Unit 12 — Limit and Continuity

INTRODUCTION

In mathematics, the concept of limit and continuity is foundational in understanding the behaviour of functions and sequences, especially when applied to real-world scenarios. This unit will introduce and explore how to demonstrate and find the limit of a sequence and a function, understand continuous and discontinuous functions, and apply these concepts in various contexts such as economics, finance, and natural sciences.

12.1 Limit of a Function

The concept of limit of a function is the basis on which the structure of calculus rests. Before the definition of the limit of a function, it is necessary to have a clear understanding of the following phrases.

12.1.1 Meaning of the Phrase "x approaches zero"

Suppose a sequence $x_n = \dfrac{1}{n^2}$ assumes a sequence of values as:

$$1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4} \cdots, \frac{1}{2^n}, \cdots$$

We can see that the sequence $x_n = \dfrac{1}{n^2}$ is becoming smaller and smaller as n increases and can be made as small as we please by taking "n" sufficiently large. In other words,

$x_n = \dfrac{1}{n^2}$ becoming closer and closer to 0 as n becoming large. This unending decrease

of x_n is denoted by $x_n \to 0$ and read as "x_n approaches zero" or "x_n tends to zero as $n \to \infty$. That is, the limit of the sequence x_n is 0.

12.1.2 Meaning of the Phrase "x approaches infinity"

Suppose a sequence $x_n = 10^n$ assumes values as 1, 10, 10^2, 10^3, …, 10^n, …

It is clear that the sequence x_n is becoming larger and larger as n increases and can be made as large as we please by taking n sufficiently large. This unending increase of the sequence x_n is symbolically written as "$x_n \to \infty$" and is read as "x_n approaches infinity" or "x_n tends to infinity" as $n \to \infty$

12.1.3 Meaning of the Phrase "x approaches a"

Symbolically it is written as $"x \to a"$ which means that x is sufficiently close to a but different from the number a, from both the left and right sides of a that is $x - a$ becomes smaller and smaller as we please but $x - a \neq 0$.

Point to remember:
The symbol $x \to 0$ is quite different from $x = 0$.
$x \to 0$ means that x is very close to zero but not actually zero.
$x = 0$ means that x is actually zero.

12.1.4 Concept of Limit of a Function

(l)   By Finding the Area of Circumscribed Regular Polygon

Consider a circle of unit radius which circumscribes a square (4-sided regular polygon) as shown in Figure 12.1.

The side of square is $\sqrt{2}$ and its area is 2 square units. It is clear that the area of inscribed 4-sided polygon is less than the area of the circum-circle $\pi = 3.142 (\pi r^2 = \pi (1)^2 = \pi = 3.142)$.
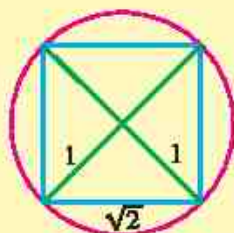

Figure 12.1: 4-sided polygon   Figure 12.2: 8-sided polygon   Figure 12.3: 16-sided polygon

Bisecting the arcs between the vertices of the square, we get an inscribed 8-sided regular polygon as shown in Figure 12.2. Its area is $2\sqrt{2} = 2.828$ square units which is closer to the area of circum-circle. A further similar bisection of the arcs gives an inscribed 16-sided regular polygon as shown in Figure 12.3 with area 3.061 square units which is more closer to the area of circum-circle.

It follows that as "n", the number of sides of the inscribed polygon increases, the area of polygon increases and becoming near to 3.142 which is the area of circle of unit radius.

We express this situation by saying that the limiting value of the area of the inscribed polygon is the area of the circle as n approaches infinity, i.e.,

Area of inscribed polygon $\to$ Area of circle as $n \to \infty$

(ii) Numerical Approach

Consider the function $f(x) = x^3$

The domain of $f(x)$ is the set of all real numbers.

Let us find the limit of $f(x) = x^3$ as x approaches 2.

The table of values of $f(x)$ for different values of x as x approaches 2 from left and right is as follows:

From left of 2 ⟶ 2 ⟶ from right of 2

x	1	1.5	1.8	1.9	1.99	1.999	1.9999	2.0001	2.001	2.01	2.1	2.2	2.5	3
$f(x)=x^3$	1	3.375	5.832	6.859	7.8806	7.8806	7.9988	8.0012	8.012	8.1206	9.261	10.648	15.625	27

The table shows that, as x gets closer and closer to 2 (sufficiently close to 2), from both sides, $f(x)$ gets closer and closer to 8.

We say that 8 is the limit of $f(x)$ when x approaches 2 and is written as:

$$f(x) \to 8 \text{ as } x \to 2 \quad \text{or} \quad \lim_{x\to 2} x^3 = 8$$

12.1.5 Limit of a Function

Let a function $f(x)$ be defined in an open interval near the number "a" (need not be at a). If, as x approaches "a" from both left and right side of "a" $f(x)$ approaches a specific number "L" then "L", is called the limit of $f(x)$ as x approaches a. Symbolically it is written as:

$\lim_{x\to a} f(x) = L$ read as "limit of $f(x)$ as $x \to a$, is L".

It is neither desirable nor practicable to find the limit of a function by numerical approach. We must be able to evaluate a limit in some mechanical way. The theorems on limits will serve this purpose. Their proofs will be discussed in higher classes.

