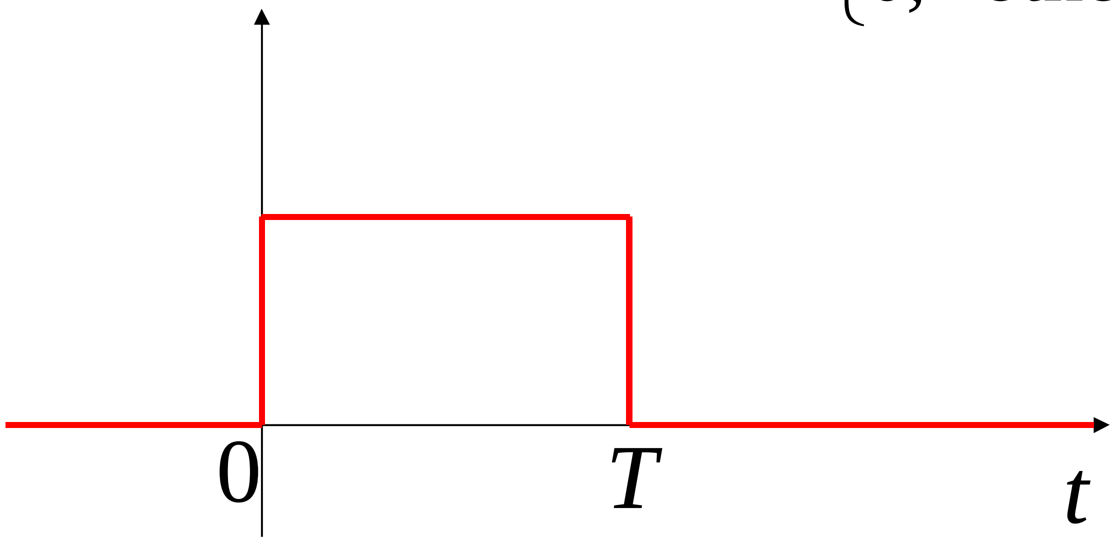
Graphical Convolution Example

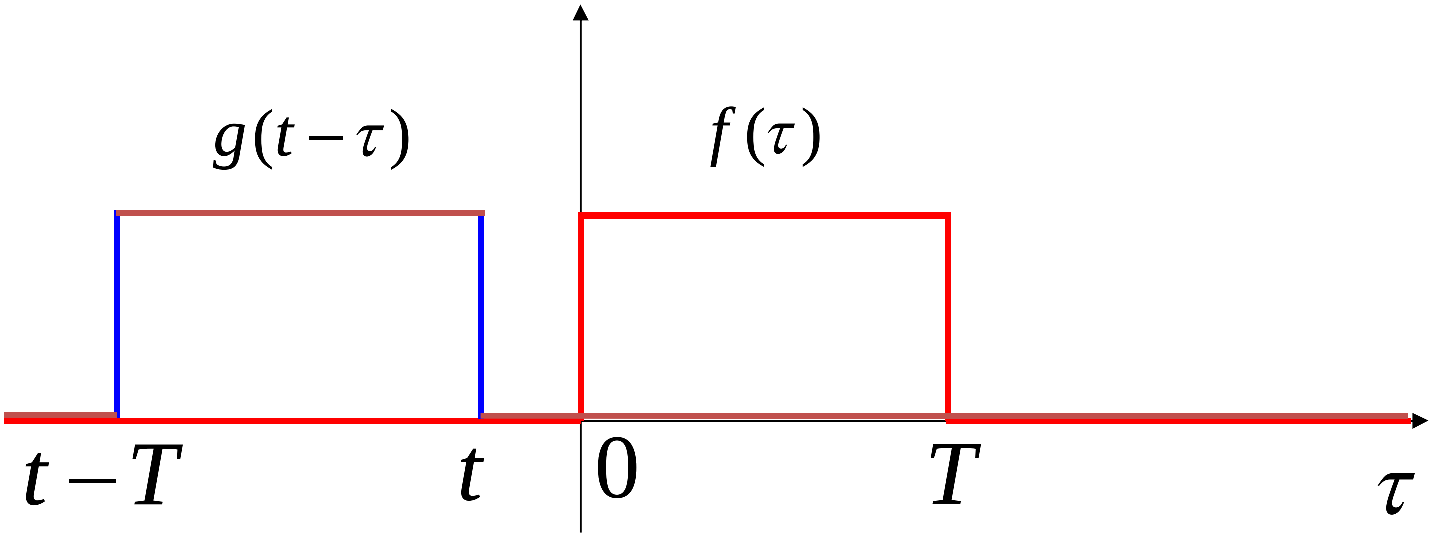
- Suppose that f(t) = g(t) where f(t) is the rectangular pulse depicted in figure, of height 1.

