Unit 11 Oscillations

Teaching Periods 11 **Weightage %** 7

Enjoying on a swing, having a smooth drive on a bike or car on a bumpy road and in today's modern world the most fearsome form of entertainment is bungee jumping. Physics involve in these thrilling phenomena, the motion of a bungee jumper is called Simple Harmonic Motion (SHM).

In this unit student should be able to:

- Describe necessary conditions for execution of simple harmonic motion.
- Investigate the motion of an oscillator using experimental and graphical methods.
- Describe that when an object moves in a circle, the motion of its projection on the diameter of circle is SHM.
- Define the terms amplitude, time period, frequency, angular frequency and phase.
- Identify and use the equation $\mathbf{a} = -\omega^2 x$ as the defining equation of SHM.
- Prove that the motion of mass attached to spring is SHM.
- Analyze the motion of a simple pendulum is SHM and calculate its time period.
- Interpret time period of the simple pendulum varies with its length.
- Describe the interchanging between kinetic energy and potential energy during SHM.
- Describe practical examples of free and forced oscillations (resonance).
- Describe graphically how the amplitude of a forced oscillation changes with frequency near to the natural frequency of the system.
- Describe practical examples of damped oscillations with particular reference to the efforts of the degree of damping and the importance of critical damping in cases such as a car suspension system.
- Describe qualitatively the factors which determine the frequency response and sharpness of the resonances.

Along with transnational and circular motion, **vibratory** or **oscillatory** motion is one of the most important kinds of motion.

Oscillatory motion is a periodic motion which repeats itself in equal interval of time.

A very common and widely experienced example of vibratory motion is sound.

Other examples of oscillatory motion are;

- Consider a bird in flight flaps its wings up and down. (Fig.11.1)
- A flat strip of metal clamped at one end on the base of table can oscillate up and down when pressed and released from unclamped end.
- A mass suspended by an elastic spring when pulled down and released
- The motion of the bob of simple pendulum when displaced from its mean position and released.
- The atoms or molecules in solid substances oscillate about their mean position. (**equilibrium**)
- Tall buildings and bridges seem to be rigid but they oscillate about their mean position

The vibrations or oscillations occur near the point of stable equilibrium of a particle or system of particles.

"An equilibrium point is stable if the net force acting on the particle for its displacement from equilibrium points back toward the equilibrium point. Such a force is called restoring force".

Since it tends to restore equilibrium of the particle (Fig.11.2).

11.1 Simple Harmonic Motion:

Simple Harmonic Motion (**SHM**) is a type of oscillation or vibratory motion produced under the action of restoring force.

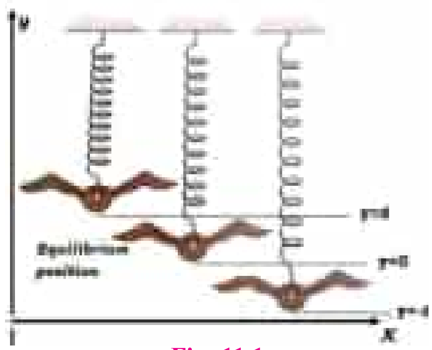


Fig: 11.1

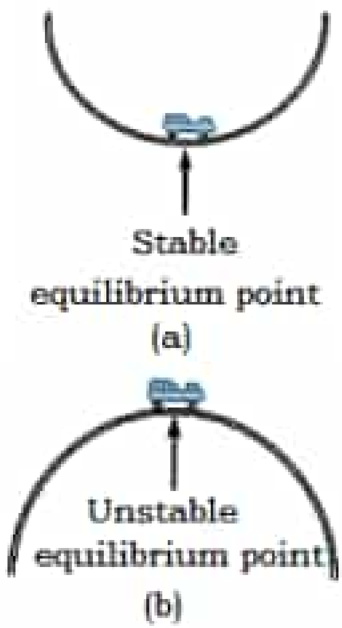


Fig: 11.2

(a) A point of stable equilibrium for a roller-coaster car. If the car is displaced slightly from its position at the bottom of the track, gravity pulls the car back toward the equilibrium point. **(b)** A point of unstable equilibrium for a roller-coaster car. If the car is displaced slightly from the top of the track the gravity pulls the car away from the equilibrium point.

11.1.1 The necessary conditions for the execution of simple harmonic motion are;

1. The restoring force shall be directly proportional to the displacement from equilibrium position.
2. The restoring force shall be proportional to the inertia (elastic limit) of the system executing SHM.
3. The displacement from the equilibrium point on either side should be small.
4. The force and displacement should follow the Hooke's law (F = -*kx*), where *k* is 'spring constant or force constant' depends upon the nature of material of spring.
5. Acceleration of the oscillating object should be proportional to the displacement $(\boldsymbol{a} \propto -\boldsymbol{x})$, the negative sign indicates that acceleration is directed towards the equilibrium position O.

11.1.2 Motion of an Oscillator:

(Experimental methods)

Consider a system of "**the mass and spring system**" oscillating in horizontal and vertical directions executing simple harmonic motion.

Horizontal Mass-Spring System:

Consider a spring with spring constant *k* and negligible mass. An object of mass *m* is attached to one end of the spring whose other end is fixed by a rigid support as shown in Fig.11.3. The spring-mass system can slide on a frictionless horizontal surface. Since the normal reaction force *R* on the object is balanced by the weight (*mg*), the net force acting on the object is due to spring. When the spring is at mean position the net force is zero; and the system object is in equilibrium.

If a force *F* is applied on the object to pull it along the horizontal surface towards right side. The object is displaced from its equilibrium point O for a distance *x*.

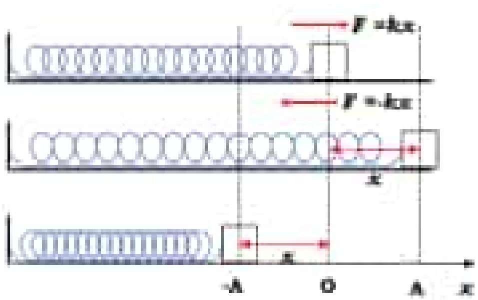


Fig: 11.3
(A horizontal spring mass system to understand the simple harmonic motion under elastic restoring force)

$$F \propto x$$

or $$F = kx \qquad \text{......(11.1)}$$

"Due to elasticity, the force (equal and opposite) obeying the Hooke's law and is stored in the spring - mass system called elastic restoring force".

The motion of an object under an elastic restoring force about a fixed point (equilibrium point O) between two extreme point (A, -A) at equal distance from O is a type of oscillatory (vibratory) motion called Simple Harmonic Motion (or SHM).

So restoring force is,

$$F_{res} = -kx \qquad \text{.....(11.2)}$$

where the negative sign indicates that the elastic restoring force is opposite in direction to the displacement from equilibrium point.
If, now the object is released from A, it begins to oscillate about the equilibrium position between two points A and –A

Using $F = ma$ from Newton's second Law of motion in equation 11.2
We have;

$$ma = -kx$$

$$\boldsymbol{a = -\frac{k}{m}x} \qquad \text{......}(11.3)$$

Thus, the acceleration and displacement are always in opposite directions.
As; $\boldsymbol{\frac{k}{m}}$ is a constant; so whenever the acceleration is a negative constant times the displacement the motion is **SHM**.

$$\boldsymbol{a \propto -x}$$

The acceleration of a body executing simple harmonic motion at any instant shall always be proportional to displacement and is always directed towards its equilibrium position.

Self-Assessment Questions:
1. Describe the motion of a mass-spring system in simple harmonic motion (SHM).
2. What are some real-world applications of mass-spring systems?

