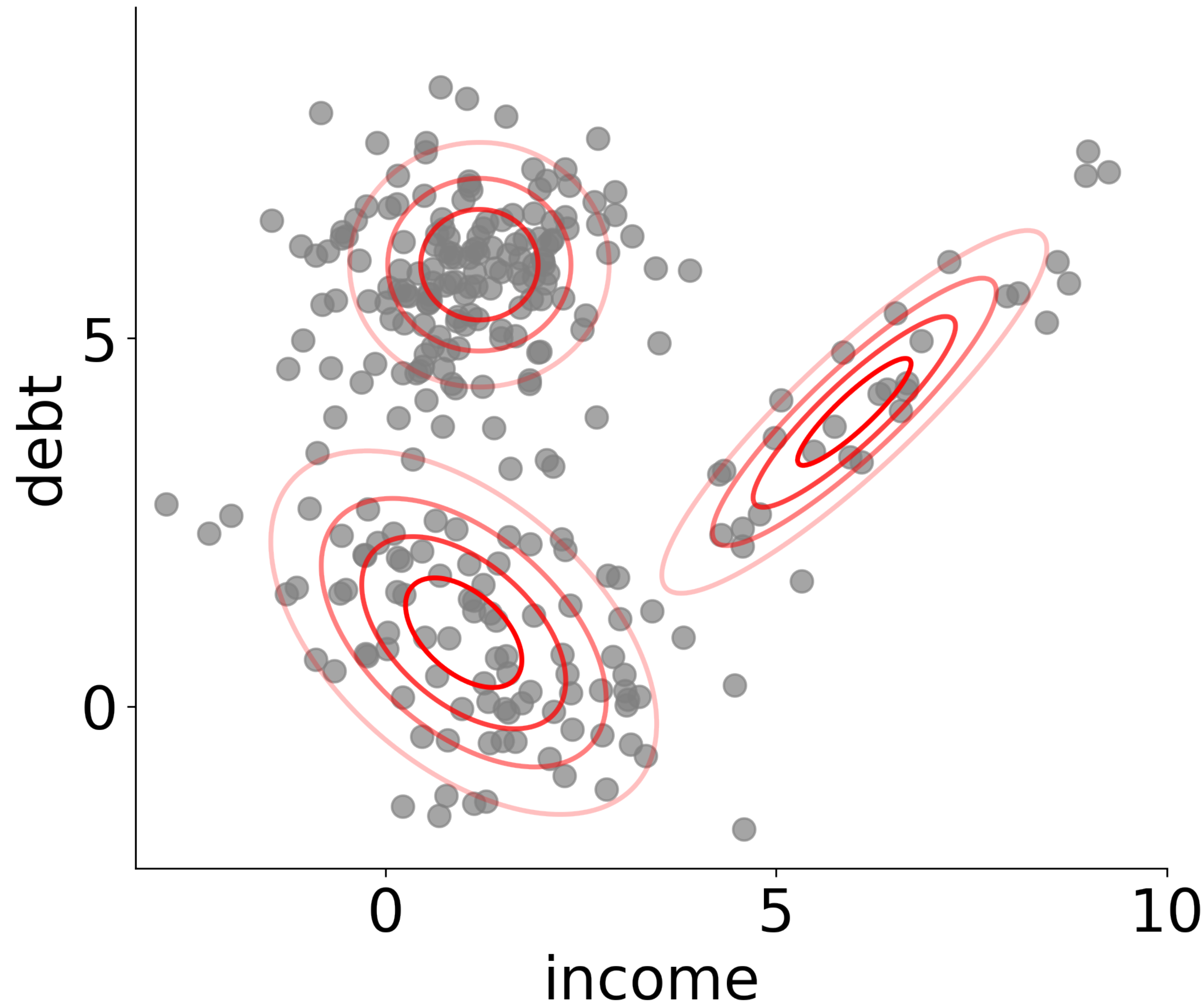
Gaussian Mixture Model (GMM)



