تمرین سری چهار

$$1-a)\quad x(t) = e^{-3|t|} \sin(2t)$$

$$e^{-3|t|} \xrightarrow{FT} \frac{6}{9+\omega^2} \qquad \sin(2t) \xrightarrow{FT} \frac{\pi}{j}\left(\delta(\omega-2) - \delta(\omega+2)\right)$$

$$x(t)\,y(t) \xrightarrow{FT} \frac{1}{2\pi} X(j\omega) * Y(j\omega)$$

$$\Rightarrow X(j\omega) = \frac{1}{2\cancel{\pi}} \times \frac{\cancel{\pi}}{j}\left(\frac{6}{9+(\omega-2)^2} - \frac{6}{9+(\omega+2)^2}\right)$$

$$1-b)\quad x(t) = \begin{cases} 1-t^2 & 0<t<1 \\ 0 & o.w. \end{cases}$$

$$\Rightarrow X(j\omega) = \int_0^1 (1-t^2)\, e^{-j\omega_0 t}\, dt$$

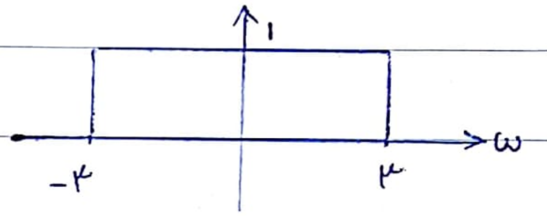
$$= \underbrace{\left[(1-t^2) \times \frac{-1}{j\omega_0} e^{-j\omega_0 t}\right]\Big|_0^1}_{0 - \left(\frac{-1}{j\omega_0}\right)} - \underbrace{\int_0^1 (-2t) \times \frac{-1}{j\omega_0} e^{-j\omega_0 t}\, dt}_{+\frac{1}{j\omega_0}\left(\left[(-2t) \times \frac{-1}{j\omega_0} e^{-j\omega_0 t}\right]\Big|_0^1 - \int_0^1 \frac{2}{j\omega} e^{-j\omega_0 t}\right)}$$

$$\frac{1}{j\omega_0} + \frac{1}{j\omega_0}\left(\frac{2}{j\omega_0} e^{-j\omega_0} + \frac{2}{j\omega_0^2}\left[e^{-j\omega_0} - 1\right]\right)$$

$$1-c)\quad x(t) = \frac{\sin 3t \cdot \cos t}{\pi t}$$

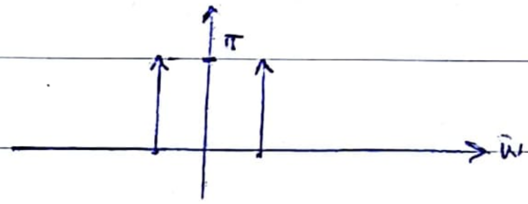
$$\frac{\sin \omega_0 t}{\pi t} \xrightarrow{FT} X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

$\Rightarrow FT\left\{\frac{\sin 3t}{\pi t}\right\}$:



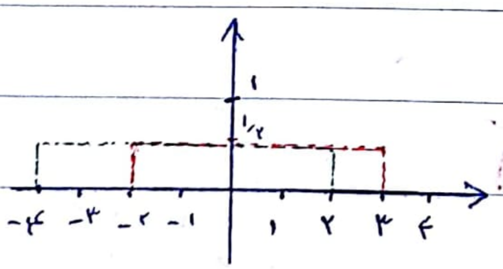
$\Rightarrow X_1(j\omega)$

$FT\{\cos t\}$:

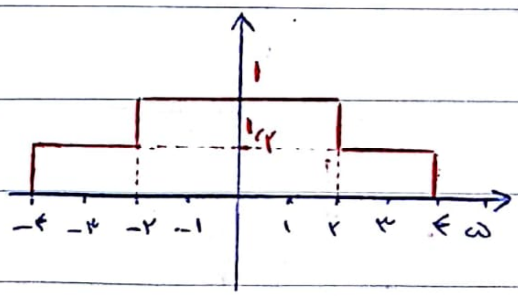


$X_2(j\omega)$

طبق خاصیت ضرب : $X(j\omega) = \frac{1}{2\pi} X_1(j\omega) * X_2(j\omega)$



$\Rightarrow$



1-d) $te^{-2|t-1|}$

$\to e^{-a|t|} \xrightarrow{FT} \frac{2a}{a^2+\omega^2} \Rightarrow e^{-2|t|} \xrightarrow{FT} \frac{4}{4+\omega^2}$

