Chapter 3 Circular and Rotational Motion

Learning Objectives

After studying this chapter, the students will be able to:

- Express angles in radians
- Define and calculate angular displacement, angular velocity and angular acceleration [This involves use of $S = r\theta$, $v = r\omega$, $\omega = 2\pi r/T$, $a = r\omega^2$, and $a = v^2/r$ to solve problems]
- Use equations of angular motion to solve problems involving rotational motions.
- Discuss qualitatively motion in a curved path due to a perpendicular force.
- Define and calculate centripetal force [Use $F_c = mr\omega^2$, $F_c = mv^2/r$]
- Analyze situations involving circular motion in terms of centripetal force [e.g. situations in which centripetal acceleration is caused by a tension force, a frictional force, a gravitational force, or a normal force.]
- Define and calculate average orbital speed for a satellite. [from the equation $v = 2\pi r/T$ where r is the average radius of the orbit and T is the orbit period; apply this equation to solve numerical problems]
- Explain why the objects in orbiting satellites appear to be weightless.
- Describe how artificial gravity is created to counter weightlessness.
- Define and calculate moments of inertia of a body and angular momentum.
- Derive and apply the relation between torque, moment of inertia and angular acceleration. Illustrate the applications of conservation of angular momentum in real life. [such as by flywheels to store rotational energy, by gyroscopes in navigation systems, by ice skaters to adjust their angular velocity]
- Describe how a centrifuge is used to separate materials using centripetal force

Among all possible motions of the material bodies, the circular motion is one that appears to be working in the most of the natural world. Satellites moving in circular orbits around the Earth, orbital and spin motion of the Earth itself, a car turning around a curved road, and a stone whirled around by a string are the familiar examples. When objects move in circular paths, their direction is continuously changing. Since velocity is a vector quantity, this change of direction means that their velocities are not constant.

In this chapter, we will study, circular motion, rotational motion, moment of inertia, angular momentum and the related topics.

3.1 ANGULAR MEASUREMENTS

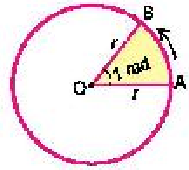


Fig. 3.1

Consider an angle subtended at the centre 'O' of a circle by an arc 'AB' as shown in Fig. 3.1. If the length of the arc 'AB' is equal to the radius 'r' of the circle, then the angle is called one radian. It is the SI unit of angular measurement and its symbol is "rad".

Angular Displacement

Consider the motion of a single particle P of mass m in a circular path of radius r. Suppose this motion is taking place by attaching the particle P at the end of a massless rigid rod of length r whose other end is pivoted at the centre O of the circular path, as shown in Fig. 3.2 (a). As the particle is moving on the circular path, the rod OP rotates in the plane of the circle. The axis of rotation passes through the pivot O and is normal to the plane of rotation. Consider a system of axes as shown in Fig. 3.2 (b). The z-axis is taken along the axis of rotation with the pivot O as origin of coordinates. Axes x and y are taken in the plane of rotation. While OP is rotating, suppose at any instant t, its position is OP_1, making angle θ with x-axis. At a later time $t + \Delta t$, let its position be OP_2 making angle $\theta + \Delta\theta$ with x-axis (Fig. 3.2-c).

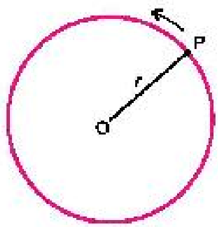


Fig. 3.2(a)

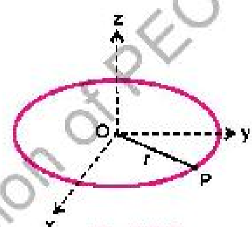


Fig. 3.2(b)

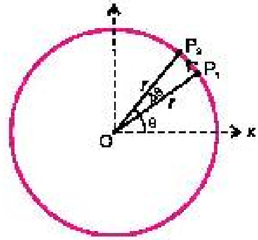


Fig. 3.2(c)

Angle $\Delta\theta$ defines the angular displacement of OP during the time interval Δt. For very small values of $\Delta\theta$, the angular displacement is a vector quantity.

The angular displacement $\Delta\theta$ is assigned a positive sign when the sense of rotation of OP is counter clock wise.

The direction associated with $\Delta\theta$ is along the axis of rotation and is given by right hand rule as shown in Fig 3.2 (d) which states that:

> Grasp the axis of rotation in right hand with fingers curling in the direction of rotation; the thumb points in the direction of angular displacement.

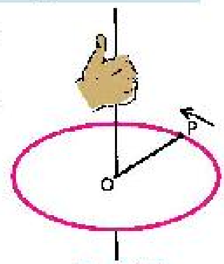


Fig. 3.2(d)

Three units are generally used to express angular displacement, namely degrees, revolution and radian. Consider an arc of length S of a circle of radius r (Fig. 3.3) which subtends an angle θ at the centre of the circle. Its value in radians (rad) is given as:

$$\theta = \frac{S}{r}$$

or $$S = r\theta \quad \text{(where } \theta \text{ is in radian)} \quad \ldots\ldots \quad (3.1)$$

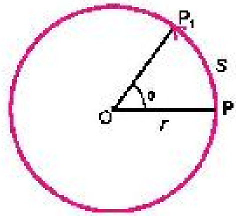


Fig. 3.3

If OP is rotating, the point P covers a distance $S = 2\pi r$ in one revolution of P. In radian, it would be:

$$\frac{S}{r} = \frac{2\pi r}{r} = 2\pi$$

So $$1 \text{ revolution} = 2\pi \text{ rad} = 360^\circ$$

or $$1 \text{ rad} = \frac{360^\circ}{2\pi} = 57.3^\circ$$

Angular Velocity

Very often, we are interested in knowing how fast or how slow a body is rotating. It is determined by its angular velocity defined as the rate at which the angular displacement is changing with time. Referring to Fig. 3.2(c), if $\Delta\theta$ is the angular displacement during the time interval Δt, the average angular velocity ω_{av} during this interval is given by

$$\omega_{av} = \frac{\Delta\theta}{\Delta t} \quad \ldots\ldots\ldots \quad (3.2)$$

The instantaneous angular velocity ω is the limit of the ratio $\Delta\theta/\Delta t$ as Δt, approaches to zero.