$$f(t) = g(t) = \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}$$



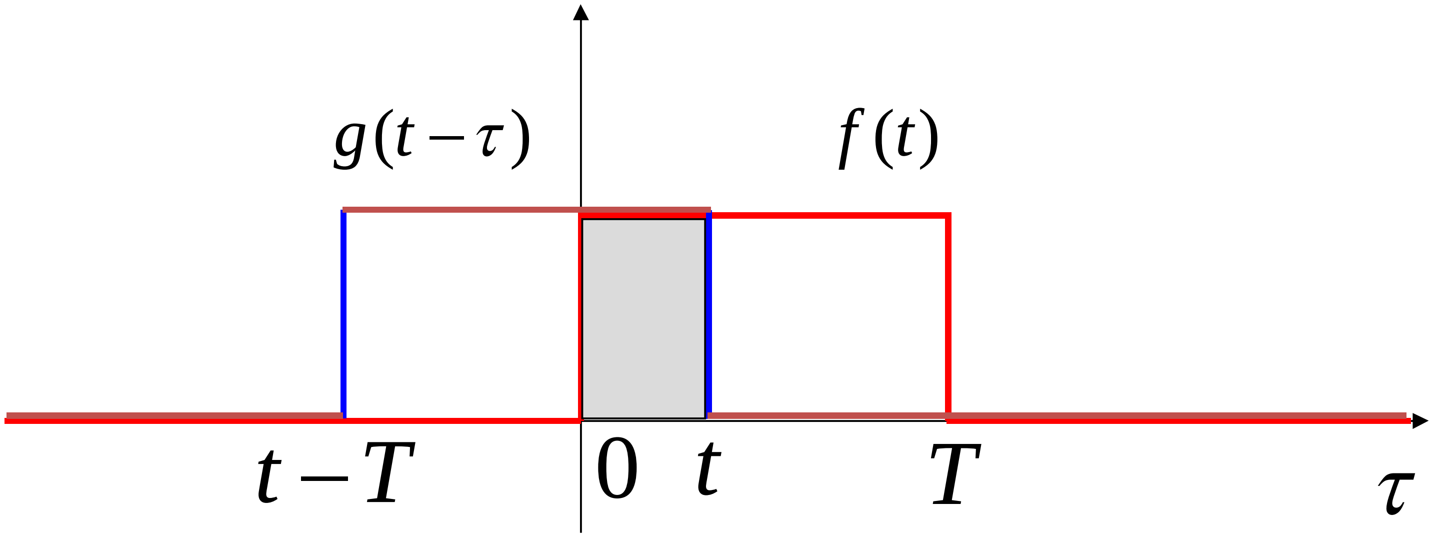
Graphical Convolution Example

- Case 1: t < 0



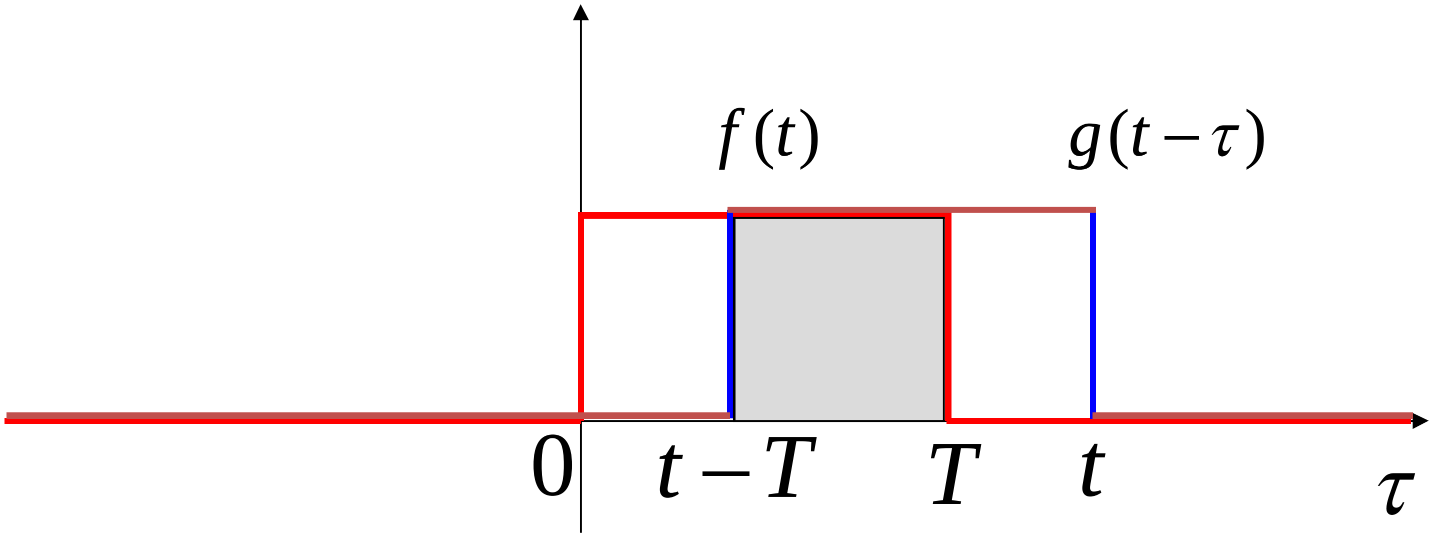
$$y(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau = 0$$