$\to x(t+t_0) \xrightarrow{FT} e^{j\omega t_0} X(j\omega) \Rightarrow e^{-2|t-1|} \xrightarrow{FT} e^{-j\omega} \times \frac{4}{4+\omega^2}$

$\to t\,x(t) \xrightarrow{FT} -\frac{1}{j} \frac{dX(j\omega)}{d\omega} \Rightarrow te^{-2|t-1|} \xrightarrow{FT} -\frac{1}{j} \frac{d\left(\frac{4e^{-j\omega}}{4+\omega^2}\right)}{d\omega}$

• عکس تبدیل فوریه

۲-a) $X(j\omega) = \omega e^{-|\omega|}$

با توجه به سؤال قبل ، $t.e^{-|t|} \xrightarrow{FT} \frac{-4j\omega}{(1+\omega^2)^2}$

دوگانی : $x(t) \to X(j\omega)$

$X(t) \to 2\pi x(-\omega) \Rightarrow x(-\omega) \xrightarrow{F^{-1}} \frac{1}{2\pi} X(t)$

$x(-\omega) = -\omega e^{-|\omega|} \xrightarrow{F^{-1}} \frac{1}{2\pi} \frac{-4jt}{(1+t^2)^2}$

$x(\omega) = \omega e^{-|\omega|} \xrightarrow{F^{-1}} \frac{1}{2\pi} \times \frac{4jt}{(1+t^2)^2}$

2-b) $X(j\omega) = \begin{cases} e^{-\omega} & \omega>0 \\ -e^{\omega} & \omega<0 \end{cases}$

$$\Rightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega = \frac{1}{2\pi}\Big[\underbrace{\int_{0}^{+\infty} e^{-\omega}e^{j\omega t}\,d\omega}_{x_1(t)} + \underbrace{\int_{-\infty}^{0} -e^{\omega}e^{j\omega t}\,d\omega}_{x_2(t)}\Big]$$

$$x_1(t) = \frac{-1}{1-jt}\, e^{-(1-jt)\omega}\Big|_{0}^{+\infty} = \frac{1}{1-jt}$$

$$x_2(t) = \frac{-1}{1+jt} \Rightarrow x(t) = \frac{1}{2\pi}\left(\frac{1}{1-jt} - \frac{1}{1+jt}\right)$$

2-c) $X(j\omega) = \dfrac{2a - j\omega}{2a + j\omega} = \overbrace{\dfrac{2a}{2a+j\omega}}^{X_1} - \overbrace{j\omega\,\dfrac{1}{2a+j\omega}}^{X_2}$

$$\rightsquigarrow e^{-2at}u(t) \rightarrow \frac{1}{2a+j\omega} \Rightarrow 2a\,e^{-2at}u(t) \xrightarrow{FT} \frac{2a}{2a+j\omega} \quad (1)$$

$$\rightsquigarrow \frac{dx}{dt} \rightarrow j\omega X(j\omega) \Rightarrow \frac{d}{dt}\left(e^{-2at}u(t)\right) \xrightarrow{FT} \frac{j\omega}{2a+j\omega}$$

$$\Rightarrow x_2(t) = -2a\,e^{-2at}u(t) + \underbrace{e^{-2at}\delta(t)}_{\delta(t)} \quad (2)$$

$$x(t) = x_1(t) - x_2(t) \xRightarrow{2,1} x(t) = e^{-2at}u(t) \times 4a - \delta(t)$$

۴)