Thus $$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} \quad \ldots\ldots\ldots \quad (3.3)$$

In the limit when Δt approaches zero, the angular displacement would be infinitesimally small. So, it would be a vector quantity and the angular velocity as defined by Eq. 3.3 would also be a vector. Its direction is along the axis of rotation and is given by right hand rule as described earlier.

Angular velocity is measured in radians per second which is the SI unit. Sometimes it is also given in terms of revolution per minute (rpm).

Angular acceleration

When we switch on an electric fan, we notice that its angular velocity goes on increasing till it becomes uniform. We say that it has an angular acceleration. We define angular acceleration as the rate of change of angular velocity. If ω_i and ω_f are the values of instantaneous velocity of a rotating body at instants t_i and t_f , the average angular acceleration during the interval $t_f - t_i$ is given by

$$\alpha_{av} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t} \quad \ldots\ldots\ldots \quad (3.4)$$

The instantaneous angular acceleration is the limit of the ratio $\frac{\Delta\omega}{\Delta t}$ as Δt approaches zero. Therefore, instantaneous angular acceleration α is given by

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} \quad \text{..........} \quad (3.5)$$

The angular acceleration is also a vector quantity whose magnitude is given by Eq. 3.5 and its direction is along the axis of rotation. Angular acceleration is expressed in rad s^{-2}.

Till now we have been considering the motion of a particle P on a circular path. The point P was fixed at the end of a rotating massless rigid rod. Now consider the rotation of a rigid body as shown in Fig. 3.4. Imagine a point P on the rigid body. Line OP is the perpendicular dropped from P on the axis of rotation usually referred as the reference line. As the body rotates, line OP also rotates with the same angular velocity and angular acceleration. Thus, the rotation of a rigid body can be described by the rotation of the reference line OP and all the terms that we defined with the help of rotating line OP are also valid for the rotational motion of a rigid body. Henceforth, while dealing with rotation of a rigid body, we will replace it by its reference line OP.

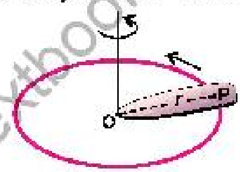


Fig. 3.4

Relation between Angular and Linear Velocities

Consider a rigid body rotating about z-axis with an angular velocity ω as shown in Fig. 3.5 (a).

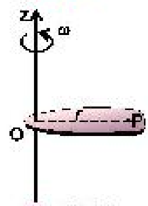


Fig. 3.5(a)

Imagine a point P in the rigid body at a perpendicular distance r from the axis of rotation. OP represents the reference line of the rigid body. As the body rotates, the point P moves along a circle of radius r with a linear velocity **v** whereas the line OP rotates with angular velocity ω as shown in Fig. 3.5 (b). We are interested in finding out the relation between ω and **v**. As the axis of rotation is fixed, so the direction of ω always remains the same and ω can be manipulated as a scalar. As regards the linear velocity of the point P, let us consider only its magnitude only which can also be treated as a scalar.

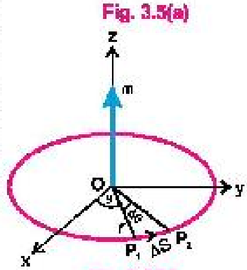


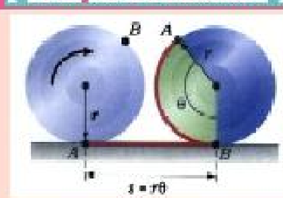
Fig. 3.5(b)

$$P_1P_2 = \Delta S$$

Suppose during the course of its motion, the point P moves through a distance $P_1P_2 = \Delta S$ in a time interval Δt during which reference line OP covers an angular displacement $\Delta\theta$ radian. So, ΔS and $\Delta\theta$ are related by Eq. 3.1as:

$$\Delta S = r\Delta\theta$$

For your information



As the wheel turns through an angle θ, it lays out a tangential distance $S = r\theta$.

Dividing both sides by Δt

$$\frac{\Delta S}{\Delta t} = r\frac{\Delta \theta}{\Delta t} \quad \text{..........} \quad (3.6)$$

In the limit when linear $\Delta t \rightarrow 0$ the ratio $\Delta S/\Delta t$ represents v, the magnitude of the linear velocity with which point P is moving on the circumference of the circle. Similarly $\Delta\theta/\Delta t$ represents the angular velocity ω of the reference line OP. So, Eq. 3.6 becomes:

$$v = r\omega \quad \text{..........} \quad (3.7)$$

Point to ponder!

You may feel scared at the top of roller coaster ride in the amusement parks but you never fall down even when you are upside down. Why?

From Fig. 3.5(b), it can be seen that the point P is moving along the arc P_1P_2. In the limit when $\Delta t \rightarrow 0$, the length of arc P_1P_2 becomes very small and its direction represents the direction of tangent to the circle at point P_1. Thus, the velocity with which point P is moving on the circumference of the circle has a magnitude v and its direction is always along the tangent to the circle at that point. That is why, the linear velocity of the point P is also known as tangential velocity.

Similarly, Eq. 3.7 shows that if the reference line OP is rotating with an angular acceleration α, the point P will also have a linear or tangential acceleration a_t. Using Eq. 3.7 it can be shown that the two accelerations are related by

$$a_t = r\alpha \quad \text{................} \quad (3.8)$$

Eqs. 3.7 and 3.8 show that on a rotating body, points that are at different distances from the axis do not have the same speed or acceleration, but all points on a rigid body rotating about a fixed axis do have the same angular displacement, angular speed and angular acceleration at any instant. Thus, by the use of angular variables, we can describe the motion of the entire body in a simple way.

Equations of Angular Motion

The equations (3.2, 3.3, 3.4 and 3.5) of angular motion are exactly analogous to those in linear motion if θ, ω and α be replaced by S, v and a, respectively. As the other equations of linear motion were obtained by algebraic manipulation of these equations, it follows that analogous equations will also apply to angular motion. Given below are angular equations together with their linear counterparts.

