BETA & GAMMA FUNC^N : Module 5

→ BETA FUNC^C :

$$\boxed{\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx}, \quad m,n > 0$$

→ Gamma FUN^C :

$$\boxed{\Gamma m = \int_0^\infty e^{-x}\, x^{m-1}\, dx}$$

⇒ $\beta(m,n) = ?$, $x = \sin^2\theta$

Soln

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} (\cos^2\theta)^{n-1} \sin 2\theta \, d\theta$$

rem.

$\cancel{\int (\sin^2\theta)^{m-1}}$

General formula

$$\boxed{2\int_0^{\pi/2} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1}\, d\theta}$$

rem

* ⇒ $$\boxed{\beta(m,n) = \frac{\Gamma m\, \Gamma n}{\Gamma m+n}}$$ ← imp

*

$$\Gamma n+1 = n\Gamma n$$

(or)

$$\Gamma n+1 = n!$$

$$\Gamma \tfrac{1}{2} = \sqrt{\pi}$$

learn proves of these formula ← very imp.

⇒ $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \cdot \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}}$

Soln: $\frac{1}{2}\left[2\int_0^{\pi/2} \sin^{1/2}\theta \cdot \cos^{2(\frac{1}{2})-1}\theta\, d\theta\right]$ (from above)

∴ $2m-1 = \frac{1}{2}$, $n = \frac{1}{2}$

$m = \frac{3}{4}$

∴ $\frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{2}\right) = \frac{1}{2}\,\frac{\Gamma \frac{3}{4}\, \Gamma \frac{1}{2}}{\Gamma \frac{5}{4}}$ (Ans)

$$\int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}}\, d\theta = \frac{1}{2}\left[2\int_0^{\pi/2} \sin^{-1/2}\theta \cos^{2(\frac{1}{2})-1} d\theta\right]$$

$2m-1 = -\frac{1}{2}$, $n = \frac{1}{2}$

$m = \frac{1}{4}$

$$\therefore \frac{1}{2}\beta\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{1}{2} \frac{\Gamma\frac{1}{4}\,\Gamma\frac{1}{2}}{\Gamma\frac{3}{4}}$$

$$\therefore \frac{1}{2}\frac{\Gamma\frac{3}{4}\,\Gamma\frac{1}{2}}{\Gamma\frac{5}{4}} \cdot \frac{1}{2}\frac{\Gamma\frac{1}{4}\,\Gamma\frac{1}{2}}{\Gamma\frac{3}{4}} = \frac{1}{4}\frac{\left(\Gamma\frac{1}{2}\right)^2 \Gamma\frac{1}{4}}{\Gamma\frac{5}{4}}$$

$$= \frac{1}{4}\frac{\pi\,\Gamma\frac{1}{4}}{\Gamma\left(\frac{1}{4}+1\right)} = \frac{1}{4}\frac{\pi\,\Gamma\frac{1}{4}}{\frac{1}{4}\Gamma\frac{1}{4}}$$

$$= \underline{\pi} \text{ (Ans)}$$

imp

$\Rightarrow$ PT : $\Gamma\frac{1}{2} = \sqrt{\pi}$

Soln

$$\Gamma m = \int_0^\infty e^{-x} x^{m-1}\, dx$$

$$\Gamma\frac{1}{2} = \int_0^\infty e^{-x} x^{-1/2}\, dx \quad , \quad \text{let } x = t^2$$

$dx = 2t\, dt$

$$\Gamma\frac{1}{2} = \int_0^\infty e^{-t^2} t^{-1}(2t)\, dt$$

$$\Gamma\frac{1}{2} = 2\int_0^\infty e^{-t^2} dt$$

Similarly : $$\Gamma\frac{1}{2} = \int_0^\infty e^{-y} y^{-1/2}\, dy$$

$$\Rightarrow \Gamma\frac{1}{2} = 2\int_0^\infty e^{-s^2} ds$$

$$\left(\Gamma\frac{1}{2}\right)^2 = 4\int_0^\infty\int_0^\infty e^{-(t^2+s^2)}\, dt\, ds.$$