$$a(t) = x(t) \times p(t) \rightsquigarrow A(\omega) = X(j\omega) * P(j\omega)$$

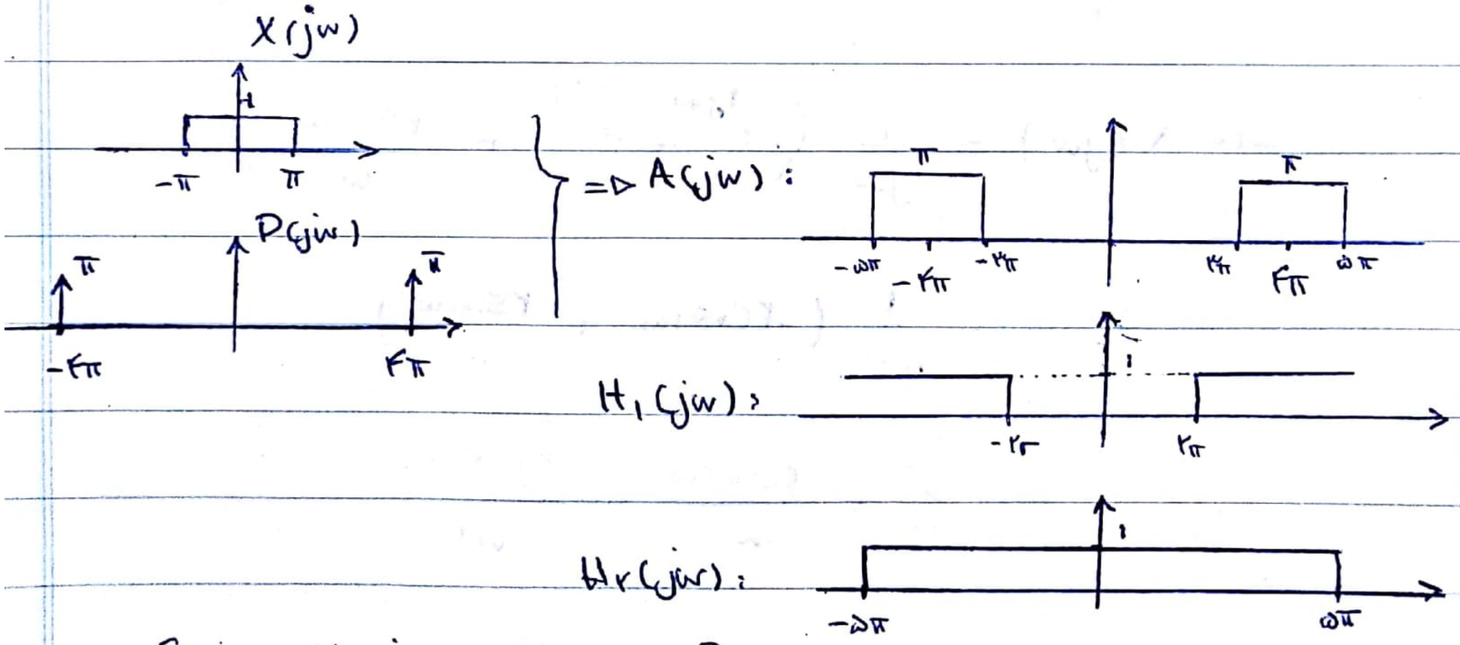
$$B(j\omega) = A(j\omega) \times H_1(j\omega)$$

$$C(j\omega) = A(j\omega) \times H_1(j\omega) \times H_۲(j\omega)$$

$$x(t) = \operatorname{Sinc}(t) \Rightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \pi \\ 0 & |\omega| > \pi \end{cases}$$

$$p(t) = \cos ۴\pi t \Rightarrow P(j\omega) = \pi\left(\delta(\omega - ۴\pi) + \delta(\omega + ۴\pi)\right)$$

$$h_۲(t) = \frac{\sin ۵\pi t}{\pi t} \Rightarrow H_۲(j\omega) = \begin{cases} 1 & |\omega| < ۵\pi \\ 0 & |\omega| > ۵\pi \end{cases}$$

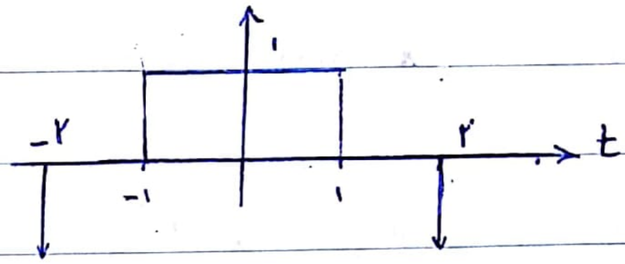


$$\Rightarrow C(j\omega) = A(j\omega) = X(j\omega) * P(j\omega)$$

$$\Rightarrow c(t) = a(t) = x(t) \times p(t) = \frac{\sin \pi t}{\pi t} \times \cos ۴\pi t$$

۴)

$$\Rightarrow \frac{dx(t)}{dt}:$$



$$\delta(t-t_0) \xrightarrow{FT} e^{-j\omega t_0}$$

و

$$x(t) = \begin{cases} 1 & |t|<T \\ 0 & |t|>T \end{cases} \xrightarrow{FT} \frac{2\sin\omega T}{\omega}$$

$$\Rightarrow \frac{dx(t)}{dt} \xrightarrow{FT} -e^{-j\omega\times 2} - e^{-j\omega\times(-2)} + \frac{2\sin\omega}{\omega}$$

