

# Unit 13

## Differentiation

### INTRODUCTION

The ancient Greeks knew the concepts of area, volume, centroids etc. which are related to integral calculus. Later on, in the seventeenth century, Sir Isaac Newton, an English mathematician (1642 – 1727) and Gottfried Wilhelm G. W. Leibniz, a German mathematician, (1646 – 1716) considered the problem of instantaneous rates of change. They reached independently to the invention of differential calculus. After the development of calculus, mathematics became a powerful tool for dealing with rates of change and describing the physical universe.

### 13.1 Tangent to a Curve at a Point

Let  $P(x, f(x))$  and  $Q(x + \delta x, f(x + \delta x))$  be two points on arc  $AB$  of graph of  $f$  defined by the equation  $y = f(x)$  as shown in Figure 13.1.

Where  $\delta x$  is the increment in the value of  $x$  (read as delta  $x$ )

The line  $PQ$  is secant of the curve and slope of secant line passing through  $P(x, f(x))$  and  $Q(x + \delta x, f(x + \delta x))$  is:

$$m_{\text{sec}} = \frac{RQ}{PR} = \frac{f(x + \delta x) - f(x)}{\delta x} \quad (1)$$

Where  $m_{\text{sec}}$  is slope of the secant line.

Revolving the secant line  $PQ$  towards  $P$ , some of its successive positions

$PQ_1, PQ_2, PQ_3, \dots$  are shown in the

Figure 13.2. Points  $Q_i (i = 1, 2, 3, \dots)$

are getting closer and closer to the point  $P$  and  $PR$  are approaching

zero.

In other words, as  $\delta x \rightarrow 0$ , the point  $Q$  approaches  $P$ , and the secant line becomes the **tangent line**. The revolving secant line becomes the tangent line  $PT$  at  $P$  while  $\delta x$

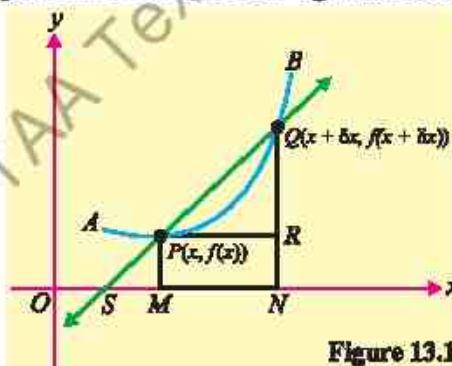


Figure 13.1

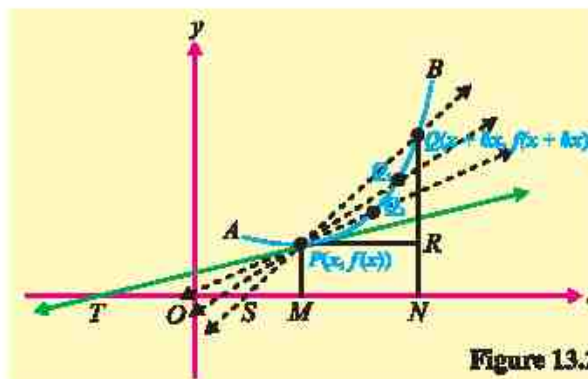


Figure 13.2

approaches zero, that is,

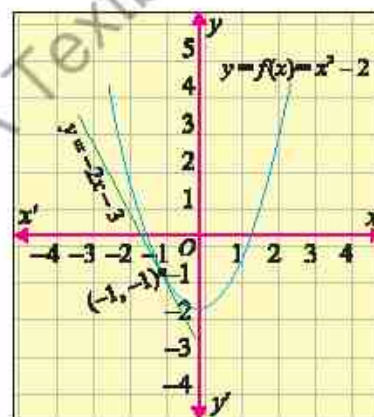
$$m_{\text{tan}} = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x} \quad (2)$$

where  $m_{\text{tan}}$  denotes the slope of tangent line. We see that  $m_{\text{tan}}$  is the limit of  $m_{\text{sec}}$  as  $Q$  approaches  $P$  along the curve  $y = f(x)$ .

**Example 1** Find the gradient and an equation of tangent line to the graph of  $f(x) = x^2 - 2$  at the point  $P(-1, -1)$ .

**Solution** To find the gradient or slope of the tangent line at point  $(-1, -1)$ , put  $x = -1$  in equation (2)

$$\begin{aligned} m_{\text{tan}} &= \lim_{\delta x \rightarrow 0} \frac{f(-1 + \delta x) - f(-1)}{\delta x} \\ &= \lim_{\delta x \rightarrow 0} \frac{(-1 + \delta x)^2 - 2 - ((-1)^2 - 2)}{\delta x} \\ &= \lim_{\delta x \rightarrow 0} \frac{1 - 2\delta x + \delta x^2 - 2 - (1 - 2)}{\delta x} \\ &= \lim_{\delta x \rightarrow 0} \frac{1 - 2\delta x + \delta x^2 - 2 + 1}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{-2\delta x + \delta x^2}{\delta x} \\ &= \lim_{\delta x \rightarrow 0} \frac{\delta x(-2 + \delta x)}{\delta x} = \lim_{\delta x \rightarrow 0} (-2 + \delta x) = -2 \end{aligned}$$



Now to find the equation of tangent line we use the point slope form of equation of line with slope  $= -2$  and point  $(-1, -1)$

$$y - (-1) = -2(x - (-1)) \Rightarrow y + 1 = -2x - 2$$

or  $y = -2x - 3$ , which is the required equation of tangent line.

The graph of  $f$  and tangent line are shown in the above figure.

### 13.2 Derivative as the Limit of a Difference Quotient

Let  $f$  be a real valued function continuous in the interval  $(x, x_1) \subseteq D_f$  (domain of  $f$ ),

then difference quotient 
$$\frac{f(x_1) - f(x)}{x_1 - x} \quad (i)$$

represents the average rate of change in the value of  $f$  with respect to the change  $x_1 - x$  in the value of independent variable  $x$ .

If  $x_1$  approaches to  $x$ , then  $\lim_{x_1 \rightarrow x} \frac{f(x_1) - f(x)}{x_1 - x}$

provided this limit exists, is called the instantaneous rate of change of  $f$  with respect to  $x$  and is written as  $f'(x)$ .

If  $x_1 = x + \delta x$  i.e.,  $x_1 - x = \delta x$ , then the expression (i) can be expressed as

$$\frac{f(x + \delta x) - f(x)}{\delta x} \quad (\text{ii})$$

and  $\lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x} \quad (\text{iii})$

provided the limit exist, is defined to be the derivative of  $f$  (or differential coefficient of  $f$ ) with respect to  $x$  and is denoted by  $f'(x)$  (read as “ $f$ -prime of  $x$ ”). The domain of  $f'$  consists of all  $x$  for which the limit exists. If  $x \in D_f$  and  $f'(x)$  exists, then  $f$  is said to be differentiable at  $x$ . The process of finding  $f'$  is called **differentiation**.

