MODULE - 6

* Vector fn.
* Scalar funᶜ
* Gradient
  - → find normal to surface ✓
  - → angle b/w the surface ✓
  - → directional derivative.
* Divergence
  - → Simply find divergence
  - → Check solanoidal / not.
* Curl.
  - → Irrotational or not
  - → Find scalar potential.

→ GRADIENT : (∇)

$$\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

⇒ If $u = x+y+z$, $v = x^2+y^2+z^2$, $w = yz + xz + xy$. PT. grad u, grad v and grad w are coplanar.

Solⁿ

$$\nabla u = \hat{i} + \hat{j} + \hat{k}$$

$$\nabla v = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\nabla w = (y+z)\hat{i} + (x+z)\hat{j} + (x+y)\hat{k}$$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & x+y \end{vmatrix} = \underline{0} \quad \text{(coplanar)}$$

⇒ Find a unit normal to surface $xy^3z^2 = 4$ at $(-1,-1,2)$

Solⁿ

$$\nabla = y^3z^2\hat{i} + 3xy^2z^2\hat{j} + 2xy^3z\hat{k}$$

$$\nabla = \hat{i} + 3\hat{j} - \hat{k}$$

$$\therefore \text{unit vector} = \frac{\hat{i} + 3\hat{j} - \hat{k}}{\sqrt{11}}$$

⇒ Find the constants 'm' & 'n' such that the surface $mx^2 - 2nyz = (m+4)x$ will be orthogonal to surface $4x^2y + z^3 = 4$ at $(1,-1,2)$.

Sol^n) ∇ gives normal to a surface.

$\therefore$ $(1,-1,2)$ lies on both surfaces :

$$m + 4n = m + 4$$

$$\Rightarrow n = 1$$

Let $\phi_1 = mx^2 - 2yz - (m+4)x$

$\phi_2 = 4x^2y + z^3 - 4$

$\therefore$ normal to $\phi_1 = \nabla\phi_1$

$$= \hat{i}(2mx - m - 4) - 2z\hat{j} - 2y\hat{k}$$

$$= (m-4)\hat{i} - 4\hat{j} + 2\hat{k}$$

normal to $\phi_2 = \nabla\phi_2 = -8\hat{i} + 4\hat{j} + 12\hat{k}$

$\therefore$ $\Delta\phi_1 \cdot \Delta\phi_2 = 0$

$$\Rightarrow m = 5 \quad , \quad n = 1$$

⇒ Find value of λ & μ so that surfaces $\lambda x^2 - \mu yz = (\lambda+2)x$, $4x^2y + z^3 = 4$ intersect orthogonally at point $(1,-1,2)$

Sol^n) ~~$\phi_1 = \lambda x^2 - \mu yz - (\lambda+2)x$~~

$\phi_1 = \lambda x^2 - \mu yz - (\lambda+2)x$

$\phi_2 = 4x^2y + z^3 - 4$

$\therefore$ $\nabla\phi_1$ at $(1,-1,2) = \hat{i}(\lambda-2) + \hat{j}\,z(2\mu) + \hat{k}\mu$.

$\nabla\phi_2$ at $(1,-1,2) = -8\hat{i} + 4\hat{j} + 12\hat{k}$

$\therefore$ $\nabla\phi_1 \cdot \nabla\phi_2 = 0 \Rightarrow 2\lambda - 5\mu = 4$

point $(1,-1,2)$ satisfy

$$\lambda(1)^2 - \mu(-1)(2) = (\lambda+2)(1) \Rightarrow \mu = 1.$$

$$\therefore\ 2\lambda - 5 = 4 \rightarrow \lambda = 9/2$$

⇒ Find the angle b/w surfaces $x^2+y^2+z^2=9$, $z = x^2+y^2-3$ at point $(2,-1,2)$