از طرفی ، $\frac{dx(t)}{dt} \xrightarrow{FT} j\omega X(j\omega)$

$$\Rightarrow X(j\omega) = \frac{1}{j\omega}\left(-e^{2j\omega} - e^{-2j\omega} + \frac{2\sin\omega}{\omega}\right)$$

$$= \frac{1}{j\omega}\left(-2\cos 2\omega + \frac{2\sin\omega}{\omega}\right)$$

$$= -\frac{2\cos 2\omega}{j\omega} - \frac{2j\sin\omega}{\omega^2}$$

۵)

الف)

$x(t)$ حقیقی : $\text{Even}\{x(t)\} \xrightarrow{FT} \text{Real}\{X(j\omega)\}$ (1)

۲ ، $\frac{1}{2\pi}\int_{-\infty}^{+\infty} \text{Real}\{X(j\omega)\} e^{j\omega t}\, d\omega = |t| e^{-|t|}$

۱ و ۲ : $\text{Even}\{x(t)\} = |t| e^{-|t|}$

ج ، $t<0 : x(t) = 0 \Rightarrow x(t) = \text{Even}\{x(t)\} + \text{Odd}\{x(t)\} = 0$ for $t \leq 0$

$$\Rightarrow \text{Odd}\{x(t)\} = \underbrace{-|t| e^{-|t|}}_{t \leq 0} = t e^{t} \quad t \leq 0$$

$\text{Odd}\{x(t)\}$: فرد $\Rightarrow -\text{Odd}\{x(-t)\} = \text{Odd}\{x(t)\}$

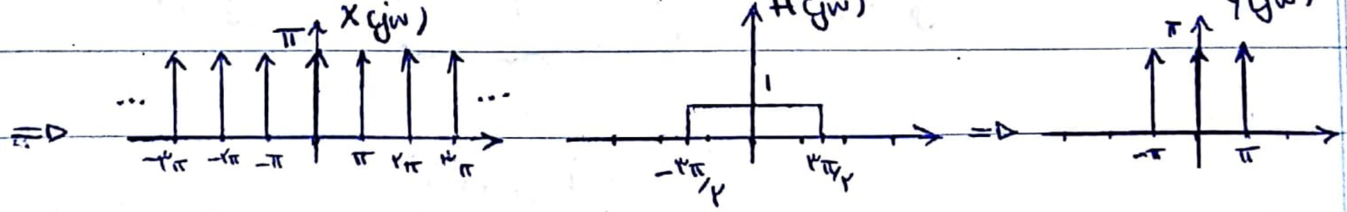
$$\Rightarrow -(-t) e^{-t} = t e^{-t} = \text{Odd}\{x(t)\}$$

$$\rightarrow x(t) = \text{Odd}\{x(t)\} + \text{Even}\{x(t)\}$$

۹) $\rightarrow X(j\omega) = \sum_{k=-\infty}^{+\infty} a_k \times 2\pi \times \delta(\omega - k\omega_0)$

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t-2k) \Rightarrow a_k = \frac{1}{2} \Rightarrow X(j\omega) = \sum_{k=-\infty}^{+\infty} \pi\, \delta(\omega - k\pi)$$

$\hookrightarrow T_0 = 2 \Rightarrow \omega_0 = \pi$



$$\Rightarrow y(t) = \cos \pi t + \frac{1}{2}$$

$\cos \omega_0 t \xrightarrow{FT} \pi(\delta(\omega-\omega_0) + \delta(\omega+\omega_0))$ چون $\hookleftarrow$ $\quad \hookrightarrow$ چون $1 \xrightarrow{FT} 2\pi\delta(\omega)$

۷) $H(j\omega) = \frac{j\omega+۲}{۶-\omega^۲+۵j\omega}$

الف)

$$۶H(j\omega) \underbrace{-\omega^۲}_{+j^۲\omega^۲} H(j\omega) + ۵j\omega H(j\omega) = j\omega+۲$$

$$\xrightarrow{F^{-1}} ۶h(t) + \frac{d^۲h(t)}{dt} + ۵\frac{dh(t)}{dt} = \frac{d\delta(t)}{dt} + ۲\delta(t)$$