### 13.2.1 Derivative as the Rate of Change of Velocity

The rate of change is a fundamental concept in describing the motion of an object moving in a straight line. In physics, this is typically analyzed using position, velocity, and acceleration, which are all related through derivatives (rates of change).

The position versus time graph provides a simple interpretation of the average velocity over a given time interval.

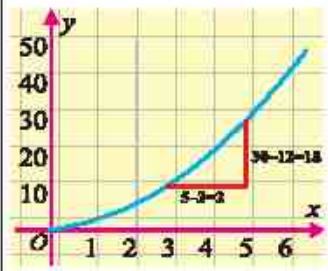
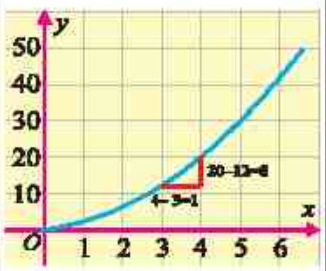
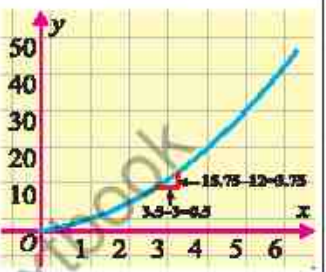
Suppose a particle moves in a straight line and its position at time  $t$  is given by the function  $s(t)$ . The average velocity over the interval from  $t$  to  $t_1$  denoted by  $v_{\text{avg}}$  is defined as:

$$v_{\text{avg}} = \frac{s(t_1) - s(t)}{t_1 - t} \quad (\text{i})$$

Equation (i) also represents the slope of secant line passing through the points  $(t, s(t))$  and  $(t_1, s(t_1))$ . If the interval  $t_1 - t$  is not small, this average velocity does not accurately represent the rate of change at time  $t$ .

To illustrate this, consider a particle whose position at time  $t$  (in seconds) is given by a function  $s(t) = t^2 + t$  in metres. The average rate of change over various time intervals

starting at  $t = 3$  seconds is shown in the table below:

Interval	$t = 3$ secs to $t = 5$ secs	$t = 3$ secs to $t = 4$ secs	$t = 3$ secs to $t = 3.5$ secs
Average velocity	$\frac{s(5) - s(3)}{5 - 3} = \frac{30 - 12}{2} = 9$	$\frac{s(4) - s(3)}{4 - 3} = \frac{20 - 12}{1} = 8$	$\frac{s(3.5) - s(3)}{3.5 - 3} = \frac{\frac{63}{4} - 12}{0.5} = 7.5$
			

We observe that these values are not closely approximate the particle's velocity at exactly 3 seconds. To obtain a better approximation of velocity at  $x = 3$ , we use smaller intervals:

Interval	Average velocity
$t = 3$ secs to $t = 3.1$ secs	$\frac{\{(3.1)^2 + 3.1\} - 12}{3.1 - 3} = \frac{0.71}{0.1} = 7.1$
$t = 3$ secs to $t = 3.01$ secs	$\frac{\{(3.01)^2 + 3.01\} - 12}{3.01 - 3} = \frac{0.0701}{0.01} = 7.01$
$t = 3$ secs to $t = 3.001$ secs	$\frac{\{(3.001)^2 + 3.001\} - 12}{3.001 - 3} = \frac{0.007001}{0.001} = 7.001$

We see as the length of the time interval decreases, the average velocity becomes instantaneous velocity at  $t = 3$ . Based on the trend, we estimate the instantaneous velocity to be approximately 7 m/sec.

Thus, over a sufficiently small interval, the velocity changes negligibly. If  $t_1$  is very close to  $t$ , the average velocity over  $t_1 - t$  approximates the instantaneous velocity at  $t$ . As  $t_1$  approaches  $t$ , the average velocity is called the instantaneous velocity.

This is similar to approximating the slope of a tangent line by calculating the slope of a secant line. Mathematically, the instantaneous velocity denoted by  $v_{inst}$  is given by the following limit:

$$v_{inst} = \lim_{t_1 \rightarrow t} \frac{s(t_1) - s(t)}{t_1 - t} \quad (\text{Provide the limit exist})$$

For convenient, if  $t_1 = t + \delta t$ , then as  $t_1 \rightarrow t \Rightarrow \delta t \rightarrow 0$ , thus above equation becomes:

$$v_{inst} = \lim_{\delta t \rightarrow 0} \frac{s(t + \delta t) - s(t)}{\delta t} \quad (\text{ii})$$

In other words, the instantaneous velocity is the derivative of the position function  $s(t)$  with respect to time.

**Example 2** A particle moves along a line such that its position after  $t$  hours is given by:  $s(t) = 4t^2 + 2t + 1$  (in miles)

(a) Find the average velocity over the interval  $[2, 5]$

(b) Find the instantaneous velocity at  $t = 3$

**Solution** (a) given position function  $s(t) = 4t^2 + 2t + 1$ , where  $2 \leq t \leq 5$

The average velocity over the interval  $2 \leq t \leq 5$  is:

$$\begin{aligned} \text{Average velocity} = v_{\text{avg}} &= \frac{s(5) - s(2)}{5 - 2} = \frac{4(5)^2 + 2(5) + 1 - [4(2)^2 + 2(2) + 1]}{3} \\ &= \frac{111 - 21}{3} = \frac{90}{3} = 30 \text{ miles/hours} \end{aligned}$$

(b) Instantaneous velocity can be found using the formula

$$\begin{aligned} \text{Instantaneous velocity} &= \lim_{\delta t \rightarrow 0} \frac{s(t + \delta t) - s(t)}{\delta t} \\ &= \lim_{\delta t \rightarrow 0} \frac{4(3 + \delta t)^2 + 2(3 + \delta t) + 1 - [4(3)^2 + 2(3) + 1]}{\delta t} \\ &= \lim_{\delta t \rightarrow 0} \frac{4(9 + 6\delta t + \delta t^2) + 6 + 2\delta t + 1 - 43}{\delta t} \\ &= \lim_{\delta t \rightarrow 0} \frac{36 + 24\delta t + 4\delta t^2 + 6 + 2\delta t + 1 - 43}{\delta t} \\ &= \lim_{\delta t \rightarrow 0} \frac{43 + 26\delta t + 4\delta t^2 - 43}{\delta t} = \lim_{\delta t \rightarrow 0} \frac{26\delta t + 4\delta t^2}{\delta t} \\ &= \lim_{\delta t \rightarrow 0} \frac{\delta t(26 + 4\delta t)}{\delta t} = \lim_{\delta t \rightarrow 0} (26 + 4\delta t) = 26 \end{aligned}$$