Introducing latent variable

$$p(x \mid \theta) = \pi_1 \mathcal{N}(x|\mu_1, \Sigma_1) + \pi_2 \mathcal{N}(x|\mu_2, \Sigma_2)$$
$$+ \pi_3 \mathcal{N}(x|\mu_3, \Sigma_3)$$

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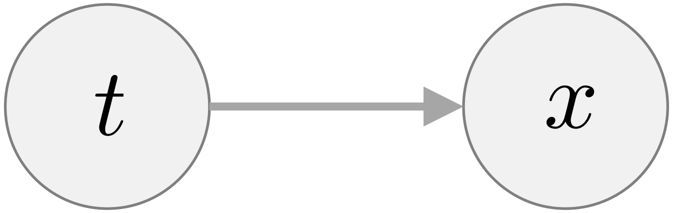

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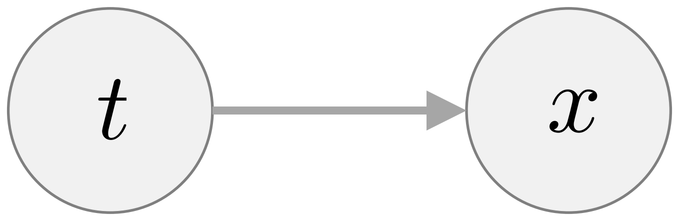

$$p(t = c \mid \theta) = \pi_c$$

Introducing latent variable

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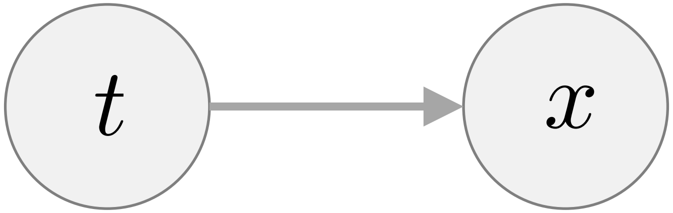

$$p(t = c \mid \theta) = \pi_c$$

$$p(x \mid t = c, \theta) = \mathcal{N}(x \mid \mu_c, \Sigma_c)$$

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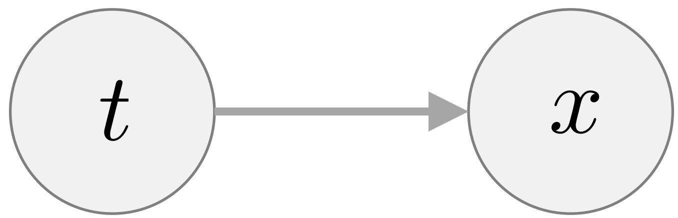
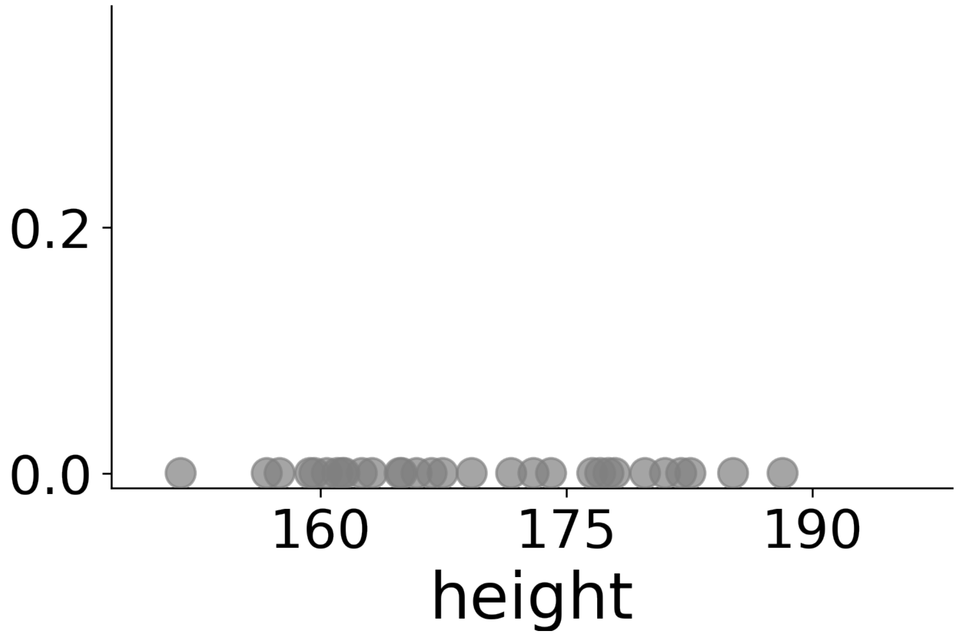


