4.4-1.

Use a recursion tree to determine a good asymptotic upper bound on the recurrence $T(n) = 3T(\lfloor n/2 \rfloor) + n$. Use the substitution method to verify your answer.

Answer.

We omit the floor function for simplicity and create a recursion tree for the recurrence $T(n) = 3T(n/2) + n$, which is shown in Figure .

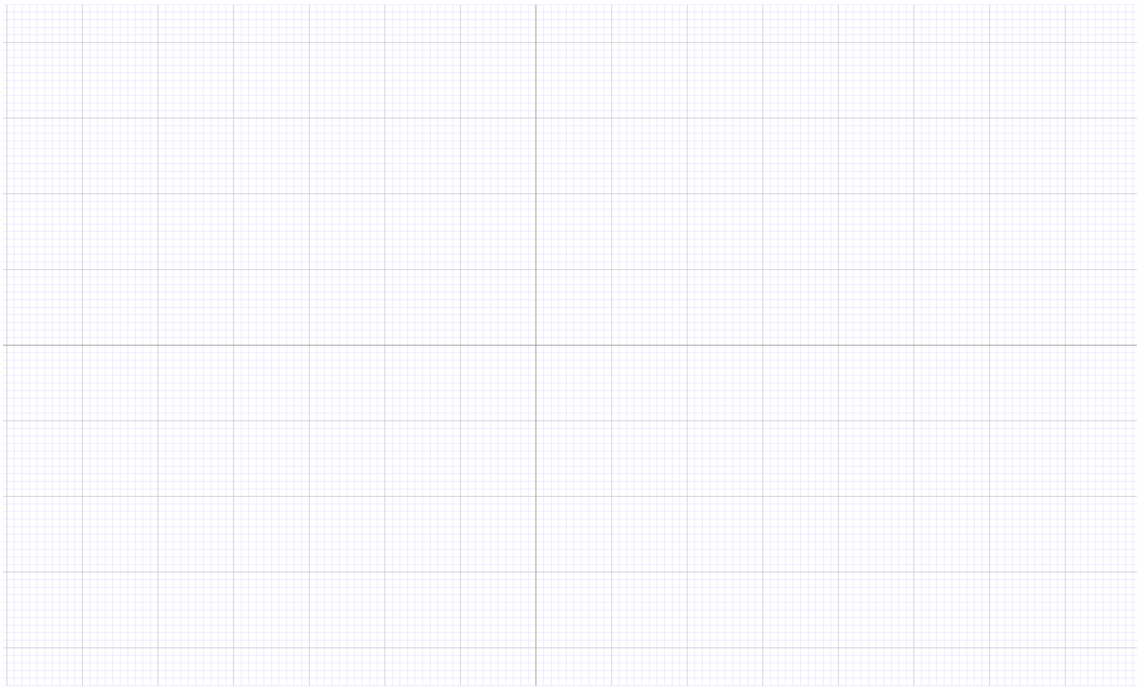

Figure 1.

Because subproblem sizes, decrease by a factor of 2 each step we go down the tree, we eventually must reach a boundary condition. The subproblem size for a node at depth i is $n/2^i$. Thus, the subproblem size hits $T(n) = 1$ when $n/2^i = 1$ or, equivalently, when $i = \log_2 n$. Thus, the tree has $\log_2 n + 1$ levels (at depth $0, 1, 2, \ldots, \log_2 n$).

Next we determine the cost at each level of the tree. Each level has three times more nodes than the level above, and so the number of nodes at depth i is 3^i. Because subproblem sizes reduce by a factor of 2 for aech level we go down from the root, each node at depth i, for $i = 0, 1, 2, \ldots$, $\log_2 n - 1$, has a cost of $n/2^i$. Multiplying, we see that total the cost over all nodes at depth i, is $(3/2)^i n$. The bottom level, at depth $\log_2 n$, has $3^{\log_2 n} = n^{\log_2 3}$ nodes, each contributing cost $T(1)$, for a total cost of $n^{\log_2 3} T(1)$, which is $\Theta(n^{\log_2 3})$, for we assume that $T(1)$ is a constant.

Now we add up the costs over all levels to determine the cost for the entire tree:

$$
\begin{aligned}
T(n) &= n + \frac{3}{2} n + \left(\frac{3}{2}\right)^2 n + \cdots + \left(\frac{3}{2}\right)^{\log_2 n - 1} n + \Theta(n^{\log_2 3}) \\
&= \sum_{i=0}^{\log_2 n - 1} \left(\frac{3}{2}\right)^i n + \Theta(n^{\log_2 3}) \\
&= \frac{(3/2)^{\log_2 n} - 1}{(3/2) - 1} n + \Theta(n^{\log_2 3})
\end{aligned}
$$

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$$
\begin{aligned}
&= 2\left[(3/2)^{\log_2 n} - 1\right] n + \Theta\left(n^{\log_2 3}\right) \\
&= 2\left[n^{\log_2 (3/2)} - 1\right] n + \Theta\left(n^{\log_2 3}\right) \\
&< 2\left(n^{\log_2 2} - 1\right) n + \Theta\left(n^{\log_2 3}\right) \\
&= 2\,(n-1)\,n + \Theta\left(n^{\log_2 3}\right) \\
&= 2\,n^2 - 2\,n + \Theta\left(n^{\log_2 3}\right) \\
&= O\left(n^2\right)
\end{aligned}
$$

Indeed, we can use the substitution method to verify that $O\left(n^2\right)$ is an upper bound for the solution to the recurrence. We show that $T(n) \leq c\,n^2 - d\,n$, where c and d are both suitable positive constants. We have

$$
\begin{aligned}
T(n) &\leq 3\,T(n/2) + n \\
&\leq 3\,c\,(n/2)^2 - d\,(n/2) + n \\
&= (3/4)\,c\,n^2 - (d/2)\,n + n \\
&\leq c\,n^2 - d\,n
\end{aligned}
$$

where the last step holds as longs $d > -2$.