Sol:

$$\nabla\phi_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k} \longrightarrow 4\hat{i} - 2\hat{j} + 4\hat{k}$$

$$\nabla\phi_2 = 2x\hat{i} + 2y\hat{j} - \hat{k} \longrightarrow 4\hat{i} - 2\hat{j} - \hat{k}$$

$$\therefore\ \nabla\phi_1 \cdot \nabla\phi_2 = \sqrt{16+4+16}\ \sqrt{16+4+1}\ \cos\theta$$

$$\Rightarrow 16+4-4 = 6\sqrt{21}\cos\theta$$

$$\Rightarrow \cos\theta = \frac{8}{3\sqrt{21}}$$

$$\Rightarrow \theta = \cos^{-1}\frac{8}{3\sqrt{21}}$$

⇒ Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,1)$ in the direction of $2\hat{i} - \hat{j} - 2\hat{k}$. Find the greatest rate of increase of ϕ

Sol:

$$\text{directional derivative} = \nabla\phi\ \frac{\vec{U}}{|\vec{U}|}$$

$$\nabla\phi = (2xyz + 4z^2)\hat{i} + (x^2z)\hat{j} + (x^2y + 8xz)\hat{k}$$

$$\nabla\phi\,(1,-2,1) = \hat{j} + 6\hat{k}$$

$$\therefore\ \hat{U} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{9}}$$

$\therefore \nabla\phi \cdot \hat{u}$

$= (\hat{j} + 6\hat{k}) \cdot \frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k}) = \frac{-13}{3}$

Greatest rate of increase of $\phi = |\hat{j} + 6\hat{k}| = \sqrt{37}$

⇛ Find the directional derivative of $(\vec{v})^2$, where $\vec{v}$.

$\vec{v} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$, at the point $(2,0,3)$ in dirn of outward normal to sphere $x^2+y^2+z^2 = 14$ at the point $(3,2,1)$.

Soln) $(\vec{v})^2 = \vec{v}\cdot\vec{v} = x^2y^4 + z^2y^4 + x^2z^4$

~~direction derivative~~ $\nabla v^2_{(2,0,3)} = 324\hat{i} + 432\hat{k}$

~~at (2,0,3)~~

Normal to sphere ⇒ $\nabla(x^2+y^2+z^2-14)$

$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

at $(3,2,1) = 6\hat{i} + 4\hat{j} + 2\hat{k}$

Unit normal vector $= \frac{6\hat{i}+4\hat{j}+2\hat{k}}{\sqrt{56}}$

$\begin{array}{r} 36 \\ 16 \\ 4 \\ \hline 56 \end{array}$

$\therefore$ Directional derivative $= 108(3\hat{i} + 4\hat{k}) \cdot \frac{6\hat{i}+4\hat{j}+2\hat{k}}{\sqrt{56}}$

$= \frac{1404}{\sqrt{14}}$ (Ans)

⇒ Find the dirⁿ derivative of function $\phi = x^2 - y^2 + 2z^2$ at point P (1,2,3) in dirⁿ of line PQ where Q is point (5,0,4)

Solⁿ
$\nabla\phi\ (1,2,3) = 2\hat{i} - 4\hat{j} + 12\hat{k}$

$\therefore \quad \vec{PQ} = 4\hat{i} - 2\hat{j} + \hat{k}$

$\therefore \quad \hat{PQ} = \dfrac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}}$

$\therefore$ Dirⁿ derivative $= (2\hat{i} - 4\hat{j} + 12\hat{k}) \dfrac{(4\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{21}} = \dfrac{28}{\sqrt{21}}$

⇒ Find the dirⁿ derivative of $\nabla(\nabla f)$ at point (1,-2,1) in dirⁿ of normal to surface $xy^2z = 3x + z^2$, where $f = 2x^3y^2z^4$.