$$p(t = c \mid \theta) = \pi_c$$

$$p(x \mid t = c, \theta) = \mathcal{N}(x \mid \mu_c, \Sigma_c)$$

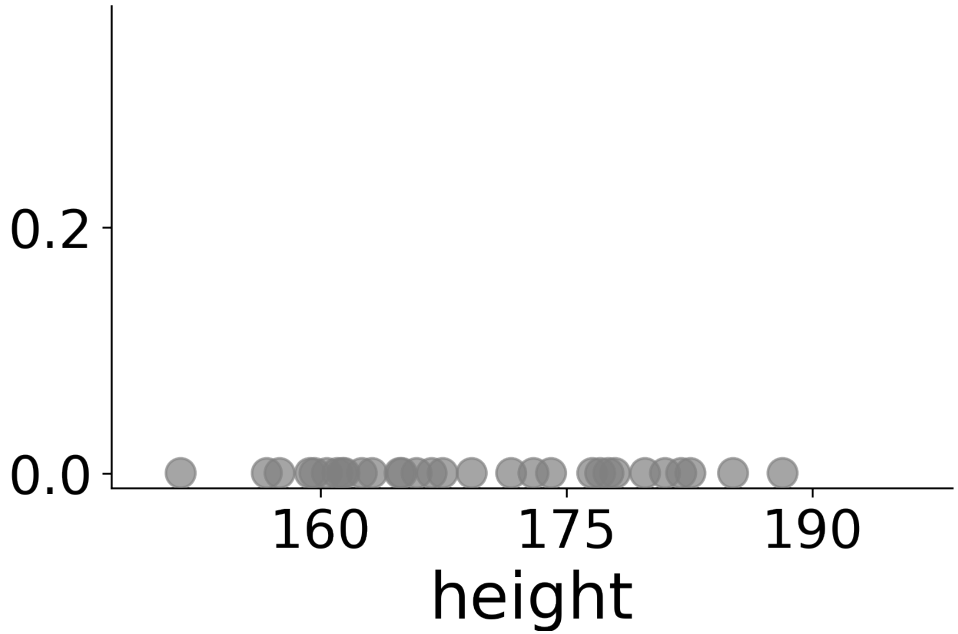
$$p(x \mid \theta) = \sum_{c=1}^{3} p(x \mid t = c, \theta) p(t = c \mid \theta)$$