$t = r\cos\theta$, $s = r\sin\theta$

$$\Rightarrow 4 \int_0^{\pi/2} \int_0^{\infty} e^{-r^2}\, r\, dr\, d\theta \quad \begin{matrix} r \to \infty & \pi/2 & r\theta \end{matrix}$$

$$\Rightarrow 4 \int_0^{\pi/2} d\theta \int_0^{\infty} e^{-r^2}\, r\, dr \quad = \quad \pi$$

$$\therefore \quad \Gamma\tfrac{1}{2} = \sqrt{\pi}$$

$\Rightarrow$ Evaluate : $\int_0^1 x^4 (1-\sqrt{x})^5\, dx$

Sol^n let $\sqrt{x} = t$

$x = t^2 \Rightarrow dx = 2t\, dt$

$$2\int_0^1 t^{\cancel{8}9} (1-t)^5\, dt$$

$\therefore$ $m-1 = 9,\ n-1 = 5$

$m = 10,\ n = 6.$

$$\therefore 2\beta(10,6) = 2\,\frac{\Gamma 10\, \Gamma 6}{\Gamma 16}$$

$$= \frac{2(9!)(5!)}{15!}$$

$$= \frac{2(5!)}{15\times14\times13\times12\times11\times10} = \frac{1}{15015}$$

$\Rightarrow$ Evaluate : $\int_0^1 (1-x^3)^{-1/2}\, dx.$

Sol^n let $x^3 = y \Rightarrow \cancel{y}\ x = y^{1/3}$.

$dx = \frac{1}{3} y^{-2/3}\, dy.$

$$\therefore \int_0^1 \tfrac{1}{3} y^{-2/3} (1-y)^{-1/2}\, dy$$

$m-1 = -2/3,\ n-1 = -1/2$

$m = 1/3,\ n = 1/2.$

$$\therefore \tfrac{1}{3}\beta(1/3, 1/2)$$

~~$\Rightarrow$ If $u^3+v^3 = x+y$, $u^2+v^2 = x^3+y^3$~~ PT: $\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{2}\frac{(y^2-x^2)}{uv(u-v)}$

DIRICHLET'S INTEGRAL :

PT: $$\iint_D x^{l-1} y^{m-1}\, dx\, dy = \frac{\Gamma l\, \Gamma m}{\Gamma l+m+1}\, h^{l+m}, \quad h>0.$$

D is the region, $x \geq 0,\ y \geq 0,\ x+y \leq h.$

$\therefore$ $x+y \leq h$

$\frac{x}{h} + \frac{y}{h} \leq 1$

put $X = \frac{x}{h},\ Y = \frac{y}{h}$

$\Rightarrow x = Xh \ ;\ y = Yh$

$dx = h\, dX,\quad dy = h\, dY$

LHS: $\iint (xh)^{l-1} (yh)^{m-1} h^2 \, dx \, dy$

$\Rightarrow h^{l+m} \iint x^{l-1} y^{m-1} \, dy \, dx$

D' is domain

$x>0, \; y>0, \; x+y \le 1$

$\Rightarrow h^{l+m} \int_{x=0}^{x=1} x^{l-1} \int_{y=0}^{1-x} y^{m-1} \, dy \, dx$

$\Rightarrow \frac{h^{l+m}}{m} \underbrace{\int_0^1 x^{l-1} (1-x)^{(m+1-1)} \, dx}_{\beta}$

$\because \frac{\Gamma m+1}{m} = \frac{m\Gamma m}{m} = \Gamma m$

$= \frac{h^{l+m}}{m} \frac{\Gamma l \, \Gamma m+1}{\Gamma l+m+1} = h^{l+m} \frac{\Gamma l \, \Gamma m}{\Gamma l+m+1}.$

$\Rightarrow$ Find the area of $x^{2/3} + y^{2/3} = a^{2/3}$. Using Beta & Gamma fn.