Thus, instantaneous velocity at  $t = 3$  is 26 miles/hour

### 13.3 Process of Finding Derivative $f'(x)$ by Definition

#### 12.3.1 Notation of Derivative

Several notations are used for derivatives. We have used the functional symbol  $f'(x)$ , for the derivative of  $f$  at  $x$ . For the function  $y = f(x)$ ,

$$y + \delta y = f(x + \delta x) \quad (\text{iv})$$

Dividing both the sides of (iv) by  $\delta x$ , we get

$$\frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x} \quad (\text{v})$$

Taking limit of both the sides of (v) as  $\delta x \rightarrow 0$ , we have

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x} \quad (\text{vi})$$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} \text{ is denoted by } \frac{dy}{dx}, \text{ so (vi) is written as } \frac{dy}{dx} = f'(x)$$

**Note**

The symbol  $\frac{dy}{dx}$  is used for the derivative of  $y$  with respect to  $x$  and here it is not a quotient of  $dy$  and  $dx$ .  $\frac{dy}{dx}$  is also denoted by  $y'$ .

Now we write, in a table the notations for derivative of  $y = f(x)$  used by different mathematicians:

Name of mathematician	Leibniz	Newton	Lagrange	Euler
Notation used for derivative	$\frac{dy}{dx}$ or $\frac{df}{dx}$	$f'(x)$ or $y'$	$f'(x)$	$Df(x)$

If we replace  $x + \delta x$  by  $x$  and  $x$  by  $a$ , then the expression  $f(x + \delta x) - f(x)$  becomes  $f(x) - f(a)$  and the change  $\delta x$  in the independent variable, in this case, is  $x - a$ .

$$\text{So, the expression } \frac{f(x + \delta x) - f(x)}{\delta x} \text{ is written as } \frac{f(x) - f(a)}{x - a} \quad (\text{vii})$$

Taking the limit of the expression (vii) when  $x \rightarrow a$ , gives  $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = f'(a)$ .

Here  $f'(a)$  is called the derivative or gradient of  $f$  at  $x = a$ .

### 13.3.2 Finding $f'(x)$ by Definition of Derivative

Given a function  $f$ , then  $f'(x)$  if it exists, can be found by the following four steps:

**Step I:** Find  $f(x + \delta x)$

**Step II:** Simplify  $f(x + \delta x) - f(x)$

**Step III:** Divide  $f(x + \delta x) - f(x)$  by  $\delta x$  to get  $\frac{f(x + \delta x) - f(x)}{\delta x}$  and simplify it.

**Step IV:** Find  $\lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}$

The method of finding derivatives by this process is called differentiation by definition or by ab-initio or from first principle.

**Example 3** Find the derivative of the following functions by definition

(a)  $f(x) = c$  (b)  $f(x) = x^2$

**Solution**

(a)  $f(x) = c$

(i)  $f(x + \delta x) = c$

(ii)  $f(x + \delta x) - f(x) = c - c = 0$

(iii)  $\frac{f(x + \delta x) - f(x)}{\delta x} = \frac{0}{\delta x} = 0$

(iv)  $\lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x} = \lim_{\delta x \rightarrow 0} (0) = 0$

Thus,  $f'(x) = 0$ , that is,  $\frac{d}{dx}(c) = 0$

(b)  $f(x) = x^2$

(i)  $f(x + \delta x) = (x + \delta x)^2$

(ii)  $f(x + \delta x) - f(x) = (x + \delta x)^2 - x^2 = x^2 + 2x\delta x + (\delta x)^2 - x^2 = (2x + \delta x)\delta x$

(iii)  $\frac{f(x + \delta x) - f(x)}{\delta x} = \frac{(2x + \delta x)\delta x}{\delta x} = 2x + \delta x, (\delta x \neq 0)$

(iv)  $\lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x} = \lim_{\delta x \rightarrow 0} (2x + \delta x) = 2x$

i.e.,  $f'(x) = 2x$

**Example 4** Find the derivative of  $\sqrt{x}$  at  $x = a$  from first principle.

**Solution** If  $f(x) = \sqrt{x}$ , then

(i)  $f(x + \delta x) = \sqrt{x + \delta x}$  and (ii)  $f(x + \delta x) - f(x) = \sqrt{x + \delta x} - \sqrt{x}$

$$= \frac{(\sqrt{x + \delta x} - \sqrt{x})(\sqrt{x + \delta x} + \sqrt{x})}{\sqrt{x + \delta x} + \sqrt{x}} \quad \left( \begin{array}{l} \text{rationalizing the} \\ \text{numerator} \end{array} \right)$$

$$= \frac{x + \delta x - x}{\sqrt{x + \delta x} + \sqrt{x}}$$

i.e.,  $f(x + \delta x) - f(x) = \frac{\delta x}{\sqrt{x + \delta x} + \sqrt{x}} \quad (1)$

(iii) Dividing both sides of (1) by  $\delta x$ , we have

$$\begin{aligned} \frac{f(x + \delta x) - f(x)}{\delta x} &= \frac{\delta x}{\delta x(\sqrt{x + \delta x} + \sqrt{x})} \\ &= \frac{1}{\sqrt{x + \delta x} + \sqrt{x}}, (\delta x \neq 0) \end{aligned}$$

(iv) Taking limit of both the sides as  $\delta x \rightarrow 0$ , we have

$$\lim_{\delta x \rightarrow 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \lim_{\delta x \rightarrow 0} \left( \frac{1}{\sqrt{x+\delta x} + \sqrt{x}} \right)$$

$$\text{i.e., } f'(x) = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}, \quad (x > 0) \text{ and } f'(a) = \frac{1}{2\sqrt{a}}$$

Alternate method: Putting  $x = a$  in  $f(x) = \sqrt{x}$ , gives  $f(a) = \sqrt{a}$

$$\text{So, } f(x) - f(a) = \sqrt{x} - \sqrt{a}$$

Using alternative form for the definition of the derivative, we have

$$\begin{aligned} \frac{f(x) - f(a)}{x - a} &= \frac{\sqrt{x} - \sqrt{a}}{x - a} \\ &= \frac{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})}{(x - a)(\sqrt{x} + \sqrt{a})} \quad (\text{rationalizing the numerator}) \\ &= \frac{x - a}{(x - a)(\sqrt{x} + \sqrt{a})} = \frac{1}{\sqrt{x} + \sqrt{a}}, \quad (x \neq a) \end{aligned} \quad (2)$$

Taking limit of both the sides of (2) as  $x \rightarrow a$ , gives

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow a} \frac{1}{\sqrt{x} + \sqrt{a}} = \frac{1}{\sqrt{a} + \sqrt{a}}$$

$$\text{i.e., } f'(a) = \frac{1}{2\sqrt{a}}$$

which is the gradient of  $f$  at  $x = a$ .