Expectation Maximization



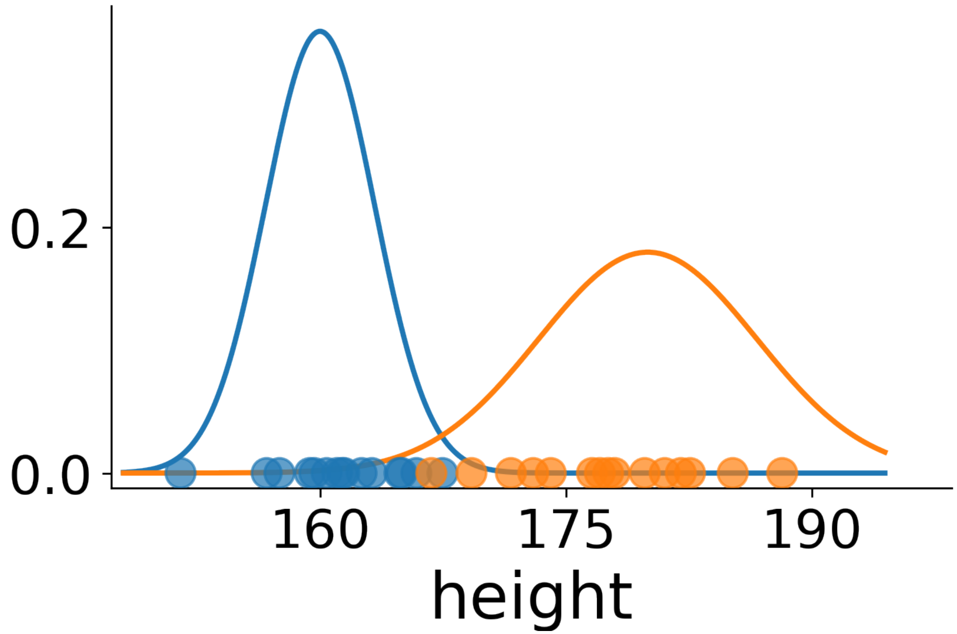
Dataset: $\{x_1, \ldots, x_N\}$

Expectation Maximization



How to estimate parameter θ?

Expectation Maximization



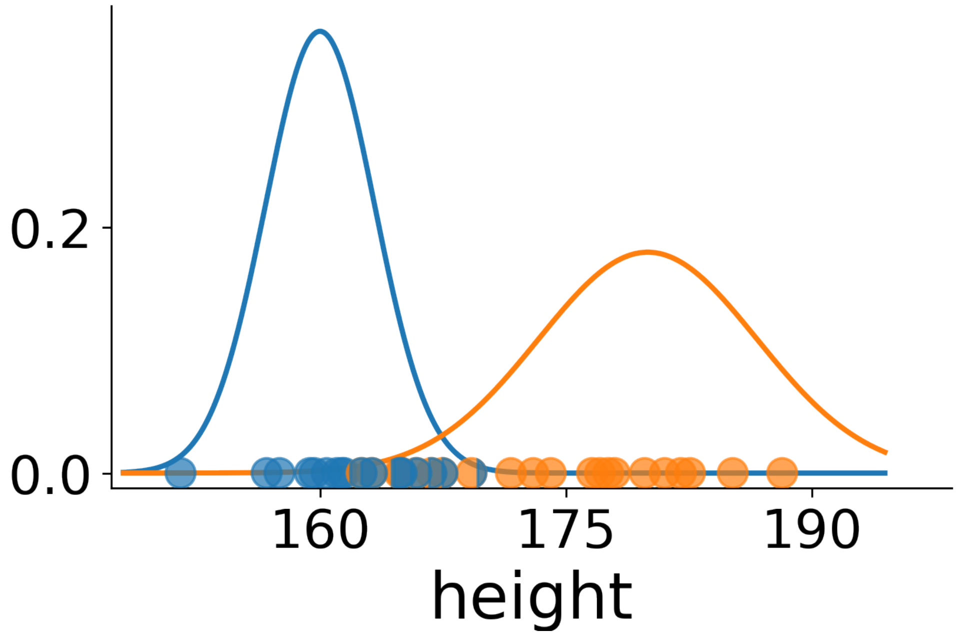
How to estimate parameter θ?

If sources t are known, easy:

$$p(x \mid t = 1, \theta) = \mathcal{N}(x \mid \mu_1, \sigma_1^2)$$

$$\mu_1 = \frac{\sum_{\text{blue } i} x_i}{\# \text{ of blue points}} \qquad \sigma_1^2 = \frac{\sum_{\text{blue } i} (x_i - \mu_1)^2}{\# \text{ of blue points}}$$