Graphical representation of SHM:
To understand the simple harmonic motion graphically, we set up an experiment (Fig. 11.4a) with an object attach to a spring. The object oscillates with a maximum displacement x_o from its equilibrium position. At the same moment a pin projected up and fitted on a horizontal circular disc of radius R = x_o is set into motion with constant angular speed ω. The two objects are set into motion at same instant.

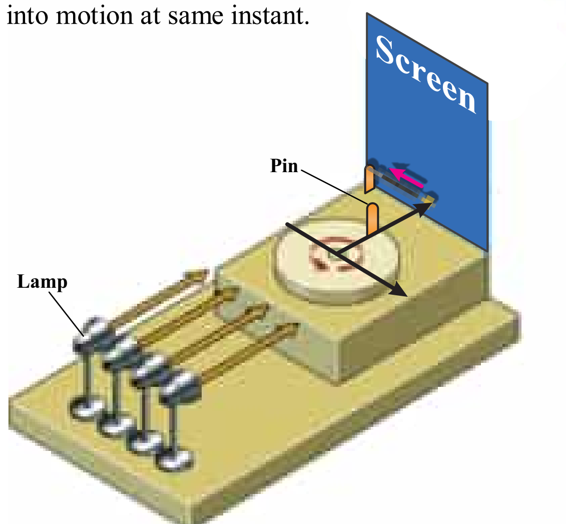


Fig: 11.4 (a) an experiment to show the relation between uniform circular motion and simple harmonic motion

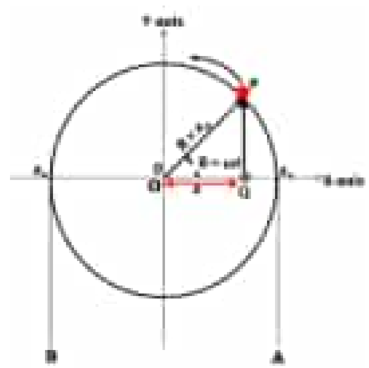


Fig: 11.4 (b)
The motion of the pin moving counter clockwise around a circle of R = x_o as the disc rotates with constant angular speed ω.

Both the pin and the object attached to the spring are illuminated so that the shadows of the pin on the rotating disc and the oscillating object are seen on the screen. The speed of the disk is adjusted until the shadows oscillate with the same period.

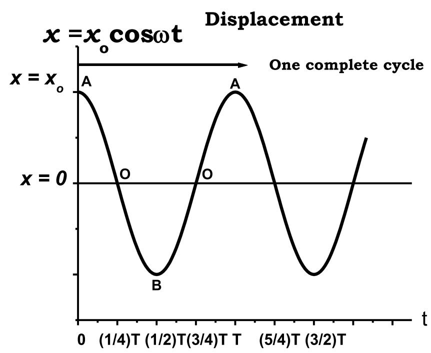


Fig: 11.5 Displacement time graph

Displacement- time (x-t) graph:

We analyze the uniform circular motion of the pin P fitted on a disc of radius x_o. Figure 11.4(a, b) shows that the pin moves counter clock wise along the circle of radius x_o with constant angular velocity ω. the displacement – time graph shown in figure 11.5. Let at an instant $t = 0$ the pin starts at A. The position of projection Q is also at A i.e. $x = x_o$. (Fig.11.4b) When pin covers the quarter of its rotation in $t = \frac{1}{4}T$ the projection Q is at $x = 0$ and reaches at B in next $\frac{1}{4}T$. Same motion is now repeated from B to A in next $\frac{1}{2}T$. Similar pattern can be seen in the motion of object attach to the spring.

At any time $t = t$ the angular displacement of the pin is given by;

$$\theta(t) = \omega t$$

The motion of pin's projection (shadow) Q on the screen has the same x-component as the pin itself. Consider a right angle triangle OQP, (Fig.11.4b) we find that

$x(t) = x_o \cos\theta = x_o \cos\omega t$...... (11.4)

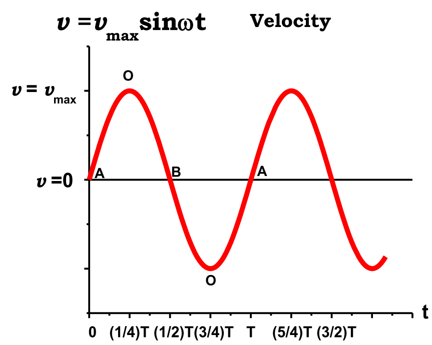


Fig: 11.6 Velocity-time graph

Velocity – time (v-t) graph:

The comparison of velocity – time graph shown in figure 11.6 and the displacement –time relation in figure 11.5 explains the interrelationship between the displacement and velocity of the object executing SHM. The shape of the curve for velocity - time relation is also same as that of displacement - time graph (Fig.11.7), but the displacement time graph is one quarter ahead of velocity time graph (Fig.11.6). At time $t = \frac{1}{4}$T the mass is at equilibrium position O and this is the point where the velocity is maximum.

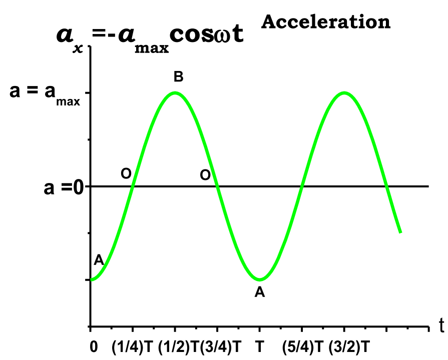


Fig: 11.7 Acceleration-time graph

Acceleration – time (a-t) graph:

In figure11.7 the relation between acceleration and time is described. At $t = \frac{1}{4}T$ the object is at O and there is no net force acting on it so its acceleration is zero, although the velocity is maximum at equilibrium position. As the object is displaced towards right hand side under the action of a net force $F = kx$, the restoring force $F = -kx$ will act towards left, produces a negative acceleration in the object.

The figure 11.8 shows that acceleration graph is 180° out of phase with displacement graph. This shows that $(a \propto -x)$

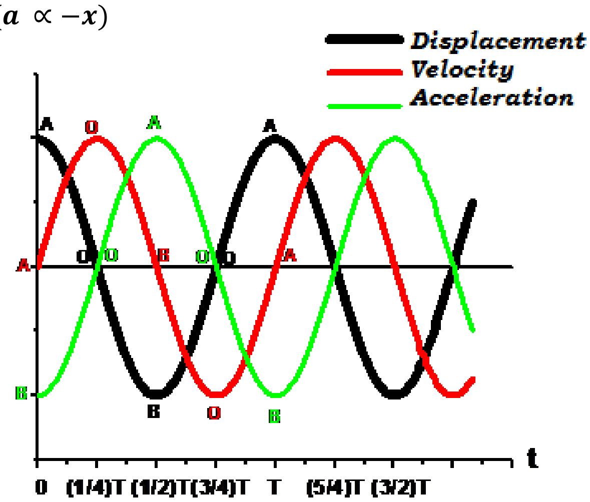


Fig: 11.8

The graph is showing the inter relationship between displacement, velocity and acceleration. At t=0 the displacement is $\frac{\pi}{2}$ radians ahead of velocity and reaches at its positive maximum one quarter cycle ahead of velocity. Likewise acceleration is following one quarter cycle to velocity and one half cycles to displacement to reach its positive maximum.

11.2 Uniform Circular Motion and SHM:

The motion of the pin in figure 11.9 a is considered as uniform circular motion. It possesses centripetal acceleration directed towards the center of circle. We know that, the magnitude of centripetal acceleration is given as

$a_P = \omega^2 r = \omega^2 x_o$ …..(11.5)

From figure (11.9) and , the motion of P is along the circumference but its projection Q is oscillating along the diameter AOB. The acceleration of projection Q will be a component of centripetal acceleration of P.