Linear Equations	Angular Equations	
$v_f = v_i + at$	$\omega_f = \omega_i + \alpha t$	(3.9)
$2aS = v_f^2 - v_i^2$	$2\alpha\theta = \omega_f^2 - \omega_i^2$	(3.10)
$S = v_i t + \frac{1}{2}at^2$	$\theta = \omega_i t + \frac{1}{2}\alpha t^2$	(3.11)

The angular equations 3.9 to 3.11 hold true only in the case when the axis of rotation is fixed, so that all the angular vectors have the same direction. Hence, they can be manipulated as scalars.

Example 3.1 An electric fan rotating at 3 rev s^{-1} is switched off. It comes to rest in 18.0 s. Assuming deceleration to be uniform, find its value. How many revolutions did it turn before coming to rest?

Solution In this problem, we have

$\omega_i = 3.0 \text{ rev s}^{-1}$, $\omega_f = 0$, $t = 18.0\text{ s}$ and $\alpha = ?$, $\theta = ?$

From Eq. 3.9, we have

$$\alpha = \frac{\omega_f - \omega_i}{t} = \frac{(0-3.0)\text{ rev s}^{-1}}{18.0\text{ s}} = -0.167 \text{ rev s}^{-2}$$

and from Eq 3.11, we have

$$0 = \omega_i t + \frac{1}{2}\alpha t^2$$

$= 3.0 \text{ rev s}^{-1} \times 18.0\text{ s} + \frac{1}{2}(-0.167 \text{ rev s}^{-2}) \times (18.0\text{ s})^2$

$= 27$ rev

Do you know?

Direction of motion changes continuously in circular motion.

3.2 CENTRIPETAL FORCE

Newton's second law of motion states that when a force acts on a body, it produces acceleration in the same direction. A force acting on a moving body along the direction of its velocity will change magnitude of the velocity (speed) keeping the direction unchanged. On the other hand, a constant force acting perpendicular to the velocity of a body moving in a circular path will change the direction but magnitude of velocity (speed) will remain the same. Such force makes the body move in a circle by producing a radial (or centripetal) acceleration and is called centripetal force (Centre seeking) force. Figure 3.6(a) shows a ball tied at the end of a string is whirled in a horizontal surface. It would not continue in a circular path if the string is snapped. Careful observation shows at once that if the string snaps, when the ball is at the point A, in Fig. 3.6 (b), the ball will follow the straight line path AB which is tangent AB at point A.

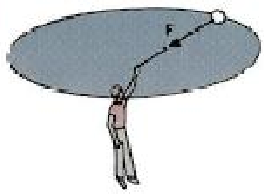


Fig. 3.6(a)

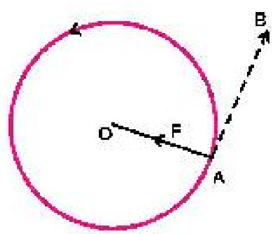


Fig. 3.6(b)

Thus, a force is needed to change the direction of velocity or motion of a body continuously at each point in circular motion moving with uniform speed. The force that does not alter speed but only direction at each point is a perpendicular force which acts along the radius of the circular path. This force always pulls the object towards the centre of the circular path. Its direction is perpendicular to the tangential velocity at each point.

The force needed to bend the straight path of the particle into a circular path is called the centripetal force.

For a body of mass m moving with velocity v in a circular path of radius r, centripetal force F_c is given by

$$F_c = ma_c = \frac{mv^2}{r} \quad \text{..........} \quad (3.12)$$

where $a = v^2/r$ is the centripetal acceleration and its direction is towards the centre of the circle. As $v = r\omega$, so the above equation becomes:

$$F_c = mr\omega^2 \quad \text{.............} \quad (3.13)$$

Example 3.2 If a CD spins at 210 rpm, what is the radial acceleration of a point on the outer rim of the CD? The CD is 12 cm in diameter.

Solution We convert 210 rpm into a frequency in revolutions per second (Hz).

Thus $f = 210\,\frac{\text{rev}}{\text{min}} \times \frac{1}{60}\,\frac{\text{min}}{\text{s}} = 3.5\,\frac{\text{rev}}{\text{s}} = 3.5\text{ Hz}$

For each revolution, the CD rotates through an angle of 2π radians. The angular velocity is:

$$\omega = 2\pi f = 2\pi\text{ rad} \times 3.5\text{ s}^{-1} = 7.0\,\pi\text{ rad s}^{-1}$$

The radial acceleration is:

$$a = \omega^2 r = (7.0\pi\text{ rad s}^{-1})^2 \times 0.06\text{ m} = 29\text{ m s}^{-2}$$

Do you know?

Curved flight at high speed requires a large centripetal force that makes the stunt dangerous even if the air planes are not so close.

Example 3.3 A ball tied to the end of a string, is swung in a vertical circle of radius r under the action of gravity as shown in Fig. 3.7. What will be the tension in the string when the ball is at the point A of the path and its speed is v at this point?

Solution For the ball to travel in a circle, the force acting on the ball must provide the required centripetal force. In this case, at point A, two forces act on the ball, the pull of the string and the weight w of the ball. These forces act along the radius at A, and so their vector sum must furnish the required centripetal force. We, therefore, have

$$T + w = \frac{mv^2}{r} \qquad \text{as} \quad w = mg, \text{ therefore,}$$

$$\therefore \quad T = \frac{mv^2}{r} - mg = m\left(\frac{v^2}{r} - g\right)$$

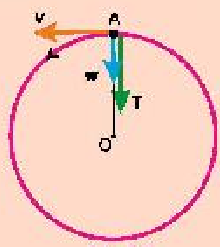


Fig. 3.7

If $\frac{v^2}{r} = g$, then T will be zero and the centripetal force is just equal to the weight.

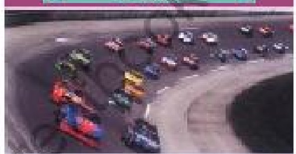
Examples of centripetal force

In every circular or orbital motion, centripetal force is needed which is provided by some agency.

1. When a ball is whirled in a horizontal circle with the help of a string, then tension in the string provides necessary centripetal force.
2. For an object placed on a turntable, the friction is the centripetal force.
3. The gravitational force is the cause of the Earth orbiting around the Sun, Moon and artificial satellites revolving around the Earth.
4. A normal or perpendicular magnetic force compels a charge particle moving along a straight path into a circular path.
5. When a vehicle takes turn on a road, it also needs centripetal force which is provided by the friction between the tyres and the road. If the road is slippery, then at high speed, the friction may not be sufficient enough to provided necessary centripetal force.