با جایگذاری $x(t)=\delta(t)$, $y(t)=h(t)$ داریم:

$$۶y(t) + \frac{d^۲y(t)}{dt} + ۵\frac{dy(t)}{dt} = \frac{dx(t)}{dt} + ۲x(t)$$

ب) $h(t) = F^{-1}\{H(j\omega)\}$

$$H(j\omega) = \frac{j\omega+۲}{(j\omega+۳)(j\omega+۲)} = \frac{1}{j\omega+۳} \xrightarrow{F^{-1}} e^{-۳t}u(t)$$

ج) $X(j\omega) = \frac{1}{j\omega+۴} - j\underbrace{\frac{d}{d\omega}\left\{\frac{1}{j\omega+۴}\right\}}_{\frac{-1}{(j\omega+۴)^۲}} = \frac{j\omega+۳}{(j\omega+۴)^۲}$

$$\Rightarrow Y(j\omega) = X(j\omega)H(j\omega) = \frac{\cancel{j\omega+۳}}{(j\omega+۴)^۲} \times \frac{\cancel{j\omega+۲}}{(\cancel{j\omega+۲})(\cancel{j\omega+۳})}$$

$$\Rightarrow Y(j\omega) = \frac{1}{(j\omega+4)^2} \Rightarrow y(t) = t e^{-4t} u(t)$$

د) $x_1(t) = e^{2t} \longrightarrow$ می دانیم e^{2t} تبدیل فوریه ندارد.

اما اگر کانولوشن را انجام دهیم، سیستم برای این ورودی، خروجی می دهد،

$$y_1(t) = x_1(t) * h(t) = \int_{-\infty}^{+\infty} e^{2\tau} e^{-3(t-\tau)} u(t-\tau)\, d\tau$$

$$= e^{-3t} \int_{-\infty}^{t} e^{5\tau}\, d\tau = e^{2t} \times \frac{1}{5}$$

در واقع برای این حالت خاص، می توانیم بگوییم $\rightarrow a>0$

$$y(t) = e^{at} * h(t) = e^{at} \cdot H(j\omega)\Big|_{j\omega=a}$$

۸)

الف) $\rightarrow x(t) = \delta(t) \Rightarrow y(t) = h(t)$

$$\rightarrow \frac{d^2 h(t)}{dt^2} + 6\frac{dh(t)}{dt} + 9h(t) = \frac{d^2\delta(t)}{dt^2} + 3\frac{d\delta(t)}{dt} + 2\delta(t)$$

$$\xrightarrow{FT} j^2\omega^2 H(j\omega) + 6j\omega H(j\omega) + 9H(j\omega)$$

$$= j^2\omega^2 + 3j\omega + 2$$

$$\Rightarrow H(j\omega) = \frac{j^2\omega^2 + 3j\omega + 2}{j^2\omega^2 + 6j\omega + 9} = 1 - \frac{3j\omega + 7}{(3+j\omega)^2}$$

$$= 1 - \frac{3}{j\omega+3} + \frac{2}{(j\omega+3)^2}$$

$$h(t) = \delta(t) - 4e^{-3t}u(t) + 2te^{-3t}u(t)$$

ب) با عوض کردن جای x و y در معادله‌ی سیستم، در واقع معادله‌ی سیستم وارون را خواهیم داشت:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{d^2\ (t)}{dt^2} + 4\frac{dx(t)}{dt} + 9x(t)$$

برای محاسبه‌ی $g(t)$: ← راه اول: مثل قسمت قبل، $x(t) = \delta(t)$ می‌گذاریم، تبدیل فوریه می‌گیریم و بعد F^{-1}.

راه کوتاه‌تر: ← راه دوم: در مورد پاسخ ضربه‌ی سیستم معکوس می‌دانیم که:

$$h(t) * g(t) = \delta(t) \Rightarrow H(j\omega) \times G(j\omega) = 1$$

$$\Rightarrow G(j\omega) = \frac{1}{H(j\omega)}$$

$$\Rightarrow G(j\omega) = \frac{j^2\omega^2 + 4j\omega + 9}{j^2\omega^2 + 3j\omega + 2} = 1 + \frac{3j\omega + 7}{(2+j\omega)(1+j\omega)}$$

$$= 1 + \frac{4}{1+j\omega} + \frac{-1}{2+j\omega} \xrightarrow{F^{-1}} \delta(t) + 4e^{-t}u(t) - e^{-2t}u(t)$$