

Consider the linear time-invariant system given by the transfer function

$$H(s) = \frac{b_ms^m + b_{m-1}s^{m-1} + \cdots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0} = \frac{N(s)}{D(s)} .$$

Recall that this system is stable if all of the poles are in the OLHP, and these poles are the roots of the polynomial $D(s)$. It is important to note that **one should not cancel any common poles and zeros of the transfer function** before checking the roots of $D(s)$. Specifically, suppose that both of the polynomials $N(s)$ and $D(s)$ have a root at $s = a$, for some complex (or real) number a . One must *not* cancel out this common zero and pole in the transfer function before testing for stability. The reason for this is that, even though the pole will not show up in the response to the input, it will still appear as a result of any initial conditions in the system, or due to additional inputs entering the system (such as disturbances). If the pole and zero are in the CRHP, the system response might blow up due to these initial conditions or disturbances, even though the input to the system is bounded, and this would violate BIBO stability.

To see this a little more clearly, consider the following example. Suppose the transfer function of a linear system is given by

$$H(s) = \frac{s-1}{s^2+2s-3} = \frac{N(s)}{D(s)} .$$

Noting that $s^2 + 2s - 3 = (s+3)(s-1)$, suppose we decided to cancel out the common pole and zero at $s = 1$ to obtain

$$H(s) = \frac{1}{s+3} .$$

Based on this transfer function, we might (erroneously) conclude that the system is stable, since it only has a pole in the OLHP. What we should actually do is look at the original denominator $D(s)$, and correctly conclude that the system is unstable because one of the poles is in the CRHP. To see why the pole-zero cancellation hides instability of the system, first write out the differential equation corresponding to the transfer function to obtain

$$\ddot{y} + 2\dot{y} - 3y = \dot{u} - u .$$

Take the Laplace Transform of both sides, taking initial conditions into account:

$$s^2Y(s) - sy(0) - \dot{y}(0) + 2sY(s) - 2y(0) - 3Y(s) = sU(s) - u(0) - U(s) .$$

Rearrange this equation to obtain

$$Y(s) = \underbrace{\frac{s-1}{s^2+2s-3}}_{H(s)} U(s) + \frac{s+2}{s^2+2s-3}y(0) + \frac{1}{s^2+2s-3}\dot{y}(0) - \frac{1}{s^2+2s-3}u(0) .$$

Note that the denominator polynomial in each of the terms on the right hand sides is equal to $D(s)$ (the denominator of the transfer function). For simplicity, suppose that $y(0) = y_0$ (for some real number y_0), $\dot{y}(0) = 0$ and $u(0) = 0$. The partial fraction expansion of the term $\frac{s+2}{s^2+2s-3}y_0$ is given by

$$\frac{s+2}{s^2+2s-3}y_0 = \frac{y_0}{4} \left(\frac{1}{s+3} + \frac{1}{s-1} \right) ,$$

and this contributes the term $\frac{y_0}{4}(e^{-3t} + e^t), t \geq 0$, to the response of the system. Note that the e^t term blows up, and thus the output of the system blows up if y_0 is not zero, even if the input to the system is bounded.

If all poles of the transfer function are in the OLHP (before any pole-zero cancellations), all initial conditions will decay to zero, and not cause the output of the system to go unbounded.

The above example demonstrates the following important fact:

Stability of a transfer function must be checked **without** canceling any common poles and zeros from the transfer function. In particular, systems with unstable pole-zero cancellations are **unstable**.