11.2.1 Motion of the Projection of a Particle Moving Along a Circular Path:

In order to establish that projection Q of the particle P, moving along the diameter AOB executes simple harmonic motion as shown in figure 11.9. We shall derive the expression for the acceleration of Q, by resolving the centripetal acceleration of particle P a_P into its rectangular components.

From eq. 11.5.

$a_x = \omega^2 x_o \cos\theta$ ...... (11.6)

and

$a_y = \omega^2 x_o \sin\theta$ ...... (11.7)

Since a_x is the component along the diameter AOB and always directed towards the equilibrium position O. At any instant t the direction of the acceleration vector is opposite to the direction of the displacement vector. Therefore;

$\boldsymbol{a_x = -\omega^2 x_o \cos\theta}$ ...... (11.8)

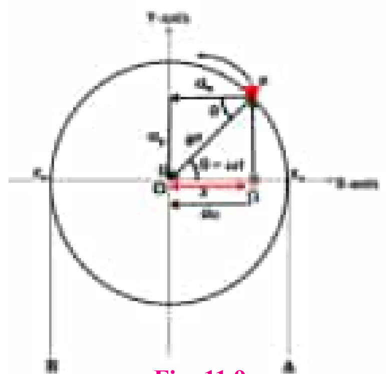


Fig: 11.9
Motion of particle moving along circular path

Consider the right angle triangle OQP (Fig.11.9)

$$\cos\theta = \frac{x}{x_o}$$

Substituting the value of $\cos\theta$ in eq. 11.8

We have; $a_x = -\omega^2 x$

or

$a_x = \omega^2(-x)$ ......(11.9)

Since; ω is a constant so ω^2 is also a constant, therefore the acceleration a_x of projection Q moving along the diameter AOB is proportional to the displacement x, complying with the characteristic property of SHM which is;

$$\boldsymbol{a_x \propto -x}$$

Velocity of Projection Q:

The linear velocity $\boldsymbol{v}$ of the particle P moving along a circular path at any instant t and at any point on the circumference shall always be tangent to the circle and perpendicular to the radius of circular path. The magnitude of linear velocity is given as;

$\boldsymbol{v_P} = \omega x_o$ ......(11.10)

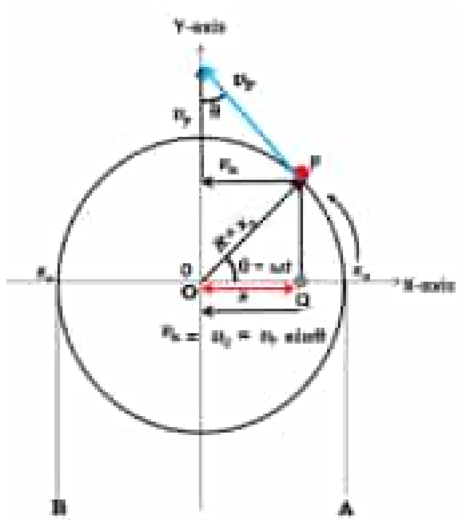


Fig: 11.10 velocity of projection

As the projection Q is oscillating along the diameter AOB and its motion is due to the motion of P. Therefore the velocity of Q can be determined by resolving $\boldsymbol{v_P}$ into its rectangular components as shown in figure.11.10.

$v_x = v_P \sin\theta$ and $v_y = v_P \cos\theta$

Therefore; $\boldsymbol{v_Q = v_P \sin\theta}$ ...... (11.11)

From eq. 11.10, $\boldsymbol{v_P} = \omega x_o$ thus $\boldsymbol{v_Q} = \omega x_o \boldsymbol{\sin\theta}$ ...... (11.12)

Consider the right angle triangle OQP.

$$\cos\theta = \frac{x}{x_o}$$

Using; $\sin^2\theta + \cos^2\theta = 1$ and $\sin^2\theta = 1 - \cos^2\theta = 1 - \frac{x^2}{x_o^2}$

Therefore; $$\sin\theta = \sqrt{1 - \frac{x^2}{x_o^2}}$$

Putting the value of $\sin\theta$ in eq.11.12

$$v_Q = \omega x_o\sqrt{1-\frac{x^2}{x_o^2}} \qquad \ldots\ldots(11.13)$$

Eq. 11.13 represents the instantaneous velocity of projection Q depending upon the position (x) of Q from equilibrium position.

For maximum velocity; $x = 0$ i.e. the projection shall be at equilibrium point O and eq. 11.13 becomes

$$v_{max} = \omega x_o \qquad \ldots\ldots\ (11.14)$$

The velocity shall be minimum if the projection is at the end points of diameter i.e. at A or B, such that $x = x_o$.

Relation between Linear and Uniform Circular Motion:

Since; an oscillating mass attached by an elastic spring (horizontal or vertical) and an object moving along a circular path with constant angular velocity ω has already been proved are executing SHM.

From eqs. (11.3 and 11.9)

$$-\omega^2 x = -\frac{k}{m}x$$

Therefore; $$\omega = \sqrt{\frac{k}{m}} \qquad \ldots\ldots(11.15)$$

11.2.2 Important terms used in SHM:

Instantaneous Displacement and Amplitude:

When an object is executing SHM, either in spring-mass system or in uniform circular motion its position from its equilibrium point is changed continuously. At any instant t the distance of the object from its equilibrium position is called **instantaneous displacement (x)**. The maximum change in its position is observed when the object is at extreme points i.e. at A or –A (Fig.11.3) or at A or B (Fig.11.5) and the displacement is now termed as **Amplitude (x_o)**.

Period, Frequency and Angular Frequency:

Simple harmonic motion is a periodic motion under the action of elastic restoring force, a body executing SHM repeats the same motion again and again. To complete one cycle or rotation of motion the object must be at the same point and in same direction as it was at the start of the cycle. The same analogy is applied for uniform circular motion.

Time period (*T*): It is the time taken by the object to complete one cycle of oscillation.

Frequency (*f*): It is the number of cycles (oscillations, vibrations, and rotations) completed per unit time.

Angular Frequency (ω): It is defined as the ratio of the angular displacement or change in angle (θ) to the time taken (t) to undergo that change. Mathematically, it is expressed as:

$\omega = \Delta\theta / \Delta t$

Since the object in Spring-mass system and the pin in uniform circular motion both executing SHM, so we can develop a relation between ω, *f* and *T*.

Using; $\theta = \omega T$

where for one complete cycle θ = 2π radians.

It gives; $T = \frac{2\pi}{\omega}$ since $f = \frac{1}{T}$

From eq.(11.18) $\omega = \sqrt{\frac{k}{m}}$

Therefore; eq. (11.19) becomes

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

(11.21)

and

$$T = 2\pi\sqrt{\frac{m}{k}} \qquad \ldots\ldots (11.16)$$

It is important to note that in spring-mass system the quantity **ω** is called the **angular frequency**. It is also be marked that in spring-mass system the **ω** depends upon mass and spring constant and independent of amplitude.

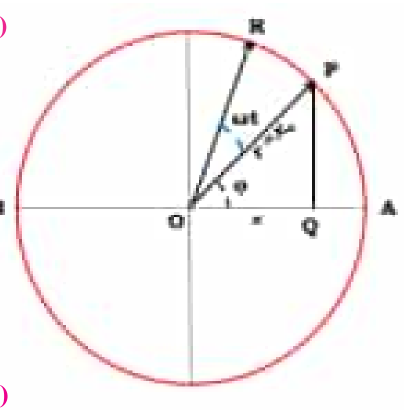
Phase:

From the discussion we made regarding the displacement and velocity of the particle executing SHM. It is clear that both displacement and velocity are the function of angle ($\theta = \omega t$).