12.1.6 Theorems on Limits of Functions

Let f and g be two functions for which $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then

Theorem 1: (i) $\lim_{x\to a} x^p = a^p$, where $p > 0$ and $p \in R$

(ii) $\lim_{x\to a} c = c$

Theorem 2: (a) The limit of the sum of two functions is equal to the sum of their limits.

$$\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M$$

For example, $\lim_{x\to 1}(x+5) = \lim_{x\to 1} x + \lim_{x\to 1} 5 = 1 + 5 = 6$

(b) **The limit of the difference of two functions is equal to the difference of their limits.**

$$\underset{x\to a}{Lim}[f(x)-g(x)]=\underset{x\to a}{Lim}f(x)-\underset{x\to a}{Lim}g(x)=L-M$$

For example, $\underset{x\to 3}{Lim}(x-5)=\underset{x\to 3}{Lim}(x)-\underset{x\to 3}{Lim}5=3-5=-2$

(c) **If k is any real number, then**

$$\underset{x\to a}{Lim}[kf(x)]=k\underset{x\to a}{Lim}f(x)=kL$$

For example, $\underset{x\to 2}{Lim}(3x)=3\underset{x\to 2}{Lim}(x)=3(2)=6$

(d) **The limit of the product of the functions is equal to the product of their limits.**

$$\underset{x\to a}{Lim}[f(x)\,g(x)]=\underset{x\to a}{Lim}f(x)\cdot\underset{x\to a}{Lim}g(x)=LM$$

For example, $\underset{x\to 1}{Lim}(2x)(x+4)=\underset{x\to 1}{Lim}(2x)\cdot\underset{x\to 1}{Lim}(x+4)=(2)(5)=10$

(e) **The limit of the quotient of the functions is equal to the quotient of their limits provided the limit of denominator is non-zero.**

$$\underset{x\to a}{Lim}\left[\frac{f(x)}{g(x)}\right]=\frac{\underset{x\to a}{Lim}f(x)}{\underset{x\to a}{Lim}g(x)}=\frac{L}{M}, \text{ provided } g(x)\neq 0 \text{ in a neighborhood of}$$

a and $M\neq 0$

For example: $\underset{x\to 2}{Lim}\left[\frac{3x+4}{x+3}\right]=\frac{\underset{x\to 2}{Lim}(3x+4)}{\underset{x\to 2}{Lim}(x+3)}=\frac{6+4}{2+3}=\frac{10}{5}=2$

(f) **Limit of $[f(x)]^n$, where n is an integer**

$$\underset{x\to a}{Lim}[f(x)]^n=[\underset{x\to a}{Lim}f(x)]^n=L^n$$

For example, $\underset{x\to 4}{Lim}(2x-3)^3=\left(\underset{x\to 4}{Lim}(2x-3)\right)^3=(5)^3=125$

We conclude from the theorems on limits that limits are evaluated by merely substituting the number that x approaches into the function.

12.2 Limits of Important Functions

If by substituting the number that x approaches into the function, we get $\left(\dfrac{0}{0}\right)$, then one possible way to evaluate the limits is as follows:

We simplify the given function by using algebraic techniques of making factors if possible and cancel the common factors. The method explained in the following important limits.

12.2.1 $\displaystyle\lim_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$ where n is a non-zero integer and $a > 0$

Case I: Suppose n is a positive integer.

By substituting $x = a$, we get $\left(\dfrac{0}{0}\right)$ form, so we make factors as follows:

$$x^n - a^n = (x-a)(x^{n-1} + ax^{n-2} + a^2 x^{n-3} + \dots + a^{n-1})$$

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{x \to a} \frac{(x-a)(x^{n-1} + ax^{n-2} + a^2 x^{n-3} + \dots + a^{n-1})}{x-a}$$

$$= \lim_{x \to a} (x^{n-1} + ax^{n-2} + a^2 x^{n-3} + \dots + a^{n-1})$$

$$= a^{n-1} + a \cdot a^{n-2} + a^2 \cdot a^{n-3} + a^3 a^{n-4} + \dots + a^{n-1}$$

$$= a^{n-1} + a^{n-1} + a^{n-1} + a^{n-1} \dots + a^{n-1} = na^{n-1}$$

Case II: Suppose n is a negative integer (Say $n = -m$) where m is a positive integer.

Now, $\dfrac{x^n - a^n}{x-a} = \dfrac{x^{-m} - a^{-m}}{x-a} = \dfrac{\dfrac{1}{x^m} - \dfrac{1}{a^m}}{x-a} = \dfrac{\dfrac{a^m - x^m}{x^m a^m}}{x-a}$

$\therefore \quad \displaystyle\lim_{x \to a} \frac{x^n - a^n}{x-a} = \lim_{x \to a} \left(\frac{-1}{x^m a^m}\right)\left(\frac{x^m - a^m}{x-a}\right)$

$$= \frac{-1}{a^m a^m}(ma^{m-1}) \qquad \text{(by Case–I)}$$

$$= -ma^{-m-1}$$

$\therefore \quad \displaystyle\lim_{x \to a}\left(\frac{x^n - a^n}{x-a}\right) = na^{n-1} \qquad \because n = -m$

12.2.2 $\lim\limits_{x \to 0} \dfrac{\sqrt{x+a} - \sqrt{a}}{x} = \dfrac{1}{2\sqrt{a}}$, where n is an integer and $a > 0$.

By substituting $x = 0$, we have $\left(\dfrac{0}{0}\right)$ form, so rationalizing the numerator.

$$\lim\limits_{x \to 0} \frac{\sqrt{x+a} - \sqrt{a}}{x} = \lim\limits_{x \to 0} \left(\frac{\sqrt{x+a} - \sqrt{a}}{x}\right)\left(\frac{\sqrt{x+a} + \sqrt{a}}{\sqrt{x+a} + \sqrt{a}}\right) = \lim\limits_{x \to 0} \frac{x + a - a}{x(\sqrt{x+a} + \sqrt{a})}$$

$$= \lim\limits_{x \to 0} \left(\frac{1}{\sqrt{x+a} + \sqrt{a}}\right)$$

$$= \lim\limits_{x \to 0} \frac{1}{\sqrt{x+a} + \sqrt{a}} = \frac{1}{\sqrt{a} + \sqrt{a}} = \frac{1}{2\sqrt{a}}$$

Example 1   Evaluate: (i) $\lim\limits_{x \to 1} \dfrac{x^2 - 1}{x^2 - x}$ (ii) $\lim\limits_{x \to 3} \dfrac{x - 3}{\sqrt{x} - \sqrt{3}}$

Solution   (i) $\lim\limits_{x \to 1} \dfrac{x^2 - 1}{x^2 - x}$ $\left(\dfrac{0}{0}\right)$ form.

