11/5

Experiment-4

Verification of Green's Theorem.

Green's theorem

Statement:

Let R be a simply connected region with smooth boundary c oriented positively and P and Q have continuous partial derivative in an open system region containing R, then

$$\oint_c P\,dx + Q\,dy = \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dx\,dy$$

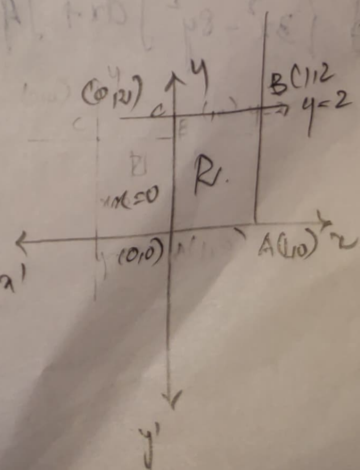
Problems

① verify Green's theorem for the closed curve $\oint_c [3x^2-8y^2]\,dx + [4y(2-3x)]\,dy$ where c is the curve which most of coordinate axes and the line $x=1$ and $y=2$.

Sol

Given $P = 3x^2 - 8y^2$ $\quad Q = 4y - 6xy$

$x=1$ & $y=2$



Evaluation of R.H.S

$P = 3x^2 - 6y^2 \qquad Q = 4y - 6xy$

$\frac{\partial P}{\partial y} = -12y \qquad \frac{\partial Q}{\partial x} = -6y$

$$\iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dx\,dy = \int_{y=0}^{y=2}\int_{x=0}^{x=1} [-6y + 16y]\,dx\,dy$$

$$= \int_0^2\int_0^1 10y\,dx\,dy$$

$$= 10\int_0^2 y\,(x)_0^1\,dy$$

$$= 10\int_0^2 y\,dy = \cancel{10}^{5}\left[\frac{y^2}{\cancel{2}}\right]_0^2 = 5 \times 4$$

$\Rightarrow 20.$

Evaluation of L.H.S

$$\oint_C P\,dx + Q\,dy = \int_{OA} P\,dx + Q\,dy + \int_{AB} P\,dx + Q\,dy +$$

$$\int_{BC} P\,dx + Q\,dy + \int_{CO} P\,dx + Q\,dy \rightarrow (1)$$

Case (i)

Along the Straight line OA

$$\oint_C [3x^2 - 6y^2]\,dx + [4y - 6xy]\,dy = \int_{x=0}^{x=1} 3x^2\,dx$$

$$= \cancel{3}\left(\frac{x^3}{\cancel{3}}\right)_0^1$$

$$= 1 \rightarrow (i)$$

case(ii)

Along the Straight line. AB

$x=1$

$dx=0$

y is curve as from 0 to 2

$y=2$

$$\oint_{AB} (3x^2-8y^2)dx+(4y-6xy)dy = \int_{y=0}^{y=2} -2y\,dy$$

$$= -2\int_0^2 y\,dy$$

$$= -2\left(\frac{y^2}{2}\right)_0^2$$

$$= -4-0 = -4$$

case(iii) Along the Straight line BC

$$\oint_{BC} 3x^2-8y^2)dx+(4y-6xy)dy = \int_{=1}^{0} 3x^2-32\,dx$$

$y=2$

$\Rightarrow dy=0.$

x is varies from 1 to 0

$$\Rightarrow \left[\frac{3x^3}{3}-32x\right]_1^0$$

$$\Rightarrow [-1+32]$$

$$\Rightarrow 31$$

case(iv) Along the Straight line CP

$x=0$

$dx=0$

y is varies from 2 to 0.

$$\oint_{CP} (3x^2-8y^2)\,dx+(4y-6xy)\,dy$$

$$= \int_2^0 4y\,dy$$

19k kabab piece
5 fry eggs

$$= \left[\frac{4y^2}{2}\right]_2^0 = -8$$

Evaluation of L.H.S

$= 1-4+31-8 = 20.$

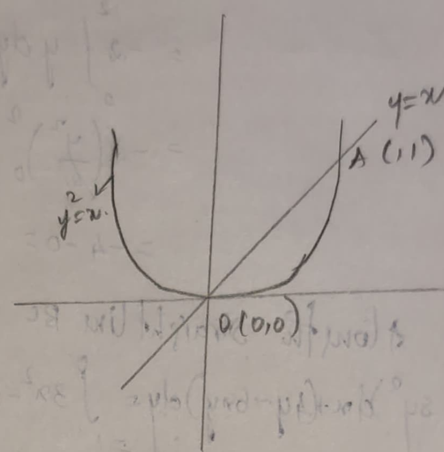
(2) Verify the Green's theorem for the closed curve $\oint_C [(xy+y^2)\,dx + x^2\,dy]$ bounded by $y=x$ and $y=x^2$