   Hence, vehicle will not be able to take turn and may skid or may even be toppled. To overcome this difficulty, the highway road is banked on turns. That is, the outer edge of the track is kept slightly higher than that of the inner edge.

Tidbits



Banked tracks are needed for turns that are taken so quickly that friction alone cannot provide energy for centripetal force.

Applications of Centripetal Force

We know that an object moves in a circle because of centripetal force. If the magnitude of applied force falls short of required centripetal force then the object will move away from the centre of the circle. The centrifuge (Fig. 3.8-a) functions on this basic principle.

Fig. 3.8 (a)

Centrifuge: It is one of the most useful laboratory device. It helps to separate out denser and lighter particles from a mixture. The mixture is rotated at high speed for a specific time. In a laboratory setup, sample tubes are used where the denser particles will settle at the bottom and lighter particles will rise to the top of the sample tubes (Fig. 3.8-b).

Fig. 3.8 (b)

The **dryer** of the washing machine also functions on the principle of centrifuge. The dryer consists of a long cylinder with hundreds of small holes on its wall. Wet clothes are piled up in this cylinder, which is then rotated rapidly about its axis. Water moves outward to the walls of the cylinder and thus, drained out through the holes. In this way, clothes become dry quickly.

Fig. 3.9

Cream separator is another practical device which is used to separate cream form the milk. In this machine, milk is whirled rapidly. Since milk is a mixture of light and heavy particles, when it is rotated, the light particles gather near the axis of rotation whereas the heavy particles will go outwards and hence, cream can easily be separated from milk.

Fig. 3.10

3.3 ARTIFICIAL SATELLITES

Satellites are objects that orbit in nearly circular path around the Earth. They are put into orbit by rockets and are held in orbits by the gravitational pull of the Earth. The low flying Earth satellites have acceleration 9.8 m s^{-2} towards the centre of the Earth. If there is no gravitational pull, they would fly off in a straight line along tangent to the orbit. When the satellite is moving in a circle, it has an acceleration $\frac{v^2}{r}$. In a circular orbit around the Earth, the centripetal acceleration is supplied by gravity and we have

$$g = \frac{v^2}{R} \qquad .... \qquad (3.14)$$

where v is the orbital velocity and R is the radius of the Earth (6400 km). From Eq. 3.14, we have

$$v = \sqrt{gR}$$

$$= \sqrt{9.8 \text{ m s}^{-2} \times 6.4 \times 10^6 \text{ m}}$$

$$= 7.9 \times 10^3 \text{ m s}^{-1} \quad = 7.9 \text{ km s}^{-1}$$

This is the minimum velocity necessary to put a satellite into the orbit, called the critical velocity. The period T is given by

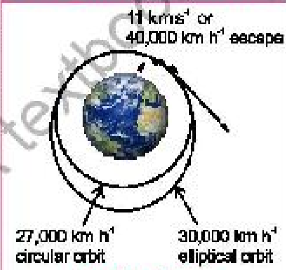
$$T = \frac{2\pi R}{v} \quad = 2 \times 3.14 \times \frac{6400 \text{ km}}{7.9 \text{ km s}^{-1}}$$

$$= 5060 \text{ s} = 84 \text{ min approx.}$$

If, however, a satellite in a circular orbit is at a distance h much greater than R above the Earth's surface, we must take into account the experimental fact that the gravitational acceleration decreases inversely as the square of the distance from the centre of the Earth (Fig. 3.11).

The higher the satellite, the slower will be the required speed and longer it will take to complete one revolution around the Earth.

For your information



Satellites Orbits

Do you know?

Newton had predicated about the artificial satellites 300 years ago. The above figure has been taken from his well-known book "Principia Mathematica". According to this book, if an object is thrown horizontally with a particular speed from a place which is sufficiently high, it will start revolving around the Earth.

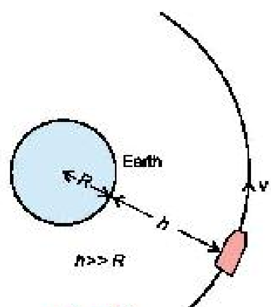


Fig. 3.11

Orbital Velocity

Figure 3.12 shows a satellite going round the Earth in a circular path. Let the mass of the satellite be m_s and v is its orbital speed. The mass of the Earth is M and r represents the radius of the orbit. A centripetal force $m_s v^2/r$ is required to hold the satellite in the orbit. This force is provided by the gravitational force of attraction between the Earth and the satellite. Equating the gravitational force to the required centripetal force, we have

$$\frac{G m_s M}{r^2} = \frac{m_s v^2}{r}$$

or $$v = \sqrt{\frac{GM}{r}} \quad \ldots\ldots\ldots \quad (3.15)$$

This shows that the mass of the satellite is unimportant in describing the satellite's orbit. Thus, any satellite orbiting at distance r from the Earth's centre must have the orbital speed given by Eq. 3.15. Any speed less than this will bring the satellite tumbling back to the Earth.

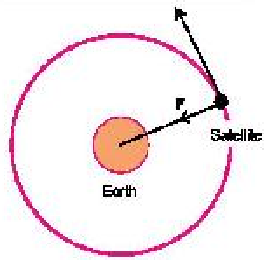


Fig. 3.12

Tidbits

The moment you switch on your mobile phone, your location can be tracked immediately by global positioning system.

Tidbits

In 1984, at a height of 100 km above Hawaii Island with a speed of 29000 km h^{-1} Bruce McCandless stepped into space from a space shuttle and became the first human satellite of the Earth.

Example 3.4 An Earth satellite is in circular orbit at distance of 384,000 km from the Earth's surface. What is its period of one revolution in days? Take mass of the Earth $M = 6.0 \times 10^{24}$ kg and its radius $R = 6400$ km.