- Case 2: 0 <= t< T



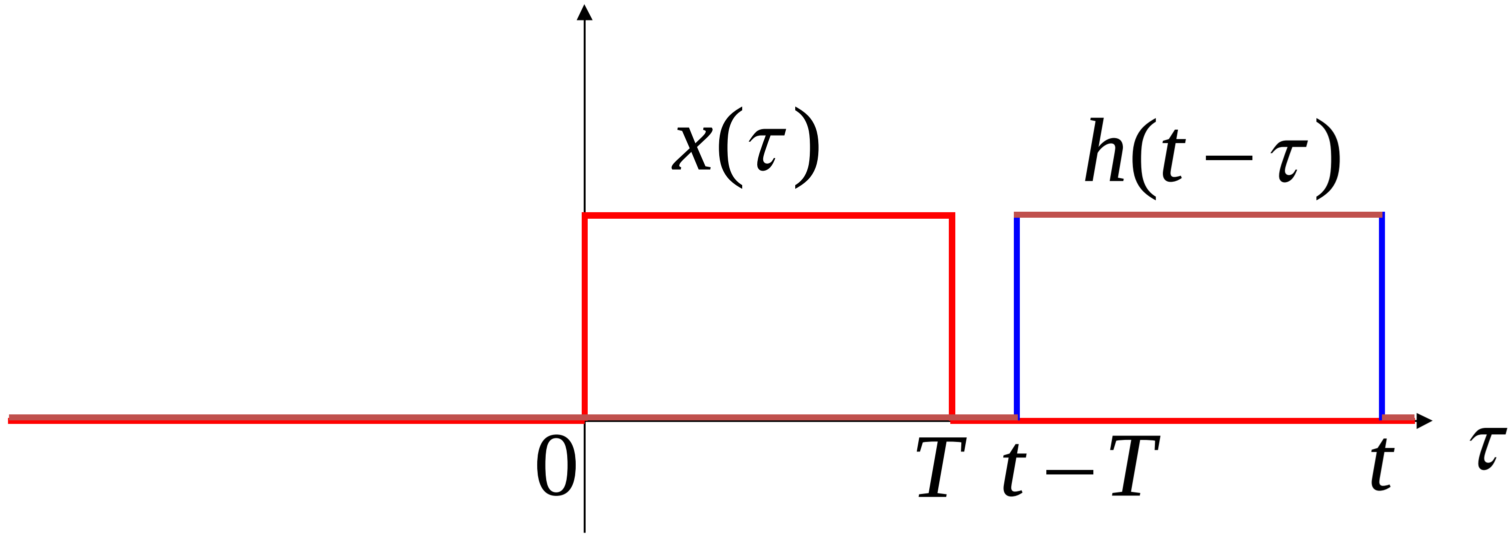
$$\int_0^t 1.1 d\tau = t$$

- Case 3: $T \le t \le 2T$



$$\int_{t-T}^{T} 1.1\,d\tau = 2T - t$$

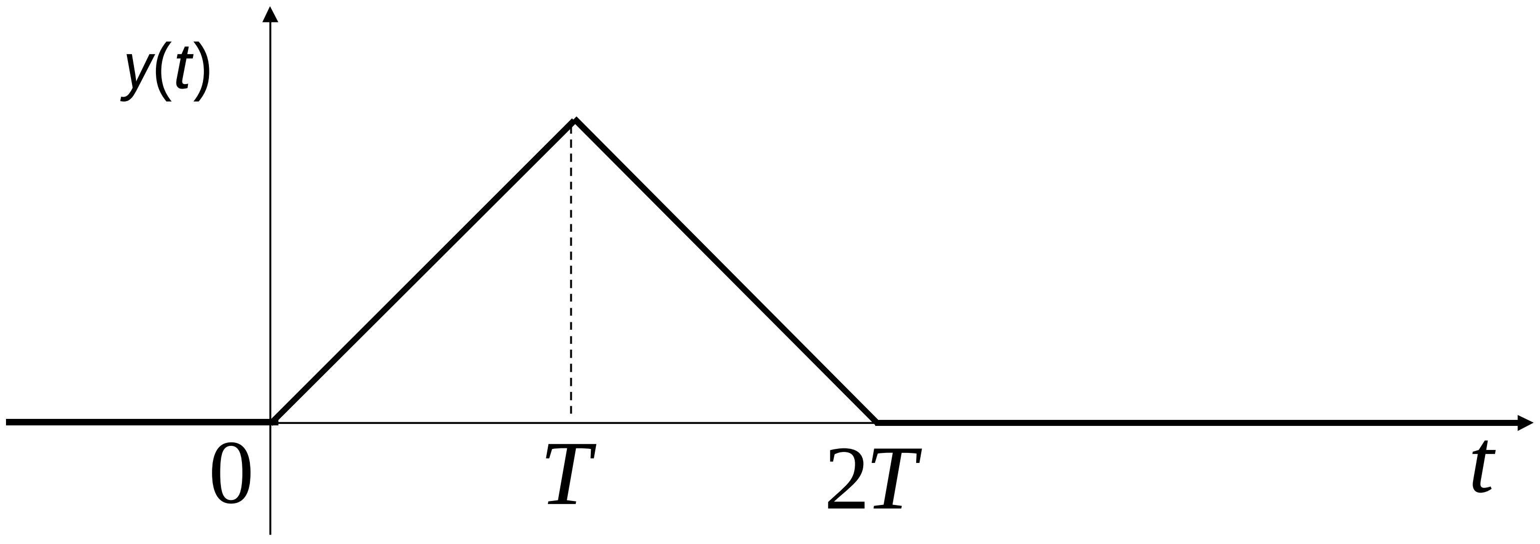
- Case 4: $t > 2T$



Output

This image cannot currently be displayed.

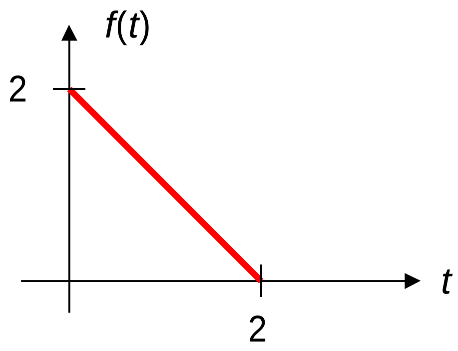
$$y(t) = \begin{cases} 0, & t < 0 \\ t, & 0 \le t < T \\ 2T - t, & T \le t \le 2T \\ 0, & t > 2T \end{cases}$$



Example of a resistor?

