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تمرین سری چهارم

جواب سؤال ۱)

a) $x(t) = e^{-3|t|} \sin 2t$

$$X(\omega) = F\{e^{-3|t|}\} * F\{\sin 2t\}$$

$$= \frac{2\alpha}{\alpha^2+\omega^2} * \frac{j}{2}\left(\delta(\omega-2) - \delta(\omega+2)\right)$$

$$= \left(\frac{6}{9+\omega^2}\right) * \frac{j}{2}\left(\delta(\omega-2) - \delta(\omega+2)\right)$$

$$= \frac{j}{2}\left(\frac{6}{9+(\omega-2)^2} - \frac{6}{9+(\omega+2)^2}\right)$$

b) $x(t) = \begin{cases} 1-t^2 & 0<t<1 \\ 0 & \text{o.w} \end{cases}$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt = \int_0^1 (1-t^2) e^{-j\omega t} dt =$$

$$\int_0^1 e^{-j\omega t} - t^2 e^{-j\omega t} dt = \frac{1}{-j\omega} e^{-j\omega t}\Big|_0^1 - \int_0^1 \underbrace{t^2}_{u} \underbrace{e^{-j\omega t} dt}_{dv} =$$

$$\frac{e^{-j\omega}}{-j\omega} + \frac{1}{j\omega} - \left[\left(\frac{e^{-j\omega t}}{-j\omega}\right) t^2\Big|_0^1 - \int_0^1 \underbrace{\left(\frac{e^{-j\omega t}}{-j\omega}\right)}_{dv} \underbrace{2t}_{u} dt\right] =$$

$$-\frac{e^{-j\omega}}{j\omega} + \frac{1}{j\omega} + \frac{e^{-j\omega}}{j\omega} - \int_0^1 \underbrace{\frac{e^{-j\omega t}}{+j\omega}}_{dv} \underbrace{(2t)}_{u} dt =$$

$$\frac{1}{j\omega} - \frac{e^{-j\omega t}}{\omega^2}(2t)\Big|_0^1 + \int_0^1 2\frac{e^{-j\omega t}}{+\omega^2} dt = \frac{1}{j\omega} - \frac{2e^{-j\omega}}{\omega^2} + \frac{2e^{-j\omega} - 2}{-j\omega^3}$$

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c) $$x(t) = \frac{\sin ۳t \ \cos t}{\pi t}$$

$$x(t) = \underbrace{\frac{۳ \sin ۳t}{۳\pi t}}_{g(t)} \ \underbrace{\cos t}_{h(t)}$$

$$G(j\omega) = \begin{cases} ۱ & |\omega| < ۳ \\ ۰ & o.w \end{cases}$$

$$H(j\omega) = \frac{۱}{۲} \left(\delta(\omega - ۱) + \delta(\omega + ۱) \right)$$

$$X(j\omega) = G(j\omega) * H(j\omega) = \frac{۱}{۲} \left[G(j(\omega - ۱)) + G(j(\omega + ۱)) \right]$$

$$X(j\omega) = \begin{cases} \frac{۱}{۲} & -۴ < \omega < -۲, \quad ۲ < \omega < ۴ \\ ۱ & -۲ \leq \omega \leq ۲ \\ ۰ & o.w \end{cases}$$

d) $$x(t) = t \, e^{-۲|t-۱|}$$

$$e^{-a|t|} \xrightarrow{F.T} \frac{۲a}{a^۲ + \omega^۲}$$

$$e^{-a|t-۱|} \xrightarrow{F.T} \frac{۲a}{a^۲ + \omega^۲} e^{-j\omega}$$

$$-jt \, e^{-a|t-۱|} \xrightarrow{F.T} \frac{d}{d\omega} \left(\frac{۲a \, e^{-j\omega}}{a^۲ + \omega^۲} \right)$$

$$t \, e^{-a|t-۱|} \xrightarrow{F.T} \frac{-۲aj \, e^{-j\omega} (a^۲ + \omega^۲) - ۴\omega a \, e^{-j\omega}}{-j \, (a^۲ + \omega^۲)^۲}$$

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$a = ۲$

$$X(\omega) = \frac{۴je^{-j\omega}(۴+\omega^{۲}) + ۸\omega e^{-j\omega}}{+j(۴+\omega^{۲})^{۲}}$$