Expectation Maximization

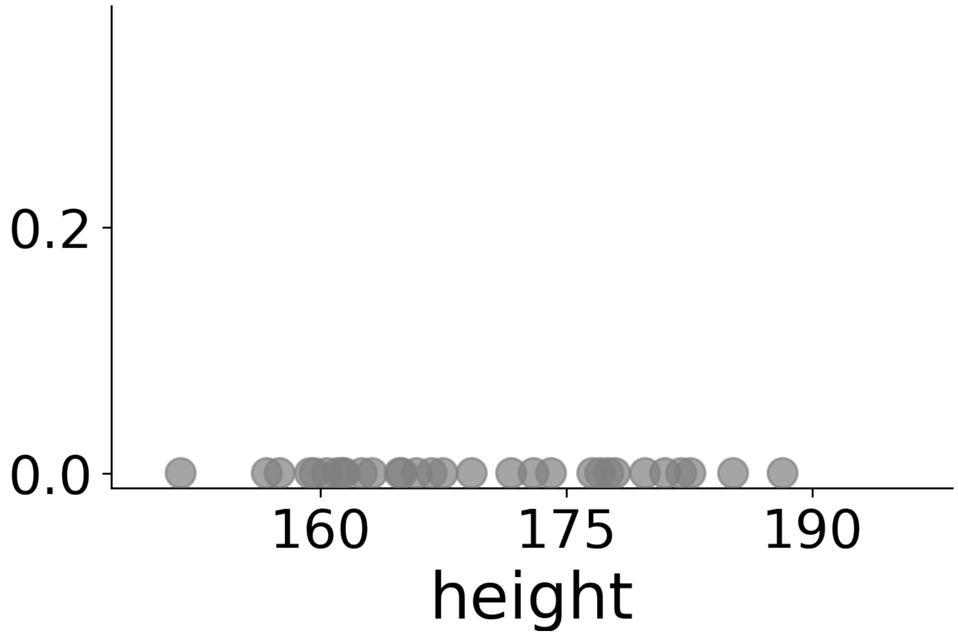


How to estimate parameter θ?

If sources t are known, easy:

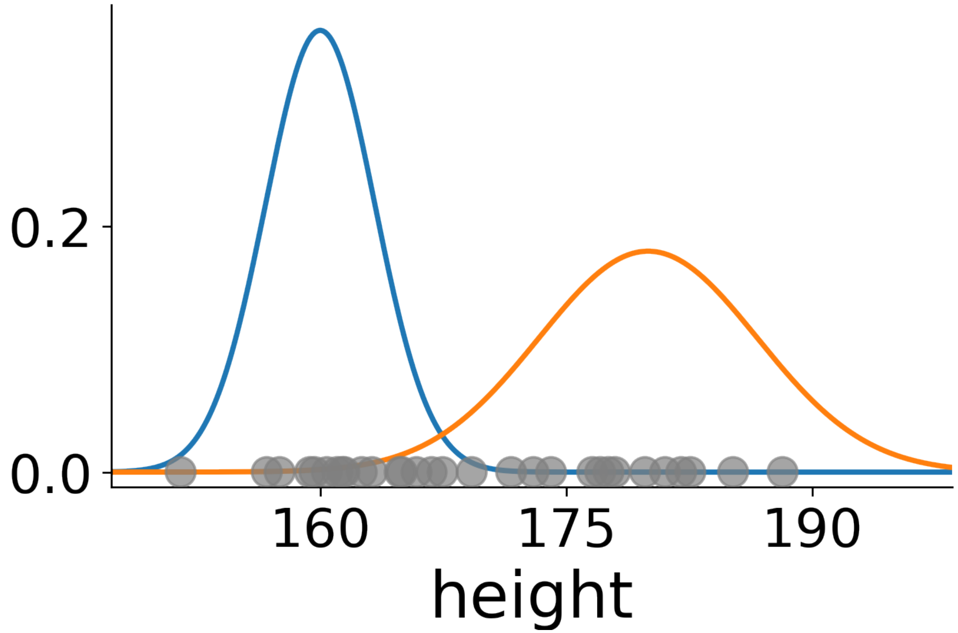
$$\mu_1 = \frac{\sum_i p(t_i = 1 \mid x_i, \theta)\, x_i}{\sum_i p(t_i = 1 \mid x_i, \theta)} \quad \sigma_1^2 = \frac{\sum_i p(t_i = 1 \mid x_i, \theta)\, (x_i - \mu_1)^2}{\sum_i p(t_i = 1 \mid x_i, \theta)}$$

Expectation Maximization



What if we don't know the sources?