Graphical Convolution Example

- Convolve the following two functions:



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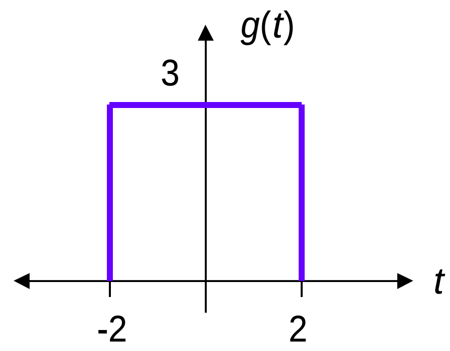
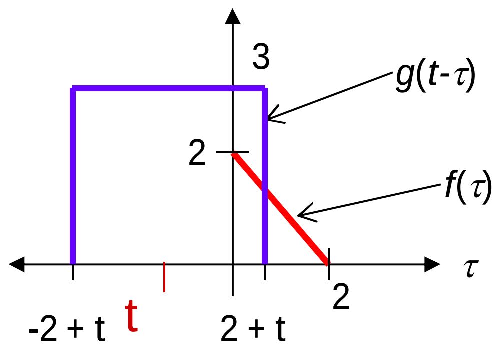


- Replace t with τ in $f(t)$ = -t +2 and $g(t)$
- Choose to flip and slide $g(t)$
- Functions overlap like this:

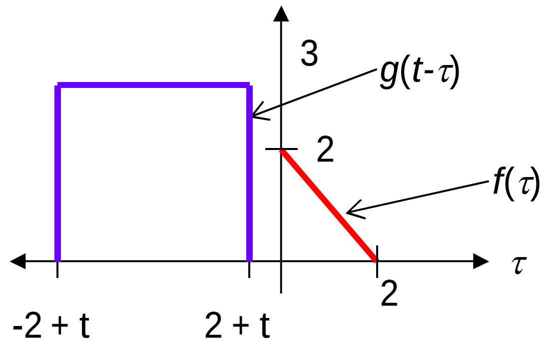
$$y(t)=\int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

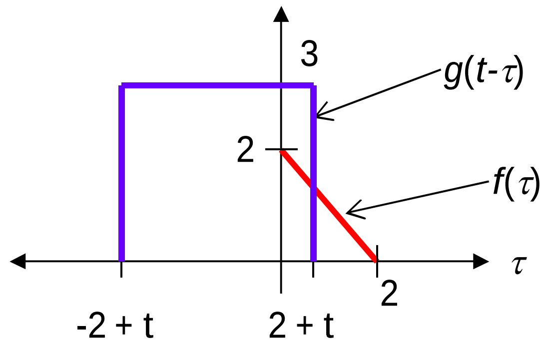


Graphical Convolution Example

- Convolution can be divided into 5 parts
  I. $t < -2$
    - Two functions do not overlap
    - Area under the product of the functions is zero

  II. $-2 \leq t < 0$
    - Part of $g(t)$ overlaps part of $f(t)$
    - Area under the product of the functions is





$$y(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau = \int_{0}^{2+t} 3(-\tau+2) d\tau = 3\left(-\frac{\tau^2}{2} + 2\tau\right)\Bigg|_{0}^{2+t}$$

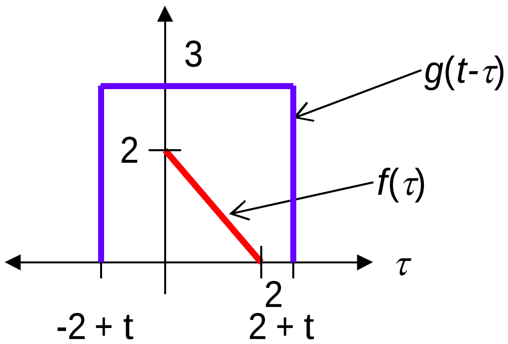
$$= -\frac{3(2+t)^2}{2} + 6(2+t) = -\frac{3t^2}{2} + 6$$