جواب سؤال ۲)

a) $$X(\omega) = \omega e^{-|\omega|}$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} \omega e^{-|\omega|} e^{j\omega t} \, d\omega =$$

$$\frac{1}{2\pi} \int_{0}^{+\infty} \underbrace{\omega}_{u} \underbrace{e^{(-1+jt)\omega}}_{dv} \, d\omega + \int_{-\infty}^{0} \omega e^{(jt+1)\omega} \, d\omega =$$

$$\frac{1}{2\pi}\left(\omega \frac{e^{-(-jt+1)\omega}}{(-1+jt)} - \int_{0}^{+\infty} \frac{e^{(-1+jt)\omega}}{(-1+jt)} d\omega + \frac{\omega e^{(jt+1)\omega}}{(jt+1)} - \int_{-\infty}^{0} \frac{e^{(jt+1)\omega}}{(jt+1)} d\omega \right)$$

$$= \frac{1}{2\pi}\left[\left(\frac{\omega e^{-(-jt+1)\omega}}{jt-1} \Big|_{0}^{+\infty} - \frac{e^{(-1+jt)\omega}}{(-1+jt)^2} \Big|_{0}^{+\infty} \right) + \left(\frac{\omega e^{(jt+1)\omega}}{(jt+1)} \Big|_{-\infty}^{0} - \frac{e^{(jt+1)\omega}}{(jt+1)^2} \Big|_{-\infty}^{0} \right)\right]$$

$$= -\frac{1}{2\pi}\left(+\frac{1}{(-jt+1)^2} - \frac{1}{(jt+1)^2} \right)$$

روش دوم این سؤال با استفاده از خواص است
که در صفحه‌ی بعد نوشته شده است

b) $$X(\omega) = \begin{cases} e^{-\omega}, & \omega > 0 \\ -e^{\omega}, & \omega < 0 \end{cases}$$

$$x(t) = \frac{1}{2\pi}\int_{0}^{+\infty} e^{-\omega} e^{j\omega t} \, d\omega + \frac{1}{2\pi}\int_{-\infty}^{0} -e^{\omega} e^{j\omega t} \, d\omega =$$

$$\frac{1}{2\pi}\left(\frac{e^{\omega(-1+jt)}}{-1+jt} \Big|_{0}^{+\infty} - \frac{e^{(jt+1)\omega}}{jt+1} \Big|_{-\infty}^{0} \right) =$$

$$\frac{1}{2\pi}\left(\frac{e^{-\omega(-jt+1)}}{-1+jt} \Big|_{0}^{+\infty} - \frac{e^{(jt+1)\omega}}{jt+1} \Big|_{-\infty}^{0} \right) = \left(\frac{1}{-jt+1} - \frac{1}{jt+1} \right) \frac{1}{2\pi}$$

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روش دوم سوال ۴ قسمت الف)

$$e^{-\alpha|t|} \xrightarrow{F.T} \frac{2\alpha}{\alpha^2+\omega^2}$$ می‌دانیم

$$jt\, e^{-\alpha|t|} \xrightarrow{F.T} \frac{d}{d\omega}\left(\frac{2\alpha}{\alpha^2+\omega^2}\right) = \frac{-4\omega\alpha}{(\alpha^2+\omega^2)^2}$$

$$t e^{-\alpha|t|} \xrightarrow{F.T} \frac{+4j\omega\alpha}{(\alpha^2+\omega^2)^2}$$ دوگانی

$$\frac{4jt\alpha}{(\alpha^2+t^2)^2} \xrightarrow{F.T} 2\pi(-\omega)\, e^{-\alpha|-\omega|}$$

$$F^{-1}\left\{\omega e^{-\alpha|\omega|}\right\} = -\frac{1}{2\pi}\frac{4jt\alpha}{(\alpha^2+t^2)^2} \overset{\alpha=1}{=}$$

$$-\frac{1}{2\pi}\frac{4jt}{(1+t^2)^2}$$

c) $$X(\omega) = \frac{۲a - j\omega}{۲a + j\omega} = \frac{۴a}{۲a + j\omega} - ۱$$

$$F^{-1}\{X(\omega)\} = -\delta(t) + ۴a\, e^{-۲at}\, u(t)$$

d) $$X(\omega) = \frac{d}{d\omega}\left\{ \frac{\sin ۲\omega - j\cos ۲\omega}{1 + \frac{j\omega}{۳}} \right\}$$

$$F^{-1}\left\{ \frac{\sin ۲\omega - j\cos ۲\omega}{1 + \frac{j\omega}{۳}} \right\} = F^{-1}\left\{ \frac{\left(\frac{e^{۲j\omega} - e^{-۲j\omega}}{۲j}\right) - j\left(\frac{e^{۲j\omega} + e^{-۲j\omega}}{۲}\right)}{1 + \frac{j\omega}{۳}} \right\} =$$

$$F^{-1}\left\{ \frac{\frac{۲e^{۲j\omega}}{۲j}}{1 + \frac{j\omega}{۳}} \right\} = -j\, F^{-1}\left\{ \frac{e^{۲j\omega}}{1 + \frac{j\omega}{۳}} \right\} = -۳j\, e^{-۳(t+۲)}\, u(t+۲)$$

$$x(t) = (-jt)(-۳j)\, e^{-۳(t+۲)}\, u(t+۲) \Rightarrow$$

$$x(t) \quad -۳t\, e^{-۳(t+۲)}\, u(t+۲)$$

جواب سؤال ۳)

