7.4-2.

Show that quicksort's best-case running time is $\Omega(n \lg n)$.

Answer.

Let $T(n)$ denote the best-case time for the procedure QUICKSORT on an input of size n. We have the recurrence

$$T(n) = \min_{0 \le q \le n-1} (T(q) + T(n-q-1)) + \Theta(n) \tag{7.3}$$

where the parameter q ranges from 0 to $n-1$. We guess that $T(n) \ge c\, n \lg n$ for some constant c. Substituting this guess into recurrence (7.3), we obtain

$$T(n) \;\ge\; \min_{0 \le q \le n-1} (c\, q \lg q + c\,(n-q-1) \lg(n-q-1)) + \Theta(n)$$

Consider the function $f(x) = c\, x \ln x + c\,(n-x-1) \ln(n-x-1)$ on the range $0 \le x \le n-1$. Its first derivative with respect to x is $f'(x) = c \ln x - c \ln(n-x-1)$, which equals to 0 iff $x = n-x-1$, that is, when $x = \frac{n-1}{2}$. Besides, the second derivative of $f(x)$ with respect to x is positive. So we can conclude that $c\, q \lg q + c\,(n-q-1) \lg(n-q-1)$ achieves a minimum over the parameter's range $0 \le q \le n-1$ at the middle point $q = \frac{n-1}{2}$, and has the bound $\min_{0 \le q \le n-1} (c\, q \lg q + c\,(n-q-1) \lg(n-q-1)) \ge c\,\frac{n-1}{2} \lg \frac{n-1}{2} + c\,\frac{n-1}{2} \lg \frac{n-1}{2} = c\,(n-1) \lg \frac{n-1}{2}$. Continuing with our bounding of $T(n)$, we obtain

$$\begin{aligned}
T(n) &\ge c\,(n-1) \lg \frac{n-1}{2} + d\, n \\
&= c\,(n-1)\,[\lg(n-1) - \lg 2] + d\, n \\
&= c\, n \lg(n-1) + d\, n - c \lg(n-1) - c\, n + c \\
&\ge c\, n \lg n
\end{aligned}$$

since we can pick the constant d large enough so that the $d\, n$ term dominates the $-c \lg(n-1) - c\, n$ term. Thus, $T(n) = \Omega(n)$.

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