Graphical Convolution Example

III. $0 \le t < 2$

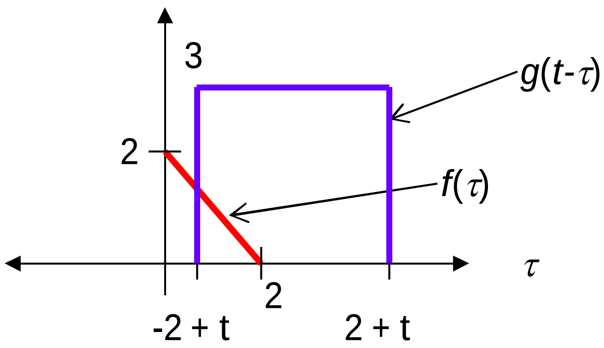
- Here, $g(t)$ completely overlaps $f(t)$
- Area under the product is just

$$\int_0^2 3(-\tau+2)\,d\tau = 3\left(-\frac{\tau^2}{2}+2\tau\right)\Bigg|_0^2 = 6$$



IV. $2 \le t < 4$

- Part of $g(t)$ and $f(t)$ overlap
- Calculated similarly to $-2 \le t < 0$
- $3t^2/2 - 12t + 24$

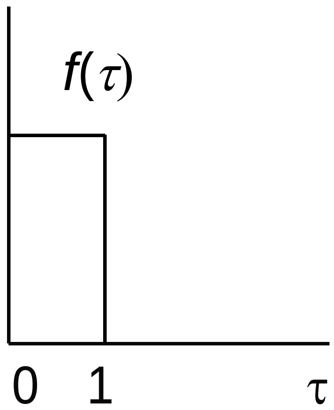


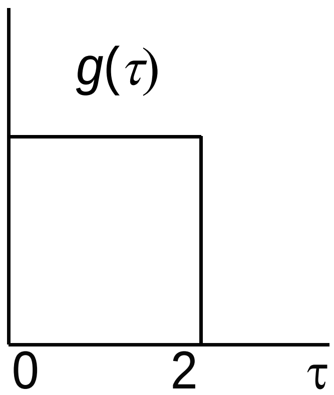
V. $t > 4$

- $g(t)$ and $f(t)$ do not overlap
- Area under their product is zero

Example

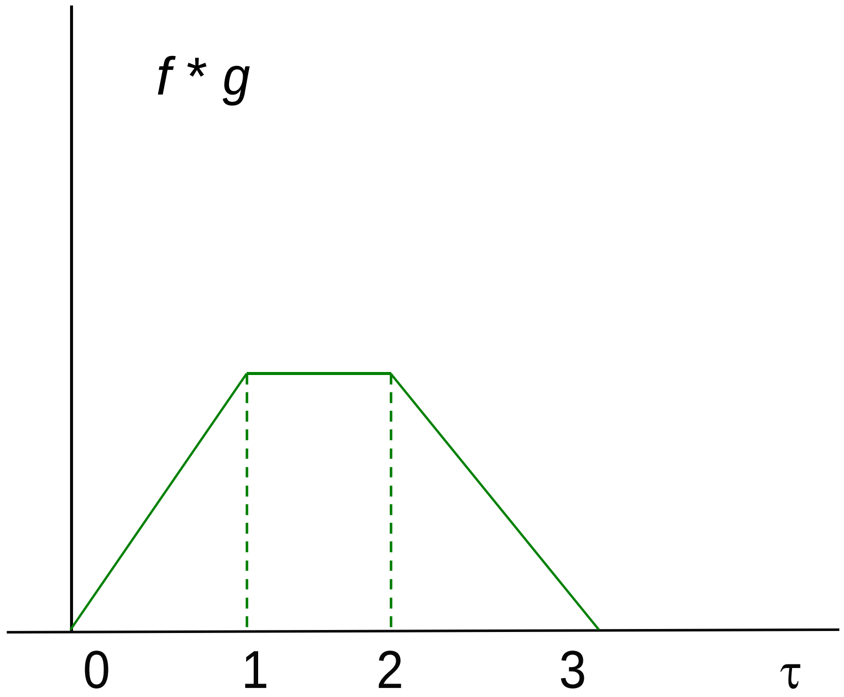
- Convolution of two gate pulses each of height 1





