Unit 11 Trigonometric Functions and their Graphs

INTRODUCTION

In this unit, students will explore key concepts essential for understanding the role of trigonometry in mathematics and its real-life applications. We will begin by learning how to determine the domain and range of trigonometric functions to understand their behavior. Next, we will discuss even and odd functions, along with their periodicity, which explains their repeating patterns.

Students will then learn how to graph and analyze sine, cosine, and tangent functions, following this, we will focus on calculating the maximum and minimum values of sinusoidal functions and examining their unique properties such as amplitude, frequency, and phase shifts.

Finally, students will apply these trigonometric concepts to solve practical problems in navigation, engineering, and physics, including calculating distances, optimizing solar panel angles, and analyzing forces in structures. Mastering these concepts will enable students to solve both theoretical and real-world problems using trigonometry.

Let us first find domains and ranges of trigonometric functions before drawing their graphs.

11.1 Domains and Ranges of Sine and Cosine Functions

We have already defined trigonometric functions $\sin\theta$, $\cos\theta$, $\tan\theta$, $\csc\theta$, $\sec\theta$ and $\cot\theta$. We know that if $P(x, y)$ is any point on unit circle with centre at the origin O such that $m\angle XOP = \theta$ in standard position, then

$$\cos\theta = x \quad \text{and} \quad \sin\theta = y$$

$\Rightarrow$ for any real number θ there is one and only one value of each x and y i.e., of each $\cos\theta$ and $\sin\theta$.

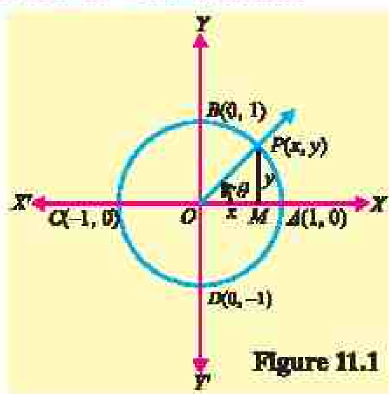


Figure 11.1

Hence $\sin\theta$ and $\cos\theta$ are the functions of θ and their domain is $\mathbb{R}$, the set of real numbers.

Since $P(x, y)$ is a point on the unit circle with centre at the origin O, therefore

$$-1 \le x \le 1 \qquad \text{and} \qquad -1 \le y \le 1$$

$$\Rightarrow \quad -1 \le \cos\theta \le 1 \qquad \text{and} \qquad -1 \le \sin\theta \le 1$$

Thus, the range of sine and cosine functions is $[-1, 1]$.

11.1.1 Domains and Ranges of Tangent and Cotangent Functions

From the Figure 11.1.

(i) $\tan\theta = \dfrac{y}{x}, x \neq 0$

$\Rightarrow$ terminal side $\overrightarrow{OP}$ should not coincide with OY or OY' (the Y-axis)

$\Rightarrow \theta \neq \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \pm\dfrac{5\pi}{2}, \ldots$

$\Rightarrow \theta \neq (2n+1)\dfrac{\pi}{2}$, where $n \in Z$

Domain of tangent function $= R - \{x \mid x = (2n+1)\dfrac{\pi}{2}, n \in Z\}$

If $y = \dfrac{1}{2}$, $\tan\theta = \dfrac{1}{2x}$ as $x \to 0$, $\dfrac{1}{2x} \to \pm\infty$ therefore the range of tangent function $= R =$ set of real numbers.

(ii) From Figure 11.1

$$\cot\theta = \frac{x}{y}, y \neq 0$$

$\Rightarrow$ terminal side $\overrightarrow{OP}$ should not coincide with OX or OX' (the X-axis)

$\Rightarrow \theta \neq 0, \pm\pi, \pm 2\pi, \ldots$

$\Rightarrow \theta \neq n\pi$, where $n \in Z$

Domain of cotangent function $= R - \{x \mid x = n\pi, n \in Z\}$

If $x = \dfrac{1}{2}$, $\cot\theta = \dfrac{1}{2y}$ as $y \to 0$, $\dfrac{1}{2y} \to \pm\infty$ therefore range of cotangent function $= R =$ set of real numbers.

11.1.2 Domain and Range of Secant Function

From the Figure 11.1

$$\sec\theta = \frac{1}{x}, x \neq 0$$

$\Rightarrow$ terminal side $\overrightarrow{OP}$ should not coincide with OY or OY' (the Y-axis)

$\Rightarrow \theta \neq \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \pm\dfrac{5\pi}{2}, \ldots$

$\Rightarrow \theta \neq (2n+1)\dfrac{\pi}{2}$, where $n \in Z$

Domain of secant function $= R - \{x \mid x = (2n+1)\dfrac{\pi}{2}, n \in Z\}$

As $0 \le x \le 1$ so, $\frac{1}{x} \ge 1$, $\sec\theta \ge 1$ and $-1 \le x \le 0$ so, $\frac{1}{x} \le -1$, $\sec\theta \le -1$

As $\sec\theta$ attains all real values except those between -1 and 1

Range of secant function $= R - \{x \mid -1 < x < 1\}$

11.1.3 Domain and Range of Cosecant Function

From the Figure 11.1

$$\csc\theta = \frac{1}{y},\ y \ne 0$$

$\Rightarrow$ terminal side $\overrightarrow{OP}$ should not coincide with OX or OX' (the X–axis)

$\Rightarrow$ $\theta \ne 0, \pm\pi, \pm 2\pi, \ldots$

$\Rightarrow$ $\theta \ne n\pi$, where $n \in Z$

Domain of cosecant function $= \text{R} - \{x \mid x = n\pi,\ n \in Z\}$

As $\csc\theta$ attains all values except those between -1 and 1

Range of cosecant function $= \text{R} - \{x \mid -1 < x < 1\}$

The following table summarizes the domains and ranges of the trigonometric functions:

Function	Domain	Range
$y = \sin x$	$(-\infty, \infty) = R$	$[-1, 1]$
$y = \cos x$	$(-\infty, \infty) = R$	$[-1, 1]$
$y = \tan x$	$R = (-\infty, \infty), x \ne (2n+1)\frac{\pi}{2}, n \in Z$	$(-\infty, \infty) = R$
$y = \cot x$	$R = (-\infty, \infty), x \ne n\pi, n \in Z$	$(-\infty, \infty) = R$
$y = \sec x$	$(-\infty, \infty), x \ne (2n+1)\frac{\pi}{2}, n \in Z$	$(-\infty, -1] \cup [1, \infty)$
$y = \operatorname{cosec} x$	$(-\infty, \infty), x \ne n\pi, n \in Z$	$(-\infty, -1] \cup [1, \infty)$

11.2 Even and Odd Functions

A function f is said to be even if $f(-x) = f(x)$, for every number x in the domain of f.