**Example 5** If  $y = \frac{1}{x^2}$ , then find  $\frac{dy}{dx}$  at  $x = -1$  by ab-initio method.

**Solution** Here,  $y = \frac{1}{x^2}$ , so (i)

$$y + \delta y = \frac{1}{(x + \delta x)^2} \quad \text{(ii)}$$

Subtracting (i) from (ii), we get

$$\begin{aligned} \delta y &= \frac{1}{(x + \delta x)^2} - \frac{1}{x^2} = \frac{x^2 - (x + \delta x)^2}{x^2(x + \delta x)^2} \\ &= \frac{\{x + (x + \delta x)\} \{x - (x + \delta x)\}}{x^2(x + \delta x)^2} \end{aligned}$$

$$= \frac{(2x + \delta x)(-\delta x)}{x^2(x + \delta x)^2} = \frac{-\delta x(2x + \delta x)}{x^2(x + \delta x)^2} \quad (\text{iii})$$

Dividing both sides of (iii) by  $\delta x$ , we have

$$\frac{\delta y}{\delta x} = \frac{-\delta x(2x + \delta x)}{x^2(x + \delta x)^2 \cdot \delta x} = \frac{-(2x + \delta x)}{x^2(x + \delta x)^2}, \quad (\delta x \neq 0)$$

Taking limit as  $\delta x \rightarrow 0$ , gives

$$\begin{aligned} \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} &= \lim_{\delta x \rightarrow 0} \frac{-(2x + \delta x)}{x^2(x + \delta x)^2} \\ &= \frac{-(2x)}{x^2(x^2)} \quad (\text{Using quotient theorem of limits}) \end{aligned}$$

$$\text{i.e., } \frac{dy}{dx} = \frac{-2}{x^3} \quad \text{and} \quad \left. \frac{dy}{dx} \right|_{x=-1} = \frac{-2}{(-1)^3} = \frac{-2}{-1} = 2$$

The gradient of  $f$  at  $x = -1$  is  $m = 2$ .

**Note** The value of  $\frac{dy}{dx}$  at  $x = -1$  is written as  $\left. \frac{dy}{dx} \right|_{x=-1}$

### 13.4 Derivation of $x^n$ where $n \in \mathbb{Z}$

(a) We find the derivative of  $x^n$  when  $n$  is positive integer.

(b) Let  $y = x^n$ . Then

$$y + \delta y = (x + \delta x)^n$$

$$\text{and} \quad \delta y = (x + \delta x)^n - x^n$$

Using the binomial theorem, we have

$$\delta y = \left[ x^n + nx^{n-1} \cdot \delta x + \frac{n(n-1)}{2!} x^{n-2} (\delta x)^2 + \dots + (\delta x)^n \right] - x^n$$

$$\text{i.e., } \delta y = \delta x \left[ nx^{n-1} + \frac{n(n-1)}{2!} x^{n-2} \cdot \delta x + \dots + (\delta x)^{n-1} \right] \quad (\text{i})$$

Dividing both sides of (i) by  $\delta x$ , gives

$$\frac{\delta y}{\delta x} = nx^{n-1} + \frac{n(n-1)}{2!} x^{n-2} \cdot \delta x + \dots + (\delta x)^{n-1} \quad (\text{ii})$$

Note that each term on the right hand side of (ii) involves  $\delta x$  except the first term, so taking the limit as  $\delta x \rightarrow 0$ , we get  $\frac{dy}{dx} = nx^{n-1}$

$$\text{As } y = x^n, \text{ so } \frac{d}{dx}(x^n) = n \cdot x^{n-1}$$

- (b) Let  $y = x^n$  where  $n$  is negative integer.  
Let  $n = -m$  ( $m$  is a positive integer). Then

$$y = x^{-m} = \frac{1}{x^m} \quad (i)$$

$$\text{and } y + \delta y = \frac{1}{(x + \delta x)^m} \quad (ii)$$

Subtracting (i) from (ii), gives

$$\begin{aligned} \delta y &= \frac{1}{(x + \delta x)^m} - \frac{1}{x^m} = \frac{x^m - (x + \delta x)^m}{x^m (x + \delta x)^m} \\ &= \frac{x^m - [x^m + mx^{m-1}\delta x + \frac{m(m-1)}{2!}x^{m-2}(\delta x)^2 + \dots + (\delta x)^m]}{x^m (x + \delta x)^m} \end{aligned}$$

(expanding  $(x + \delta x)^m$  by binomial theorem)

$$= -\delta x \frac{\left( mx^{m-1} + \frac{m(m-1)}{2!}x^{m-2}\delta x + \dots + (\delta x)^{m-1} \right)}{x^m (x + \delta x)^m}$$

$$\text{and } \frac{\delta y}{\delta x} = \frac{-1}{x^m (x + \delta x)^m} \cdot \left( mx^{m-1} + \frac{m(m-1)}{2!}x^{m-2}\delta x + \dots + (\delta x)^{m-1} \right)$$

Taking limit when  $\delta x \rightarrow 0$ , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{-1}{x^m \cdot x^m} \cdot mx^{m-1} \quad (\text{all terms containing } \delta x \text{ vanish}) \\ &= -mx^{m-1} \cdot x^{-2m} \\ &= -mx^{(-m)-1} \\ &= \frac{dx}{dx} = nx^{n-1} \quad [\because -m = n] \end{aligned}$$

$$\text{or } \frac{d(x)^n}{dx} = nx^{n-1}$$

#### Note

$\frac{d}{dx}(x^n) = nx^{n-1}$  is called power rule. Where  $n \in \mathbb{R}$

So, we have proved that  $\frac{d}{dx}(x^n) = nx^{n-1}$ , if  $n \in \mathbb{Z}$

The above rule also holds if  $n \in \mathbb{Q} - \mathbb{Z}$ , i.e. for rational powers.

$$\text{For example, } \frac{d}{dx}(x^{\frac{2}{3}}) = \frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3x^{\frac{1}{3}}}$$

The proof of  $\frac{d}{dx}(x^n) = nx^{n-1}$  when  $n \in \mathbb{Q} - \mathbb{Z}$  is left as an exercise.

### 13.5 Connection Between Derivatives and Continuity

Calculus is a powerful branch of mathematics that allows us to study change and motion. Two of its foundational concepts of continuity and derivatives are deeply connected. While each concept has its own definition and application, understanding how they relate to each other is essential for solving real-world problems in mathematics.