Sorry

Beta and Gamma

Gamma Funcⁿ :

$$\Gamma m = \int_0^\infty e^{-x} x^{m-1} dx$$

* Prove : $\Gamma 1 = 1$

Sol^n $$\Gamma 1 = \int_0^\infty e^{-x} dx$$

$$\Gamma 1 = -\left[e^{-x}\right]_0^\infty = \underline{\underline{1}}.$$

* Prove : $\Gamma n+1 = n\Gamma n$

Sol^n $\cancel{\Gamma n+1 = \int_0^\infty e^{-x} x^n dx}$

$$\lim_{x\to 0} \frac{x^{n-1}}{e^x} = \lim_{x\to 0} 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \dots + \frac{x^n}{\lfloor n} + \dots + x^{n-1}$$
$$\| \quad \underline{\underline{0}}.$$

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$$

Integration by parts :

$$\Gamma n = \left[x^{n-1}\left(\frac{e^{-x}}{-1}\right)\right]_0^\infty - (n-1)\int_0^\infty x^{n-2}\left(\frac{e^{-x}}{-1}\right) dx$$

$$\Gamma n = (n-1)\int_0^\infty x^{n-2}(e^{-x}) dx$$

$$\Gamma n = (n-1)\Gamma n-1$$

replace, $n \to n+1$

$$\underline{\underline{\Gamma n+1 = n\Gamma n}} \quad (\text{Ans}).$$

* Prove $\Gamma n+1 = \lfloor n$

Sol^n: in pre que:

$$\Gamma n = (n-1)\,\Gamma n-1$$

replace, $n \to n-1$

$$\Gamma n-1 = (n-2)\,\Gamma n-2$$

$$\therefore \Gamma n = (n-1)\,\Gamma n-1 = (n-1)(n-2)\,\Gamma n-2$$

$$= (n-1)(n-2)(n-3)\,\Gamma n-3$$

$$= (n-1)(n-2)(n-3)(n-4) \cdots\cdots \Gamma 1$$

$$\Gamma n = \lfloor n-1$$

$$\Gamma n+1 = \lfloor n \quad \text{(Proved).}$$

$\Rightarrow I = \int_0^\infty x^{1/4} e^{-\sqrt{x}}\, dx.$

Sol^n: $\sqrt{x} = t$, $x = t^2$, $dx = 2t\,dt$

$$I = \int_0^\infty t^{1/2} e^{-t} (2t\,dt).$$

$$I = 2\int_0^\infty t^{3/2} e^{-t}\, dt = 2\int_0^\infty e^{5/2-1} e^{-t}\, dt$$

$$I = 2\,\Gamma 5/2$$

$$I = 2\left(\frac{3}{2}\right)\Gamma 3/2 = 2\left(3/2\right)\left(1/2\right)\Gamma 1/2$$

$$= \frac{3\sqrt{\pi}}{2}$$

$\Rightarrow \int_0^\infty x^{n-1} e^{-h^2x^2}\, dx$

Sol^n: $h^2x^2 = t$

$2h^2x\, dx = dt$

$$\Rightarrow \int_0^\infty \left(\frac{\sqrt{t}}{h}\right)^{n-1} e^{-t} \left(\frac{dt\,h}{2h^2\sqrt{t}}\right)$$

$$\Rightarrow \frac{1}{2h^{n-1}.h} \int_0^\infty t^{\frac{n-2}{2}} e^{-t}\, dt$$

$$\Rightarrow \frac{1}{2h^n}\int_0^\infty e^{-t}\, t^{\frac{n}{2}-1}\, dt$$

$m-1 = \frac{n-2}{2}$

$m = \frac{n-2}{2}+1$

$\frac{n}{2}$

$$= \frac{1}{2h^n}\Gamma\frac{n}{2} \quad (\text{Ans}).$$

$$\Rightarrow \int_0^\infty \frac{x^a}{a^x}\, dx$$

Sol^n $a^x = e^t$

$x \log a = t$

$\log a \, dx = dt$

$$\Rightarrow \int_0^\infty \left(\frac{t}{\log a}\right)^a \cdot e^{-t} \left(\frac{dt}{\log a}\right)$$