$$y = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$$

Example



Case 1: 0 <= t < 1
Case 2: 1<= t <=2
Case 3: 2 <= t <= 3
Case $: t > 3

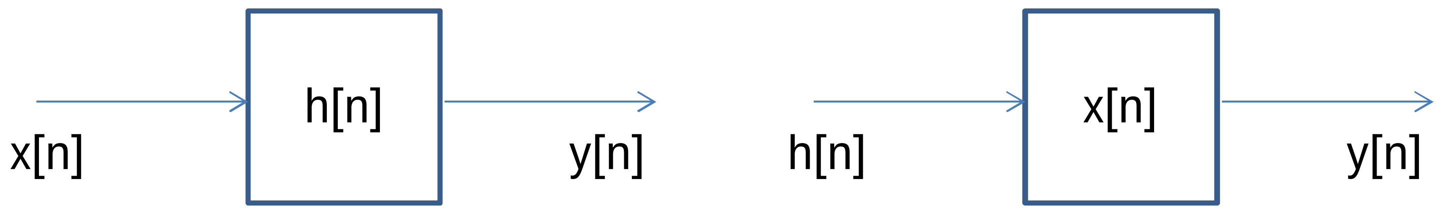
Properties of LTI systems

- Commutative Property: Roles of the input and impulse response can be interchanged
- CT Systems: $y(t) = x(t) * h(t) = h(t) * x(t)$

