8\pi}$



↓ rotated.



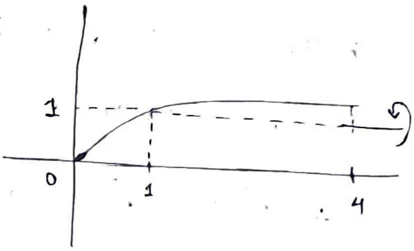
⇒ Find the volume generated by revolving the region bounded by $y = \sqrt{x}$ & lines $y = 1$, $x = 4$ about line $y = 1$.

Sol^n) at $y = 1$, $x = 1$

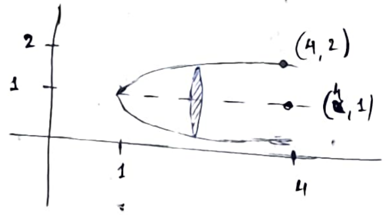
at $x = 4$, $y = 2$

$V = \int_1^4 \pi [\sqrt{x} - 1]^2 \, dx$

$V = \underline{\frac{7\pi}{6}}$



⇓



$y = \sqrt{x}$.

transform the graph

$\therefore y + 1 = \sqrt{x}$

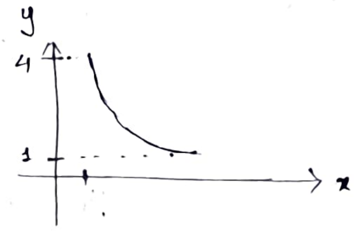
$\underline{y = \sqrt{x} - 1}$

⇒ Find the volume of solid generated by revolving the region b/w the y-axis & curve $x = 2/y$, $y \in [1,4]$ about y-axis.

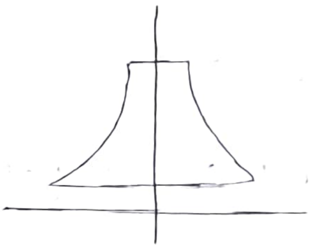
Sol^n) $V = \int_1^4 \pi \left(\frac{2}{y}\right)^2 dy.$

$V = \underline{\underline{3\pi}}$

$x = 2/y$



⇓



→ Solid Revolution (Washer Method).

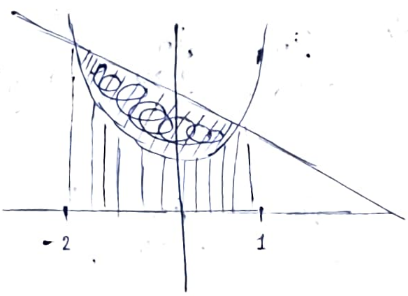
$\text{Area} = \pi\left[R(x)\right]^2 - \pi\left[r(x)\right]^2$ where $R(x) \rightarrow$ outer radius

$r(x) \rightarrow$ inner radius

$\text{Volume} = \int_a^b \pi\left[\left[R(x)\right]^2 - \left[r(x)\right]^2\right] dx.$

⇒ The region bounded by the curve, $y = x^2+1$, $y = -x+3$ is revolved about the x-axis to generate a solid. Find the volume of solid.

Sol^n) $V = \int_{-2}^{1} \pi\left[(-x+3)^2 - (x^2+1)^2\right] dx$



$V = \underline{\underline{\frac{117\pi}{5}}}$