Solⁿ
$\nabla f = 6x^2y^2z^4\ \hat{i} + 4x^3yz^4\ \hat{j} + 8x^3y^2z^3\ \hat{k}$

$\cancel{\nabla(\nabla f) = 12xy^2z^4}$

$\Rightarrow$ P.T. $(\nabla^{2} f(r))^{2} = f''(r) + \frac{2}{r} f'(r)$

Sol^n) $\nabla f(r) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) f(r)$. imp

$r^2 = x^2 + y^2 + z^2$

$\therefore 2r \frac{dr}{dx} = 2x \Rightarrow \frac{dr}{dx} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$

$\therefore \hat{i} f'(r) \cdot \frac{dr}{dx} + \hat{j} f'(r) \frac{\partial r}{\partial y} + \hat{k} f'(r) \frac{\partial r}{\partial z}$.

$\Rightarrow f'(r) \left[\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r}\right]$

$\therefore \nabla^2 f(r) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left[f'(r)\, \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r}\right]$

$= \frac{\partial}{\partial x}\left[f'(r)\frac{x}{r}\right] + \frac{\partial}{\partial y}\left[f'(r)\frac{y}{r}\right] + \frac{\partial}{\partial y}\left[f'(r)\frac{z}{r}\right]$

$= f''(r) \frac{dr}{dx} + f'(r)\left(\frac{r - x\frac{\partial r}{\partial x}}{r^2}\right) + \ldots\ldots$

$= f''(r)\left[\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2}\right] + f'(r)\left[\frac{y^2+z^2}{r^3} + \frac{z^2+x^2}{r^3} + \frac{x^2+y^2}{r^3}\right]$

$= f''(r)\left[\frac{x^2+y^2+x^2}{r^2}\right] + f'(r)\,\frac{2(x^2+y^2+z^2)}{r^3}$

$= f''(r) + f'(r)\frac{2}{r}$

DIVERGENCE

* Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\cdot\left[F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right]$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

* if $\vec{\nabla}\cdot\vec{F} = 0$ (solenoidal).

means no source nor sink.

CURL :

* $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

* if $\vec{\nabla}\times\vec{F} = 0$ (irrotational)

↳ its free from rotational motion incase of fluid flow.

⇒ $\vec{v} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$, find div $\vec{v}$.

Sol:

$$\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(\vec{v})$$

$$\frac{\partial\vec{v}}{\partial x} + \frac{\partial\vec{v}}{\partial y} + \frac{\partial\vec{v}}{\partial z} \Rightarrow \frac{2}{\sqrt{x^2+y^2+z^2}}$$

$\Rightarrow$ Find the value of 'n' for which the vector $r^n \vec{r}$ is solenoidal, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Soln)

$$\vec{\nabla} \cdot \vec{F} = \nabla r^n \vec{r} = \nabla \cdot (x^2+y^2+z^2)^{n/2} (x\hat{i}+y\hat{j}+z\hat{k})$$

$$\Rightarrow \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right] \cdot (x^2+y^2+z^2)^{n/2} (x\hat{i}+y\hat{j}+z\hat{k})$$

$$= (n+3)(x^2+y^2+z^2)^{n/2} = 0 \longrightarrow \text{for solenoidal}$$

$$\Rightarrow \underline{\underline{n = -3}} \text{ (Ans).}$$

$\Rightarrow$ P.T. $\text{div}(\text{grad } r^n) = n(n+1) r^{n-2}$, where $r = \sqrt{x^2+y^2+z^2}$

Soln)

$$\text{grad } r^n = \hat{i}\frac{\partial r^n}{\partial x} + \hat{j}\frac{\partial r^n}{\partial y} + \hat{k}\frac{\partial r^n}{\partial z}$$

$$= \hat{i}\, n r^{n-1}\frac{\partial r}{\partial x} + \hat{j}\, n r^{n-1}\frac{\partial r}{\partial y} + \hat{k}\, n r^{n-1}\frac{\partial r}{\partial z}$$

$$= n r^{n-2}(x\hat{i}+y\hat{j}+z\hat{k}) = \underline{n r^{n-2}\vec{r}}$$

$$\text{div}\left(n r^{n-2}(x\hat{i}+y\hat{j}+z\hat{k})\right) = \underline{n(n+1) r^{n-2}} \text{ (Ans)}$$