$$\lim\limits_{x \to 1} \frac{x^2 - 1}{x^2 - x} = \lim\limits_{x \to 1} \frac{(x-1)(x+1)}{x(x-1)} = \lim\limits_{x \to 1} \frac{x+1}{x}$$

$$= \frac{1+1}{1} = \frac{2}{1} = 2$$

(ii) $\lim\limits_{x \to 3} \dfrac{x - 3}{\sqrt{x} - \sqrt{3}} = \lim\limits_{x \to 3} \dfrac{(\sqrt{x} + \sqrt{3})(\sqrt{x} - \sqrt{3})}{(\sqrt{x} - \sqrt{3})} = \lim\limits_{x \to 3}(\sqrt{x} + \sqrt{3}) = \sqrt{3} + \sqrt{3} = 2\sqrt{3}$

12.2.3   Limit at Infinity

We have studied the limits of the functions $f(x)$, $f(x)$. $g(x)$ and $\dfrac{f(x)}{g(x)}$, when $x \to c$(a number)

Let us see what happens to the limit of the function $f(x)$ if c is $+\infty$ or $-\infty$ (limits at infinity) i.e., when $x \to +\infty$ or $x \to -\infty$.

(a)   Limit as $x \to +\infty$

Let $f(x) = \dfrac{1}{x}$, when $x \neq 0$

This function has the property that the value of $f(x)$ can be made as close as we please to zero when the number x is sufficiently large.

We express this phenomenon by writing $\lim\limits_{x \to \infty} \dfrac{1}{x} = 0$

(b) Limit as $x \to -\infty$

This type of limits are handled in the same way as limits as $x \to +\infty$.

i.e., $\displaystyle\lim_{x \to -\infty} \frac{1}{x} = 0$, where $x \neq 0$.

The following theorem is useful for evaluating limit at infinity.

Theorem: Let p be a positive rational number. If x^p is defined, then

$\displaystyle\lim_{x \to +\infty} \frac{a}{x^p} = 0$ and $\displaystyle\lim_{x \to -\infty} \frac{a}{x^p} = 0$, where a is any real number.

For example, $\displaystyle\lim_{x \to \pm\infty} \frac{6}{x^3} = 0$ and $\displaystyle\lim_{x \to +\infty} \frac{-5}{\sqrt[3]{x}} = 0$

12.2.4 Limit of a Sequence

Let $\{a_n\}$ be a sequence, the limit of a sequence $\{a_n\}$ is the value L that the terms of the sequence approach as $n \to \infty$, that is,

$$\lim_{n \to \infty} a_n = L$$

If such an L exists, the sequence is said to converge to L and $\{a_n\}$ is called convergent sequence. If no such L exists, the sequence is said to diverge.

For example, consider the sequence $\left\{a_n = \dfrac{1}{n}\right\}$: As $n \to \infty$, $\dfrac{1}{n} \to 0$

So, we write $\displaystyle\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = 0.$

Example 2 Find the limit of the sequence $a_n = \dfrac{2n+3}{n+1}$.

Solution We can simplify the sequence:

$$a_n = \frac{2n+3}{n+1} = \frac{n\left(2 + \dfrac{3}{n}\right)}{n\left(1 + \dfrac{1}{n}\right)} = \frac{2 + \dfrac{3}{n}}{1 + \dfrac{1}{n}}$$

As $n \to \infty$, $\dfrac{3}{n} \to 0$ and $\dfrac{1}{n} \to 0$, so we are left with: $\displaystyle\lim_{n \to \infty} a_n = \frac{2+0}{1+0} = 2$

Thus, $\displaystyle\lim_{n \to \infty} \frac{2n+3}{n+1} = 2,$

Divergent Sequences: A sequence is **divergent** if it is not convergent. Divergence can occur in the following ways:

- The sequence may increase or decrease without bound (e.g., $a_n = n^2$ diverges to infinity).

- The sequence may oscillate between different values and not settle near any one value (e.g., $a_n = (-1)^n$ oscillates between -1 and 1, so it does not converge).

12.2.5 Methods for Evaluating the Limits at Infinity

In this case we first divide each term of both the numerator and the denominator by the highest power of x that appears in the denominator and then use the theorems on limit.

Example 3 Evaluate $\displaystyle\lim_{x \to +\infty} \frac{5x^4 - 10x^2 + 1}{-3x^3 + 10x^2 + 50}$

Solution Dividing numerator and denominator by x^3, we get

$$\lim_{x \to +\infty} \frac{5x^4 - 10x^2 + 1}{-3x^3 + 10x^2 + 50} = \lim_{x \to +\infty} \frac{5x - \dfrac{10}{x} + \dfrac{1}{x^3}}{-3 + \dfrac{10}{x} + \dfrac{50}{x^3}} = \frac{\infty - 0 + 0}{-3 + 0 + 0} = -\infty \quad \because \lim_{x \to \infty} \frac{a}{x^p} = 0$$

Example 4 Evaluate $\displaystyle\lim_{x \to -\infty} \frac{4x^4 - 5x^3}{-3x^5 + 2x^2 + 1}$

Solution Dividing numerator and denominator by x^5, we get

$$\lim_{x \to -\infty} \frac{4x^4 - 5x^3}{-3x^5 + 2x^2 + 1} = \lim_{x \to -\infty} \frac{\dfrac{4}{x} - \dfrac{5}{x^2}}{-3 + \dfrac{2}{x^3} + \dfrac{1}{x^5}} = \frac{0 - 0}{-3 + 0 + 0} = 0$$

Example 5 Evaluate:

  (i) $\displaystyle\lim_{x \to -\infty} \frac{2 - 3x}{\sqrt{3 + 4x^2}}$ (ii) $\displaystyle\lim_{x \to +\infty} \frac{2 - 3x}{\sqrt{3 + 4x^2}}$