For example: $f(x) = x^2$ is even function of x. Here

$$f(-x) = (-x)^2 = x^2 = f(x)$$

Remember!

The graph of even function is always symmetric about y-axis

A function f is said to be **odd** if $f(-x)=-f(x)$, for every number x in the domain of f.

For example: $f(x)=x^3$ is an odd function of x.

Here $f(-x)=(-x)^3=-x^3=-f(x)$

Remember!

The graph of odd function is always symmetric about the origin.

The function $f(\theta)=\cos\theta$ for all $\theta\in R$ is an even function (see figure 11.2).

Here $f(-\theta)=\cos(-\theta)=\cos\theta=f(\theta)$.

Thus, $f(\theta)=\cos\theta$ is an even function.

Similarly, the function $f(\theta)=\sin\theta$ for all $\theta\in R$ is an odd function.

Here $f(-\theta)=\sin(-\theta)=-\sin\theta=-f(\theta)$.

Thus, $f(\theta)=\sin\theta$ is an odd function.

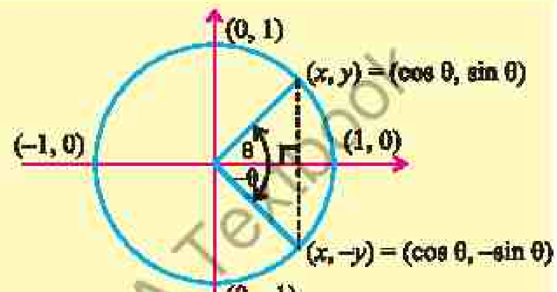


Figure 11.2

Note In both the cases, for each x in the domain of f, $-x$ must also be in the domain of f.

11.3 Period of Trigonometric Functions

All the six trigonometric functions repeat their values for each increase or decrease of 2π in θ therefore, the values of trigonometric functions for θ and $\theta\pm 2n\pi$, where $\theta\in$ R and $n\in Z$, are the same. This behaviour of trigonometric functions is called **periodicity**.

Period of a trigonometric function is the smallest positive number which, when added to the original circular measure of the angle, gives the same value of the function. A function is periodic, if $f(\theta+p)=f(\theta)$, for all θ in domain of function and the least positive value of p is called the period of the function.

Now, let us discover the periods of the trigonometric functions.

Theorem 11.1: Sine is a periodic function and its period is 2π.

Proof: Suppose p is the period of sine function such that

$$\sin(\theta+p)=\sin\theta \text{ for all } \theta\in R \qquad \text{(A)}$$

Now put $\theta=0$, we have

$$\sin(0+p)=\sin 0$$

$$\Rightarrow \quad \sin p=0$$

$$\Rightarrow \quad p=0, +\pi, +2\pi, +3\pi, \ldots$$

(i) If $p = \pi$, then from (A)

$\sin(\theta + \pi) = \sin\theta$ (not true) $\because \sin(\pi + \theta) = -\sin\theta$

Thus π is not the period of $\sin\theta$

(ii) If $p = 2\pi$, then from (A)

$\sin(\theta + 2\pi) = \sin\theta$, which is true $\because \sin(\theta + 2\pi) = \sin\theta$

As 2π is the smallest positive real number for which

$\sin(\theta + 2\pi) = \sin\theta$

2π is the period of $\sin\theta$.

Theorem 11.2: Tangent is a periodic function and its period is π.

Proof: Suppose p is the period of tangent function such that

$\tan(\theta + p) = \tan\theta$ for all $\theta \in R$ (B)

$p = 0, \pi, 2\pi, 3\pi, \ldots$

(i) If $p = \pi$, then from (B) $\tan(\theta + \pi) = \tan\theta$, which is true

As π is the smallest positive number for which

$\tan(\theta + \pi) = \tan\theta$

Therefore, π is the period of $\tan\theta$.

Note:
By adopting the procedure used in finding the periods of sine and tangent, we can prove that
(i) 2π is the period of $\cos\theta$
(ii) 2π is the period of $\csc\theta$
(iii) 2π is the period of $\sec\theta$
(iv) π is the period of $\cot\theta$.

Example 1 Find the periods of: (i) $\sin 2x$ (ii) $3 + \tan\dfrac{x}{3}$

Solution (i) We know that the period of sine is 2π

$\therefore \sin(2x + 2\pi) = \sin 2x \Rightarrow \sin 2(x + \pi) = \sin 2x$

It means that the value of $\sin 2x$ repeats when x is increased by π.

Hence π is the period of $\sin 2x$.

(ii) To find the period of $3 + \tan\dfrac{x}{3}$, consider only $\tan\dfrac{x}{3}$.

We know that the period of tangent is π

$$\tan\left(\frac{x}{3} + \pi\right) = \tan\frac{x}{3} \Rightarrow \tan\frac{1}{3}(x + 3\pi) = \tan\frac{x}{3}$$

It means that the value of $\tan\dfrac{x}{3}$ repeats when x is increased by 3π.

Hence the period of $3 + \tan\dfrac{x}{3}$ is 3π. The addition of constant number 3 to the tangent function does not affect the period.

