

## **Numerical Integration:**

The process of evaluating a definite integral from a set of tabulated values of  $f(x)$  is called numerical integration. This process when applied to a function of a single variable is known as quadrature. In numerical integration,  $f(x)$  is represented by an interpolation formula and then it is integrated between the given limits. In this way, the quadrature formula is derived for approximate integration of a function defined by a set of numerical values only.

### **Newton-Cotes Methods:**

Let the function  $y = f(x)$  takes values  $y_0, y_1, y_2, \dots, y_n$  for  $x_0, x_1, x_2, \dots, x_n$  respectively. Let  $I = \int_a^b f(x)dx$ . Dividing the interval  $(a, b)$  into  $n$  sub-intervals of width  $h$  such that  $x_0 = a, x_1 = x_0 + h, x_2 = x_0 + 2h, \dots, x_n = x_0 + nh = b$

$$\therefore I = \int_a^b f(x)dx = \int_{x_0}^{x_0+nh} f(x)dx$$

Putting  $x = x_0 + rh, dx = hdr$ ,

When  $x = x_0, r = 0$ , When  $x = x_0 + nh, r = n$

$$\int_a^b f(x)dx = h \int_0^n f(x_0 + rh)dr$$

$$= h \int_0^n \left[ y_0 + r\Delta y_0 + \frac{r(r-1)}{2!} \Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!} \Delta^3 y_0 + \dots \right] dr$$

[By Newton's forward interpolation formula]

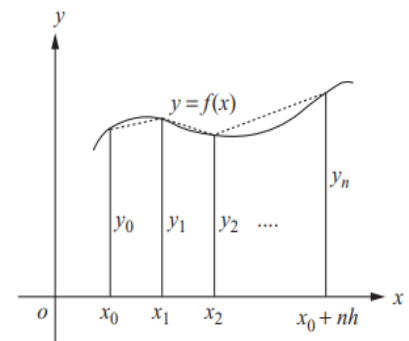
$$= h \left[ ry_0 + \frac{r^2}{2} \Delta y_0 + \left( \frac{\frac{r^3}{3} - \frac{r^2}{2}}{2!} \right) \Delta^2 y_0 + \left( \frac{\frac{r^4}{4} - r^3 + r^2}{3!} \right) \Delta^3 y_0 + \dots \right]_0^n$$

$$\therefore \int_{x_0}^{x_0+nh} f(x)dx = hn \left[ y_0 + \frac{n}{2} \Delta y_0 + \frac{n(2n-3)}{12} \Delta^2 y_0 + \frac{n(n-2)^2}{24} \Delta^3 y_0 + \dots \right]$$

This equation is known as the Newton-Cotes quadrature formula.

### **1) Trapezoidal Rule:**

By the Newton-Cotes quadrature formula,



$$\int_{x_0}^{x_0+nh} f(x)dx = hn \left[ y_0 + \frac{n}{2} \Delta y_0 + \frac{n(2n-3)}{12} \Delta^2 y_0 + \frac{n(n-2)^2}{24} \Delta^3 y_0 + \dots \right]$$

Putting  $n = 1$  in above equation and ignoring the differences of order higher than one.

$$\therefore \int_{x_0}^{x_0+h} f(x)dx = h \left[ y_0 + \frac{1}{2} \Delta y_0 \right] = h \left[ y_0 + \frac{1}{2} (y_1 - y_0) \right] = \frac{h}{2} (y_0 + y_1)$$

Similarly,

$$\begin{aligned} \int_{x_0+h}^{x_0+2h} f(x)dx &= \frac{h}{2} (y_1 + y_2) \\ &\vdots \\ \int_{x_0+(n-1)h}^{x_0+nh} f(x)dx &= \frac{h}{2} (y_{n-1} + y_n) \end{aligned}$$

Adding all these integrals,

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{2} [(y_0 + y_n) + 2(y_1 + y_2 + \dots + y_{n-1})] = \frac{h}{2} [X + 2R]$$

Where  $X$  = extreme terms,  $R$  = remaining terms

This is known as the trapezoidal rule.