Solution   (i)   Here $\sqrt{x^2} = |x| = -x$ as $x < 0$

$\therefore$ Dividing numerator and denominator by $-x$, we get

$$\lim_{x \to -\infty} \frac{2 - 3x}{\sqrt{3 + 4x^2}} = \lim_{x \to -\infty} \frac{-\dfrac{2}{x} + 3}{\sqrt{\dfrac{3}{x^2} + 4}} = \frac{0 + 3}{\sqrt{0 + 4}} = \frac{3}{2}$$

(ii)   Here $\sqrt{x^2} = |x| = x$ as $x > 0$

∴   Dividing numerator and denominator by x, we get

$$\underset{x \to +\infty}{\text{Lim}} \frac{2-3x}{\sqrt{3+4x^2}} = \underset{x \to +\infty}{\text{Lim}} \frac{\frac{2}{x}-3}{\sqrt{\frac{3}{x^2}+4}} = \frac{0-3}{\sqrt{0+4}} = \frac{-3}{2}$$

12.2.6 $\underset{n \to +\infty}{\text{Lim}}\left(1+\dfrac{1}{n}\right)^n = e$

By the binomial theorem, we have

$$\left(1+\frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + \dots$$

$$= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\dots$$

When $n \to +\infty$, $\dfrac{1}{n}, \dfrac{2}{n}, \dfrac{3}{n}, \dots$ all tends to zero, therefore

∴ $\underset{x \to +\infty}{\text{Lim}}\left(1+\dfrac{1}{n}\right)^n = 1+1+\dfrac{1}{2!}+\dfrac{1}{3!}+\dfrac{1}{4!}+\dots$

$$= 1+1+0.5+0.166667+0.0416667+\dots = 2.718281\dots$$

As approximate value of e is 2.718281.

∴ $\underset{n \to +\infty}{\text{Lim}}\left(1+\dfrac{1}{n}\right)^n = e$

> **Note**   We can also show that
> $$\underset{n \to -\infty}{\text{Lim}}\left(1+\frac{1}{n}\right)^n = e$$

Deduction: $\underset{x \to 0}{\text{Lim}}(1+x)^{\frac{1}{x}} = e$

We know that $\underset{n \to \infty}{\text{Lim}}\left(1+\dfrac{1}{n}\right)^n = e$ (i)

Put $n = \dfrac{1}{x}$ in (i)   then $x = \dfrac{1}{n}$

When $n \to \infty, x \to 0$ so, $\underset{n \to +\infty}{\text{Lim}}\left(1+\dfrac{1}{n}\right)^n = \underset{x \to 0}{\text{Lim}}(1+x)^{\frac{1}{x}}$

$e = \underset{x \to 0}{\text{Lim}}(1+x)^{\frac{1}{x}}$ ∵ $\underset{n \to +\infty}{\text{Lim}}\left(1+\dfrac{1}{n}\right)^n = e$

Hence $\underset{x \to 0}{\text{Lim}}(1+x)^{\frac{1}{x}} = e$

12.2.7 $\displaystyle\lim_{x\to0}\frac{a^x-1}{x}=\log_e a$

Put $a^x-1=y$ (i)

then $a^x=1+y$

So, $x=\log_a(1+y)$

From (i) when $x\to0,\ y\to0$

$\therefore\quad\displaystyle\lim_{x\to0}\frac{a^x-1}{x}=\lim_{y\to0}\frac{y}{\log_a(1+y)}=\lim_{y\to0}\frac{1}{\frac{1}{y}\log_a(1+y)}$

$$=\lim_{y\to0}\frac{1}{\log_a(1+y)^{\frac{1}{y}}}=\frac{1}{\log_a e}=\log_e a \quad\left(\lim_{y\to0}(1+y)^{\frac{1}{y}}=e\right)$$

Deduction: $\displaystyle\lim_{x\to0}\left(\frac{e^x-1}{x}\right)=\log_e e=1$

We know that $\displaystyle\lim_{x\to0}\left(\frac{a^x-1}{x}\right)=\log_e a$ (i)

Put $a=e$ in (i) we know $\displaystyle\lim_{x\to0}\left(\frac{e^x-1}{x}\right)=\log_e e=1$

Important Results to Remember

(i) $\displaystyle\lim_{x\to\infty}e^x=\infty$ (ii) $\displaystyle\lim_{x\to-\infty}e^x=\lim_{x\to\infty}\left(\frac{1}{e^x}\right)=0$

Example 6 Express each limit in terms of e.

(i) $\displaystyle\lim_{n\to+\infty}\left(1+\frac{3}{n}\right)^{2n}$ (ii) $\displaystyle\lim_{n\to0}(1+2n)^{\frac{1}{n}}$

Solution (i) Observe the resemblance of the limit with $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$

$$\left(1+\frac{3}{n}\right)^{2n}=\left[\left(1+\frac{3}{n}\right)^{\frac{n}{3}}\right]^6=\left[\left(1+\frac{1}{\frac{n}{3}}\right)^{\frac{n}{3}}\right]^6$$

$\therefore\quad\displaystyle\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^{2n}=\lim_{m\to+\infty}\left[\left(1+\frac{1}{m}\right)^m\right]^6=e^6$ where, $m=\dfrac{n}{3}$

(ii) Observe the resemblance of the limit with $\underset{x\to 0}{\text{Lim}}(1+x)^{\frac{1}{x}}=e$

$$\underset{n\to 0}{\text{Lim}}(1+2n)^{\frac{1}{n}}=\underset{n\to 0}{\text{Lim}}\left[(1+2n)^{\frac{1}{2n}}\right]^2$$

put $m=2n$, when $n\to 0$, $m\to 0$

$$\underset{n\to 0}{\text{Lim}}(1+2n)^{\frac{1}{n}}=\underset{m\to 0}{\text{lim}}\left[(1+m)^{\frac{1}{m}}\right]^2=e^2$$

12.2.8 The Sandwich Theorem

Let f, g and h be functions such that $f(x)\le g(x)\le h(x)$ for all numbers x in some open interval containing "c", except possibly at c itself.