$\Rightarrow$ P.T. $\nabla\left[\dfrac{(\vec{a}\cdot\vec{r})}{r^n}\right] = \dfrac{\vec{a}}{r^n} - \dfrac{n(\vec{a}\cdot\vec{r})\vec{r}}{r^{n+2}}$

Soln)

$$\frac{\vec{a}\cdot\vec{r}}{r^n} = \frac{(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\cdot(x\hat{i}+y\hat{j}+z\hat{k})}{(x^2+y^2+z^2)^{n/2}} = \frac{a_1x+a_2y+a_3z}{r^n}$$

$$\Rightarrow \frac{a_1x+a_2y+a_3z}{r^n}$$

$$\frac{\partial \phi}{\partial x} = \frac{r^n a_1 - (a_1 x + a_2 y + a_3 z)\, n r^{n-1} (\partial r/\partial x)}{r^{2n}}$$

$$r^2 = x^2 + y^2 + z^2 \Rightarrow 2r \frac{dr}{dr} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}.$$

$$\frac{\partial \phi}{\partial x} = \frac{a_1}{r^n} - \frac{n(a_1 x + a_2 y + a_3 z)x}{r^{n+2}}$$

$$\therefore \nabla \phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$= \frac{\vec{a}}{r^n} - \frac{n}{r^{n+2}}(\vec{a}\cdot\vec{r})\vec{r} \quad \text{(Ans)}.$$

⇒ Find the dir^n derivative of div$(\vec{u})$ at point $(1,2,2)$ in dir^n of outer normal of sphere $x^2+y^2+z^2 = 9$ for $\vec{U} = x^4\hat{i} + y^4\hat{j} + z^4\hat{k}$

Sol^n div $\vec{u} = \vec{\nabla}\cdot\vec{U}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(x^4\hat{i} + y^4\hat{j} + z^4\hat{k}\right)$$

$$= 4x^3 + 4y^3 + 4z^3$$

$\therefore$ normal to sphere : $\nabla(x^2+y^2+z^2-9)$

$$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

at $(1,2,2)$ $= 2\hat{i} + 4\hat{j} + 4\hat{k}$

$$\therefore \nabla(\vec{\nabla}\cdot\vec{U}).$$

$$\nabla(4x^3 + 4y^3 + 4z^3) \cdot \frac{2\hat{i} + 4\hat{j} + 4\hat{k}}{\sqrt{36}}$$

$$= (12x^2\hat{i} + 12y^2\hat{j} + 12z^2\hat{k}) \cdot \frac{(2\hat{i} + 4\hat{j} + 4\hat{k})}{6}$$

$$= \underline{68}$$

⇒ if $\vec{V} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$, $\vec{\nabla} \times \vec{V}$.

Sol^n)

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{x}{\sqrt{x^2+y^2+z^2}} & \dfrac{y}{\sqrt{x^2+y^2+z^2}} & \dfrac{z}{\sqrt{x^2+y^2+z^2}} \end{vmatrix} = \underline{\underline{0}}.$$

⇒ Determine a, b, such that curl of vector

$$\vec{A} = (2xy + 3yz)\hat{i} + (x^2 + axz - 4z^2)\hat{j} - (3xy + byz)\hat{k} = 0$$

Sol^n)

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy + 3yz & x^2 + axz - 4z^2 & -(3xy + byz) \end{vmatrix} = 0$$

$$\hat{i}\left[-x(3+a) + z(8-b)\right] + 6y\,\hat{j} + z(-3+a)\,\hat{k} = 0.$$

$\therefore$ $a+3=0$, $a-3=0$, $8-b=0$

$\underline{\underline{a = -3, 3}}$, $\underline{\underline{b = 8}}$.

