4.3-2.

Show that the solution of $T(n) = T(\lceil n/2 \rceil) + 1$ is $O(\lg n)$.

Answer.

Using substitution method, we are required to prove that $T(n) \leq c \lg n$ for an appropriate choice of the constant $c > 0$. Substituting the guess into the recurrence yields

$$\begin{aligned} T(n) &\leq c \lg \lceil n/2 \rceil + 1 \\ &= c(\lg \lceil n \rceil - \lg \lceil 2 \rceil) + 1 \\ &= c(\lg n - 1) + 1 \\ &= c \lg n - c + 1 \\ &\leq c \lg n \end{aligned}$$

where the last step holds as long as $c \geq 1$.

To examine the boundary condition, observe that for $n = 1$ the bound $T(n) \leq c \lg n$ yields $T(1) \leq c \lg 1 = 0$, which contradicts with $T(1) = 1$. Consequently, $T(1)$ fails to hold as the base case in the inductive proof. However, we can work for a little more effort to replace $T(2)$ as a base case, letting $n_0 = 2$. With $T(1) = 1$, we derive from the recurrence that $T(2) = 2$. Now we can complete the proof that $T(n) \leq c \lg n$ for some constant $c \geq 1$ by choosing c large enough so that $T(2) \leq c \lg 2$. As it turns out, any choice of $c \geq 2$ suffices for the base cases of $n = 2$ to hold. Therefore, the solution of $T(n) = T(\lceil n/2 \rceil) + 1$ is $O(\lg n)$.

Email address: informlarry@gmail.com