If $\underset{x\to c}{\text{Lim}}f(x)=L$ and $\underset{x\to c}{\text{Lim}}h(x)=L$, then $\underset{x\to c}{\text{Lim}}g(x)=L$

Many limit problems arise that cannot be directly evaluated by algebraic techniques. They require geometric arguments, so we evaluate an important theorem.

12.2.9 If θ is measured in radian, then $\underset{\theta\to 0}{\text{Lim}}\dfrac{\sin\theta}{\theta}=1.$

Proof: To evaluate this limit, we apply a new technique. Take θ be positive acute central angle of a sector of a circle with radius $r=1$. As shown in the figure, OAB represents a sector of a circle. Join A and B and extend OB to D such that $\overline{OA}\perp\overline{AD}$. Also draw $\overline{BC}\perp\overline{OC}$ on $\overline{OA}$.

Given $|OA|=|OB|=1$ (radii of unit circle)

In the right ΔOCB, $\sin\theta=\dfrac{|\overline{BC}|}{|\overline{OB}|}=|\overline{BC}|$

In the right ΔOAD, $\tan\theta=\dfrac{|\overline{AD}|}{|\overline{OA}|}=|\overline{AD}|$

(i) Area of $\Delta OAB=\dfrac{1}{2}|\overline{OA}||\overline{BC}|=\dfrac{1}{2}(1)(\sin\theta)=\dfrac{1}{2}\sin\theta$

(ii) Area of sector $OAB=\dfrac{1}{2}r^2\theta=\dfrac{1}{2}(1)(\theta)=\dfrac{1}{2}\theta$ and

(iii) Area of $\Delta OAD=\dfrac{1}{2}|\overline{OA}||\overline{AD}|=\dfrac{1}{2}(1)(\tan\theta)=\dfrac{1}{2}\tan\theta$

From the figure we see that
Area of $\Delta OAB<$ Area of sector $OAB<$ Area of ΔOAD

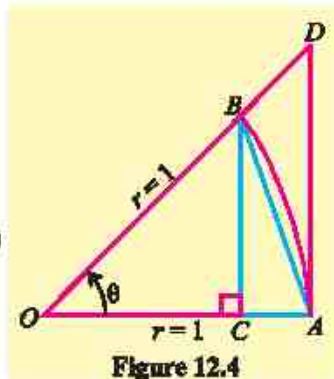$\Rightarrow \quad \dfrac{1}{2}\sin\theta<\dfrac{\theta}{2}<\dfrac{1}{2}\tan\theta$

Figure 12.4

As $\sin \theta$ is positive, so on division by $\frac{1}{2}\sin \theta$, we get

$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta} \qquad \left(0 < \theta < \frac{\pi}{2}\right)$$

i.e., $\quad 1 > \dfrac{\sin \theta}{\theta} > \cos \theta \quad$ or $\quad \cos \theta < \dfrac{\sin \theta}{\theta} < 1$

when $\theta \to 0$, $\cos \theta \to 1$

Since $\dfrac{\sin \theta}{\theta}$ is sandwiched between 1 and a quantity approaching 1 itself. So, by the sandwich theorem, it must also approach 1 that is, $\displaystyle \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

Note
The same result holds for $\dfrac{-\pi}{2} < \theta < 0$

Example 7   Evaluate $\displaystyle \lim_{\theta \to 0} \frac{\sin 7\theta}{\theta}$

Solution   Let $x = 7\theta$, so that $\theta = \dfrac{x}{7}$

when $\theta \to 0$ we have $x \to 0$

$\because \quad \displaystyle \lim_{\theta \to 0} \frac{\sin 7\theta}{\theta} = \lim_{x \to 0} \frac{\sin x}{\dfrac{x}{7}} = 7 \lim_{x \to 0} \frac{\sin x}{x} = (7)(1) = 7$

Example 8 Evaluate $\displaystyle \lim_{\theta \to 0} \frac{1-\cos \theta}{\theta}$

Solution $\dfrac{1-\cos \theta}{\theta} = \dfrac{1-\cos \theta}{\theta} \cdot \dfrac{1+\cos \theta}{1+\cos \theta} = \dfrac{1-\cos^2 \theta}{\theta(1+\cos \theta)}$

$$= \frac{\sin^2 \theta}{\theta(1+\cos \theta)} = \sin \theta \left(\frac{\sin \theta}{\theta}\right)\left(\frac{1}{1+\cos \theta}\right)$$

$$\lim_{\theta \to 0}\left(\frac{1-\cos \theta}{\theta}\right) = \lim_{\theta \to 0} \sin \theta \cdot \lim_{\theta \to 0}\frac{\sin \theta}{\theta} \cdot \lim_{\theta \to 0}\left(\frac{1}{1+\cos \theta}\right) = (0)(1)\left(\frac{1}{1+1}\right) = 0$$

EXERCISE 12.1

1.   Find the limit of the following sequences if exists:

    (i) $a_n = \dfrac{2n+3}{n+1}$ (ii) $b_n = \dfrac{2n+3}{n^2+1}$ (iii) $c_n = \dfrac{5n^2}{2n+3}$ (iv) $d_n = \dfrac{n^2-3n+1}{2n^2+n+4}$

2. Evaluate each limit by using theorems of limits:

(i) $\displaystyle\lim_{x\to 3}(2x+4)$ (ii) $\displaystyle\lim_{x\to 1}(3x^2-2x+4)$ (iii) $\displaystyle\lim_{x\to 5}\sqrt{x^2+x+4}$

(iv) $\displaystyle\lim_{x\to 2}\sqrt{x^2+4}$ (v) $\displaystyle\lim_{x\to 2}(\sqrt{x^3+1}-\sqrt{x^2+5})$ (vi) $\displaystyle\lim_{x\to -2}\frac{2x^3+5x}{3x-2}$

3. Evaluate each limit by using algebraic techniques:

(i) $\displaystyle\lim_{x\to -1}\frac{x^3-x}{x+1}$ (ii) $\displaystyle\lim_{x\to 3}\left(\frac{x^2-5x+6}{x^2-2x-3}\right)$ (iii) $\displaystyle\lim_{x\to 2}\left(\frac{x^3-8}{x^2-5x+6}\right)$