→ Scalar potential funcⁿ (ϕ) :

$$\vec{F} = \nabla\phi \quad , \quad \text{find } \phi \,?$$

$$\Rightarrow d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$\therefore \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\left(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz\right)$$

$$\therefore d\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi \,(d\vec{r})$$

$$d\phi = \nabla\phi\, d\vec{r}$$

$$d\phi = F\, d\vec{r}$$

$$\phi = \int \vec{F}\cdot d\vec{r} \quad , \quad \text{let } \vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$$

$$\therefore \boxed{\phi = \int F_1\,dx + \int F_2\,dy + \int F_3\,dz}$$

$$\Rightarrow \int y\,dz + z\,dy \longrightarrow d(yz) \longrightarrow yz + c$$

⇒ ~~when~~

⇒ If vector field is given by :

$\vec{F} = (x^2 - y^2 + x)\,\hat{i} - (2xy + y)\,\hat{j}$, Is $\vec{F}$ irrotational . If so, find scalar potential ?

Solⁿ)
$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-y^2+x & -(2xy+y) & 0 \end{vmatrix} = 0$$ (Hence irrotational)

$$\vec{F} = \nabla\phi$$

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$\phi = \int \vec{F}\cdot d\vec{r}$$

$$\phi = \int \left[(x^2 + y^2 + x)\,\hat{i} - (2xy + y)\,\hat{j} \right] \cdot \left[\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz \right]$$

$$\phi = \int x^2\,dx + x\,dx - y\,dy \underbrace{- y^2\,dx - 2xy\,dy}_{-d(xy^2)}$$

$$\phi = \frac{x^3}{3} + \frac{x^2}{2} - \frac{y^2}{2} - xy^2 + c$$

⇒ Find scalar potential of $\vec{A} = y^2\,\hat{i} + 2xy\,\hat{j} - z^2\,\hat{k}$

Sol^n

$$\phi = \int y^2\,dx + 2xy\,dy - z^2\,dz$$

$$\phi = xy^2 - \frac{z^3}{3} + c$$

Always ~~find~~ to find scalar $\nabla \times \vec{F}$ must be zero

so

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & 2xy & -z^2 \end{vmatrix} = 0$$

⇒ $\vec{V} = 2xyz\,\hat{i} + (x^2 z + 2y)\,\hat{j} + x^2 y\,\hat{k}$ ⟶ Find irrotational / not and scalar func^n.

Sol^n

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz & x^2 z + 2y & x^2 y \end{vmatrix} = 0 \text{ (irrotational)}$$

$$\therefore \phi = \int \vec{v} \cdot d\vec{r} = \int 2xyz\,dx + x^2 z\,dy + 2y\,dy + x^2 y\,dz$$

$$= x^2 yz + y^2 + c$$

$\Rightarrow$ $\vec{v} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$, irrotational / not ?, velocity potential

Sol^n
$$\nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix} = 0 \quad (\text{irrotational})$$

$$\therefore \quad \phi = \int \vec{F}\cdot d\vec{r} = \int y\,dx + z\,dx + z\,dy + x\,dy + x\,dz + y\,dz$$

$$\phi = \int d(xy) + \int d(xz) + \int d(yz)$$

$$\phi = xy + xz + yz + c.$$

$\Rightarrow$ P.T. $\vec{F} = r^2\vec{r}$ is conservative. Find scalar potential.