Solution

As $\quad r = R + h = (6400 + 384000)\text{ km} = 390400\text{ km}$

Using $$v = \sqrt{\frac{GM}{r}} = \sqrt{\frac{6.67\times10^{-11}\text{ N m}^2\text{ kg}^{-2} \times 6\times10^{24}\text{ kg}}{390400\text{ km}}}$$

$$= 1.025\text{ km s}^{-1}$$

Also

$$T = \frac{2\pi R}{v} = 2\times3.14\times390400\text{ km}\times\frac{1}{1.025\text{ km s}^{-1}}\times\frac{1\text{ day}}{60\times60\times24\text{ s}}$$

$$= 27.7\text{ days}$$

Weightlessness in Satellites

When a satellite is launched by a rocket in its desired orbit around the Earth, then it has been observed practically that everything inside the satellite experiences weightlessness because the satellite is accelerating towards the centre of the Earth as a freely falling body.

Do you know?

Your weight slightly changes when the velocity of the elevator changes at the start and end of a ride, not during the rest of the ride when that velocity is constant.

Consider a satellite of mass M revolving in its orbit of radius r around the Earth. A body of mass m inside the satellite suspended by a spring balance from the ceiling of the satellite is under the action of two forces. That is, its weight mg acting downward, while the supporting force, called normal force F_N or tension in the spring acting upward, as shown in Fig. 3.13. Their resultant force is equal to the centripetal force required by the mass m which is acting towards the centre of the Earth, and is expressed as:

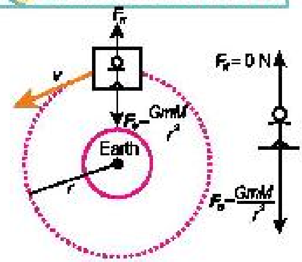


Fig. 3.13

Can you tell?

When a bucket full of water is rapidly whirled in a vertical circular path, water does not fall out even if the bucket is inverted at the maximum height. Why is it so?

$$F_c = mg - F_N \quad \text{........} \quad (3.16)$$

where $\quad F_c = \frac{mv^2}{r}$

Hence $\quad \frac{mv^2}{r} = mg - F_N \quad \text{........} \quad (3.16\text{-a})$

It may be noted that the centripetal force responsible for the revolution of the satellite of mass M around the Earth is provided by the gravitational force of attraction between the Earth and the satellite.

$$F_g = F_c$$

$$Mg = \frac{Mv^2}{r}$$

$$g = \frac{v^2}{r}$$

Hence, Eq. 3.16(a) becomes:

$$mg = mg - F_N$$

or $\quad F_N = 0$


Astronaut floating inside the cabin of a spaceship.

This shows that the supporting force which is acting on a body inside the satellite is zero. Therefore, the bodies as well as the astronauts in a satellite find themselves in a state of apparent weightlessness.

Artificial Gravity

In a gravity free space, there will be no force that will push anybody to any side of the spacecraft. If this spacecraft is to stay in the orbit over an extended period of time, the weightlessness may affect the performance of the astronauts present in that spacecraft. To overcome this difficulty, an artificial gravity can be created in the spacecraft. This could enable the crew of the space ships to function in an almost normal manner. For this situation to prevail, the spaceship is set into rotation around its own axis. The astronaut then is pressed towards the outer rim and exerts a force on the 'floor' of the spaceship in much the same way as on the Earth.

Consider a spacecraft of the shape as shown in Fig. 3.14. The outer radius of the spaceship is R and it rotates around its own central axis with angular speed ω, then its angular acceleration a_c is

$$a_c = R\,\omega^2$$

But $\omega = \dfrac{2\pi}{t}$ where t is the period of revolution of spaceship

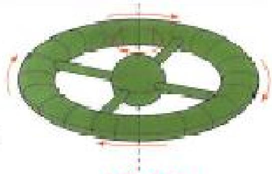


Fig. 3.14

Hence
$$a_c = R\frac{(2\pi)^2}{t^2} = R\frac{4\pi^2}{t^2}$$

As frequency $f = 1/t$, therefore,

$$a_c = R\,4\pi^2 f^2$$

or
$$f^2 = \frac{a_c}{4\pi^2 R} \qquad \text{or} \qquad f = \frac{1}{2\pi}\sqrt{\frac{a_c}{R}}$$

As described above, the force of gravity provides the required centripetal acceleration, therefore,

$$a_c = g$$

So
$$f = \frac{1}{2\pi}\sqrt{\frac{g}{R}} \qquad \text{..........} \qquad (3.17)$$

When the spaceship rotates with this frequency, the artificial gravity like the Earth is provided to the inhabitants of the spaceship.

Do you know?

The surface of the rotating space-ship pushes on an object with which it is in contact and thereby provides the centripetal force needed to keep the object moving on a circular path.

3.4 MOMENT OF INERTIA

Consider a mass m attached to the end of a massless rod as shown in Fig. 3.15. Assume that the bearing at the pivot point O is frictionless. Let the system be in a horizontal plane. A force F is acting on the mass perpendicular to the rod and hence, this will accelerate the mass according to:

$$F = ma$$

In doing so, the force will cause the mass to rotate about O. Since tangential acceleration a_T is related to angular acceleration α by the equation.

$$a_t = r\alpha$$

So
$$F = mr\alpha$$

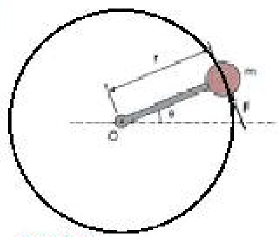


Fig. 3.15

The force F causes a torque about the axis O and gives the mass m an angular acceleration α about the pivot point.

As turning effect is produced by torque τ, it would, therefore, be better to write the equation for rotation in terms of torque. This can be done by multiplying both sides of the above equation by r. Thus,

$$rF = \tau = \text{torque} = mr^2\alpha$$

which is rotational analogue of the Newton's second law of motion, $F = ma$.

Here F is replaced by τ, a by α and m by mr^2. The quantity mr^2 is known as the moment of inertia and is represented by I. The moment of inertia plays the same role in angular motion as the mass in linear motion. It may be noted that moment of inertia depends not only on mass m but also on r^2.