(iv) $\displaystyle\lim_{x\to 1}\frac{x^3-3x^2+3x-1}{x^3-x}$ (v) $\displaystyle\lim_{x\to 2}\left(\frac{x^3-6x^2+12x-8}{x^3-4x}\right)$ (vi) $\displaystyle\lim_{x\to 1}\left(\frac{x^4-1}{x^2-3x+2}\right)$

(vii) $\displaystyle\lim_{x\to 2}\frac{x-2}{\sqrt{x+2}-\sqrt{6-x}}$ (viii) $\displaystyle\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$ (ix) $\displaystyle\lim_{x\to a}\frac{x^n-a^n}{x^m-a^m}$

4. Evaluate the following limits:

(i) $\displaystyle\lim_{x\to 0}\frac{\sin 5x}{x}$ (ii) $\displaystyle\lim_{x\to 0}\frac{\sin x^\circ}{x}$ (iii) $\displaystyle\lim_{\theta\to 0}\frac{1-\cos\theta}{\sin\theta}$

(iv) $\displaystyle\lim_{x\to \frac{\pi}{4}}\frac{\sin x-\cos x}{x-\frac{\pi}{4}}$ (v) $\displaystyle\lim_{x\to 0}\frac{\cos ax-\cos bx}{x^2}$ (vi) $\displaystyle\lim_{x\to \frac{\pi}{4}}\frac{\tan x-1}{x-\frac{\pi}{4}}$

(vii) $\displaystyle\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$ (viii) $\displaystyle\lim_{x\to 0}\frac{\cos ax-\cos bx}{\cos cx-\cos dx}$ (ix) $\displaystyle\lim_{x\to 1}\frac{x^3-1}{x^2-1}$

(x) $\displaystyle\lim_{x\to 3}\frac{x^2-x\log x+3\log x-9}{x-3}$ (xi) $\displaystyle\lim_{x\to 0}\frac{x(2^x-1)}{1-\cos x}$

5. Express each limit in terms of e.

(i) $\displaystyle\lim_{n\to +\infty}\left(1+\frac{1}{n}\right)^{2n}$ (ii) $\displaystyle\lim_{n\to +\infty}\left(1+\frac{1}{n}\right)^{\frac{n}{2}}$ (iii) $\displaystyle\lim_{n\to +\infty}\left(1-\frac{1}{n}\right)^{n}$

(iv) $\displaystyle\lim_{n\to +\infty}\left(1+\frac{1}{3n}\right)^{n}$ (v) $\displaystyle\lim_{n\to +\infty}\left(1+\frac{4}{n}\right)^{n}$ (vi) $\displaystyle\lim_{x\to 0}(1+3x)^{\frac{2}{x}}$

(vii) $\displaystyle\lim_{x\to 0}(1+2x^2)^{\frac{1}{x^2}}$ (viii) $\displaystyle\lim_{x\to 0}\frac{e^{ax}-e^{bx}}{abx}$ (ix) $\displaystyle\lim_{x\to \infty}\left(\frac{x}{1+x}\right)^{x}$

(x) $\displaystyle\lim_{x\to 0}\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1},\ x<0$ (xi) $\displaystyle\lim_{x\to 0}\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1},\ x>0$ (xii) $\displaystyle\lim_{x\to 2}\frac{e^x-e^2}{x-2}$

12.3 Continuity and Discontinuity of Functions

12.3.1 One-Sided Limits

In defining $\underset{x\to c}{\text{Lim}}\, f(x)$, we restricted x in an open interval containing c i.e., we studied the behaviour of f on both sides of c. However, in some cases it is necessary to investigate one sided limits that is, the left hand limit and the right hand limit.

(i) The Left Hand Limit

$\underset{x\to c^{-}}{\text{Lim}}\, f(x) = L$ is read as the limit of $f(x)$ is equal to L as x approaches c from the left i.e., for all x sufficiently close to c, but less than c, the value of $f(x)$ can be made as close as we please to L.

(ii) The Right Hand Limit

$\underset{x\to c^{+}}{\text{Lim}}\, f(x) = M$ is read as the limit of $f(x)$ is equal to M as x approaches c from the right i.e., for all x sufficiently close to c, but greater than c, the value of $f(x)$ can be made as close as we please to M.

> **Note** The rules for calculating the left hand and the right hand limits are the same as we studied to calculate limits in the preceding section.

12.3.2 Criterion for Existence of Limit of a Function

$$\underset{x\to c}{\text{Lim}}\, f(x) = L \text{ if and only if } \underset{x\to c^{-}}{\text{Lim}}\, f(x) = \underset{x\to c^{+}}{\text{Lim}}\, f(x) = L$$

Example 9   Determine whether $\underset{x\to 2}{\text{Lim}}\, f(x)$ and $\underset{x\to 4}{\text{Lim}}\, f(x)$ exist, when

$$f(x) = \begin{cases} 2x+1 & \text{if} & 0 \le x \le 2 \\ 7-x & \text{if} & 2 < x < 4 \\ x & \text{if} & 4 \le x \le 6 \end{cases}$$

Solution (i) $\underset{x\to 2^{-}}{\text{Lim}}\, f(x) = \underset{x\to 2^{-}}{\text{Lim}}(2x+1) = 4+1 = 5$

$\underset{x\to 2^{+}}{\text{Lim}}\, f(x) = \underset{x\to 2^{+}}{\text{Lim}}(7-x) = 7-2 = 5$

Since $\underset{x\to 2^{-}}{\text{Lim}}\, f(x) = \underset{x\to 2^{+}}{\text{Lim}}\, f(x) = 5$

$\Rightarrow$ $\underset{x\to 2}{\text{Lim}}\, f(x)$ exists and is equal to 5.