(3) Verify Green's theorem in the plane $\oint_C [3x^2 - 8y^2]\,dx + [y - 6xy]\,dy$ where c is the region bounded by the parabolas $y^2 = x$ and $y = x^2$

Sol

(2)

$y = x$ & $y = x^2$



$x = x^2$

$x - x^2 = 0$

$\Rightarrow x(1-x) = 0$

$x = 0$ (or) $1 - x = 0$

$\Rightarrow 1 = x$

$\oint_C P\,dx + Q\,dy = \int_{OA} P\,dx + Q\,dy + \oint_{AO} P\,dx + Q\,dy$

Case(i) along the curve OA

R.H.S

$P = xy + y^2$ $\qquad Q = x^2$

$\frac{\partial Q}{\partial x} = 2x$

$$\iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dx\,dy = \int_0^1\int_0^1 (2x - x - 2y)\,dx\,dy$$

$$= \int_0^1\int_0^1 (x - 2y)\,dx\,dy$$

$$= \int_0^1 \left[xy - \frac{2y^2}{2}\right]_{x^2}^{x} dx$$

$$= \int_0^1 \left[x^2 - \frac{2x^2}{2} - x^3 + \frac{2x^4}{2}\right] dx$$

$$= \left[\frac{x^3}{3} - \frac{2x^3}{6} - \frac{x^4}{4} + \frac{2x^5}{10}\right]_0^1$$

$$= \left[\frac{1}{3} - \frac{2}{6} - \frac{1}{4} + \frac{2}{10}\right]$$

$$= \left[\frac{1}{3} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5}\right]$$

$$= \left[-\frac{1}{20}\right]$$

$$\oint_C P\,dx + Q\,dy = \int_{OA} P\,dx + Q\,dy + \int_{AO} P\,dx + Q\,dy$$

Case (i)

along the curve OA

$y = x^2$

$dy = 2x\,dx$

x is varies from 0 to 1

$$\oint (xy+y^2)\,dx + x^2\,dy = \int_0^1 (x^3+x^4)\,dx + 2x^3\,dx$$

$$= \int_0^1 (x^3 + x^4 + 2x^3)\,dx$$

$$= \left[\frac{x^4}{4} + \frac{x^5}{5} + \frac{2x^4}{4}\right]_0^1$$

$$\Rightarrow \left[\frac{1}{4} + \frac{1}{5} + \frac{2}{4}\right] = \left[\frac{1}{4} + \frac{1}{5} + \frac{1}{2}\right]$$

$$\Rightarrow \left[\frac{38}{40}\right] = \left[\frac{19}{20}\right]$$

Case (ii)

Along the straight line AO

$$\oint (xy+y^2)\,dx + x^2\,dy = \int_1^0 (x^2+y^2)\,dx + x^2\,dy$$

$$= \int_1^0 3x^3\,dx$$

$$= \left[\frac{3x^3}{3}\right]_1^0$$

$$= \left[x^3\right]_1^0 = -1$$

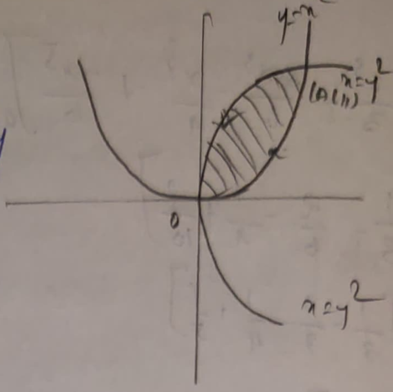
$$= \frac{19}{20} - 1 = \frac{19-20}{20} = \frac{-1}{20}$$