Most rigid bodies have different mass concentration at different distances from the axis of rotation, which means the mass distribution is not uniform. As shown in Fig. 3.16(a), the rigid body is made up of n small pieces of masses m_1, m_2,.... at distances r_1 , r_2 ...... from the axis of rotation O. Let the body be rotating with the angular acceleration α, so the magnitude of the torque acting on m_1 is

$$\tau_1 = m_1 r_1^2 \alpha_1$$

Similarly, the torque on m_2 is

$$\tau_2 = m_2 r_2^2 \alpha_2$$

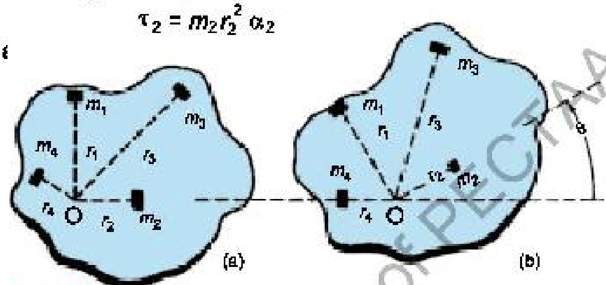


Fig. 3.16

Each small piece of mass within a large, rigid body undergoes the same angular acceleration about the pivot point.

Since the body is rigid, so all the masses are rotating with the same angular acceleration α.

Total torque τ_{total} is then given by

$$\tau_{total} = (m_1 r_1^2 + m_2 r_2^2 + \ldots + m_n r_n^2)\,\alpha$$

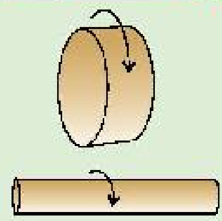
$$= \left(\sum_{i=1}^{n} m_i r_i^2\right)\alpha$$

or $\quad \tau = I\alpha \quad$ .... (3.18)

where I is the moment of inertia of the body and is expressed as

$$I = \sum_{i=1}^{n} m_i r_i^2 \quad \ldots\ldots\ldots \quad (3.19)$$

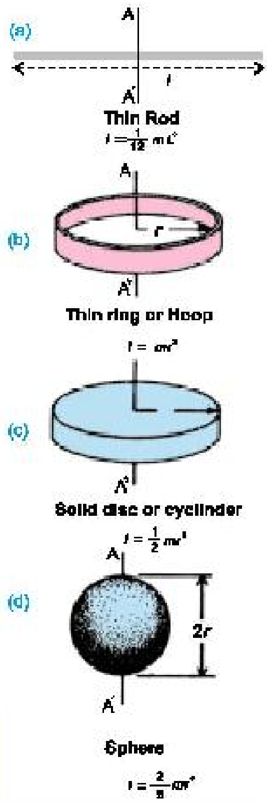
Do you know?



Two cylinders of equal mass. The one with the larger diameter has the greater rotational inertia.

For your information

Moments of inertia of various bodies about axis AA'.



3.5 ANGULAR MOMENTUM

We have already seen that linear momentum plays an important role in translational motion of bodies. Similarly, another quantity known as angular momentum has important role in the study of rotational motion.

A particle is said to possess an angular momentum about a reference axis if it so moves that its angular position changes relative to that reference axis.

The angular momentum **L** of a particle of mass m moving with velocity **v** and momentum **p** (Fig. 3.17) relative to the origin O is defined as:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \qquad .... \qquad (3.20)$$

Fig. 3.17

where **r** is the position vector of the particle at that instant relative to the origin O. Angular momentum is a vector quantity. Its magnitude is:

$$L = rp \sin\theta = m\, rv \sin\theta$$

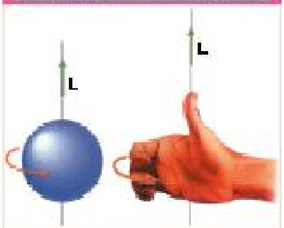
where θ is the angle between **r** and **p**. The direction of **L** is perpendicular to the plane formed by **r** and **p** and its sense is given by the right hand rule of vector product. SI unit of angular momentum is kg m^2 s^{-1} or J s.

If the particle is moving in a circle of radius r with uniform angular velocity ω, then angle between r and tangential velocity is 90°. Hence,

$$L = mrv \sin 90° = mrv$$

But $\qquad v = r\omega \qquad$ Hence $\qquad L = mr^2\omega$

For your information



(a) (b)

The sphere in (a) is rotating in the sense given by the gold arrow. Its angular velocity and angular momentum are taken to be upward along the rotational axis, as shown by the right-hand rule in (b).

Now consider a symmetric rigid body rotating about a fixed axis through the centre of mass as shown in Fig 3.18. Each particle of the rigid body rotates about the same axis in a circle with an angular velocity ω. The magnitude of the angular momentum of the particle of mass m_i is $m_i v_i r_i$ about the origin O. The direction of **L** is the same as that of ω. Since $v_i = r_i\,\omega$, the angular momentum of the ith particle is $m_i\, r_i^2\omega$. Summing this over all particles gives the total angular momentum of the rigid body.

$$\mathbf{L} = \left(\sum_{i=1}^{n} m_i\, r_i^2\right)\, \omega = I\omega$$

where I is the moment of inertia of the rigid body about the axis of rotation.

Fig. 3.18

Example 3.5 The mass of Earth is 6.00 x 10^{24} kg. The distance r from Earth to the Sun is 1.50 x 10^{11} m. As seen from the direction of the North Star, the Earth revolves counter-clockwise around the Sun. Determine the orbital angular momentum of the Earth about the Sun, assuming that it traverses a circular orbit about the Sun once a year (3.16x10^7s).

Solution To find the Earth's orbital angular momentum, we must first know its orbital speed from the given data. When the Earth moves around a circle of radius r, it travels a distance of $2\pi r$ in one year. Its orbital speed v_o is thus, $v_o = \frac{2\pi r}{t}$

Orbital angular momentum of the Earth = $L_o = m v_o r$

$$L = \frac{2\pi r^2 m}{t}$$

$$= \frac{2\pi(1.50 \times 10^{11} \text{m})^2 \times (6.00 \times 10^{24} \text{kg})}{3.16 \times 10^7 \text{s}}$$

$$L = 2.67 \times 10^{40} \text{ kg m}^2 \text{ s}^{-1}$$

The sign is positive because the revolution is counter clockwise.