$$= \frac{1}{(\log a)^{a+1}}\int_0^\infty t^a \cdot e^{-t}\, dt$$

$$= \frac{1}{(\log a)^{a+1}}\Gamma{a+1} \quad (\text{Ans})$$

Transformation of Gama Func^n :

* $\int_0^\infty e^{-ky}\, y^{n-1}\, dy = \frac{\Gamma n}{k^n}$ (✓)

* $\Gamma\frac{1}{2} = \sqrt{\pi}$ (✓) * $\int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy = \Gamma n$ (✗)

* Let $x = ky$.

$dx = k\,dy$

$$\int_0^\infty e^{-ky}\,(ky)^{n-1} \cdot (k\,dy) = \Gamma n$$

$$\Rightarrow k^n \int_0^\infty e^{-ky}\,(y)^{n-1}\, dy = \Gamma n$$

$$\Rightarrow \int_0^\infty e^{-ky}\,(y)^{n-1}\, dy = \frac{\Gamma n}{k^n}$$

*

* Let $x^n = y$:

$nx^{n-1}\, dx = dy$.

$$\Gamma n = \int_0^\infty e^{-x}\, x^{n-1}\, dx$$

$$\Gamma n = \int_0^\infty e^{-y^{1/n}} \left(\frac{dy}{n}\right) = \frac{1}{n}\int_0^\infty e^{-y^{1/n}}\, dy$$

$$\therefore \Gamma \tfrac{1}{2} = \frac{1}{1/2}\int_0^\infty e^{-y^2}\, dy = 2\int_0^\infty e^{-y^2}\, dy$$

$$= 2\left[\frac{\sqrt{\pi}}{2}\right] = \sqrt{\pi}$$

$$\therefore \Gamma \tfrac{1}{2} = \sqrt{\pi}$$

BETA FUNC^N :

$$\beta(l,m) = \int_0^1 x^{l-1}(1-x)^{m-1}\, dx$$

* $\beta(l,m) = \dfrac{\Gamma l\, \Gamma m}{\Gamma l+m}$

* Property : $\beta(l,m) = \beta(m,l)$.

$\Rightarrow \displaystyle\int_0^1 x^4 (1-\sqrt{x})^5\, dx$

Sol^n/ $\sqrt{x} = t$

$x = t^2$

$dx = 2t\, dt$

$$\int_0^1 t^8 (1-t)^5\, 2t\, dt$$

$$2\int_0^1 t^9 (1-t)^5\, dt$$

$$2\beta(10,6) = 2\frac{\Gamma 10\, \Gamma 6}{\Gamma 16} = \frac{2\,(9!)\,(5!)}{15!} = \frac{1}{15015}$$

Trasnformation of Beta Func^n :

$$\beta(l,m) = \int_0^1 x^{l-1}(1-x)^{m-1}\,dx. \qquad \alpha$$

$$\therefore \beta(l,m) = \int_0^\infty \frac{y^{l-1}}{(1+y)^{m+l}}\,dy = \int_0^\infty \frac{x^{l-1}}{(1+x)^{m+l}}\,dx$$

$$\Rightarrow \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx \qquad \alpha.$$

Sol^n)