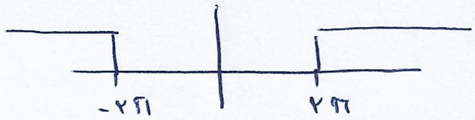
داریم $x(t) = \frac{W}{\pi} \frac{\sin Wt}{Wt} \longrightarrow FT = X(\omega) = \begin{cases} 1 & |\omega| \leq W \\ 0 & o.w \end{cases}$

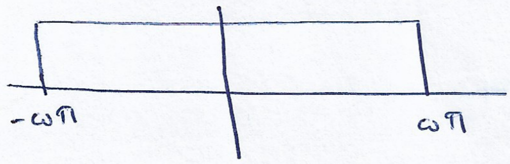
$$h_2(t) = \frac{\sin 5\pi t}{\pi t} = 5 \frac{\sin 5\pi t}{5\pi t} \qquad W = 5\pi$$

$$H_2(\omega) = \begin{cases} 1 & |\omega| \leq 5\pi \\ 0 & o.w \end{cases}$$

$$h_T(t) = h_1(t) * h_2(t) \longrightarrow H_T(\omega) = H_1(\omega) \cdot H_2(\omega)$$

$$H_1(\omega) = \begin{cases} 1 & |\omega| \geq 2\pi \\ 0 & |\omega| < 2\pi \end{cases}$$





$$H_T(\omega) = \begin{cases} 1 & -5\pi < \omega \leq -2\pi \\ 1 & 2\pi \leq \omega < 5\pi \\ 0 & o.w \end{cases}$$

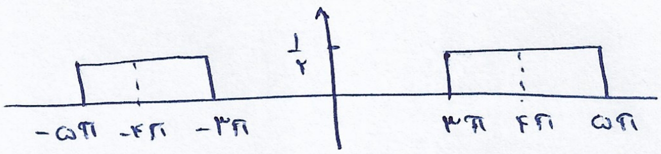
$$a(t) = x(t) \cdot p(t) = \frac{\sin \pi t}{\pi t} \cos 4\pi t$$

$$c(t) = a(t) * h_T(t) \longrightarrow C(\omega) = A(\omega) H_T(\omega)$$

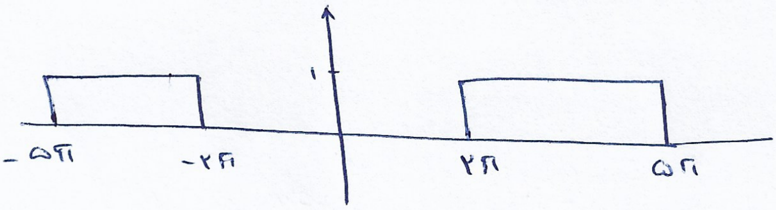
$$X(\omega) = \begin{cases} 1 & |\omega| \leq \pi \\ 0 & o.w \end{cases} * \frac{1}{2}\left(\delta(\omega - 4\pi) + \delta(\omega + 4\pi)\right)$$

$$A(\omega) = \frac{1}{2} X(\omega - 4\pi) + \frac{1}{2} X(\omega + 4\pi)$$

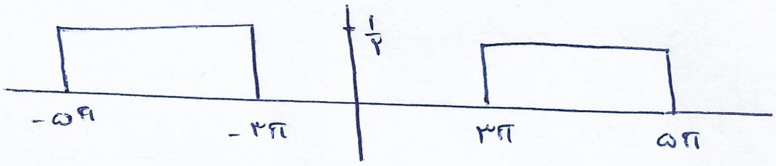
$A(\omega)$:



$H_T(\omega)$:



$C(\omega) =$



$$C(\omega) = \begin{cases} \frac{1}{2} & -5\pi < \omega < -3\pi \\ \frac{1}{2} & 3\pi < \omega < 5\pi \\ 0 & \text{o.w} \end{cases}$$

$$c(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} C(\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{2}\, e^{j\omega t}\, d\omega =$$

$$\frac{1}{4\pi} \left(\frac{e^{j\omega t}}{jt} \Bigg|_{-5\pi}^{-3\pi} + \frac{e^{j\omega t}}{jt} \Bigg|_{3\pi}^{5\pi} \right) =$$

$$\frac{1}{4\pi} \left(- \frac{e^{-5\pi jt}}{jt} + \frac{e^{-3\pi jt}}{jt} + \frac{e^{5\pi jt}}{jt} - \frac{e^{3\pi jt}}{jt} \right) =$$

$$\frac{1}{2\pi t} \left(\sin 5\pi t - \sin 3\pi t \right) = \frac{2 \cos 4\pi t \,\sin \pi t}{2\pi t} = \frac{\cos 4\pi t \,\sin \pi t}{\pi t}$$

جواب سؤال ۴)

$$x(t) = \begin{cases} t & , -1 < t < 1 \\ u(2-t) & t > 1 \\ -u(t+2) & t < -1 \end{cases}$$

$$\frac{dx(t)}{dt} = \begin{cases} 1 & , -1 < t < 1 \\ -\delta(2-t) & t > 1 \\ -\delta(t+2) & t < -1 \end{cases}$$

$$\frac{dx(t)}{dt} = \Pi\left(\frac{t}{2}\right) \longrightarrow F\left\{\frac{dx(t)}{dt}\right\} = F\left\{\Pi(t)\right\} = T \operatorname{sinc}\frac{\omega}{\pi} \times \frac{T}{2} =$$

$$2 \operatorname{sinc}\frac{\omega}{\pi} = 2 \underbrace{\left(\frac{T}{2}\right)}_{1} \frac{\sin\omega}{\omega} = 2\frac{\sin\omega}{\omega}$$

$$F\left\{\frac{dx(t)}{dt}\right\} = j\omega X(\omega)$$

$$F\left\{-\delta(2-t) - \delta(t+2)\right\} = -e^{-2j\omega} - e^{2j\omega} = -2\cos 2\omega$$

$$\frac{2\sin\omega}{\omega} - 2\cos 2\omega = j\omega X(\omega)$$

$$X(\omega) = -\frac{j2\sin\omega}{\omega^2} - \frac{2\cos 2\omega}{j\omega}$$

جواب سؤال ۵)

الف) تبدیل فوریه $x(t)$ برابر است با $X(j\omega)$ $\quad \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{j\omega t}\, d\omega = x(t)$

ب) $x(t)$ حقیقی است

بخش حقیقی X تقارن زوج دارد $\quad Re\{X(j\omega)\} = Re\{X(-j\omega)\}$

بخش موهومی X تقارن فرد دارد $\quad Im\{X(j\omega)\} = -Im\{X(-j\omega)\}$

ج) $t \leqslant 0 \;:\; x(t) = 0 \;\Rightarrow\; x(t)$ علی است

د) $\frac{1}{2\pi}\int_{-\infty}^{+\infty} Real\{X(j\omega)\}\, e^{j\omega t}\, d\omega = |t|\, e^{-|t|}$

تبدیل فوریه قسمت زوج سیگنال برابر با قسمت حقیقی تبدیل فوریه

$$x_{even} = \frac{x(t) + x(-t)}{2} \xrightarrow{\text{تبدیل فوریه}} Real\{X(j\omega)\}$$

$$x_{even} = \frac{x(t) + x(-t)}{2} = |t|\, e^{-|t|}$$

برای $t > 0$ $\quad x(t) = 0 \;\text{ for }\; t \leqslant 0 \;\Rightarrow\; x(-t) = 0$

$$x_{even} = \frac{x(t)}{2} = |t|\, e^{-|t|} \;\Rightarrow\; x(t) = 2|t|\, e^{-|t|}$$

برای $t > 0$

$$x(t) = 2|t|\, e^{-|t|}\, u(t)$$

جواب سؤال ۶)

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t-2k)$$

متناوب است با دوره تناوب $T = 2$

$$\Rightarrow a_k = \frac{1}{2} \int_{-1}^{1} \delta(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{2}$$

$$X(\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k\, \delta(\omega - k\omega_0) = \pi \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_0)$$

$$Y(\omega) = H(\omega)\, X(\omega) = \pi \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_0)$$

$$y(t) = \frac{1}{2} \sum_{k=-\infty}^{+\infty} e^{jk\omega_0 t}$$

جواب سؤال ۷)