$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

- DT systems: $y[n] = x[n] * h[n] = h[n] * x[n]$

$$= \sum_{-\infty}^{\infty} x[k]h[n-k] = \sum_{-\infty}^{\infty} h[k]x[n-k]$$

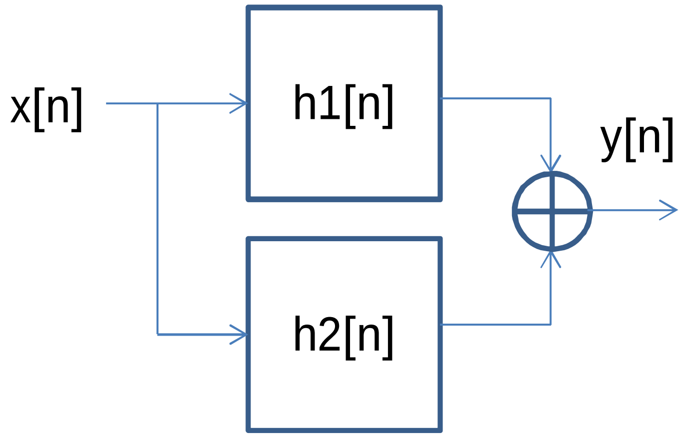


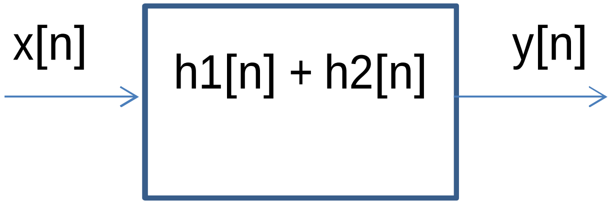
Properties of LTI systems

- Distributive

- CT systems $$y(t) = x(t) * (h1(t) + h2(t))$$ $$= x(t) * h1(t) + x(t) * h2(t)$$

- DT systems $$y[n] = x[n] * (h1[n] + h2[n])$$ $$= x[n] * h1[n] + x[n] * h2[n]$$



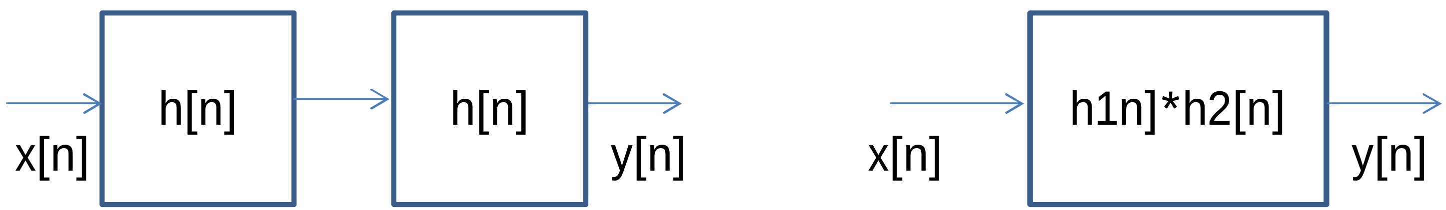


Properties of LTI systems

- Associative

- CT systems
$$y(t) = x(t) * (h1(t) * h2(t))$$
$$= (x(t) * h1(t)) * h2(t)$$
$$= h1(t) * (x(t) * h2(t))$$

- DT systems
$$y[n] = x[n] * (h1[n] * h2[n])$$
$$= (x[n] * h1[n]) * h2[n]$$
$$= h1[n] * (x[n] * h2[n])$$



- Memoryless: A LTI system is memoryless if its impulse response is

$$h[n] = K\delta[n]\,; h(t) = K\delta(t)$$

- Causality: An LTI system is causal if its output does not depend on future values of input. Or, output at [n] must not depend on k>n.

$$h[n] = 0 \ \text{ for } n < 0; h(t) = 0 \ \text{ for } t < 0$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \ = \sum_{k=-\infty}^{n} x[k]h[n-k]$$

- Stability: A LTI system is stable if its impulse response

$$\sum_{k=-\infty}^{\infty} | h[k] | < \infty; \int_{-\infty}^{\infty} | h(t) | \, dt < \infty$$

Example

- Find if memoryless, causal and stable?

a) $h(t) = u(t+1) - u(t-1)$

b) $h(t) = u(t) - 2u(t-1)$

c) $h(t) = e^{-2|t|}$

d) $h(t) = e^{at}u(t)$

e) $h[n]=2^{n}u[-n]$

f) $h[n] = e^{2n}u[n - 1]$

- Invertibility: A LTI system is invertible if