Relation B/w Beta And Gamma :

$$* \ \beta(l,m) = \frac{\Gamma l\,\Gamma m}{\Gamma l+m}.$$

$$* \int_0^{\pi/2} \sin^p\theta \cos^q\theta\,d\theta = \frac{\Gamma\left[\frac{p+1}{2}\right]\Gamma\left[\frac{q+1}{2}\right]}{2\Gamma\left[\frac{p+q+2}{2}\right]}.$$

$$\Rightarrow \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Let $x = \sin^2\theta$, $\int_0^{\pi/2} (\sin\theta)^{2m-2}(\cos\theta)^{2n-2}\, 2\sin\theta\cos\theta\,d\theta$

$dx = \sin 2\theta\,d\theta$

$$\Rightarrow 2\int_0^{\pi/2} (\sin\theta)^{2m-1}(\cos\theta)^{2n-1}\,d\theta$$

Let $2m-1 = p$, $2n-1 = q$.

$$\Rightarrow \int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta = \frac{\beta(m,n)}{2} = \frac{\Gamma m\,\Gamma n}{2\Gamma m+n}$$

$$= \frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\Gamma\frac{p+q+2}{2}} \qquad \text{(Proved)}$$

$\Rightarrow \Gamma\frac{1}{2} = \sqrt{\pi}$ (Prove)

Sol^n $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \dfrac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}}$.

Let $p=0$, $q=0$.

$\int_0^{\pi/2} d\theta = \dfrac{\left(\Gamma\frac{1}{2}\right)^2}{2} \Rightarrow \left(\frac{\pi}{2}\right)2 = \left(\Gamma\frac{1}{2}\right)^2$

$\Rightarrow \Gamma\frac{1}{2} = \sqrt{\pi}$

$\Rightarrow$ Show that : $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \frac{1}{2}\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}$

Sol^n $\cot^{1/2}\theta = \dfrac{\cos^{1/2}\theta}{\sin^{1/2}\theta}$

$\int_0^{\pi/2} \cos^{1/2}\theta \sin^{-1/2}\theta \, d\theta = \dfrac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{p+q+2}{2}} = \dfrac{\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}}{2\,\Gamma 1} = \frac{1}{2}\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}$

$\Rightarrow \int_{-1}^{1} (1+x)^{p-1}(1-x)^{q-1}\, dx.$

Sol^n $x = \cos 2\theta$.

$dx = -\sin 2\theta\,(2)\, d\theta$

$dx = -2\sin 2\theta\, d\theta$

$\int_{\pi/2}^{0} (2\cos^2\theta)^{p-1}(2\sin^2\theta)^{q-1}(-2\sin 2\theta)\, d\theta$

$2^{p-1+q-1+1+1} \int_0^{\pi/2} \cos^{2p-1}\theta \sin^{2q-1}\theta \, d\theta$

$\dfrac{\Gamma\frac{2p-1+1}{2}\,\Gamma\frac{2q-1+1}{2}}{2\,\Gamma\frac{2p-1+2q-1+2}{2}} = \dfrac{\Gamma p\,\Gamma q}{2\,\Gamma p+q}$ (Ans)

$\Rightarrow$ ~~$\Gamma n \, \Gamma 1-n = \frac{\pi}{\sin n\pi}$~~

Sol^n^

$\Rightarrow \int_0^1 \frac{dx}{(1-x^n)^{1/n}}$

Sol^n^ $\quad x^n = \sin^2\theta$

$n x^{n-1} \, dx = 2 \sin\theta \cos\theta \, d\theta$

$\int_0^{\pi/2} \frac{2 \sin\theta \cos\theta \, d\theta}{(\cos\theta)^{2/n} \, (n) \, (\sin^{2/n}\theta)^{n-1}}$

$\Rightarrow \frac{2}{n} \int_0^{\pi/2} (\sin\theta)^{1-\left(\frac{2n-2}{n}\right)} (\cos\theta)^{1-2/n} \, d\theta$

$\Rightarrow \frac{1}{n} \frac{\Gamma\frac{1-\left(\frac{2n-2}{2}\right)+1}{2} \; \Gamma\frac{1-2/n+1}{2}}{\Gamma\frac{1-\left(\frac{2n-2}{2}\right)+1-2/n+2}{2}} = \frac{\Gamma\frac{1}{n}\,\Gamma\frac{n-1}{n}}{n}$ (Ans)