3.6 LAW OF CONSERVATION OF ANGULAR MOMENTUM

The law of conservation of angular momentum states that if no external torque acts on a system, the total angular momentum of the system remains constant.

$$\mathbf{L}_{total} = \mathbf{L}_1 + \mathbf{L}_2 + \ldots = \text{constant}$$

The law of conservation of angular momentum is one of the fundamental principles of Physics. It has been verified from the cosmological to the sub microscopic level. The effect of the law of conservation of angular momentum is readily apparent if a single isolated spinning body alters its moment of inertia.

If a body of moment of inertia I_1 spinning with angular speed ω_1 alters its moment of inertia to I_2 , then its angular speed ω_2 also changes so that its angular momentum remains constant.

Hence $I_1 \omega_1 = I_2 \omega_2$

The angular momentum is a vector quantity with direction along the axis of rotation. Hence, the direction of angular momentum along the axis of rotation also remains fixed. This is illustrated by the fact given below:

The axis of rotation of an object will not change its orientation unless an external torque causes it to do so.

Do you know?

If you try to sit on a bike at rest, it falls. But if the bike is moving, the angular momentum of the spinning wheel resists any tendency to change and helps to keep the bike upright and stable.

Do you know?

The ball's speed increases as the string wraps around the finger.

This fact is of great importance for the Earth as it moves around the Sun. No other sizeable torque is experienced by the Earth, because the major force acting on it is the pull of the Sun. The Earth's axis of rotation, therefore, remains fixed in one direction with reference to the universe around us.

Examples of conservation of angular momentum

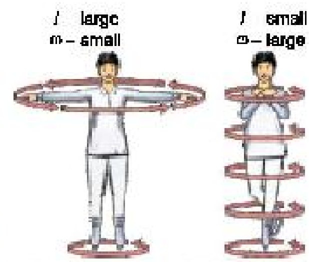
A man diving from a diving board

A diver jumping from a springboard has to take a few somersaults in air before touching the water surface, as shown in Fig.3.19. After leaving the springboard, he curls his body by rolling arms and legs in. Due to this, his moment of inertia decreases, and he spins in midair with a large angular velocity. When he is about to touch the water surface, he stretches out his arms and legs. He enters the water at a gentle speed and gets a smooth dive. This is an example of the law of conservation of angular momentum.

Fig. 3.19
A man diving from a diving board.

The spinning ice skater

An ice skater as shown in Fig.3.20 can increase his angular velocity by folding arms and bringing the stretched leg close to the other leg. By doing so, he decreases his moment of inertia. As a result, angular speed increases. When he stretches his hands and a leg outward, the moment of inertia increases and hence angular velocity decreases.



$$\text{Angular momentum} = \left(\text{Moment of Inertia}\right)\left(\text{Angular velocity}\right)$$
$$L = I \times \omega$$

Fig. 3.20
An ice skater using angular momentum

A person holding some weight in his hands standing on a turntable.

A person is standing on a turntable with heavy mass (dumb-bell) in his hands stretched out on both sides as shown in Fig. 3.21. As he draws his hands inward, his

(a) (b)

Fig. 3.21
Man with masses in his hands on a turntable. Conservation of angular momentum requires that as the man pulls his arms in, the angular velocity increases.

For your information

It has been noticed that when ice on the polar caps of Earth melts and water flows away in the form of river; the moment of inertia of water and hence that of Earth about its axis of rotation increases due to conservation of angular momentum. Hence, the angular velocity of Earth will decrease, therefore, the duration of day increases slightly.

angular speed at once increases. This is because the moment of inertia decreases on drawing the hands inwards, which increases the angular speed.

Flywheel

Flywheel is a mechanical device which consists of a heavy wheel with an axle (Fig. 3.22). It is used to store rotational energy, smooth out output fluctuations and provides stability in a wide range of applications such as bicycles and other vehicles, industrial machinery, gyroscopes, ships and spacecrafts.

Do you know?

The flywheel called the balance wheel regulates the time keeping mechanism in mechanical clocks and watches by maintaining controlled oscillations rate.

Fig. 3.22: A flywheel

Point to ponder!

Why does the coasting rotating system slow down as water drips into the beaker?

When a fly wheel spins, its angular momentum resists changes to its orientations, maintaining stability. This is useful in systems that need precise control over their orientation without external interference.

The Gyroscope

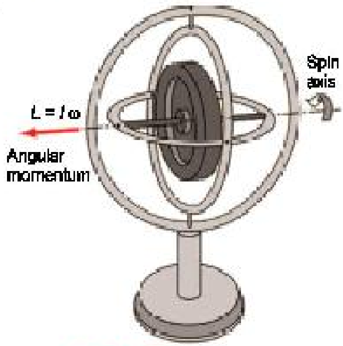


Fig. 3.23: The gryscope

A gyroscope is a device which is used to maintain its orientation relative to the Earth's axis or resists changes in its orientation. It consists of a mounted flywheel pivoted in supporting rings as shown in Fig. 3.23. It works on the basis of law of conservation of angular momentum due to its large moment of inertia. When the gyroscope spins at a large angular speed, it gains large angular momentum. It is then difficult to change the orientation of the gyroscope's rotational axis due to its large moment of inertia. A change in orientation requires a change in its angular momentum. To change the direction of a large angular momentum, a corresponding large torque is required. Even if gyroscope is tilted (Fig. 3.24), it still keeps levitated without falling. Hence, it is a reason why a gyroscope can be used to maintain orientation. The

Fig. 3.24

main applications of gyroscope are in the guiding system of aeroplanes, submarines and space vehicles in order to maintain a specific direction in space to keep steady course.

Point to ponder!

Planets move around the Sun in elliptical orbits with Sun situated at one of its foci, thus, distance of a planet from the Sun is not constant when it is nearer the Sun. Its orbital velocity increases automatically. Why?

QUESTIONS

Multiple Choice Questions

Tick (✓) the correct answer.