L.H.S = R.H.S

hence proved

(3)

$$\oint P\,dx+Q\,dy = \int_{OA} P\,dx+Q\,dy + \int_{AO} P\,dx+Q\,dy$$



LHS

$$\oint P\,dx+Q\,dy = \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dx\,dy$$

$$Q = 4y-6xy$$

$$\frac{\partial Q}{\partial x} = -6y \quad , P = 3x^2-8y$$

$$\frac{\partial P}{\partial y^2} = -16y$$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} [-6y+16y]\,dx\,dy$$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} [10y]\,dx\,dy$$

$$= \int_0^1 [10xy]_{y^2}^{\sqrt{y}}\,dy$$

$$= \int_0^1 [10y^{1/2}\cdot y - 10y^2 y^{2/2}]\,dy$$

$$= \int_0^1 [10y^{3/2} - 10y^3]\,dy$$

$$= 10\left[\frac{2y^{5/2}}{5} - \frac{y^4}{4}\right]_0^1 dy$$

$$\Rightarrow 10\left[\frac{2}{5}-\frac{1}{4}\right] = \left[\frac{20}{5} - \frac{10}{4}\right] = \frac{8-5}{2} = \frac{3}{2}.$$

Case(1) $\int_{OA} P\,dx + Q\,dy = \int_{OA} [3x^2-8y^2]\,dx + [4y-6xy]\,dy$

$$= \int_0^1 [3x^2-8x^4]\,dx + [4x^2-6x^3]\,2x\,dx \qquad y = x^2$$

$$dy = 2x\,dx$$

$$= \int_0^1 3x^2\,dx - \int_0^1 8x^4\,dx + \int_0^1 8x^3\,dx - \int_0^1 12x^4\,dx$$

$$= \left[\frac{3x^3}{3}\right]_0^1 - \left[\frac{8x^5}{5}\right]_0^1 + \left[\frac{8x^4}{4}\right]_0^1 - \left[\frac{12x^5}{5}\right]_0^1$$

$$= 1 - \frac{8}{5} + 2 - \frac{12}{5}$$

$$= 3 - \frac{20}{} \Rightarrow -1$$

case (ii)

$$\int_{AO} P\,dx + Q\,dy = \int_{AO} [3x^2 - 8y^2]\,dx + [4y - 6xy]\,dy$$

$$= \int_1^0 [3y^4 - 8y^2]\,2y\,dy + [4y - 6y^3]\,dy \qquad x = y^2,\ dx = 2y\,dy$$

$$= \int_1^0 6y^5\,dy - \int_1^0 16y^3\,dy + \int_1^0 4y\,dy - \int_1^0 6y^3\,dy$$

$$= \left[\frac{6y^6}{6}\right]_1^0 - \left[\frac{16y^4}{4}\right]_1^0 + \left[\frac{4y^2}{2}\right]_1^0 - \left[\frac{6y^4}{4}\right]_1^0$$

$$= -1 + 4 - 2 + \frac{3}{2}$$

$$= -3 + \frac{8+3}{2}$$

$$= -3 + \frac{11}{2}$$

$$= \frac{-6+11}{2} \Rightarrow \frac{5}{2}$$

$$\oint_C P\,dx + Q\,dy = \int_{OA} P\,dx + Q\,dy + \int_{AO} P\,dx + Q\,dy$$

$$= -1 + \frac{5}{2}$$

$$= \frac{-2+5}{2}$$

$$= \frac{3}{2}$$

L.H.S = R.H.S.

Program

```
from Sympy import *
x,y,a,b = Symbols("x,y,a,b")
def greens_theorem (P,Q,a,b):
 def line_integral (P,Q,f,g,a,b):
  if g== None :
    return integrate [Q. subs([(x,f)]),(y,a,b))
  if f== None :
    return integrate (P. subs([(y,g)]),(x,a,b))
I1= line_integral (P,Q, None, 0,0,a)
I2 = line_integral( P,Q, a, None,0,b)
I3 = line_integral ( P,Q, None , b,a,0)
I4= line_integral( P,Q,a,None, b,0)
I_line = I1+I2+I3+I4
I_double = integrate (diff (Q,x) - diff (P,y), (x,0,a)
                (y,0,b))
print ("value of closed line integral is", I_line )
Print("value of double integral is",I_double)
if I_line=I_double :
  Print ("Hence, Green's theorem is satisfied ")
else:
 Print ("Hence, Green's theorem is not Satisfied")
```

greens_theorem (3*...

output

value of closed in...

value of double in...

Hence, Green's theorem...

14/5/22

greens_theorem (3*x**2 - 8*y**2, 4*y - 6*x*y, 1, 2)

output
value of closed integral is 20
value of double integral is 20

hence, Green's theorem is verified

14/5/22