> **Remember!**
> $\underset{x\to 0^{-}}{\lim}\dfrac{1}{x} = -\infty$
> $\underset{x\to 0^{+}}{\lim}\dfrac{1}{x} = +\infty$

(ii) $\underset{x\to 4^{-}}{\text{Lim}}\, f(x) = \underset{x\to 4^{-}}{\text{Lim}}(7-x) = 7-4 = 3$

$\underset{x\to 4^{+}}{\text{Lim}}\, f(x) = \underset{x\to 4^{+}}{\text{Lim}}(x) = 4$

Since $\underset{x\to 4^{-}}{\text{Lim}}\, f(x) \ne \underset{x\to 4^{+}}{\text{Lim}}\, f(x)$

Therefore, $\underset{x\to 4}{\text{Lim}}\, f(x)$ does not exist.

12.3.3 Continuity of a Function at a Point

(a) Continuous Function

A function f is said to be continuous at a number "c" if and only if the following three conditions are satisfied.

(i) $f(c)$ is defined   (ii) $\lim\limits_{x \to c} f(x)$ exists   (iii) $\lim\limits_{x \to c} f(x) = f(c)$

(b) Discontinuous Function

If one or more of these three conditions fail to hold at "c", then the function f is said to be discontinuous at "c".

Example 10 Consider the function $f(x) = \dfrac{x^2 - 1}{x - 1}$, discuss the continuity of f at $x = 1$.

Solution Here $f(1)$ is not defined.

$\Rightarrow f(x)$ is discontinuous at 1.

Example 11 For $f(x) = 3x^2 - 5x + 4$, discuss continuity of f at $x = 1$.

Solution $\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1}(3x^2 - 5x + 4) = 3 - 5 + 4 = 2$ and $f(1) = 3 - 5 + 4 = 2$

$\Rightarrow \lim\limits_{x \to 1} f(x) = f(1)$

Therefore, $f(x)$ is continuous at $x = 1$

Example 12 Discuss the continuity of the functions $f(x)$ and $g(x)$ at $x = 3$

(a) $f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$

(b) $g(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3} & \text{if } x \neq 3 \end{cases}$

Solution (a) $f(3) = 6$

Now, $\lim\limits_{x \to 3} f(x) = \lim\limits_{x \to 3} \dfrac{x^2 - 9}{x - 3}$

$= \lim\limits_{x \to 3} \dfrac{(x + 3)(x - 3)}{(x - 3)}$

$= \lim\limits_{x \to 3}(x + 3) = 3 + 3 = 6$

As $\lim\limits_{x \to 3} f(x) = 6 = f(3)$

Figure 12.5

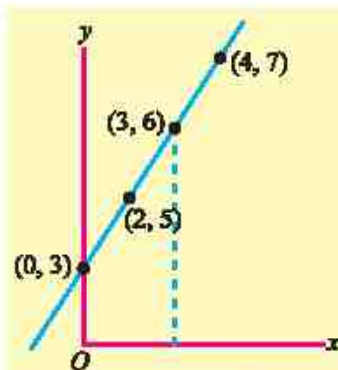$f(x)$ is continuous at $x = 3$. It is noted that there is no break in the graph.

(b) $g(x) = \dfrac{x^2 - 9}{x - 3}$ if $x \neq 3$

As $g(x)$ is not defined at $x = 3$

$\Rightarrow$ $g(x)$ is discontinuous at $x = 3$

It is noted that there is a break in the graph at $x = 3$ near $x = 3$ as shown in the Figure 12.6.

Example 13   Discuss continuity of $f(x)$ at $x = 3$, when

$$f(x) = \begin{cases} x - 1 & , \text{ if } x < 3 \\ 2x + 1 & , \text{ if } x \geq 3 \end{cases}$$

Solution A sketch of the graph of f is shown in the Figure 12.7. We can see that there is a break in the graph at a point when $x = 3$.

Now $f(3) = 2(3) + 1 = 7$

$\Rightarrow$ Condition (i) is satisfied.

$\displaystyle \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x - 1) = 3 - 1 = 2$

$\displaystyle \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (2x + 1) = 6 + 1 = 7$

$\displaystyle \lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$

i.e., condition (ii) is not satisfied.

$\therefore$ $\displaystyle \lim_{x \to 3} f(x)$ does not exist.
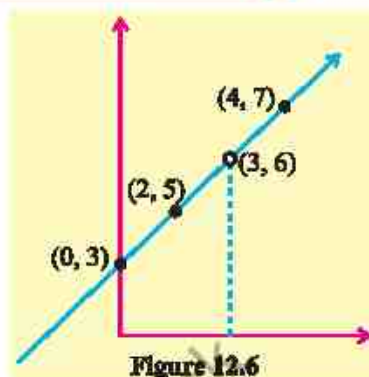
Hence, $f(x)$ is not continuous at $x = 3$

Figure 12.6
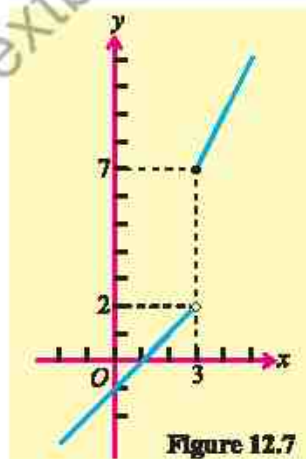
Figure 12.7

EXERCISE 12.2

1. Determine the left hand limit and the right hand limit and then, find limit of the following functions when $x \to c$.

    (i) $f(x) = 2x^2 + x - 5$, $c = 1$ (ii) $f(x) = \dfrac{x^2 - 9}{x - 3}$, $c = -3$

    (iii) $f(x) = |x - 5|$, $c = 5$

2. Discuss the continuity of $f(x)$ at $x = c$

    (i) $f(x) = \begin{cases} 2x + 5 & \text{if } x \leq 2 \\ 4x + 1 & \text{if } x > 2 \end{cases}$, $c = 2$ (ii) $f(x) = \begin{cases} 3x - 1 & \text{if } x < 1 \\ 4 & \text{if } x = 1 \\ 2x & \text{if } x > 1 \end{cases}$, $c = 1$

3. If $f(x) = \begin{cases} 3x & \text{if} & x \leq -2 \\ x^2 - 1 & \text{if} & -2 < x < 2 \\ 3 & \text{if} & x \geq 2 \end{cases}$ Discuss continuity at $x = 2$ and $x = -2$.