EXERCISE 11.1

1. Determine whether the following functions are even, odd or neither odd nor even.

(i) $\sin^2 x$ (ii) $\sin x + \cos x$ (iii) $\sin^4 x + \cos^4 x$

(iv) $\tan x + \sec x$ (v) $\dfrac{1}{\operatorname{cosec}^3 x}$ (vi) $\dfrac{\sin x + \sin 3x}{\cos x + \cos 3x}$

(vii) $\dfrac{1}{\sec x + \sec^3 x}$ (viii) $\dfrac{1}{\sec x + \cot^2 x}$

2. Find the periods of the following functions:

(i) $\sin 5x$ (ii) $\cos 7x$ (iii) $\tan 3x$ (iv) $\cot \dfrac{x}{2}$

(v) $19\sin\left(\dfrac{\pi}{20}x\right)$ (vi) $\operatorname{cosec}\left(\dfrac{2x}{5}\right)$ (vii) $\dfrac{1}{2}\sin\left(\dfrac{3x}{2}-\dfrac{\pi}{2}\right)$

(viii) $-5-3\sec\left(7\pi x+\dfrac{\pi}{4}\right)$ (ix) $12+10\tan\left(\dfrac{\pi}{30}x\right)$

(x) $6-4\cot\left(\dfrac{7x}{4}+\dfrac{\pi}{4}\right)$ (xi) $9+30\sec\left(\dfrac{x}{15}+\dfrac{2\pi}{15}\right)$

11.4 Values of Trigonometric Functions

We know the values of trigonometric functions for angles of measure $0°$, $30°$, $45°$, $60°$, and $90°$. We have also established the following identities:

$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$	$\tan(-\theta) = -\tan\theta$
$\sin(\pi-\theta) = \sin\theta$	$\cos(\pi-\theta) = -\cos\theta$	$\tan(\pi-\theta) = -\tan\theta$
$\sin(\pi+\theta) = -\sin\theta$	$\cos(\pi+\theta) = -\cos\theta$	$\tan(\pi+\theta) = \tan\theta$
$\sin(2\pi-\theta) = -\sin\theta$	$\cos(2\pi-\theta) = \cos\theta$	$\tan(2\pi-\theta) = -\tan\theta$

By using the above identities, we can easily find the values of trigonometric functions of the angles of the following measures:

$-30°, -45°, -60°, -90°$ $\pm 120°, \pm 135°, \pm 150°, \pm 180°$

$\pm 210°, \pm 225°, \pm 240°, \pm 270°$ $\pm 300°, \pm 315°, \pm 330°, \pm 360°$

11.4.1 Graphs of Trigonometric Functions

To plot the graph we shall follow these steps:

(i) Table of ordered pairs (x, y) is constructed, when x is the measure of the angle and y is the value of the trigonometric function for the angle of measure x.

(ii) The measures of the angles are taken along the X-axis.

(iii) The values of the trigonometric functions are taken along the Y-axis.

(iv) The points corresponding to the ordered pairs are plotted on the graph paper.
(v) These points are joined with the help of **smooth curves.**

11.4.2 Graph of $y = \sin x$ from -2π to 2π

We know that the period of sine function is 2π so, we will first draw the graph for the interval from $0°$ to $360°$ (from 0 to 2π).

To graph the sine function, first, recall that $-1 \le \sin x \le 1$ for all $x \in R$.

We know the range of the sine function is $[-1, 1]$, so the graph will be between the horizontal lines $y = +1$ and $y = -1$

The table of the ordered pairs satisfying $y = \sin x$ is as follows:

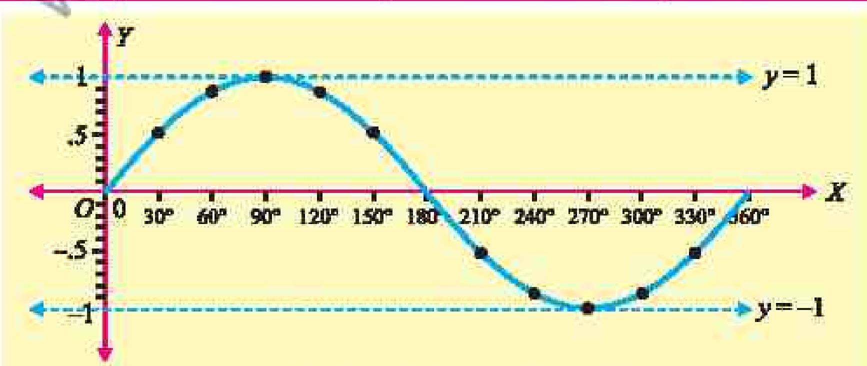
x	0 or $0°$	$\frac{\pi}{6}$ or $30°$	$\frac{\pi}{3}$ or $60°$	$\frac{\pi}{2}$ or $90°$	$\frac{2\pi}{3}$ or $120°$	$\frac{5\pi}{6}$ or $150°$	π or $180°$	$\frac{7\pi}{6}$ or $210°$	$\frac{4\pi}{3}$ or $240°$	$\frac{3\pi}{2}$ or $270°$	$\frac{5\pi}{3}$ or $300°$	$\frac{11\pi}{6}$ or $330°$	2π or $360°$
$\sin x$	0	0.5	0.87	1	0.87	0.5	0	-0.5	-0.87	-1	-0.87	-0.5	0

To draw the graph:

(i) Take a convenient scale $\begin{cases} 1 \text{ side of small square on the } x\text{-axis} = 10° \\ 1 \text{ side of big square on the } y\text{-axis} = 1 \text{ unit} \end{cases}$

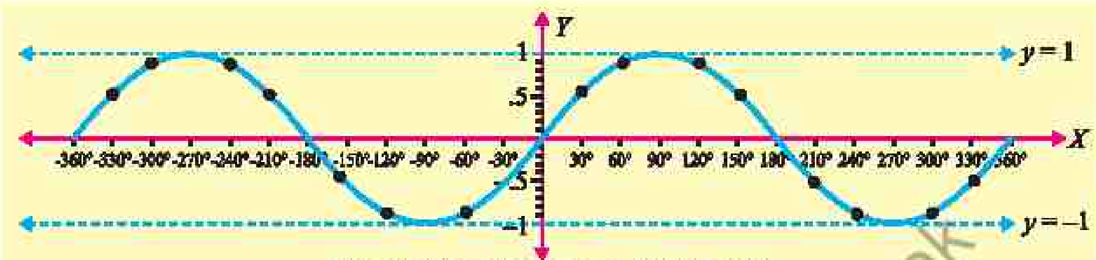
(ii) Draw the coordinate axes.
(iii) Plot the points corresponding to the ordered pairs in the table above i.e., $(0, 0)$, $(30°, 0.5)$, $(60°, 0.87)$ and so on.
(iv) Join the points with the help of a smooth curve as shown. So, we get the graph of $y = \sin x$ from 0 to $360°$ i.e., from 0 to 2π.