The Phase is the state of motion of a vibrating object in terms of position and direction.

As the particle is rotating along the circumference, its projection Q moves back and forth along the diameter AOB. At t = 0 the angle between OP and the reference radial line OA is φ, which is called the initial phase. At some later instant t the angle between OR and OA would be ωt + φ. The phase is Zero at starting point i.e. at equilibrium position A (Fig. 11.11 b) and it would be $\frac{\pi}{2}$ at extreme point O.

(a)



(b)

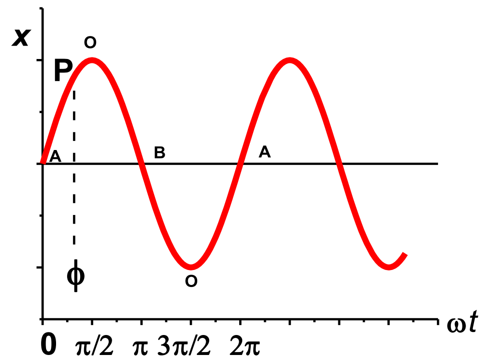


Fig: 11.11

(a) Motion of a particle along a circular path

and

(b) The waveform of the particle motion with an illustration of PHASE.

Fig: 11.12

Astronaut **Millie Hughes-Fulford** a body-mass measurement device developed for determining mass in orbit.

Consider the right angle triangle OQP from figure 11.11 a, the displacement of point Q from the mean position with respect to position R is given by

$x = x_o \cos(\omega t + \varphi)$ ......(11.17)

Worked Example 11.1

The spring used in one such device shown in Fig.11.12 has a spring constant of 606 N/m, and the mass of the chair is 12.0 kg. The measured oscillation period is 2.41 s. Find the mass of the astronaut.

Step 1: Write the known quantities and point out the quantities to be found.

Spring Constant; k = 606 Nm^{-1}

Mass of chair; m_{chair} = 12.0 Kg

Time period; T = 2.41 s

Step 2: Write the formula and rearrange if necessary.

Since the astronaut and chair are oscillating in simple harmonic motion, the total mass ($m_{total} = m_{chair} + m_{astronaut}$) of the two is related to the angular frequency ω which is related (Eq.11.20) to period of oscillation as

$$T = 2\pi\sqrt{\frac{m_{total}}{k}}$$

Squaring on both sides and rearranging the above equation.

$$m_{total} = \frac{T^2 k}{4\pi^2}$$

Hence the expression for mass of astronaut;

$$m_{astronaut} = \frac{T^2 k}{4\pi^2} - m_{chair}$$

Step 3: Put the values in the formula and calculate.

$$m_{astronaut} = \frac{2.41^2 \times 606}{4 \times 3.142^2} - 12.0 = \mathbf{77.2\ Kg}$$

DO YOU KNOW?

Astronauts who spend a long time in orbit measure their body masses as part of their health-maintenance programs. On earth, it is simple to measure body weight by a scale. However, this procedure does not work in orbit, because both the scale and the astronaut are in free fall and cannot press against each other. Instead, astronauts use a body-mass measurement device, as shown in Figure. The device consists of a spring-mounted chair in which the astronaut sits. The chair is then started oscillating in simple harmonic motion. The period of the motion is measured electronically and is automatically converted into a value of the astronaut's mass, after the mass of the chair is taken into account.

Worked Example 11.2

A block is kept on a horizontal table. The table is undergoing simple harmonic motion of frequency 3 Hz in a horizontal plane. The coefficient of static friction between the block and the table surface is 0.72. Find the maximum amplitude of the table at which the block does not slip on the surface.

Step 1: Write the known quantities and point out the quantities to be found.

Frequency; $f = 3$ Hz

Coefficient of friction; $\mu = 0.72$

Amplitude; $x_o = ?$

Step 2: Write the formula and rearrange if necessary.

Since $a = \omega^2 x_o$

Maximum force of static friction is given as

$F = \mu mg$

In case that the body does not slip;

$ma = \mu mg$

or $m\,\omega^2 x_o = \mu mg$

and $x_o = \mu g/\omega^2 = \mu g/(2\pi f)^2$

Step 3: Put the values in the formula and calculate.

Amplitude $(x_o) = 0.72 \times 9.8/(2 \times 3.14 \times 3)^2 =$ **0.0198m** or 1.98cm

Self-Assessment Questions:

1. Figure A shows a 10-coil spring that has a spring constant k. When this spring is cut in half, so there are two 5-coil springs, is the spring constant of each of the shorter springs remains same or changed?
2. The drawing in figure B shows plots of the displacement x versus the time t for three objects undergoing simple harmonic motion. Which object I, II, or III has the greatest maximum velocity?

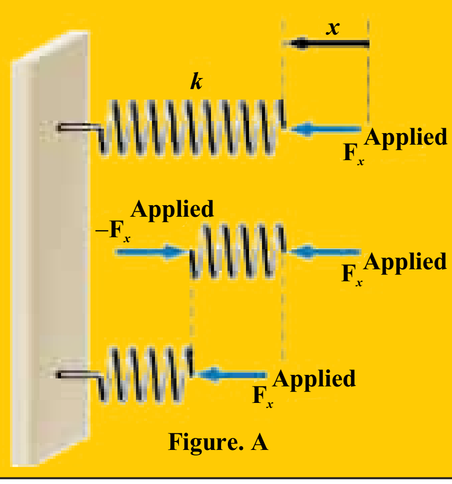


Figure. A

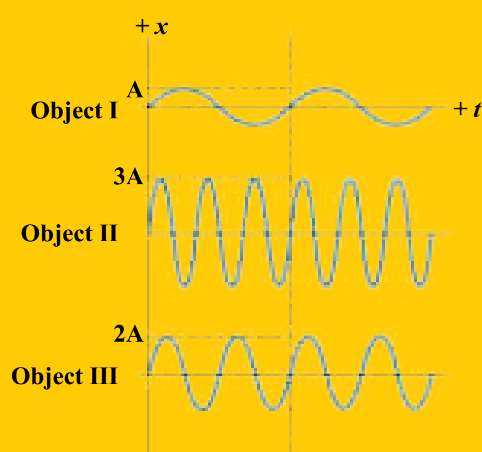


Figure. B

11.3 Practical SHM Systems:

11.3.1 Motion of a Mass Attached to a Spring:

An oscillating mass on a vertical spring also exhibits SHM. The main difference between horizontal and vertical examples is that, in vertical motion of spring-mass system the equilibrium point is moved down under the gravity. Fig. (11.13 a, b, c)

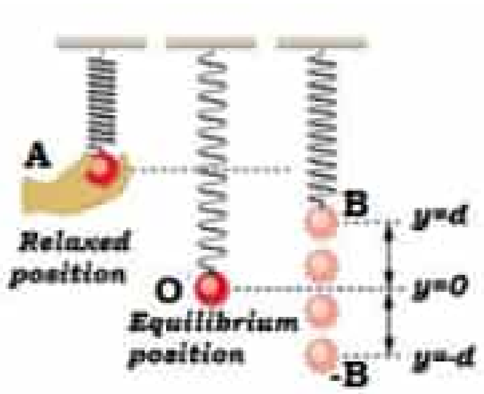


Fig: 11.13

a) A relaxed spring of spring constant k at A.

b) The spring is stretched downward under the gravity and reaches at equilibrium point O.

c) The spring is raised to a distance y

In this example we will consider an ideal spring of negligible mass that obeys Hooke's law. Suppose an object of mass m and weight mg is hung from the spring with spring constant k. The spring is stretched downward under the gravity to a distance d from its relaxed point A and settled at O the equilibrium position. Taking y - axis in upward direction the net force acting on mass at equilibrium is

$$F_{net} = kd - mg = 0 \qquad \ldots\ldots(11.18)$$

If the object is raised from the equilibrium point O to a position B up to a distance of y, the spring force is less than kd.