4. If $f(x) = \begin{cases} x+2 & x \le -1 \\ c+2 & x > -1 \end{cases}$

   find "c" so that $\underset{x \to -1}{\text{Lim}} f(x)$ exists.

5. Find the values of m and n, so that given function f is continuous at $x = 3$

   (i) $f(x) = \begin{cases} mx & \text{if } x < 3 \\ n & \text{if } x = 3 \\ -2x+9 & \text{if } x > 3 \end{cases}$ (ii) $f(x) = \begin{cases} mx & \text{if } x < 3 \\ x^2 & \text{if } x \ge 3 \end{cases}$

6. $f(x) = \begin{cases} \dfrac{\sqrt{2x+5} - \sqrt{x+7}}{x-2} & , \quad x \ne 2 \\ k & , \quad x = 2 \end{cases}$

   Find value of k so that f is continuous $x = 2$.

7. Given the function $f(x) = \begin{cases} 2x+3, & x \le 1 \\ -x+4, & x > 1 \end{cases}$

   Discuss the limit and continuity at $x = 1$.

12.4 Application of Transcendental Functions to Limits and Continuity on Real World Problems

Limit and continuity of transcendental functions are fundamental concepts in calculus with numerous real-world applications.

These concepts help us model, analyze and solve problems in various fields such as growth and decay, finance, economics, surveying and predicting long-term stock prices.

Example 14 Growth and Decay (Radioactive Decay)

The radioactive decay of a substance is given by the function $A(t) = A_0 e^{-kt}$, where A_0 is the initial amount of the substance, k is the decay constant, and t is the time in years. Find the limit of the amount of substance as $t \to \infty$.

Solution

We need to compute the limit: $\underset{t \to \infty}{\text{Lim}} A(t) = \underset{t \to \infty}{\text{Lim}} A_0 e^{-kt}$

As $t \to \infty, e^{-kt} \to 0$, so $\underset{t \to \infty}{\text{Lim}} A_0 e^{-kt} = A_0 \times 0 = 0$

Thus, the amount of radioactive substance approaches 0 as time increases indefinitely.

Example 15 Finance (Compound Interest)

The value of an investment grows according to the formula for continuous compounding $A(t) = P_0 e^{rt}$, where P_0 is the initial principal, r is the annual interest rate, and t is the time in years. What happens to the value of the investment as $t \to \infty$?

Solution We need to compute the limit: $\text{Lim}_{t \to \infty} A(t) = \text{Lim}_{t \to \infty} P_o e^{rt}$

Since $e^{rt} \to \infty$ as $t \to \infty$ for any positive r, the value of the investment grows without bound:

$$\text{Lim}_{t \to \infty} P_o e^{rt} = \infty$$

Thus, the value of the investment increases indefinitely as time approaches infinity.

Example 16 Economics (Supply and Demand)

In economics, the demand function $D(p)$ decreases as the price p increases. Suppose the demand function is given by $D(p) = \dfrac{100}{p+1}$, where p is the price in dollars. Find the limit of the demand as the price becomes very large, i.e., $\text{Lim}_{p \to \infty} D(p)$.

Solution $\text{Lim}_{p \to \infty} D(p) = \text{Lim}_{p \to \infty} \dfrac{100}{p+1}$

As $p \to \infty$, the denominator becomes very large, so $\text{Lim}_{p \to \infty} \dfrac{100}{p+1} = 0$

Thus, as the price becomes very large, the demand approaches 0.

Example 17 Astronomy

The apparent brightness $B(d)$ of a star decreases as the distance from Earth increases following the inverse square law $B(d) = \dfrac{L}{d^2}$, where L is the star's luminosity. Find the limit of the brightness as $d \to \infty$.

Solution $\text{Lim}_{d \to \infty} B(d) = \text{Lim}_{d \to \infty} \dfrac{L}{d^2}$

As $d \to \infty$ the denominator becomes very large, so:

$$\text{Lim}_{d \to \infty} \dfrac{L}{d^2} = 0$$

Thus, as the distance increases indefinitely, the apparent brightness of the star approaches 0.

EXERCISE 12.3

1. A substance decays exponentially following the formula $A(t) = A_o e^{-0.1t}$, where A_o is the initial amount. Find the limit of $A(t)$ as $t \to \infty$.

2. A town's population is modeled by $P(t) = \dfrac{100,000}{1 + 9e^{-0.5t}}$. What is the long-term population as $t \to \infty$.

3. A company's weekly sales (in thousands) follow the function $S(t) = \dfrac{500t}{t+10}$. What is the limit of $S(t)$ as $t \to \infty$ and what does it represent?

4. Signal strength $S(d)$ at a distance d from a tower is modeled as $S(d) = \dfrac{1000}{d^2}$.

    (i)   What is the signal strength at $d = 10$?

    (ii) What happens to signal strength as $d \to \infty$?

5. A stock price grows according to the function $P(t) = 50e^{0.05t}$.

    (i)   Find the limit of $P(t)$ as $t \to \infty$.

    (ii) Calculate the price after 10 years.

6. The factory's cost function is given as:

$$C(x) = \begin{cases} 10x + 500 & \text{if} & x \le 100 \\ 12x + 300 & \text{if} & x > 100 \end{cases}$$

Is the cost function continuous at $x = 100$?

7. Inflation is modeled by $I(t) = I_0\, e^{0.03t}$, where I_0 is the initial price index and t is the number of years

    (i)   Find the inflation rate after 5 years if $I_0 = 100$.

    (ii) What is the expected price index after 10 years?

8. The cost to produce x units is:

$$C(x) = \begin{cases} 5x + 20 & \text{if} & x \le 10 \\ 6x + 10 & \text{if} & x > 10 \end{cases}$$

Is the cost function continuous at $x = 10$?