As discussed in previous units, a function is continuous at a point if its graph has no breaks, jumps, or holes at that point. On the other hand, the derivative of a function at a point measures the instantaneous rate of change or equivalently, the slope of the tangent line at that point. However, this definition depends on the function being well-behaved around the point. This leads to a well-known result:

If a function is differentiable at a point, it must also be continuous there. This means that differentiability implies continuity, but the reverse is not necessarily true. For example, consider the function  $f(x) = |x|$ , clearly this function is continuous at  $x = 0$  (see Figure 13.3). Now we check the differentiability of  $f(x) = |x|$  at  $x = 0$ .

$$f(x) = |x|$$

$$f(0) = |0| = 0 \text{ and,}$$

$$f(0 + \delta x) = |0 + \delta x| = |\delta x|$$

$$\text{so } f(0 + \delta x) - f(0) = |\delta x| - 0$$

$$\text{and } \frac{f(0 + \delta x) - f(0)}{\delta x} = \frac{|\delta x|}{\delta x}$$

$$\text{Thus } f'(x) = \lim_{\delta x \rightarrow 0} \frac{|\delta x|}{\delta x}$$

Because  $|\delta x| = \delta x$  when  $\delta x > 0$

and  $|\delta x| = -\delta x$  when  $\delta x < 0$ ,

so, we consider one-sided limits

$$\lim_{\delta x \rightarrow 0^+} \frac{|\delta x|}{\delta x} = \lim_{\delta x \rightarrow 0^+} \frac{\delta x}{\delta x} = 1 \quad \text{and} \quad \lim_{\delta x \rightarrow 0^-} \frac{|\delta x|}{\delta x} = \lim_{\delta x \rightarrow 0^-} \frac{-\delta x}{\delta x} = -1$$

The right hand and left hand limits are not equal, therefore, the  $\lim_{\delta x \rightarrow 0} \frac{|\delta x|}{\delta x}$  does not exist.

This implies that derivative of  $f$  at  $x = 0$  does not exist, and thus, there is no tangent line to the graph of  $f$  at this point (see Figure 13.3). However, the derivative exists at all other points of  $f$  i.e., it is 1 on the right side and  $-1$  on the left side. A function can be continuous at a point but not necessarily differentiable there.

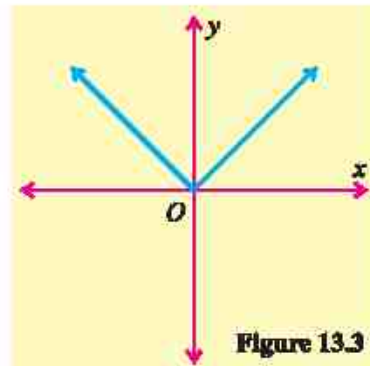


Figure 13.3

## EXERCISE 13.1

1. Find by definition, the derivatives w.r.t. 'x' of the following functions defined as:

(i)  $2x^2 + 1$       (ii)  $2 - \sqrt{x}$       (iii)  $\frac{1}{\sqrt{x}}$       (iv)  $x(x-3)$

2. Find  $\frac{dy}{dx}$  from first principle and find gradient of the curve at the given point:

(i)  $\sqrt{x+2}$  at  $x = 6$       (ii)  $\frac{1}{\sqrt{x+a}}$  at  $x = a$

3. (i) Find the derivative of  $x^{\frac{2}{3}}$  at  $x = 8$  from the first principle.  
(ii) Find the derivative of  $x^2 + 2x + 3$  by definition.  
4. Find from first principle, the derivatives of the following expressions w.r.t. their respective independent variables:

(i)  $(3x-2)^{-2}$       (ii)  $(2t+3)^4$       (iii)  $(aw+b)^7$

5. Find the gradient and equation of the tangent line to  $y = 3x^2 - 4x + 1$  at  $x = 2$ .  
6. For the function  $f(x) = 2x^3 + x$ , calculate the equation of the tangent line at  $x = -1$ .  
7. Find the coordinates of the point of tangency and the equation of the tangent line for  $f(x) = x^3 - 2x + 1$  at  $x = 1$ .  
8. Find the gradient of the curve  $f(x) = 3x^2 + 2x$  at  $x = 1$ .  
9. Find the gradient and an equation of tangent line to the graph of  $f(x) = \sqrt{x}$  at  $x = 9$ .  
10. The position of a car after  $t$  hours is given by:  $s(t) = 2t^3 - 3t^2 + t$  (in kilometres)  
(i) Find the average velocity over the interval  $[1, 4]$   
(ii) Find the instantaneous velocity at  $t = 2$   
11. A stone is thrown upwards and its height after  $t$  seconds is given by:  
 $s(t) = -16t^2 + 32t + 10$  (in feet), Find the instantaneous velocity at  $t = 1$   
12. The outdoor temperature (in  $^{\circ}\text{C}$ ) over time is modeled by:  $T(t) = -t^2 + 12t + 10$ , where  $t$  is the time in hours. Find the instantaneous rate of change at  $t = 2$ .

## 13.6 Theorems on Differentiation

We have, so far, proved the following two formulas:

1.  $\frac{d}{dx}(c) = 0$  that is, the derivative of a constant function is zero.
2.  $\frac{d}{dx}(x^n) = nx^{n-1}$ , power formula (or rule) when  $n$  is any real number.

Now we will prove other important formulas (or rules) which are used to determine derivatives of different functions efficiently. Henceforth, in all subsequent discussion,  $f, g, h$  etc, all denote functions differentiable at  $x$ , unless stated otherwise.

### 3. Derivative of $y = cf(x)$

**Proof:** Let  $y = cf(x)$ , Then

(i)  $y + \delta y = cf(x + \delta x)$  and

(ii)  $y + \delta y - y = cf(x + \delta x) - cf(x)$

or  $\delta y = c[f(x + \delta x) - f(x)]$  (Factoring out  $c$ )

(iii)  $\frac{\delta y}{\delta x} = c \left[ \frac{f(x + \delta x) - f(x)}{\delta x} \right]$

Taking limit when  $\delta x \rightarrow 0$

(iv)  $\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \left[ c \cdot \frac{f(x + \delta x) - f(x)}{\delta x} \right] = c \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}$

A constant factor can be taken out from a limit sign.

Thus,  $\frac{dy}{dx} = cf'(x)$ , that is  $[cf(x)]' = cf'(x)$  or  $\frac{d}{dx}[cf(x)] = c \frac{d}{dx}[f(x)]$

**Example 6** Calculate  $\frac{d}{dx}(3x^{\frac{4}{3}}) = 3 \frac{d}{dx}(x^{\frac{4}{3}})$  (Using Formula 3)

**Solution**  $= 3 \times \frac{4}{3} x^{\frac{4}{3}-1} = 4x^{\frac{1}{3}}$  (Using power rule)

### 4. Derivative of a sum or a difference of functions

If  $f$  and  $g$  are differentiable at  $x$ , then  $f + g, f - g$  are also differentiable at  $x$  and

$[f(x) + g(x)]' = f'(x) + g'(x)$ , that is,  $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$

Also  $[f(x) - g(x)]' = f'(x) - g'(x)$ , that is,  $\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$