Note As we see that the graphs of trigonometric functions are smooth curves and none of them is line segment or has sharp corners or breaks within their domain. This behaviour of the curve is called continuity. It means that the trigonometric functions are continuous, wherever they are defined. Moreover, as the trigonometric functions are periodic so their curves repeat after fixed intervals.



Graph of $y = \sin x$ from $0°$ to $360°$

In the similar way, we can draw the graph for the interval from 0° to −360°. This will complete the graph of $y = \sin x$ from −360° to 360° (from -2π to 2π), which is given below:



Graph of $y = \sin x$ from − 360° to 360°

The graph in the interval $[0, 2\pi]$ is called a **cycle** and the maximum height of the wave from its mid line is called **amplitude.** Since the period of sine function is 2π, so the sine graph can be extended on both sides of x-axis through every interval of 2π.

Properties of graph of sine function ($y = \sin x$)

(i) The domain is the set of real numbers ($-\infty < x < \infty$).

(ii) The range includes all real numbers from −1 to 1, inclusive, $[-1, 1]$.

(iii) The graph of sine function is continuous for all real numbers.

(iv) The period of sine function is 2π. Mathematically, we can express it as $\sin(\theta + 2\pi) = \sin\theta$.

(v) The sine function is an odd function. As the graph of sine function is symmetric about the origin. Mathematically, it can be written as $\sin(-\theta) = -\sin\theta$.

(vi) The maximum value of $y = \sin x$ is 1 when $x = \frac{\pi}{2} + 2\pi n$, where $n \in Z$.

(vii) The minimum value of $y = \sin x$ is −1 when $x = \frac{3\pi}{2} + 2\pi n$, where $n \in Z$.

(viii) The x-intercepts of the sine function occurs at $x = \pi n$, where $n \in Z$.

(ix) The y-intercept of the sine function is 0.

(x) The amplitude of sine function is 1.

(xi) In unit circle $\sin\theta$ is equal to the y-coordinate of the given point.

11.4.3 Graph of $y = \cos x$ from -2π to 2π

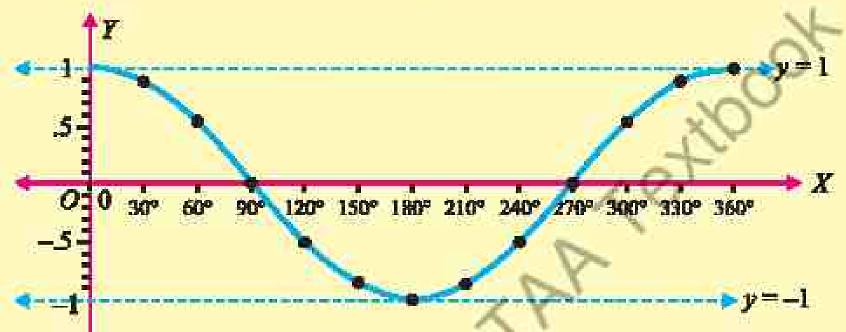
We know that the period of cosine function is 2π so, we will first draw the graph for the interval from 0° to 360° (from 0 to 2π).

We know the range of the cosine function is $[-1, 1]$, so the graph will be between the horizontal lines $y = +1$ and $y = -1$.

The table of the ordered pairs satisfying $y = \cos x$ is as follows:

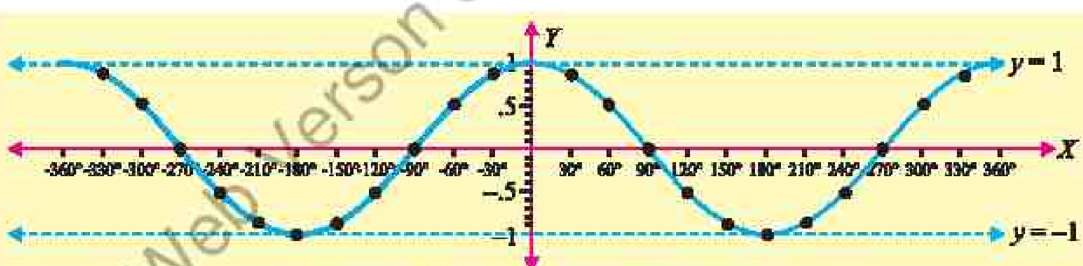
x	0 or $0°$	$\frac{\pi}{6}$ or $30°$	$\frac{\pi}{3}$ or $60°$	$\frac{\pi}{2}$ or $90°$	$\frac{2\pi}{3}$ or $120°$	$\frac{5\pi}{6}$ or $150°$	π or $180°$	$\frac{7\pi}{6}$ or $210°$	$\frac{4\pi}{3}$ or $240°$	$\frac{3\pi}{2}$ or $270°$	$\frac{5\pi}{3}$ or $300°$	$\frac{11\pi}{6}$ or $330°$	2π or $360°$
$\cos x$	1	0.87	0.5	0	−0.5	−0.87	−1	−0.87	−0.5	0	0.5	0.87	1

The graph of $y = \cos x$ from 0° to 360° is given below:



Graph of $y = \cos x$ from 0° to 360°

In the similar way, we can draw the graph for the interval from 0° to −360°. This will complete the graph of $y = \cos x$ from −360° to 360° i.e. from -2π to 2π, which is given below:



Graph of $y = \cos x$ from − 360° to 360°

As in the case of *sine* graph, the *cosine* graph is also extended on both sides of x-axis through an interval of 2π.

