Experiment-1: Evaluation of line Integrals.

Problems

1. Evaluate $\int x\,dy - y\,dx$ along the curve $y = x^2$ from the origin to the point $(1,1)$.

Solution:

The curve given is $y = x^2$

So, $dy = 2x\,dx$ along the curve.

$$\int_{y=x^2} x\,dy - y\,dx = \int_{x=0}^{1} x(2x\,dx) - (x^2)\,dx$$

$$= \int_{x=0}^{1} (2x^2 - x^2)\,dx$$

$$= \int_{x=0}^{1} x^2\,dx$$

$$= \left.\frac{x^3}{3}\right|_0^1 = 1/3.$$

2. Evaluate $\int (3x+y)\,dx + (2y-x)\,dy$ along the curve $y = x^2+1$ from $(0,1)$ to $(3,10)$

Solution

The curve given is $y = x^2+1$.

So, $dy = 2x\,dx$

$$\int (3x+y)\,dx + (2y-x)\,dy = \int_{x=0}^{3} (3x + x^2+1)\,dx + (2(x^2+1)-x)\,2x\,dx.$$

$$= \int_0^3 (3x+x^2+1)\,dx + (2x^2 - x + 2)\,2x\,dx$$

$$= \int_0^3 (3x+x^2+1)\,dx + (4x^3 - 2x^2 + 4x)\,dx$$

$$= \int_0^3 (4x^3 - x^2 + 7x + 1)\,dx$$

$$= \left[\frac{4x^4}{4} - \frac{x^3}{3} + \frac{7x^2}{2} + x\right]_0^3$$

$$= \left[\frac{4(81)}{4} - \frac{27}{3} + \frac{7(9)}{2} + 3 - 0\right]$$

$$= \left[81 - 9 + \frac{63}{2} + 3\right]$$

$$= \left[75 + \frac{63}{2}\right]$$

$$= \left[\frac{150 + 63}{2}\right]$$

$$= \left[\frac{213}{2}\right]$$

$\sin 2t = 2\sin t \cos t$

$\cos 2t = \cos^2 t - \sin^2 t$.

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

Problems where the curve is specified in parameter form:

(1) Evaluate $\int_c (xy\,dx + x^2 z\,dy + xyz\,dz)$

where c is given by,

$x = e^t$

$y = e^{-t}$

$z = t^2$

Solution

$dx = e^t . dt$

$dy = -e^{-t} . dt$.

$dz = 2t . dt$.

$\frac{dx}{dt} = e^t$.

$\frac{dy}{dt} = e^{-t}$.

$\frac{dz}{dt} = t^2$.

$$= \int_{t=0}^{t=1} (\cancel{e^{t}} \cdot \cancel{e^{-t}} \cdot e^{t}\, dt) + (e^{2t} \cdot t^2 (-e^{-t} \cdot dt)) + (\cancel{e^{t}} \cdot \cancel{e^{-t}} \cdot t^2 \cdot 2t \cdot dt)$$

$$+ e^{2t} - e^{-t}$$

$$= \int_{t=0}^{t=1} e^t\, dt - e^t t^2 \cdot dt + t^2 \cdot 2t \cdot dt$$

$$= \int_0^1 e^t\, dt - t^2 e^t\, dt + t^2 \cdot 2t \cdot dt$$

$$= \int_0^1 e^t\, dt - \underbrace{t^2 e^t\, dt}_{\downarrow} + 2t^3\, dt$$

$$= \left[e^t \overbrace{- t^2 e^t + 2te^t - 2e^t} + \frac{\cancel{2}t^4}{\cancel{4}\,2} \right]_0^1$$

$$= \left[\cancel{e} - \cancel{e} + \cancel{2e} - \cancel{2e} + \frac{1}{2} \right] - [1 - 2]$$

$$= \frac{1}{2} + 1$$

$$= \frac{3}{2}.$$

ILATE

$-t^2 e^t$

$\int uv\,dx = uv_1 - \int u' v_1\, dx$

$u = -t^2 \quad v = e^t$

$u_1 = -2t\, dt$

$u_2 = -2$

$= -t^2 e^t - (-2t) e^t + (-2) e^t$

$\int uv\,dx = u \int v\,dx - \int \left[\frac{d}{dx}(u) \int v\,dx \right] dx$

(2) Evaluate $\int_C (x+2y)\,dx + (4-2x)\,dy$ around the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ in the counter clockwise direction.

Solution

The parametric form of the given curve is

$x = 4\cos t$
$y = 3\sin t$, $0 \le t \le 2\pi$

$dx = -4\sin t\, dt$
$dy = 3\cos t\, dt$

$= \int_0^{2\pi} = \int_C (x+2y)\,dx + (4-2x)\,dy$

$= \int_{t=0}^{t=2\pi} [(4\cos t + 2(3\sin t))(-4\sin t\, dt) +$
$[(4-2(4\cos t))(3\cos t)\, dt]$

$= \int_0^{2\pi} [-16\cos t\sin t - 24\sin^2 t + 12\cos t - 24\cos^2 t]\, dt$

$= \int_0^{2\pi} (-8\sin 2t - 24 + 12\cos t)$ $\qquad 2\cos t\sin t = \sin 2t$

$= -8\int_0^{2\pi} \sin 2t - 24\int_0^{2\pi} dt + 12\int_0^{2\pi} \cos t\, dt$

$= -8\left[-\frac{\cos 2t}{2}\right]_0^{2\pi} - 24[t]_0^{2\pi} + 12[\sin t]_0^{2\pi}$

$= 4(\cos 4\pi - \cos 0) - 48\pi + 12(\sin 2\pi - \sin 0)$

$= 4(1-1) - 48\pi + 12(0-0)$

$= -48\pi$

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