**Proof:** Let  $\phi(x) = f(x) + g(x)$ . Then

(i)  $\phi(x + \delta x) = f(x + \delta x) + g(x + \delta x)$  and

(ii)  $\phi(x + \delta x) - \phi(x) = f(x + \delta x) + g(x + \delta x) - [f(x) + g(x)]$   
 $= [f(x + \delta x) - f(x)] + [g(x + \delta x) - g(x)]$  (rearranging the terms)

(iii)  $\frac{\phi(x + \delta x) - \phi(x)}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x} + \frac{g(x + \delta x) - g(x)}{\delta x}$

Taking the limit when  $\delta x \rightarrow 0$

$$\begin{aligned}
 \text{(iv)} \quad \lim_{\delta x \rightarrow 0} \frac{\phi(x+\delta x) - \phi(x)}{\delta x} &= \lim_{\delta x \rightarrow 0} \left[ \frac{f(x+\delta x) - f(x)}{\delta x} + \frac{g(x+\delta x) - g(x)}{\delta x} \right] \\
 &= \lim_{\delta x \rightarrow 0} \frac{f(x+\delta x) - f(x)}{\delta x} + \lim_{\delta x \rightarrow 0} \frac{g(x+\delta x) - g(x)}{\delta x}
 \end{aligned}$$

(The limit of a sum is the sum of the limits)

$$\phi'(x) = f'(x) + g'(x), \text{ that is } [f(x) + g(x)]' = f'(x) + g'(x)$$

$$\text{or } \frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

The proof for the second part is similar.

**Note**

Sum or difference formula can be extended to find derivative of more than two functions.

**Example 7** Find the derivative of  $y = \frac{3}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + 2x + 5$  w.r.t.  $x$ .

**Solution**  $y = \frac{3}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + 2x + 5$

Differentiating with respect to  $x$ , we have

$$\frac{dy}{dx} = \frac{d}{dx} \left[ \frac{3}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + 2x + 5 \right] = \frac{d}{dx} \left( \frac{3}{4}x^4 \right) + \frac{d}{dx} \left( \frac{2}{3}x^3 \right) + \frac{d}{dx} \left( \frac{1}{2}x^2 \right) + \frac{d}{dx}(2x) + \frac{d}{dx}(5)$$

(Using formula 4)

$$= \frac{3}{4} \frac{d}{dx}(x^4) + \frac{2}{3} \frac{d}{dx}(x^3) + \frac{1}{2} \frac{d}{dx}(x^2) + 2 \frac{d}{dx}(x) + 0 \quad (\text{Using formula 3 and 1})$$

$$= \frac{3}{4}(4x^{4-1}) + \frac{2}{3}(3x^{3-1}) + \frac{1}{2}(2x^{2-1}) + 2(1 \cdot x^{1-1}) \quad (\text{By power formula})$$

$$= 3x^3 + 2x^2 + x + 2$$

**Example 8** Find the derivative of  $y = (x^2 + 5)(x^3 + 7)$  with respect to  $x$ .

**Solution**  $y = (x^2 + 5)(x^3 + 7) = x^5 + 5x^3 + 7x^2 + 35$

Differentiating with respect to  $x$ , we get

$$\frac{dy}{dx} = \frac{d}{dx}[x^5 + 5x^3 + 7x^2 + 35]$$

$$= \frac{d}{dx}(x^5) + 5 \frac{d}{dx}(x^3) + 7 \frac{d}{dx}(x^2) + \frac{d}{dx}(35) \quad (\text{Using formulas 3 and 4})$$

$$= 5x^{5-1} + 5 \times 3x^{3-1} + 7 \times 2x^{2-1} + 0$$

$$= 5x^4 + 15x^2 + 14x$$

**Example 9** Find the derivative of  $y = (2\sqrt{x} + 2)(x - \sqrt{x})$

**Solution**  $y = (2\sqrt{x} + 2)(x - \sqrt{x})$

$$= 2(\sqrt{x} + 1) \cdot \sqrt{x}(\sqrt{x} - 1) = 2\sqrt{x}(\sqrt{x} + 1)(\sqrt{x} - 1)$$

$$= 2\sqrt{x}(x - 1) = 2(x^{\frac{3}{2}} - x^{\frac{1}{2}})$$

Differentiating with respect to  $x$  we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} [2(x^{\frac{3}{2}} - x^{\frac{1}{2}})] \\ &= 2 \left[ \frac{d}{dx} x^{\frac{3}{2}} - \frac{d}{dx} x^{\frac{1}{2}} \right] = 2 \left[ \frac{3}{2} x^{\frac{3}{2}-1} - \frac{1}{2} x^{\frac{1}{2}-1} \right] \\ &= 3x^{\frac{1}{2}} - x^{-\frac{1}{2}} = 3\sqrt{x} - \frac{1}{\sqrt{x}} = \frac{3x-1}{\sqrt{x}}\end{aligned}$$

### 5. Derivative of a Product (The Product Rule)

If  $f$  and  $g$  are differentiable at  $x$ , then  $fg$  is also differentiable at  $x$  and

$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$ , that is

$$\frac{d}{dx} [f(x)g(x)] = \left[ \frac{d}{dx} [f(x)] \right] g(x) + f(x) \left[ \frac{d}{dx} [g(x)] \right]$$

**Proof:** Let  $\phi(x) = f(x)g(x)$ . Then

(i)  $\phi(x + \delta x) = f(x + \delta x)g(x + \delta x)$  and

(ii)  $\phi(x + \delta x) - \phi(x) = f(x + \delta x)g(x + \delta x) - f(x)g(x)$

Subtracting and adding  $f(x)g(x + \delta x)$  in step (ii), gives

$$\begin{aligned}\phi(x + \delta x) - \phi(x) &= f(x + \delta x)g(x + \delta x) - f(x)g(x + \delta x) + f(x)g(x + \delta x) - f(x)g(x) \\ &= [f(x + \delta x) - f(x)]g(x + \delta x) + f(x)[g(x + \delta x) - g(x)]\end{aligned}$$

(iii)  $\frac{\phi(x + \delta x) - \phi(x)}{\delta x} = \left[ \frac{f(x + \delta x) - f(x)}{\delta x} \right] g(x + \delta x) + f(x) \left[ \frac{g(x + \delta x) - g(x)}{\delta x} \right]$