The spring force will be

$$F_{res} = k(d - y)$$

If upward direction of y is taken as positive then the net force acting on the mass at B is

$$F_{net} = k(d - y) - mg$$

$$F_{net} = kd - ky - mg$$

From (11.1) $\quad kd = mg$; therefore,

$$\mathbf{F_{net} = -ky} \qquad \ldots\ldots (11.19)$$

As k is a spring or force constant, therefore;

$$\mathbf{F_{net} \propto -y}$$

The restoring force is directly proportional to the displacement from equilibrium point and directed towards the equilibrium position.

Therefore, the vertical mass – spring system exhibits SHM.

Simple Pendulum:

A simple pendulum consists of a point mass suspended from a fixed point by a mass less inextensible string of length L. At equilibrium the weight (W = mg) balances the tension T along the string. If the mass is displaced to one side and then released, we assume that for *small amplitude*, the mass moves back and forth along the x-axis.

Suppose at any instant the pendulum makes an angle θ with verticle axis. At A the weight mg may be resolved into its rectangular components

The radial component of the weight along the string **$mg\cos\theta$** balances the tension T along the string (Fig.11.14). The tangential component **$mg\sin\theta$** provides the necessary restoring force.

Hence $F = -\,mg\sin\theta$ …… (11.20)

We expect the restoring force to be proportional to the displacement for small oscillations. Note that the restoring force is proportional to $\sin\theta$ rather to θ. Moreover if the displacement is large i.e. θ is large the motion is no longer be SHM.

However if θ is small and taken in radians then

$\sin\theta \approx \theta$

Hence eq. (11.20) can be re-write as

$F = -\,mg\,\theta$

Considering $F = ma$ from Newton's second law of motion and substituting in above equation we have

$$\boldsymbol{a = -\,g\,\theta} \qquad (11.21)$$

Since point O is very close to point A, therefore OA is considered to be a straight line. Using

s = r θ from figure 11.14

Where s = x the arc length OA and r = L

Then $\theta = \frac{x}{L}$

Substituting θ in eq.(11.21), we have

$$\boldsymbol{a = -\left(\frac{g}{L}\right)x} \qquad (11.22)$$

Since $\frac{g}{L}$ is a constant

Hence;

$$\boldsymbol{a} = \text{constant}\ (-\,\boldsymbol{x}\,)$$

Or $\boldsymbol{a \propto -x}$

Since; acceleration is directly proportional to displacement and directed towards the equilibrium position, Hence motion of simple pendulum is SHM for small amplitudes.

Time Period of Simple Pendulum:

Time period of the simple pendulum is the time required to complete one oscillation.

To identify the angular frequency we recall that equation 11.9

$a = -\omega^2 x$ (11.9)

Comparing equations 11.9 and 11.22

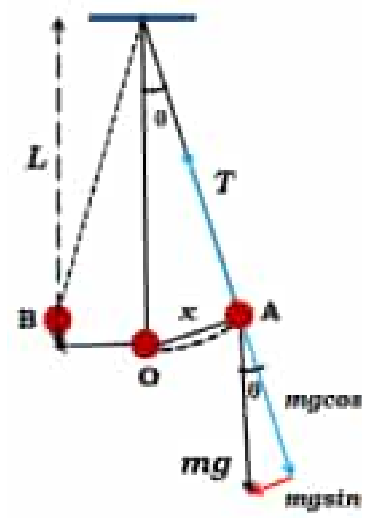
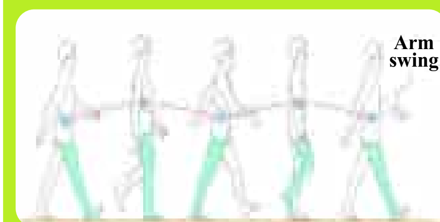


Fig: 11.14
Motion of simple pendulum.

DO YOU KNOW?

Physical Pendulum



Contrary to simple pendulum, where the mass is concentrate at a point. If the mass is uniformly distributed like arms swing and legs movement of a man walking shown in Fig.11.18 then it is called **Physical Pendulum**. When we walk, our legs alternately swing forward about the hip joint as a pivot. In this motion the leg is acting approximately as a physical pendulum.

$$\omega = \sqrt{\frac{g}{L}}$$

Therefore the time period is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \qquad \ldots\ldots (11.23)$$

Self-Assessment Questions:

1. Describe the relationship between the period and the gravitational acceleration (g) for a simple pendulum.
2. How does the mass of the pendulum bob affect the period?

11.4 Energy Conservation in SHM :

Consider a spring-mass system, when the mass is pulled towards right and released it moves towards the equilibrium position. The figures 11.3 – 11.15 suggest that the speed is greatest as the object passes through the equilibrium position. The object slows down as it reaches to the end points. This phenomenon indicates the inter conversion of kinetic and potential energies of the system at different points. We will see that these conversions will support the law of conservation of energy.

The total mechanical energy of the system at any instant shall remain constant.

E = kinetic energy + Potential energy

11.4.1 Inter Conversion of Kinetic and Potential Energies during SHM :

Kinetic Energy (K) of the Oscillator :

Since the kinetic energy at any instant of the system is given by

$$K = \frac{1}{2}mv^2 \qquad \ldots\ldots (11.24)$$

Where; v is the instantaneous velocity of the system. From eqs. (11.13) and (11.15)

$$K = \frac{1}{2}m\left[x_o\sqrt{\frac{k}{m}}\sqrt{1-\frac{x^2}{x_o^2}}\right]^2$$

$$K = \frac{1}{2}k(x_o^2 - x^2) \qquad (11.25)$$

DO YOU KNOW?

It is important to note down that, in case of simple pendulum the **ω is considered as the constant angular frequency of simple pendulum, rather than the angular velocity** (rate of change of angular displacement will change with time from zero to maximum). Even both have the same SI unit (radians/s).

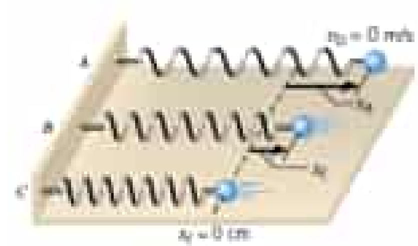


Fig: 11.15 (*A*) The total mechanical energy of this system is entirely elastic potential energy (*B*) partly elastic potential energy and partly kinetic energy (*C*) and entirely kinetic energy.

Eq.11.25 represents the instantaneous kinetic energy of the object executing simple harmonic motion.

Since the speed is maximum at equilibrium position i.e. at $x = 0$. Therefore eq. (11.25) for maximum kinetic energy of the object at equilibrium point.

$$\mathbf{K_{max} = \frac{1}{2}\,k x_o^2}$$

As the object is instantaneously at rest on extreme position, where $v = 0$ and $x = x_o$.