قسمت الف)

$$H(j\omega) = \frac{j\omega + 2}{6 - \omega^2 + 5j\omega}$$

$$X(j\omega)\, H(j\omega) = Y(j\omega)$$

$$(j\omega + 2)\quad X(j\omega) = (6 + (j\omega)^2 + 5j\omega)\; Y(j\omega)$$

$$j\omega\, X(j\omega) + 2\, X(j\omega) = 6\, Y(j\omega) + 5j\omega\, Y(j\omega) + (j\omega)^2\, Y(j\omega)$$

$$\frac{d^2 y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6\, y(t) = \frac{dx(t)}{dt} + 2x(t)$$

قسمت ب)

$$H(j\omega) = \frac{j\omega + 2}{(j\omega + 2)(j\omega + 3)} = \frac{1}{j\omega + 3}$$

$$h(t) = F^{-1}\left\{\frac{1}{j\omega + 3}\right\} = e^{-3t}\, u(t)$$

قسمت ج)

$$X(j\omega) = \frac{1}{4 + j\omega} - \frac{1}{(4 + j\omega)^2}$$

$$Y(j\omega) = H(j\omega)\, X(j\omega) = \left(\frac{1}{j\omega + 3}\right)\left(\frac{1}{4 + j\omega} - \frac{1}{(4 + j\omega)^2}\right)$$

$$\left(\frac{1}{(j\omega + 3)}\right)\left(\frac{3 + j\omega}{(4 + j\omega)^2}\right) = \frac{1}{(4 + j\omega)^2}$$

$$F^{-1}\left\{\frac{1}{(4 + j\omega)^2}\right\} = ?$$

$$e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{j\omega+a}$$

$$-jte^{-at}u(t) \longleftrightarrow \frac{d}{d\omega}\left(\frac{1}{j\omega+a}\right)$$

$$-jte^{-at}u(t) \longleftrightarrow \frac{j}{(j\omega+a)^2}$$

$$y(t) = t\,e^{-4t}u(t)$$

تمرین ۲

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} e^{-3\tau}u(\tau)\,e^{2(t-\tau)}\,d\tau =$$

$$\int_{-\infty}^{+\infty} u(\tau)\,e^{-5\tau+2t}\,d\tau = e^{2t}\int_{0}^{+\infty} e^{-5\tau}\,d\tau =$$

$$e^{2t}\int_{0}^{+\infty} e^{-5\tau}\,d\tau = e^{2t} \times \frac{e^{-5\tau}}{5}\Big|_{0}^{+\infty} = \frac{e^{2t}}{5}$$

جواب سؤال ۸)

قسمت الف)

$$\frac{d^2y(t)}{dt^2}+6\frac{dy(t)}{dt}+9y(t)=\frac{d^2x(t)}{dt^2}+3\frac{dx(t)}{dt}+2x(t)$$

$$Y(j\omega)\left((j\omega)^2+6j\omega+9\right)=X(j\omega)\left((j\omega)^2+3(j\omega)+2\right)$$

$$H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{(j\omega)^2+3j\omega+2}{(j\omega)^2+6j\omega+9}=\frac{(j\omega+2)(j\omega+1)}{(j\omega+3)^2}$$

$$H(j\omega)=1-\left(\frac{A}{(j\omega+3)}+\frac{B}{(j\omega+3)^2}\right)=1-\frac{3j\omega+7}{(j\omega+3)^2}$$

$$A(j\omega+3)+B=3j\omega+7 \qquad B=-2 \quad A=3$$

$$H(j\omega)=1-\frac{3}{j\omega+3}+\frac{2}{(j\omega+3)^2}$$

$\downarrow$ تبدیل فوریه معکوس

$$h(t)=\delta(t)-3e^{-3t}+2te^{-3t}u(t)$$

قسمت ب)

$$G(j\omega)=\frac{1}{H(j\omega)}=\frac{(j\omega+3)^2}{(j\omega+2)(j\omega+1)}$$

$$G(j\omega)=\frac{-\omega^2+6j\omega+9}{-\omega^2+3j\omega+2}=\frac{Y(j\omega)}{X(j\omega)}$$

$$\frac{d^2x(t)}{dt^2}+6\frac{dx(t)}{dt}+9x(t)=\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)$$

$$g(t) = F^{-1}\{G(j\omega)\} = F^{-1}\left\{1 + \frac{A}{j\omega+۲} + \frac{B}{j\omega+1}\right\}$$

$$Aj\omega + A + Bj\omega + ۲B = ۷ + ۳j\omega$$

$$\begin{cases} A+B = ۳ \\ A+۲B = ۷ \end{cases} \qquad B = ۴ \qquad A = -1$$

$$g(t) = \delta(t) - e^{-۲t}u(t) + ۴e^{-t}u(t)$$