Properties of graph of cosine function ($y = \cos x$)

(i) The domain is the set of real numbers ($-\infty < x < \infty$).

(ii) The range includes all real numbers from −1 to 1, inclusive, [−1, 1].

(iii) The graph of cosine function is continuous for all real numbers.

(iv) The period of cosine function is 2π. Mathematically, we can express it as $\cos(\theta + 2\pi) = \cos\theta$.

(v) The cosine function is an even function, as the graph of cosine function is symmetric about the y-axis. Mathematically, it can be written as $\cos(-\theta)=\cos\theta$.

(vi) The maximum value of $y=\cos x$ is 1 when $x=\pi n$, where n is an even integer.

(vii) The minimum value of $y=\cos x$ is -1 when $x=\pi n$, where n is an odd integer.

(viii) The x-intercepts of the cosine function occurs at $x=\frac{\pi}{2}+\pi n$, where $n\in Z$.

(ix) The y-intercept of the cosine function is 1.

(x) The amplitude of cosine function is 1.

(xi) In unit circle $\cos\theta$ is equal to the x-coordinate of the given point.

11.4.4 Graph of $y=\tan x$ from $-\pi$ to π

We know that $\tan(-x) = -\tan x$ and $\tan(\pi - x) = -\tan x$, so the values of $\tan x$ for $x = 0°, 30°, 60°, 90°$ can help us in making the table.

Also, we know that $\tan x$ is undefined at $x = \pm 90°$, when

(i) x approaches $\frac{\pi}{2}$ from left $x\to\left(\frac{\pi}{2}\right)^{-}$, tan x decreases indefinitely in Quard I.

(ii) x approaches $\frac{\pi}{2}$ from right i.e., $x\to\left(\frac{\pi}{2}\right)^{+}$, $\tan x$ decreases indefinitely in Quard IV.

(iii) x approaches $-\frac{\pi}{2}$ from left i.e., $x\to\left(-\frac{\pi}{2}\right)^{-}$, $\tan x$ increases indefinitely in Quard II.

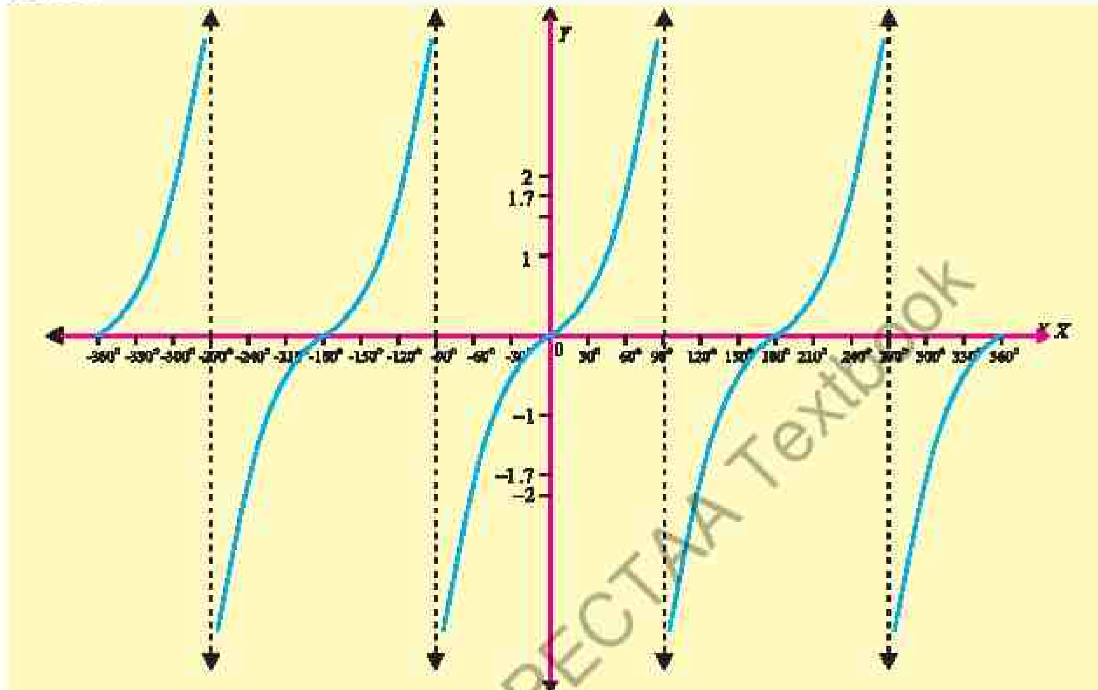
(iv) x approaches $-\frac{\pi}{2}$ from right i.e., $x\to\left(-\frac{\pi}{2}\right)^{+}$, $\tan x$ decreases indefinitely in Quard III.

We know that the period of tangent is π, so we shall first draw the graph for the interval from 0 to π (from 0° to 180°).

$\therefore$ The table of ordered pairs satisfying $y=\tan x$ is given below:

x	0 or 0	$\frac{\pi}{6}$ or $30°$	$\frac{\pi}{3}$ or $60°$	$\frac{\pi}{2}-0$ or $90°-0$	$\frac{\pi}{2}+0$ or $90°+0$	$\frac{2\pi}{3}$ or $120°$	$\frac{5\pi}{6}$ or $150°$	π or $180°$
$\tan x$	0	0.58	1.73	$+\infty$	$-\infty$	-1.73	-0.58	0

Since the period of tan x is π, so we have the following graph of $y = \tan x$ from $-360°$ to $360°$.



Graph of $y = \tan x$ from $-360°$ to $360°$

Properties of graph of tangent function ($y = \tan x$)

(i) The domain is the set of real numbers except the values where function is undefined domain of $\tan x = (-\infty, \infty)$, $x \neq (2n+1)\dfrac{\pi}{2}$, where $n \in Z$

(ii) The range includes all real numbers $(-\infty, \infty)$

(iii) The graph of $\tan x$ is not continuous for all real numbers. It breaks at $x = (2n+1)\dfrac{\pi}{2}$, where $n \in Z$

(iv) The period of tan function is π. Mathematically, we can express it as $\tan(\theta + \pi) = \tan\theta$

(v) The tan function is an odd function, as the graph of tan function is symmetric about the origin. Mathematically, it can be written as $\tan(-\theta) = -\tan\theta$

(vi) The x-intercepts of the tangent function occurs at $x = \pi n$, where $n \in Z$.