Therefore;

$$K = \frac{1}{2}k(x_o^2 - x_o^2) = 0$$

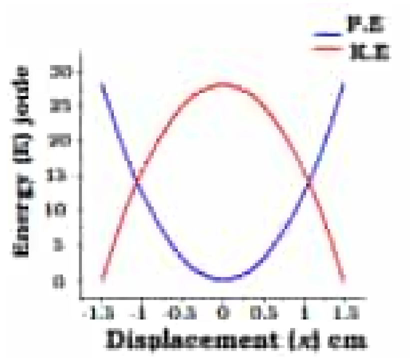


Fig: 11.16 The graph shows the elastic potential energy and kinetic energy as a function of position, for a mass oscillating on a spring.

Potential Energy (U) of the Oscillator:

Figure 11.15 suggests that the net force on the oscillator at O is F_o = zero and at extreme point i.e. at A, it is $F_A = kx$. The average applied force exerted on the system in displacing it from O to A is

$$F_{av} = \frac{F_o + F_A}{2} = \frac{0 + kx}{2} = \frac{1}{2}kx \quad \text{...... (11.26)}$$

The work done in moving the object from O to A, against the elastic restoring force

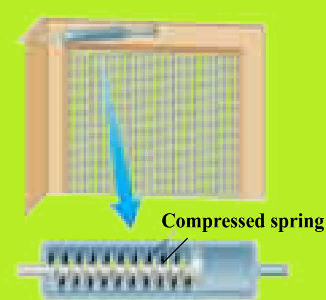
$$W = F_{av}.x = \frac{1}{2}kx^2 \quad \text{...... (11.27)}$$

This work is stored in the spring-mass system as its elastic potential energy U as shown in figure (11.16).

Eq. 11.41 is re-written as

$$U = \frac{1}{2}kx^2 \quad \text{...... (11.28)}$$

DO YOU KNOW?



A door-closing unit. The elastic potential energy stored in the compressed spring is used to close the door.

Eq.11.27 expresses the instantaneous elastic potential energy of the object executing simple harmonic motion.

Mathematically, in general and practically eq. 11.28 depend upon x, the instantaneous displacement. Hence the potential energy shall be maximum at ($x = \pm x_o$) are the extreme positions.

$$U_{max} = \frac{1}{2}kx_o^2 \quad \text{...... (11.29)}$$

and minimum at O where $x = 0$

Total Energy (E) of the Oscillator:

Using equations 11.25, 11.28 and expression of total energy of a system executing SHM can be found as shown in figure 11.17.

$$E = \frac{1}{2}k(x_o^2 - x^2) + \frac{1}{2}kx^2 \qquad \text{...... (11.30)}$$

It gives the total energy of a body executing SHM at a distance *x*, from the equilibrium position.

Simplifying eq. 11.30

$$E = \frac{1}{2}kx_o^2 \qquad \text{...... (11.31)}$$

The graphs show that the elastic potential energy is zero where the displacement is zero and maximum at extreme position. Contrary to P.E, kinetic energy is maximum at zero displacement i.e. at equilibrium position and minimum at extreme positions.

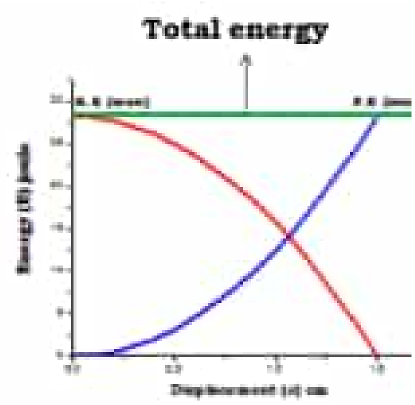


Fig: 11.17

The total energy of a body executing SHM at any point is constant.

11.5.1 Free and Forced Oscillations:

In simple harmonic motion, we assume that no dissipative forces such as friction or viscous drag of air exist. Since the mechanical energy is constant, the oscillations supposed to be continued forever with maximum amplitude.

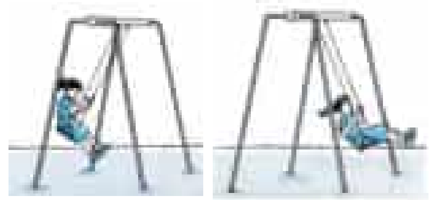


Fig: 11.18

Damped Oscillation. A girl is swinging on a swing. Damping occurs and the swing will oscillate with smaller and smaller amplitude and eventually stop

Free Oscillations:

The observation of the motion of an object executing SHM indicates a gradual dying out of amplitude of oscillations.

The amplitude of each cycle is a little smaller than that of the previous one (Fig.11.18), This motion is called damped oscillation.

The word damped is used in the meaning of extinguished or restrained (locked up). For a small amount of damping, oscillations occur at approximately the same frequency as if there were no restrained forces fig.11.19a. An increase in damping decreases the frequency (Fig.11.19b,c) even more damping prevents oscillations from occurring at all (Fig11.19 d).

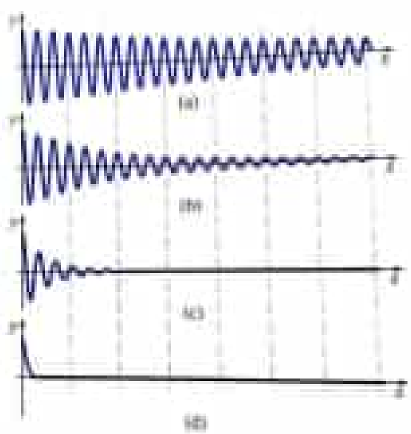


Fig: 11.19

Graphs of displacement x(t) for a system executing SHM, with increasing amount of damping a),b),c) and d).

Graph d) shows that the damping is quite large to stop the oscillation.

11.5.2 Forced Oscillations and Resonance :

When damping forces are present, the only way to keep the amplitude of oscillations from diminishing is to replace the dissipated energy from some other source. When a child is being pushed on a swing, (Fig.11.20) the parent replaces the energy dissipated with a small push. This push keeps the amplitude of motion constant. Every time the parent gives a little pushes once per cycle that compensates the dissipated amount of energy in one cycle. The frequency of the driving force (*the parent's push*) matches the natural frequency (The frequency at which it would oscillate on its own) of the system.

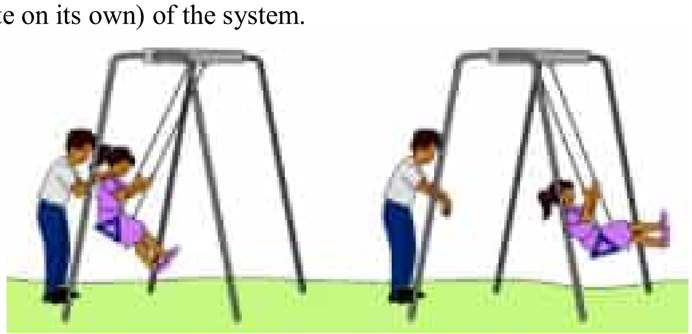


Fig: 11.20 Forced Oscillation

The applied a certain force to keep the amplitude of oscillation constant.

Resonance:

Forced oscillations occur when a periodic external driving force (push of parent) acts on a system that can oscillate. The frequency of driving force does not have to match the natural frequency of the system. With this driving force the system starts to oscillate with a frequency of driving force although it is far from its natural frequency. However the amplitude is not greatly affected as long as the frequency of driving force is closed to the natural frequency of the system.

If the frequency of external driving force f is continued to increase and if it becomes equal or integral multiple of natural frequency f_o of the system such that

$$f_{external} = f_1 \text{ or} = 2f_1 \text{ or} = 3f_1 \ldots\ldots \text{ or} = nf_1 \quad \ldots (11.32)$$

The amplitude of the motion is maximum, this condition is called **resonance**.