DIRICHLET'S INTEGRAL :

$$\iiint_V x^{l-1} y^{m-1} z^{n-1} \, dx \, dy \, dz = \frac{\Gamma l \, \Gamma m \, \Gamma n}{\Gamma l+m+n+1}$$, where $x \geq 0, y \geq 0$, $z \geq 0$, $x+y+z \leq 1$.

proof : $\int_0^1 x^{l-1} dx \int_0^{1-x} y^{m-1} dy \int_0^{1-x-y} z^{n-1} dz$.

$\Rightarrow \int_0^1 x^{l-1} dx \int_0^h \int_0^{h-y} y^{m-1} z^{n-1} \, dy \, dz$. from next prove. que.

$= \int_0^1 x^{l-1} dx \left[\frac{\Gamma m \, \Gamma n}{\Gamma m+n+1} h^{m+n} \right]$

$= \frac{\Gamma m \, \Gamma n}{\Gamma m+n+1} \int_0^1 x^{l-1} (1-x)^{m+n} \, dx$

$\frac{\Gamma m \, \Gamma n}{\Gamma m+n+1} \; \frac{\Gamma l \; \Gamma m+n+1}{\Gamma l+m+n+1} = \frac{\Gamma l \, \Gamma m \, \Gamma n}{\Gamma l+m+n+1}$ (Proved)

* $$\iiint_V x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma l+m+n+1}\, h^{l+m+n}$$ → for 3 variable

where $x \geq 0,\ y \geq 0,\ z \geq 0,\ x+y+z \leq h$.

⇒ Prove : $$\iint_D x^{l-1} y^{m-1}\, dx\, dy = \frac{\Gamma l\, \Gamma m}{\Gamma l+m+1}\, h^{l+m}$$ → for 2 variable.

where $x \geq 0,\ y \geq 0,$ and $x+y \leq h$.

Sol^n^ Let $x = Xh,\ y = Yh,$

$$dx\, dy = h^2\, dX\, dY$$

$$\iint_D (Xh)^{l-1} (Yh)^{m-1} h^2\, dX\, dY \quad , \quad X \geq 0,\ Y \geq 0,\ X+Y \leq 1$$

$$= h^{l+m} \int_0^1 \int_0^{1-X} X^{l-1} Y^{m-1}\, dX\, dY$$

$$= h^{l+m} \int_0^1 X^{l-1}\, dX \int_0^{1-X} Y^{m-1}\, dY$$

$$= h^{l+m} \int_0^1 X^{l-1}\, dX \left[\frac{Y^m}{m}\right]_0^{1-X}$$

$$= \frac{h^{l+m}}{m} \int_0^1 X^{l-1} (1-X)^m\, dX$$

$$= \frac{h^{l+m}}{m}\, \beta(l, m+1) = \frac{h^{l+m}}{m}\, \frac{\Gamma l\, \Gamma m+1}{\Gamma l+m+1}$$

$$= \frac{h^{l+m}}{m}\, \frac{m\, \Gamma l\, \Gamma m}{\Gamma l+m+1} = h^{l+m}\, \frac{\Gamma l\, \Gamma m}{\Gamma l+m+1}$$ (Proved) :

$\Rightarrow$ Prove : $\beta(l,m) = \beta(m,l)$.

Sol^n) $\beta(l,m) = \int_0^1 x^{l-1} (1-x)^{m-1} \, dx$

we know : $\int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx$

$\therefore \beta(l,m) = \int_0^1 (1-x)^{l-1} (1-1+x)^{m-1} \, dx$

$\beta(l,m) = \int_0^1 x^{m-1} (1-x)^{l-1} \, dx = \underline{\underline{\beta(m,l)}}$.