Taking limit when  $\delta x \rightarrow 0$

$$\begin{aligned}
 \text{(iv)} \quad & \lim_{\delta x \rightarrow 0} \frac{\phi(x + \delta x) - \phi(x)}{\delta x} \\
 &= \lim_{\delta x \rightarrow 0} \left[ \frac{f(x + \delta x) - f(x)}{\delta x} \cdot g(x + \delta x) + f(x) \cdot \frac{g(x + \delta x) - g(x)}{\delta x} \right] \\
 &= \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x} \cdot \lim_{\delta x \rightarrow 0} g(x + \delta x) + \lim_{\delta x \rightarrow 0} f(x) \cdot \lim_{\delta x \rightarrow 0} \frac{g(x + \delta x) - g(x)}{\delta x} \\
 &\quad \text{(Using limit theorem)}
 \end{aligned}$$

$$\text{Thus } \phi'(x) = f'(x) g(x) + f(x) g'(x) \quad \left[ \lim_{\delta x \rightarrow 0} g(x + \delta x) = g(x) \right]$$

$$\text{or } \frac{d}{dx} [f(x) \cdot g(x)] = \frac{d}{dx} [f(x)] \cdot g(x) + f(x) \left[ \frac{d}{dx} g(x) \right]$$

**Example 10** Find derivative of  $y = (2\sqrt{x} + 2)(x - \sqrt{x})$  with respect to  $x$ .

**Solution**  $y = (2\sqrt{x} + 2)(x - \sqrt{x})$

$$= 2(\sqrt{x} + 1)(x - \sqrt{x})$$

Differentiating with respect to  $x$ , we get

$$\begin{aligned}
 \frac{dy}{dx} &= 2 \frac{d}{dx} [(\sqrt{x} + 1)(x - \sqrt{x})] \\
 &= 2 \left[ \left( \frac{d}{dx} (\sqrt{x} + 1) \right) (x - \sqrt{x}) + (\sqrt{x} + 1) \frac{d}{dx} (x - \sqrt{x}) \right] \\
 &= 2 \left[ \left( \frac{1}{2} x^{\frac{1}{2}-1} + 0 \right) (x - \sqrt{x}) + (\sqrt{x} + 1) \times \left( 1 - \frac{1}{2} x^{\frac{1}{2}-1} \right) \right] \\
 &= 2 \left[ \frac{1}{2\sqrt{x}} (x - \sqrt{x}) + (\sqrt{x} + 1) \times \left( 1 - \frac{1}{2\sqrt{x}} \right) \right] \\
 &= 2 \left[ \frac{x - \sqrt{x}}{2\sqrt{x}} + (\sqrt{x} + 1) \left( \frac{2\sqrt{x} - 1}{2\sqrt{x}} \right) \right] \\
 &= \frac{1}{\sqrt{x}} [x - \sqrt{x} + 2x - \sqrt{x} + 2\sqrt{x} - 1] \\
 &= \frac{3x - 1}{\sqrt{x}}
 \end{aligned}$$

### 6. Derivative of a Quotient (The Quotient Rule)

If  $f$  and  $g$  are differentiable at  $x$  and  $g(x) \neq 0$ , for any  $x \in D(g)$  then  $\frac{f}{g}$  is differentiable

$$\text{at } x \text{ and } \left( \frac{f(x)}{g(x)} \right)' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$\text{that is } \frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = \frac{\left[ \frac{d}{dx} [f(x)] \right] g(x) - f(x) \left[ \frac{d}{dx} [g(x)] \right]}{[g(x)]^2}$$

**Proof:** Let  $\phi(x) = \frac{f(x)}{g(x)}$ . Then

$$(i) \quad \phi(x + \delta x) = \frac{f(x + \delta x)}{g(x + \delta x)} \text{ and}$$

$$(ii) \quad \phi(x + \delta x) - \phi(x) = \frac{f(x + \delta x)}{g(x + \delta x)} - \frac{f(x)}{g(x)} = \frac{f(x + \delta x)g(x) - f(x)g(x + \delta x)}{g(x)g(x + \delta x)}$$

Subtracting and adding  $f(x)g(x)$  in the numerator of step (ii), gives

$$\begin{aligned} \phi(x + \delta x) - \phi(x) &= \frac{f(x + \delta x)g(x) - f(x)g(x) - f(x)g(x + \delta x) + f(x)g(x)}{g(x)g(x + \delta x)} \\ &= \frac{1}{g(x)g(x + \delta x)} [(f(x + \delta x) - f(x))g(x) - f(x)(g(x + \delta x) - g(x))] \end{aligned}$$

$$(iii) \quad \frac{\phi(x + \delta x) - \phi(x)}{\delta x} = \frac{1}{g(x)g(x + \delta x)} \left[ \frac{f(x + \delta x) - f(x)}{\delta x} \cdot g(x) - f(x) \cdot \frac{g(x + \delta x) - g(x)}{\delta x} \right]$$

Taking limit when  $\delta x \rightarrow 0$

$$\begin{aligned} (iv) \quad \lim_{\delta x \rightarrow 0} \frac{\phi(x + \delta x) - \phi(x)}{\delta x} &= \lim_{\delta x \rightarrow 0} \left[ \frac{1}{g(x)g(x + \delta x)} \left( \frac{f(x + \delta x) - f(x)}{\delta x} \cdot g(x) - f(x) \cdot \frac{g(x + \delta x) - g(x)}{\delta x} \right) \right] \end{aligned}$$

Using limit theorems, we have

$$\phi'(x) = \frac{1}{g(x) \cdot g(x)} [f'(x)g(x) - f(x)g'(x)] \quad \left( \because \lim_{\delta x \rightarrow 0} g(x + \delta x) = g(x) \right)$$

Thus  $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

or  $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\left[\frac{d}{dx}(f(x))\right]g(x) - f(x)\left[\frac{d}{dx}(g(x))\right]}{[g(x)]^2}$

**Example 11** Differentiate  $\frac{2x^3 - 3x^2 + 5}{x^2 + 1}$  with respect to  $x$ .

**Solution** Let  $\phi(x) = \frac{2x^3 - 3x^2 + 5}{x^2 + 1}$ . Then

$$f(x) = 2x^3 - 3x^2 + 5 \quad \text{and} \quad g(x) = x^2 + 1$$

Now  $f'(x) = \frac{d}{dx}[2x^3 - 3x^2 + 5] = 2(3x^2) - 3(2x) + 0 = 6x^2 - 6x$

and  $g'(x) = \frac{d}{dx}[x^2 + 1] = 2x + 0 = 2x$

Using the quotient formula  $\phi'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$ , We obtain

$$\begin{aligned} \frac{d}{dx}\left[\frac{2x^3 - 3x^2 + 5}{x^2 + 1}\right] &= \frac{(6x^2 - 6x)(x^2 + 1) - (2x^3 - 3x^2 + 5)(2x)}{(x^2 + 1)^2} \\ &= \frac{6x^4 - 6x^3 + 6x^2 - 6x - (4x^4 - 6x^3 + 10x)}{(x^2 + 1)^2} \\ &= \frac{6x^4 - 6x^3 + 6x^2 - 6x - 4x^4 + 6x^3 - 10x}{(x^2 + 1)^2} \\ &= \frac{2x^4 + 6x^2 - 16x}{(x^2 + 1)^2} \end{aligned}$$

