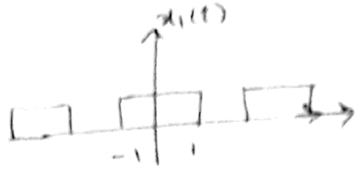
4) (الف) $a_k = \frac{1}{T}\int_{-T_1}^{T_1} x(t)\, e^{-j\omega_0 k t}\, dt = \frac{1}{T}\int_{-T_1}^{T_1} e^{-j\omega_0 k t}\, dt = \frac{1}{T}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_{-T_1}^{T_1}$

$= \frac{1}{T}\left[\frac{e^{-jk\omega_0 T_1} - e^{jk\omega_0 T_1}}{-jk\omega_0}\right] = \frac{2}{T}\,\frac{\sin(k\omega_0 T_1)}{k\omega_0}$

ب) $T=4 \Rightarrow \omega_0 = \frac{\pi}{2}$

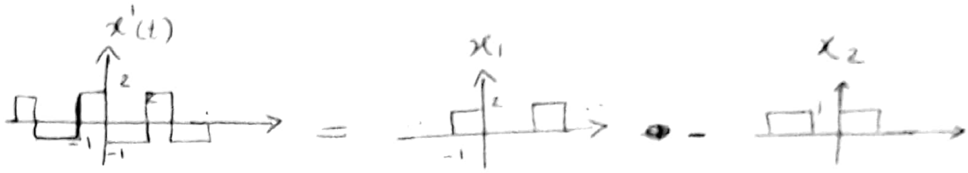


$T_1 = 1 \Rightarrow a'_k = \frac{1}{2}\,\frac{\sin k\omega_0}{k\omega_0}$

$x(t) = x_1(t-1) \Rightarrow a_k = e^{-j\omega_0} a'_k = \frac{e^{-j\omega_0}}{2}\,\frac{\sin k\omega_0}{k\omega_0}$

ج) $x'(t) = \frac{dx(t)}{dt} \rightarrow$

$T=3$



$x_1: \ a_k = e^{-j(1/2)\omega_0} \times \frac{2}{3}\,\frac{\sin(\frac{1}{2}k\omega_0)}{k\omega_0} \times 2) \rightarrow$ دامنه $\underline{2}$

$x_2: \ b_k = e^{-j\omega_0} \times \frac{2}{3}\,\frac{\sin(k\omega_0)}{k\omega_0} \Rightarrow x'(t) \rightarrow c_k = a_k + b_k$

$\Rightarrow x(t) \rightarrow X_k = \frac{c_k}{j\omega_0 k}$

5) قسمت ب سوال : $\sum_{k=-\infty}^{+\infty} |a_k|^2 = \frac{1}{T}\int_{-T/2}^{T/2} |x(t)|^2\, dt$, $T = \frac{\pi}{3}$

$\Rightarrow \sum_{k=-\infty}^{+\infty} |a_k|^2 = \frac{3}{\pi}\int_0^{\pi/3} |\sin(3t)|^2\, dt = \frac{3}{\pi}\int_0^{\pi/3} \frac{1-\cos 6t}{2}\, dt$

$= \frac{3}{\pi}\left[\frac{t}{2} - \frac{1}{12}\sin 6t\right]_0^{\pi/3} = \frac{3}{\pi}\left(\frac{\pi}{6} - \frac{1}{12}\cancel{\sin 2\pi}\right) = \frac{1}{2}$

6) 1) $T=6 \Rightarrow \omega_0 = \frac{2\pi}{6} = \frac{\pi}{3}$ 2) $a_k = a^*_{-k}$

3) $x(t) = -x(t-3) \Rightarrow a_k = -a_k\, e^{-j3(\pi/3)}$

$\Rightarrow a_k(1+(-1)^k) = 0 \Rightarrow$

به ازای k زوج باید , $a_k = 0$

4) $a_0 = 0 \xrightarrow{3} a_2 = 0$

$a_{k>2} = 0$ $\Rightarrow$ فقط a_1 و a_{-1} باقی می‌ماند.

5) a_1 حقیقی و مثبت $\xrightarrow{2} a_1 = a_{-1}^{*} \Rightarrow$ کسینوسی $\Rightarrow x = a\cos\omega_0 t$

6) $\int_{-3}^{3} |x(t)|^2 dt = 3 \Rightarrow a\int_{-3}^{3} |\cos\omega_0 t|^2 dt = 3 \Rightarrow a\int_{-3}^{3} \frac{1-\cos 2\omega_0 t}{2} dt = 3$

$= a\left[\frac{t}{2} - \frac{1}{4\omega_0}\sin 2\omega_0 t\right]_{-3}^{3} = 3 \Rightarrow 2a\left[\frac{3}{2} - 0\right] = 3 \Rightarrow \underline{a = 1} \Rightarrow \underline{x(t) = \cos\frac{\pi}{3}t}$

7) $\frac{dy(t)}{dt} + y(t) = x(t) \Rightarrow j\omega Y + Y = X \Rightarrow \frac{Y}{X} = \frac{1}{1 + j\omega k}$

$X_k = \begin{cases} \frac{1}{2} & k = -1, 1 \\ 0 & o.w \end{cases} \Rightarrow Y_k = \begin{cases} \frac{\frac{1}{2}}{1 + j\omega k} & k = -1, 1 \\ 0 & o.w \end{cases}$