3.1 The ratio of angular speed of minute's hand and hour's hand of a watch is:

(a) 1 : 6 (b) 6 : 1
(c) 1 : 12 (d) 12 : 1

3.2 A body travelling in a circle at constant speed:

(a) has constant velocity (b) has an inward radial acceleration
(c) is not accelerated (d) has an outward radial acceleration

3.3 A stone at the end of long string is whirled in vertical circle at a constant speed. The tension in the string will be minimum when the stone is:

(a) at the top of the circle (b) half way down
(c) at the bottom of circle (d) any where in the circle

3.4 Every point of rotating rigid body has:

(a) same angular velocity (b) same linear velocity
(c) same linear acceleration (d) same linear distance

3.5 The minimum velocity necessary to put a satellite into the orbit is called:

(a) terminal velocity (b) critical velocity
(c) artificial velocity (d) angular velocity

3.6 An astronaut is orbiting around the Earth in a large capsule. Then,

(a) he will be in a state of weightlessness with respect to capsule
(b) he is freely falling towards the Earth
(c) a ball projected at an angle has a straight line path as observed by him
(d) all the above

3.7 An object in uniform circular motion makes 10 revolutions in 2 seconds. Which of the following statement is true?

(a) Its period is 2.0 s (b) Its period is 20 s
(c) Its frequency is 5 Hz (d) Its frequency is 0.2 Hz

3.8 A man inside the artificial satellite feels weightlessness because the force of attraction due to the Earth is:

(a) zero at pole
(b) balanced by the force of attraction due to the moon
(c) equal to the centripetal force
(d) non-effective due to some particular design of the satellite

3.9 A bottle of soda water is grasped from the neck and swung briskly in a vertical circle. Near which portion of the bottle do the bubbles collect?

(a) Near the bottom (b) In the middle of bottle
(c) Bubbles remain distributed throughout the volume of the bottle.
(d) Near the neck of the bottle

3.10 The moment of inertia of body depends upon:

(a) mass of the body and its distribution about axis of rotation
(b) volume of the body
(c) kinetic energy of the body
(d) angular momentum of the body

Short Answer Questions

3.1 State second law of motion in case of rotation.
3.2 What is the effect of changing the position of a diver while diving in the pool?
3.3 How do we get butter from the milk by using centrfuge?
3.4 Mass is a measure of inertia in linear motion. What is its analogue in rotational motion? Describe briefly.
3.5 Why is it harder for a car to take turn at higher speed than at lower speed?
3.6 What are the benefits of using rear wheels of heavy vehicles consisted of double tyres?
3.7 When a moving car turns around a corner to the left, in what direction do the occupants tend to fall? Explain briefly.
3.8 Why is the acceleration of a body moving uniformly in a circle, directed towards the centre?
3.9 How does an astronaut feel weightlessness while orbiting from the Earth in a spaceship?

Constructed Response Questions

3.1 If angular velocity of different particles of a rigid body is constant, will the linear velocity of these particles be also constant?
3.2 A loaf of bread is lying on rotating plate. A crow takes away the loaf of bread and the rotation of the plate increases. Why?

3.3 Why do we tumble when we take the sharp turn with large speed?

3.4 What will be time period of a simple pendulum in an artificial satellite at a certain height?

3.5 Is the motion of a satellite in its orbit, uniform or accelerated?

3.6 What are the advantages that radian has been preferred as SI unit to degree?

3.7 In uniform circular motion, what are the average velocity and average acceleration for one revolution? Explain.

3.8 In a rainstorm with a strong wind, what determines the best position to hold an umbrella?

3.9 A ball is just supported by a string without breaking. If it is whirled in a vertical circle, it breaks. Explain why.

3.10 How is the centripetal force supplied in the following cases:

(a) a satellite orbiting around the Earth?

(b) a car taking a turn on a level road?

Comprehensive Questions

3.1 What is meant by angular momentum? Explain the law of conservation of angular momentum with daily life examples.

3.2 Show that orbital angular momentum is; $L = I\omega$.

3.3 Define moment of inertia. Prove that torque acting on rotating rigid body is equal to the product of its moment of inertia and angular acceleration.

3.4 What are artificial satellites? Calculate the minimum time period necessary to put a satellite into the orbit near the surface of the Earth.

3.5 Define orbital velocity and derive an expression for the same.

3.6 Write a note on artificial gravity. Derive an expression for frequency with which the spaceship rotates to provide artificial gravity.

3.7 Prove that; (i) $v = r\omega$ and (ii) $a = r\alpha$

Numerical Problems

3.1 A laser beam is directed from the Earth to the moon. The beam spreads over a diameter of 2.50 m at the moon surface. What is divergence angle of the beam? The distance of moon from the Earth is 3.8×10^8 m. (Ans: 6.6×10^{-9} rad)

3.2 A car is moving with a speed of 108 km h^{-1}. If its wheel has a diameter of 60 cm, find its angular speed in rad s^{-1} and rev s^{-1}. (Ans: 100 rad s^{-1}, 16 rev s^{-1})

3.3 An electric motor is running at 1800 rev min^{-1}. On switching off, it comes to rest in 20 s. If angular retardation is uniform, find the number of revolutions it makes before stopping. (Ans: 300 rev)

3.4 A string 0.5 m long holding a stone can withstand maximum tension of 35.6 N. Find the maximum speed at which a stone of 0.5 kg mass can be whirled with it in a vertical circle. (Ans: 5.5 m s^{-1})

3.5 The flywheel of an engine is rotating at 2100 rev min^{-1} when the power source is shut off. What torque is required to stop it in 3 minutes? The moment of inertia of the flywheel is 36 kg m^2. (Ans: 44 N m)

3.6 What is the moment of inertia of a 200 kg sphere whose diameter is 60 cm?

(Ans: 7.2 kg m^2)

3.7 A satellite is orbiting the Earth at an altitude of 200 km. Assuming the Earth's radius is 6400 km, calculate the orbital speed of the satellite. (Ans: 7.77 km s^{-1})

3.8 A space station has a radius of 20 m and rotates at an angular velocity of 0.5 rad s^{-1}. What is the artificial gravity experienced by the astronauts on the space station? (Ans: 5 m s^{-2})

3.9 A bicycle wheel has an angular momentum of 10 kg m^2 s^{-1} and angular velocity of 2 rad s^{-1}. Find the value of its moment of inertia. (Ans: 5 kg m^2)

3.10 A diver comes off a board with arms straight up and legs straight down, giving him a moment of inertia of 18 kg m^2 about his rotation axis. Then tucks into a small ball, decreasing his moment of inertia to 3.6 kg m^2. While tucked, he makes two complete rotations in 1.0 second. If he had not tucked at all, how many revolutions would he have made in 1.5 s from board to water?

(Ans: 0.6 rev)