(vii) The y-intercept of the tangent function is 0

(viii) The amplitude of tangent function is undefined because it has no maximum or minimum values.

EXERCISE 11.2

1. Draw the graph of each of the following function for the intervals mentioned against each:

(i) $y = -\sin 2x$, $x \in [-2\pi, 2\pi]$ (ii) $y = 2\cos 2x$, $x \in [-2\pi, 2\pi]$

(iii) $y = \tan 2x$, $x \in [-\pi, \pi]$ (iv) $y = \tan \frac{x}{2}$, $x \in [-2\pi, 2\pi]$

(v) $y = \sin \frac{\pi}{2}x$, $x \in [0, 2\pi]$ (vi) $y = \cos \frac{\pi}{2}x$, $x \in [-\pi, \pi]$

2. On the same axes and to the same scale, draw the graphs of the following functions for their complete period:

(i) $y = \sin x$ and $y = \sin 2x$ (ii) $y = \cos x$ and $y = \cos 2x$

3. Solve graphically:

(i) $\sin x = \cos x$, $x \in [0, \pi]$ (ii) $\sin x = x$, $x \in [0, \pi]$

11.5 Maximum and Minimum Values of Given Functions of the Type

- $a + b\sin\theta$
- $a + b\cos\theta$
- $a + b\sin(c\theta + d)$
- $a + b\cos(c\theta + d)$
- **The reciprocals of the above, where *a*, *b*, *c* and *d* are real numbers.**

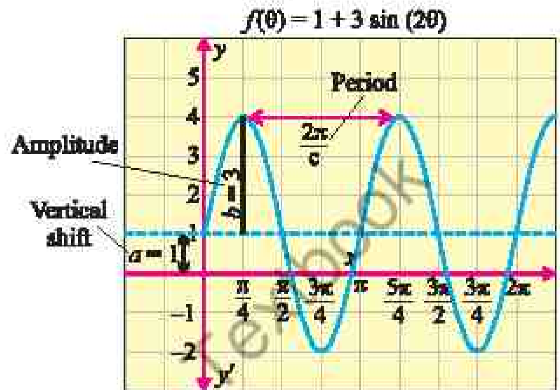
The trigonometric functions like sine and cosine are periodic function because the values of these function repeat over regular intervals. These functions are fundamental in mathematics because of the repetition of their values at definite cycles and are used to model various real-life situations, such as radio waves, light wave, and alternating current in electricity and are also known as a specific case of sinusoidal functions.

The functions of the form $f(\theta) = a + b\sin\theta$, $g(\theta) = a + b\cos\theta$, $f_1(\theta) = a + b\sin(c\theta + d)$ and $g_1(\theta) = a + b\cos(c\theta + d)$ are the types of sinusoidal functions.

Now consider the general form of sinusoidal function $f_1(\theta) = a + b\sin(c\theta + d)$ ...(i)

here 'a' represent the **vertical shift** refers to the vertical translation of the graph of the function, achieved by shifting the entire graph upward or downward. This shift, also known as the vertical displacement, moves the function's position along the y-axis without altering its shape or period. **Amplitude** $|b|$ is the maximum height of a wave measured from its midline. The **period** of (i) is equal to $\dfrac{2\pi}{c}$. **Phase shift 'd'** indicates the horizontal translation of the graph of the function, determining how far the wave is shifted left or right along the x-axis. A positive d shifts the graph to the left, while a negative d shifts it to the right, altering the starting point of the wave without changing its shape or period.

For Example, consider the function $f(\theta) = 1 + 3\sin(2\theta)$. Here $a = 1$ is vertical shift, amplitude $= |b| = |3| = 3$ and period$= \dfrac{2\pi}{2} = \pi$ as shown in the adjacent figure.



Now, finding the maximum and minimum values of the functions $f(\theta) = a + b\sin(c\theta + d)$ and $g(\theta) = a + b\cos(c\theta + d)$ is not a difficult task. We know that the maximum absolute values of sine and cosine are equal to 1, so the maximum value of the product $b\sin\theta$ is $|b|$.

Thus, the maximum value of $f(\theta)$ or $g(\theta)$ is : $M = a + |b|$, whenever $\sin\theta = 1$ or $\cos\theta = 1$ where M denotes the maximum value of the function.

The minimum value of $f(\theta)$ or $g(\theta)$ function is $m = a - |b|$, whenever $\sin\theta = -1$ or $\cos\theta = -1$ and m denotes the minimum value of the function.

Note The absolute value of b is called the Amplitude of $f(\theta) = a + b\sin\theta$. The value of the amplitude can also be determined using the formula

$$\text{Amplitude} = \frac{\text{Maximum value} - \text{Minimum value}}{2}$$

Example 2 Find the maximum and minimum values of the following functions:

(i) $2 + 3\sin x$ (ii) $5 - 2\cos 3x$ (iii) reciprocal of (ii)

Solution (i) Let $f(x) = 2 + 3\sin x$

The maximum value of $f(x)$ will occur when $\sin x = 1$. Here $a = 2$ and $b = 3$,

Maximum value of the function: $M = a + |b| = 2 + 3 = 5$

The minimum value of the function will occur when $\sin x = -1$.

Minimum value of the function: $m = a - |b| = 2 - 3 = -1$

Thus, maximum value of the function is 5 and the minimum value is -1

(ii) Let $f(x) = 5 - 2\cos 3x$

The maximum value of f(x) will occur when $\cos 3x = 1$. Here $a = 5$ and $b = -2$,

Maximum value of the function: $M = a + |b| = 5 + |-2| = 5 + 2 = 7$.

The minimum value of the function will occurs when $\cos 3x = -1$.

Minimum value of the function: $m = a - |b| = 5 - |-2| = 5 - 2 = 3$.