At resonance the driving force is always in the same direction as the object's velocity. Since the driving force is always doing positive work, the energy of the

DO YOU KNOW?

MRI magnetic Resonance Imaging system.

Resonance has a very wide range of use in the medical science. Only MRI provided sufficient information about the patient's disease then it is a useful tool.it allows us to see inside the human body with amazing detail, by using magnets and radio waves.it uses magnetic fields and radio waves to measures how much water is in different tissues of the body, maps the location of the water and then uses this information to generate a detailed image. The images are so detailed because our bodies are made up of around 65% water,

oscillator builds up until the dissipation of energy balances the energy added by the driving force. For an oscillator with little damping, the amplitude becomes large (Fig.11.21). When the driving force is not at resonance, some negative work is stored in the system. Hence the net work done by the driving force decreases as the driving frequency moves away from the resonance. Therefore the oscillator's energy and amplitude is smaller than at resonance.

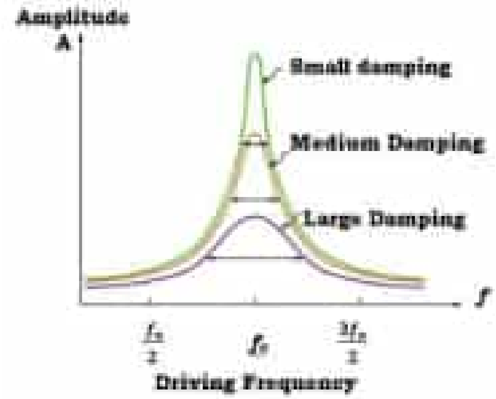


Fig: 11.21

The curves show a relation between amplitude and driven frequency of a harmonic oscillator. The curves represent the same oscillator with the same natural frequency but with different amounts of damping. Resonance occurs when the driving frequency equals the natural frequency.

11.5.3 Practical Example of Damped Oscillations :

Damping is not always disadvantageous. An example of damped oscillation can be seen in a shock absorber used in vehicles as shown in figure 11.22. A shock absorber is a device that is integrated into the suspension system of a car or motorcycle. Its primary purpose is to dampen the oscillations caused by irregularities in the road surface or when the vehicle encounters bumps. In order to compress or expand the shock absorber viscous oil must flow through the holes in the piston. The viscous force dissipates energy regardless of which direction the piston moves. The shock absorber enables the spring to smoothly return to its equilibrium length without oscillating up and down.

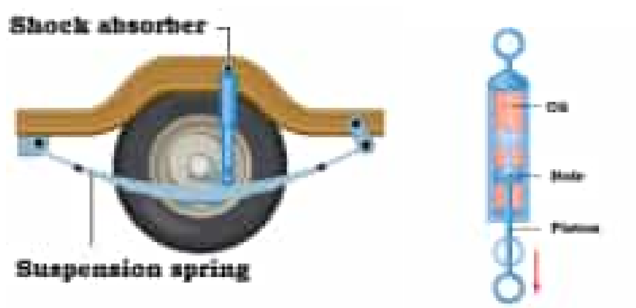


Fig: 11.22

shock absorbers mounted in the suspension system of an automobile, with a simplified cut away view of the shock absorber.

11.5.4 Frequency response and Sharpness of Resonance (Q-Factor):

In most physics and engineering problems the oscillators are analyzed in the limit of small amplitudes. In mechanics problems, an oscillating spring or other structural element has some nonlinearity in its stress-strain curve as the driving force increases and reaches close to the elastic limit.

An oscillating system does not like that an external force resonates with its natural frequency. If you do this the system responds and sometimes its response is catastrophic, the collapse of Tacoma Narrows Bridge is a textbook example of this fact.

On the contrary, when the damping forces are sufficiently strong to restrict the oscillation's amplitude at resonance, the oscillator behaves linearly. This behavior arises because, at resonance, the energy supplied to the oscillator from an external source precisely matches the energy loss due to work done against the damping forces. Increasing damping diminishes the sharpness of resonance (Fig. 11.21), thereby reducing its strength.

The sharpness of resonance depends mainly on two factors: amplitude and damping. The Q-factor quantifies the sharpness of resonance. It signifies the reduction of the oscillation's amplitude over time, which corresponds to the decay of energy in an oscillating system. It is approximately defined as the number of free oscillations the oscillator undergoes before its amplitude decays to zero. In the case of light damping, the Q-factor will be large, whereas it will be small for significant damping. Mathematically, the Q-factor is the ratio of energy stored to energy lost per oscillation, and it is a dimensionless quantity.

$$Q = E_{stored}/E_{lost} \quad \text{......(11.33)}$$

DO YOU KNOW?

In 1940 **Tacoma Narrows Bridge in Washington USA** was collapsed due to increase in amplitude as heavy wind blowing across the bridge resonated with the natural frequency of oscillation of the bridge. This decreases the damping and with the increasing amplitude enormous amount of energy is stored in it which causes the bridge to collapse.

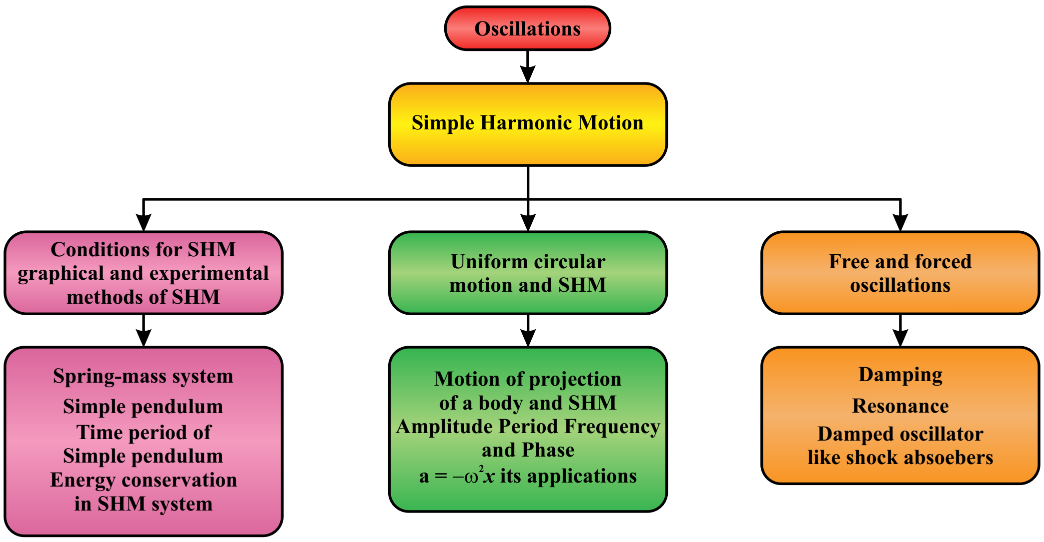
The collapsed, Tacoma Narrows Bridge.

The newly built Tacoma Narrows bridges opened in 1950 (right) and 2007 (left). These bridges are built with much higher resonant frequencies.