Expectation Maximization

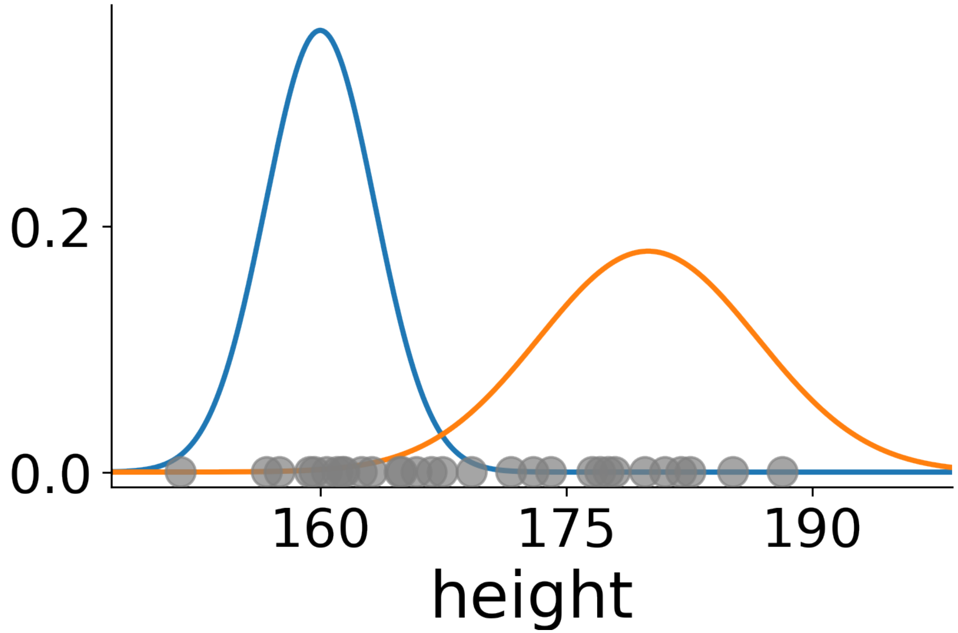


What if we don't know the sources?

Given: $$p(x \mid t = 1, \theta) = \mathcal{N}(-2, 1)$$

Find: $$p(t = 1 \mid x, \theta)$$

Expectation Maximization



What if we don't know the sources?

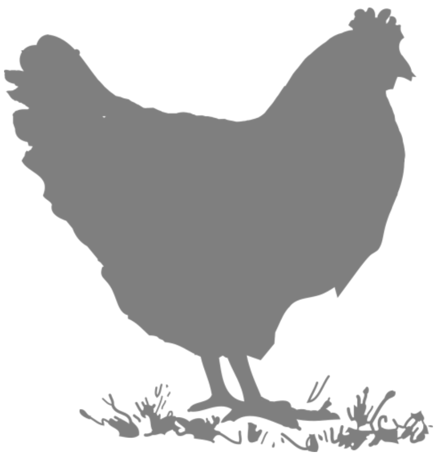
If we know parameters θ, easy:

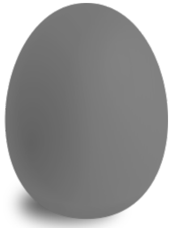
$$p(t = 1 \mid x, \theta) = \frac{p(x \mid t = 1, \theta)\, p(t = 1 \mid \theta)}{Z}$$

Expectation Maximization

Chicken and egg problem

- Need Gaussian parameters to estimate sources
- Need sources to estimate Gaussian parameters





Expectation Maximization

Chicken and egg problem

- Need Gaussian parameters to estimate sources
- Need sources to estimate Gaussian parameters

EM algorithm

1. Start with 2 randomly placed Gaussians parameters θ
2. Until convergence repeat:
   a) For each point compute $p(t = c \mid x_i, \theta)$: does x_i look like it came from cluster c?
   b) Update Gaussian parameters θ to fit points assigned to them