Thus, maximum value of the function is 7 and the minimum value is 3.

(iii) reciprocal of part (ii)

The reciprocal of $5 - 2\cos 3x$ is $\dfrac{1}{5 - 2\cos 3x}$

Let $g(x) = \dfrac{1}{5 - 2\cos 3x}$

To find the maximum and minimum values of $g(x)$, first we will find the maximum and minimum values of $5 - 2\cos 3x$, which are 7 and 3 respectively.

After finding the maximum and minimum values take their reciprocal. The reciprocal of the maximum value is the minimum of $g(x)$ and the reciprocal of the

minimum value is the maximum of $g(x)$.

Maximum value of $g(x) = \dfrac{1}{m} = \dfrac{1}{3} = 0.33$

Minimum value of $g(x) = \dfrac{1}{M} = \dfrac{1}{7} = 0.14$

11.5.1 Real World Applications

Ferris Wheel Problems

The first Ferris wheel was invented by George W. Ferris. He built the first one for 1893 World's Fair. A Ferris wheel is an important example of periodic motion that can be described using trigonometric functions, specifically sinusoidal functions. When we model the height of a rider on a Ferris wheel over time, we can use these functions to capture the periodic nature of the motion. The motion of Ferris wheel can be modeled by $f(t) = a + b\sin(ct + d)$ or $f(t) = a + b\cos(ct + d)$.

Example 3 A Ferris wheel with a radius of 45 feet has its lowest point located 5 feet above the ground. It completes one full revolution every 60 seconds in counter clock wise direction. Model an equation that describes the height of a rider on the Ferris wheel as a function of time t. How high is the rider from the ground after 40 seconds?. Also graph the model equation.

Solution Since it takes 60 seconds for the Ferris wheel to complete one full revolution (one cycle), which is the period of the Ferris wheel, that is period = 60

$$\frac{2\pi}{c}=60 \quad \Rightarrow \quad c=\frac{2\pi}{60} \quad \Rightarrow \quad c=\frac{\pi}{30}$$

The amplitude b which is equal to the radius of a ferris wheel (in this case $b = 45$).
The vertical shift a is the height of the center of the Ferris wheel above the ground.
Since the lowest point is 5 feet above the ground, so $a = 5 + b = 5 + 45 = 50$.
we can model the height of a rider using (sine or cosine), because it reflects the periodic nature of the motion. We usually choose a cosine function if the rider starts at the maximum or minimum height, or a sine function if the rider starts at the midpoint.
Since the rider starts at the lowest point and goes up, we can easily model the required equation as a negative cosine function so,
$h(t)=-b\cos(ct)+a$, where t is time and h is height.

Now substituting the above values we get the function $h(t)=-45\cos\left(\frac{\pi}{30}t\right)+50$,

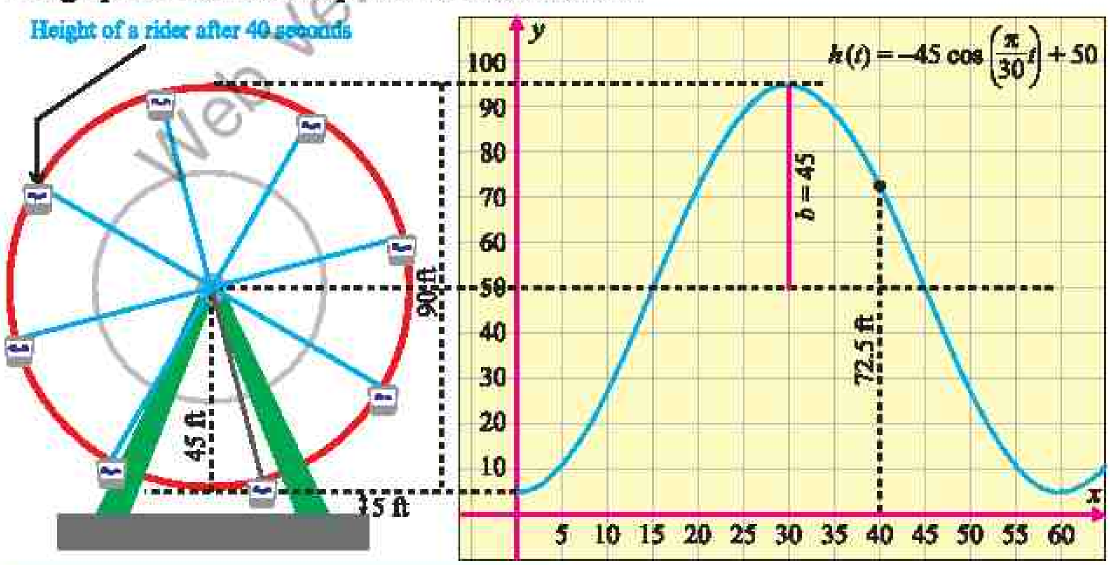
which is the required equation of Ferris wheel.
Next, we find the height of the rider at $t = 40$ seconds.

$$h(t)=-45\cos\left(\frac{\pi}{30}t\right)+50$$

For $t = 40$, we have

$$h(40)=-45\cos\left(\frac{\pi}{30}.40\right)+50 = 72.5 \text{ feet}$$

Thus, height of rider after 40 second is 72.5 feet.
The graph of the model equation is shown below.



Example 4 The water level L (in feet) of a tidal river varies throughout the day. Suppose the level of the tidal river can be modeled by the equation: $L(t)=8+4\sin\left(\frac{\pi}{6}t\right)$, where t denotes the time (in hours). The water level oscillates 4 feet above and below an average level of 8 feet.

(a) Find the water level at $t = 3$ hours?

(b) What is the minimum water level?