Sol^n $\vec{F} = \underbrace{(x^2+y^2+z^2)}_{r^2}(x\hat{i} + y\hat{j} + z\hat{k})$.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^2x & r^2y & r^2z \end{vmatrix} = \hat{i}\left[\frac{\partial r^2 z}{\partial y} - \frac{\partial r^2 y}{\partial z}\right] - \hat{j}(\text{—}) + \hat{k}(\text{—})$$

$$r^2 = x^2 + y^2 + z^2$$

$$2r\frac{dr}{dx} = 2x \Rightarrow \frac{dr}{dx} = \frac{x}{r}.$$

$$\hat{i}\left[2rz\frac{y}{r} - 2ry\frac{z}{r}\right] - \hat{j}(\text{—}) + \hat{k}(\text{—}) = \underline{\underline{0}}$$

conservative $\rightarrow$ $\nabla \times \vec{F} = 0$

$$\therefore \quad \phi = \int \vec{F}\cdot d\vec{r} = \int r^2x\,dx + r^2y\,dy + r^2z\,dz$$

$$= \int r^3\,dr + \int r^3\,dr + \int r^3\,dr$$

$$\frac{3r^4}{4} + c$$

$$= \frac{3(x^2+y^2+z^2)^4}{4} + c.$$

$$\phi = \int \vec{F}\cdot d\vec{r} = \int (x^2+y^2+z^2)(x\hat{i}+y\hat{j}+z\hat{k})\cdot(\hat{i}dx + y\hat{j}dy + \hat{k}dz)$$

$$\phi = \int x^3 dx + \int y^3 dy + \int z^3 dz + \int (x\,dx)y^2 + (y\,dy)x^2$$

$$+ \int (x\,dx)z^2 + (z\,dz)x^2 + \int \text{———}$$

$$\phi = \frac{1}{4}\left(x^4\, y^4 + z^4 + 2x^2y^2 + 2x^2z^2 + 2y^2z^2\right) + C$$

⇒ Find a, b, c, so that

$\vec{F} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$ is irrotational

also find scalar funcn.

Soln $\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+2y+az) & (bx-3y-z) & (4x+cy+2z) \end{vmatrix} = 0$

$$= \hat{i}(c+1) - \hat{j}(4-a) + \hat{k}(b-2) = 0$$

$$\therefore\ c = -1,\ a = 4,\ b = 2$$

$$\vec{F} = (x+2y+4z)\hat{i} + (2x-3y-z)\hat{j} + (4x-y+2z)\hat{k}$$

$$\phi = \int \vec{F}\cdot d\vec{r} = \int x\,dx + \int 2y\,dx + \int 4z\,dx + \int 2x\,dy - \int 3y\,dy - \int z\,dy$$

$$+ \int 4x\,dz + - \int y\,dz + \int 2z\,dz$$

$$\phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4xz - yz + C$$

$\Rightarrow$ P.T. $\vec{F} = \dfrac{\vec{r}}{|\vec{r}|^3}$ is irrotational and solenoidal. Scalar potential $= ?$

Sol^n $\vec{F} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^{3/2}}$

$\nabla \times \vec{F} = \left| \quad \right| = 0$ (Proved)

$\nabla \cdot \vec{F} = \text{———} = 0$ (Proved)

$\therefore \phi = \int \vec{F} \cdot d\vec{r}$

$$\phi = \frac{x}{(x^2+y^2+z^2)^{3/2}}dx + \frac{y}{(x^2+y^2+z^2)^{3/2}}dy + \frac{z}{(x^2+y^2+z^2)^{3/3}}dz$$

Let $x^2 + y^2 + z^2 = t$

$2x\,dx = dt$, $2y\,dy = dt$, $2z\,dz = dt$

imp.

$$\phi = \frac{1}{2}\left[\frac{2x\,dx + 2y\,dy + 2z\,dz}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$\phi = \frac{1}{2}\left[\frac{dt}{t^{3/2}} + \frac{dt}{t^{3/2}} + \frac{dt}{t^{3/2}}\right]$$

$$\phi = \frac{1}{2}\left[-2t^{-1/2} - 2t^{-1/2} - 2t^{-1/2}\right]$$

$$\phi = \frac{-1}{\sqrt{x^2+y^2+z^2}} = \frac{-1}{|\vec{r}|}$$ (Ans)