$$h[n] * h^{-1}[n] = \delta[n]$$

$$h(t) * h^{-1}(t) = \delta(t)$$

- Find a causal inverse system of y[n] = x[n] + ax[n-1]. Recall the echo problem or multipath communication problem. A signal may travel through different paths.

- Also find if inverse system is stable?

RADAR Range measurement

- Suppose we transmit an RF pulse and determine the round trip time delay

$$x(t) = \begin{cases} \sin(\omega t), 0 \le t \le T \\ 0, otherwise \end{cases}$$

We need to compute the received signal and channel's impulse response.

Matched filter

- We need to compute β, towards this we need to match the received signal with the transmitted signal.

- We can build an LTI system, such that the impulse response is

$$h(-t) = \begin{cases} -\sin \omega t, -T \le t \le 0 \\ 0, otherwise \end{cases}$$

Example

$$If\ \ x(t) = e^{-at}u(t), a > 0$$

$$and\ h(t) = u(t)$$

$$for\ t < 0, \text{then product of }\ x(\tau) \text{ and } h(t\text{-}\tau) = 0$$

$$x(\tau)h(t\text{-}\tau) = \begin{cases} e^{-a\tau}, 0 < \tau < t \\ 0, \text{otherwise} \end{cases}$$

$$y(t) = \int_0^t e^{-a\tau} d\tau = -\frac{1}{a} e^{-a\tau}\Big|_0^t = \frac{1}{a}(1 - e^{-at})$$

thus for all t, y(t) is

$$\text{y(t)} = \frac{1}{a}(1 - e^{-at})u(t)$$

Example

- If y(t) denote convolution of following two signals

$$x(t) = e^{2t}u(-t)$$

$$h(t) = u(t-3)$$

$$when\ (\mathrm{t}-3) \le 0\text{, the product of } \mathrm{x}(\tau) \text{ and } \mathrm{h}(\mathrm{t}-\tau) \text{ is nonzero}$$

$$\text{for } -\infty < \tau < \mathrm{t}-3$$

$$\text{for } \mathrm{t} > 3\text{, product is non zero for } -\infty < \tau < 0$$