Solution (a) Given equation of water level: $L(t)=8+4\sin\left(\frac{\pi}{6}t\right)$

To find the water level, substitute $t = 3$ into the equation

$$L(3)=8+4\sin\left(\frac{\pi}{6}\cdot 3\right)=8+4\sin\left(\frac{\pi}{2}\right)$$

$$L(3) = 8 + 4(1) = 12$$

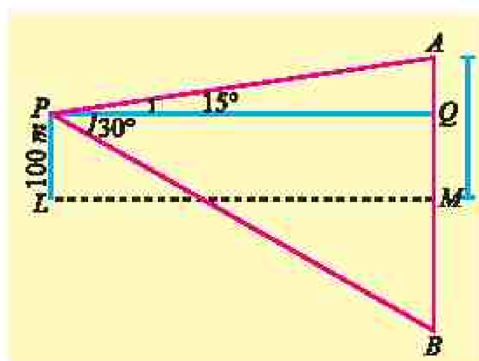
Thus, water level at t=3 hours is 12 feet.

(b) Now, to find the minimum water level, we need to determine when the sine function attains its minimum value. We know that the minimum value of $\sin t = -1$, substitute the $\sin\left(\frac{\pi}{6}t\right)=-1$ into the equation

$$L(t)=8+4\sin\left(\frac{\pi}{6}t\right)=8+4(-1)=8-4=4$$

Thus, minimum water level of the tidal river is 4 feet.

Example 5 From a point 100 m above the surface of a lake, the angle of elevation of a peak of a cliff is found to be 15° and the angle of depression of the image of the peak is 30°. Find the height of the peak.



Solution Let A be the top of the peak $\overline{AM}$ and $\overline{MB}$ be its image. Let P be the point of observation and L be the point just below P (on the surface of the lake).

From P, draw $\overline{PQ} \perp \overline{AM}$.

Let $m\overline{PQ} = y$ metres and $m\overline{AM} = h$ metres.

$\therefore \quad m\overline{AQ} = h - m\overline{QM} = h - m\overline{PL} = h - 100$

From the figure,

$$\tan 15° = \frac{AQ}{PQ}=\frac{h-100}{y} \quad \text{and} \quad \tan 30° = \frac{BQ}{PQ}=\frac{100+h}{y}$$

By division, we get

$$\frac{\tan 15^\circ}{\tan 30^\circ} = \frac{h-100}{h+100}$$

By Componendo and Dividendo, we have

$$\frac{\tan 15^\circ + \tan 30^\circ}{\tan 15^\circ - \tan 30^\circ} = \frac{h-100+h+100}{h-100-h-100} = \frac{2h}{-200} = \frac{h}{-100}$$

$$\therefore \quad h = \frac{\tan 30^\circ + \tan 15^\circ}{\tan 30^\circ - \tan 15^\circ} \times 100 = \left[\frac{0.5774+0.2679}{0.5774-0.2679}\right] \times 100$$

$\Rightarrow \quad h = 273.1179.$

Hence height of the peak = 273 m. (approximately)

EXERCISE 11.3

1. **Find the maximum and minimum values of the following functions:**

(i) $3-\sin 3x$ (ii) $3+\sin 2x$ (iii) $\frac{1}{2}+\sin(5x+\pi)$

(iv) $\frac{3}{2}+\cos\left(x-\frac{\pi}{4}\right)$ (v) $1-3\cos 2x$ (vi) $1+2\sin\left(x+\frac{\pi}{6}\right)$

(vii) $\frac{1}{10-2\sin 3x}$ (viii) $\frac{1}{7+3\cos(-2x)}$ (ix) $\frac{1}{5-3\cos(3x-1)}$

2. The temperature T in degrees Celsius of a certain city varies throughout the day according to the equation $T(t)=\frac{13}{2}\sin\left(\frac{\pi}{6}t-\frac{\pi}{9}\right)+15$, where t is the time in hours, with $t = 0$ corresponding to midnight.

(a) Find the maximum and minimum temperature during the day.

(b) Find the temperature at $t = 9$ hours (9:00 a.m.).

3. A man on the top of a 100 m high light-house is in line with two ships on the same side of it, whose angles of depression from the man are 17° and 19° respectively. Find the distance between the ships.

4. P and Q are two points in line with a tree. If the distance between P and Q be 30 m and the angles of elevation of the top of the tree at P and Q are 12° and 15° respectively, find the height of the tree.

5. A giant Ferris wheel has a diameter of 60 feet. The lowest point of the wheel is located 6 feet above the ground. The wheel completes one full revolution every 80 seconds.

(a) Model an equation that represent the height $h(t)$ of a rider on the Ferris wheel at any given time t.

(b) Find the maximum height of the rider.

(c) Find the height of the rider from the ground after 35 seconds.

6. A child is playing on a swing in a playground. The height $h(t)$ of the swing seat above the ground (in metres) at time t (in seconds) is modeled by the function:

$h(t) = 1.5 + 1.2\sin(3\pi t)$

(a) What is the maximum height reached by the swing seat?

(b) What is the minimum height reached by the swing seat?

(c) How long does it take for the swing to complete one full back-and-forth motion (period)?

(d) At what time(s) does the swing seat first reach a height of 2.12 metres?

7. A carnival ride consists of a vertical wheel with a diameter of 40 feet. The centre of the wheel is 28 feet above the ground. The wheel rotates at a constant speed and takes 120 seconds to make one complete revolution. Model an equation that describes the height h(t) of a rider on the wheel as a function of time t. How high is the rider from the ground after 90 seconds? At what times will the rider be 36 feet above the ground?

8. Suppose the temperature T in degrees Fahrenheit of Lahore city in a month of December throughout the day can be modeled by the equation: $T = 64 + 8\sin\left(\frac{\pi}{12}(t-8)\right)$, where t is the time in hours. The temperature oscillates 8 degrees above and below an average temperature of 64 degrees.

(a) Find the temperature at $t = 9$ hours?

(b) At what time the temperature will be maximum?

(c) Calculate the maximum temperature.

9. Suppose the population of a coastal city follows a sinusoidal pattern due to seasonal migration. The population of the city over the course of a year can be modeled by the equation: $P(t) = 70000 + 10000\cos\left(\frac{\pi}{6}t - \frac{\pi}{2}\right)$, $P(t)$ is the population at time t (t is the time in months, with $t = 0$ corresponding to January 1st). where t denoted the months in a year.

(a) Find the population of the city at $t = 7$ months.

(b) Find the maximum population.