### EXERCISE 13.2

1. Differentiate w.r.t 'x'.

(i)  $x^4 + 2x^3 + x^2$

(ii)  $x^{-3} + 2x^{\frac{3}{2}} + 3$

(iii)  $\frac{2x-3}{2x+1}$

(iv)  $\frac{(1+\sqrt{x})(x-x)}{\sqrt{x}}$

(v)  $\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2$

(vi)  $(x-5)(3-x)$

(vii)  $\frac{(x^2+1)^2}{x^2-1}$

(viii)  $\frac{x^2+1}{x^2-3}$

(ix)  $\frac{2x-1}{\sqrt{x^2+1}}$

(x)  $\sqrt{\frac{a-x}{a+x}}$

(xi)  $\frac{\sqrt{x^2+1}}{\sqrt{x^2-1}}$

2. Find  $\frac{dy}{dx}$  if  $y = \frac{(\sqrt{x}+1)(x^{\frac{3}{2}}-1)}{x^{\frac{1}{2}}-1}$ , ( $x \neq 1$ )

3. Differentiate  $\frac{(\sqrt{x}+1)(x^{\frac{3}{2}}-1)}{x^{\frac{3}{2}}-x^{\frac{1}{2}}}$  with respect to  $x$ .

4. If  $y = \sqrt{x} - \frac{1}{\sqrt{x}}$ , show that  $2x \frac{dy}{dx} + y = 2\sqrt{x}$

5. If  $y = x^4 + 2x^2 + 2$ , prove that  $\frac{dy}{dx} = 4x\sqrt{y-1}$ .

### 13.7 Application of Differentiation

We will apply concept of differentiation to real-world problems such as (profits on diminishing returns, environmental factors, financial investments, population growth, spread of diseases, movement of particles, time-speed in transportation, structural stress, material required that is changes in construction).

#### Profits on Diminishing Returns

**Example 12** A company's profit function is given by  $P(x) = 100x - 5x^2$ , where  $x$  is the number of units produced. Determine the marginal profit when  $x = 8$  units.

**Solution** The marginal profit is the derivative of the profit function with respect to  $x$ .

$$P'(x) = \frac{d}{dx}(100x - 5x^2) = 100 - 10x$$

Now, substitute  $x = 8$ :  $P'(8) = 100 - 10(8) = 20$

So, the marginal profit is 20 when 8 units are produced (in the given currency).

#### Movement of Particles

**Example 13** A particle moves along a line according to the position function  $s(t) = 4t^3 - 3t^2 + 2t$ , where  $s(t)$  is the position in metres and  $t$  is the time in seconds. Find the velocity and acceleration at  $t = 2$  seconds.

**Solution** Velocity is the derivative of the position function:

$$v(t) = \frac{d}{dt}(4t^3 - 3t^2 + 2t) = 12t^2 - 6t + 2$$

Substitute  $t = 2$ :

$$v(2) = 12(2)^2 - 6(2) + 2 = 48 - 12 + 2 = 38$$

So, the velocity at  $t = 2$  is 38 m/s.

Acceleration is the derivative of the velocity function:

$$a(t) = \frac{d}{dt}(12t^2 - 6t + 2) = 24t - 6$$

Substitute  $t = 2$

$$a(2) = 24(2) - 6 = 48 - 6 = 42$$

So, the acceleration at  $t = 2$  is 42 m/s<sup>2</sup>.

### Financial Investments

**Example 14** A bank offers a compound interest rate on an investment, and the value of the investment after  $t$  years is given by  $V(t) = 5000(1 + 0.04t)^2$ . Find the rate of change of the investment value after 10 years.

**Solution** The rate of change of the investment is the derivative of  $V(t)$  with respect to  $t$ .

$$V'(t) = \frac{d}{dt}(5000(1 + 0.04t)^2) = 5000(2)(1 + 0.04t)(0.04)$$

$$V'(t) = 400(1 + 0.04t)$$

Substitute  $t = 10$ :

$$V'(10) = 400(1 + 0.04 \times 10) = 400(1 + 0.40) = 400 \times 1.4 = 560$$

So, the investment is growing at a rate of Rs.560 per year after 10 years.

### Structural Stress

**Example 15** The stress on a beam under a varying load is modeled by  $S(x) = 500x - 2x^3$ , where  $S(x)$  is the stress in pascals (Pa) and  $x$  is the distance (in metres) from the beam's fixed end. Find the rate of change of stress at  $x = 5$  metres.

**Solution** The rate of change of stress is the derivative of  $S(x)$  with respect to  $x$ .

$$S'(x) = \frac{d}{dx}(500x - 2x^3) = 500 - 6x^2$$

Substitute  $x = 5$ :

$$S'(5) = 500 - 6(5)^2 = 500 - 6 \times 25 = 500 - 150 = 350$$

So, the stress is increasing at a rate of 350 Pa per metre at  $x = 5$  metres.

## EXERCISE 13.3

1. A car's position at time  $t$  is given by  $s(t) = 5t^3 - 3t^2 + t$ . Find the velocity by differentiating the position function with respect to time.
2. Structural stress on a bridge is modeled by the function  $S(x) = 100 - 5x^2$ , where  $x$  is the distance from the center of the bridge. Calculate the rate of change of stress at that point.

3. A company's revenue function is given by  $R(x) = 1000x - 10x^2$ , where  $x$  is the number of units produced. The cost function is  $C(x) = 300x + 2000$ .
- Find the profit function  $P(x)$
  - Determine the marginal profit when  $x = 15$
4. An investment grows according to the function  $A(t) = 10000(1 + 0.05t)^3$ , where  $A(t)$  is the value of the investment and  $t$  is the time in years.
- Find the rate of change of the investment after 8 years.
  - What is the investment value after 8 years?
5. The position of a particle moving along a line is given by  $s(t) = 5t^3 - 12t^2 + 8t$ , where  $s(t)$  is the position in meters and  $t$  is the time in seconds.
- Determine the velocity of the particle at  $t = 4$  seconds
  - Find the acceleration at  $t = 4$  seconds
  - When is the particle at rest?
6. The position of a car traveling along a straight highway is given by  $x(t) = 30t^2 - 4t^3$ , where  $x(t)$  is the distance traveled in kilometres and  $t$  is the time in hours.
- Find the car's velocity at  $t = 3$  hours.
  - Determine the car's acceleration at  $t = 3$  hours
7. The stress on a beam under a varying load is given by  $S(x) = 400x - x^3$ , where  $S(x)$  is the stress in pascals (Pa) and  $x$  is the distance from the fixed end in metres. Calculate the rate of change of stress at 6 metres.
8. The cost  $C(r)$  to construct a cylindrical tank depends on the radius of the base, and is given by  $C(r) = 8000\pi r^2 + \frac{150000}{r}$ , where the first term represents the cost of the base and the second term represents the cost of the walls. Determine the rate of change of the cost at  $r = 4$  metres.