SUMMARY

- An equilibrium point is stable if the net force on an object when it is displaced from equilibrium position back towards equilibrium point.
- Vibration occurs in the vicinity of a point of stable equilibrium.
- A force which restores the equilibrium state of a system is called elastic restoring force.
- Simple harmonic motion is periodic motion that occurs whenever the restoring force is proportional to the displacement from equilibrium.
- The acceleration is proportional to and in opposite direction of the displacement: $a_x(t) = -\omega^2 x(t)$.
- In SHM position, velocity and acceleration as a functions of time are sinusoidal (i.e. sine or cosine function).
- An oscillatory motion is approximately SHM if the amplitude is small.
- The maximum velocity and acceleration in SHM are $v_m = \omega x_o$ and $a_m = \omega^2 x_o$.
- The angular frequency for a mass-spring system is: $\boldsymbol{\omega} = \sqrt{\frac{k}{m}}$; where k is spring or force constant.
- The angular frequency for a simple pendulum is: $\boldsymbol{\omega} = \sqrt{\frac{g}{L}}$; where L is length of simple pendulum.
- In the absence of any resistive (dissipative) forces the total mechanical energy of a simple harmonic oscillator at any point is constant and proportional to the square of the amplitude: $E = \frac{1}{2}kx_o^2$
- Resistive forces take out energy from an oscillating system. This takeoff is called Damping.
- Damping causes the amplitude to decrease with time.
- Resonance is a phenomenon exhibited by an oscillating system, when the system is vibrating under an external driven force close to its natural frequency.
- Q-Factor is the measurement of sharpness of resonance.

CONCEPT MAP


Oscillations
Simple Harmonic Motion
Conditions for SHM graphical and experimental methods of SHM
Uniform circular motion and SHM
Free and forced oscillations
Spring-mass system
Simple pendulum
Time period of Simple pendulum
Energy conservation in SHM system
Motion of projection of a body and SHM
Amplitude Period Frequency and Phase
$a = -\omega^2 x$ its applications
Damping
Resonance
Damped oscillator like shock absoebers

EXERCISE

Section (A): Multiple Choice Questions (MCQs)

1. Two simple pendulums A and B with same lengths, and equal amplitude of vibrations, but the mass of A is twice the mass of B, their period are T_A and T_B and energies are E_A and E_B respectively. Choose the correct statement.
   a) $T_A = T_B$ and $E_A > E_B$ b) $T_A < T_B$ and $E_A > E_B$
   c) $T_A > T_B$ and $E_A < E_B$ d) $T_A = T_B$ and $E_A < E_B$
2. In order to double the period of a simple pendulum:
   a) Its length should be doubled b) Its length should be quadrupled
   c) The mass should be doubled d) The mass should be quadrupled
3. A simple harmonic oscillator has amplitude A and time period t. Its maximum speed is:
   a) $\frac{4A}{t}$ b) $\frac{2A}{t}$ c) $\frac{4\pi A}{t}$ d) $\frac{2\pi A}{t}$
4. A spring attached by a load of weight W is vibrating with a period T. If the spring is divided in four equal parts and the same load is suspended from one of these parts, the new time period is:
   a) $\frac{T}{4}$ b) 2T c) $\frac{T}{2}$ d) 4T
5. The total energy of a particle executing simple harmonic motion is proportional to:
   a) frequency of oscillation b) maximum velocity of motion
   c) amplitude of motion d) square of amplitude of motion
6. A child swinging on a swing in sitting position, stands up, then the time period of the swing will:
   a) Increase b) decrease c) remains the same
   d) increases if the child is long and decreases if the child is short
7. If a body oscillates at the angular frequency ω_d of the driving force, then the oscillations are called:
   a) Forced oscillations b) Coupled oscillations
   c) Free oscillations d) Maintained oscillations
8. A simple harmonic oscillator with a natural frequency ω_N is force to oscillate with a driving frequency ω_d. The Resonance occurred when:
   a) $\omega_N < \omega_d$ b) $\omega_N > \omega_d$ c) $\omega_N = \omega_d$ d) $\omega_N \approx \omega_d$
9. In vehicles, shock absorber reduced the jerks:
   a) The shock absorber is the application of damped oscillations.
   b) Damping effect is due to the fractional loss of energy
   c) Shock absorbers in vehicles reduced jerk
   d) All of these
10. A heavily damped system has a fairly flat resonance curve in:
    a) An acceleration time graph b) An amplitude frequency graph
    c) Velocity time graph d) Distance-time graph

Section (B): Structured Questions

CRQs:

1. Explain the concept of periodic motion to oscillatory motion. Discuss the terms period, frequency, and amplitude.
2. Explain the concept of phase and phase difference in oscillatory motion. Discuss how phase is related to the position and time in an oscillating system.
3. Explain the concept of damping and its effects on oscillatory motion. Discuss the types of damping, such as over-damping, under-damping, and critical-damping
4. Discuss the concept of resonance frequency and its relationship to the natural frequency of an oscillating system.
5. Discuss the factors that affect the period of a simple pendulum. Explain how the length of the pendulum, the acceleration due to gravity, mass and the amplitude of oscillation influence the period.

ERQs:

1. Define simple harmonic motion (SHM). Discuss the key characteristics of a system undergoing SHM.
2. Derive the equation of motion for a mass-spring system in SHM, illustrating each step of the derivation.
3. Discuss the concept of energy in SHM. Explain how kinetic energy and potential energy vary throughout the motion of a particle in SHM and how the total mechanical energy is conserved.
4. Discuss the concept of resonance in simple harmonic motion. Explain how resonance occurs and its effects on the amplitude and energy transfer in a driven oscillating system.
5. Discuss the factors that affect the sharpness of resonance in an oscillatory system. Explain how damping and quality factor influence the width and peak of the resonance curve.
6. Explain how the period of a mass-spring system can be independent of amplitude, even though the distance travelled during each cycle is proportional to amplitude.
7. A mass hanging vertically from a spring and a simple pendulum both have a period of oscillation of 1s on Earth. The two devices are sent to another planet, where gravitational field is stronger than that of Earth. For each of the two systems, state whether the period is now longer than 1s, shorter than 1s, or equal to 1s. Explain your reasoning.

Numericals:

1. The period of oscillation of an object in an ideal spring and mass system is 0.50 s and the amplitude is 5.0 cm. what is the speed at the equilibrium point? and the acceleration at the point of maximum extension of the spring. **(62.8 cm/s 7.9 m/s^2)**
2. A sewing machine needle moves with a rapid vibratory motion, like SHM, as it sews a seam. Suppose the needle moves 8.4 mm from its highest to its lowest position and it makes 24 stitches in 9.0s. What is the maximum needle speed? **(7.0 cm/s)**
3. An ideal spring with a spring constant of 15 N/m is suspended vertically. A body of mass 0.60 kg is attached to the upstretched spring and released.
   (a) What is the extension of the spring when the speed is a maximum?
   (b) What is the maximum speed? **[(a) 0.39 m (b) 2.0 m/s]**
4. A body is suspended vertically from an ideal spring of spring constant 2.5 N/m. the spring is initially in its relaxed position. The body is then released and oscillates about its equilibrium position. The motion is described by $y = (4.0\text{cm}) \sin[(0.70\text{rad/s})\, t]$. What is the maximum kinetic energy of the body? **(2.0 mJ)**
5. The period of oscillation of a simple pendulum does not depend on the mass of the bob. By contrast the period of a mass-spring system does depend on mass. Explain the apparent condition.
6. What is the period of a simple pendulum of a 6.0 kg mass oscillating on a 4.0 m long string? **(4.01s)**
7. A pendulum of length 75 cm and mass 2.5 kg swings with a mechanical energy of 0.015 J. what is its amplitude? **(3.0 cm)**
8. A pendulum of length L_1 has a period of $T_1 = 0.950$ s. the length of the pendulum is adjusted to a new value L_2 such that $T_2 = 1.00$ s what is the ratio L_2/L_1 **(1.11)**
9. A wire is hanging from the top of a tower such that the top is not visible due to darkness. How do you calculate the height of tower?
10. The amplitude of oscillation of a pendulum decays by a factor of 20.0 in 120 s. By what factor has its energy decayed in that time
    (The energy has decreased by a factor of 400)