**Example-1:** Evaluate  $\int_0^1 e^x dx$ , with  $n = 10$  using the trapezoidal rule.

**Solution:**  $a = 0, b = 1, n = 10, h = \frac{b-a}{n} = \frac{1-0}{10} = 0.1, y = f(x) = e^x$

$x$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$f(x)$	1	1.1052	1.2214	1.3499	1.4918	1.6487	1.8221	2.0138	2.2255	2.4596	2.7183
	$y_0$	$y_1$	$y_2$	$y_3$	$y_4$	$y_5$	$y_6$	$y_7$	$y_8$	$y_9$	$y_{10}$

By the trapezoidal rule,

$$\begin{aligned} \int_0^1 e^x dx &= \frac{h}{2} [(y_0 + y_{10}) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9)] \\ &= \frac{0.1}{2} [(1 + 2.7183) + 2(1.1052 + 1.2214 + 1.3499 + 1.4918 + 1.6487 \\ &\quad + 1.8221 + 2.0138 + 2.2255 + 2.4596)] \\ &= 1.7196 \end{aligned}$$

**Example-2:** Given the data a below, find the isothermal work done on the gas if it is compressed from

$v_1 = 22 \text{ L}$  to  $v_2 = 2 \text{ L}$ . Use  $W = - \int_{v_1}^{v_2} p dv$ . (**Winter 22-23**)

$v, \text{ L}$	2	7	12	17	22
$P, \text{ atm}$	12.20	3.49	2.049	1.44	1.11

**Solution:**  $v_1 = 22, v_2 = 2, h = 5$

By the trapezoidal rule,  $W = -\int_{v_1}^{v_2} p dv = -\int_{22}^2 p dv = \int_2^{22} p dv$

$$= \frac{h}{2} [(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= \frac{5}{2} [(12.20 + 1.11) + 2(3.49 + 2.049 + 1.44)]$$

$$= 68.17$$

## 2) Simpson's $\frac{1}{3}$ Rule:

By the Newton–Cotes quadrature formula,

$$\int_{x_0}^{x_0+nh} f(x) dx = hn \left[ y_0 + \frac{n}{2} \Delta y_0 + \frac{n(2n-3)}{12} \Delta^2 y_0 + \frac{n(n-2)^2}{24} \Delta^3 y_0 + \dots \right]$$

Putting  $n = 2$  in above equation and ignoring the differences of order higher than 2,

$$\int_{x_0}^{x_0+2h} f(x) dx = 2h \left[ y_0 + \Delta y_0 + \frac{1}{6} \Delta^2 y_0 \right]$$

$$= 2h \left[ y_0 + (y_1 - y_0) + \left( \frac{y_2 - 2y_1 + y_0}{6} \right) \right]$$

$$= \frac{h}{3} (y_0 + 4y_1 + y_2)$$

Similarly,

$$\int_{x_0+2h}^{x_0+4h} f(x) dx = \frac{h}{3} (y_2 + 4y_3 + y_4)$$

$$\vdots$$

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x) dx = \frac{h}{3} (y_{n-2} + 4y_{n-1} + y_n)$$

Adding all these integrals,

$$\int_{x_0}^{x_0+nh} f(x) dx = \frac{h}{3} [(y_0 + y_n) + 4(y_1 + y_3 + \dots + y_{n-1}) + 2(y_2 + y_4 + \dots + y_{n-2})]$$

$$= \frac{h}{3} [X + 4O + 2E]$$

Where  $X$  = extreme terms,  $O$  = odd terms,  $E$  = even terms

This is known as Simpson's 1/3 rule.

**Note:** To apply this rule, the number of sub-intervals must be a multiple of 2.

**Example-1:** Consider the following values:

$x$	10	11	12	13	14	15	16
$y$	1.02	0.94	0.89	0.79	0.71	0.62	0.55

Find  $\int_{10}^{16} y dx$  by Simpson's 1/3 rule.

**Solution:**  $a = 10, b = 16, h = 1$

By Simpson's 1/3 rule,

$$\begin{aligned}\int_{10}^{16} y dx &= \frac{h}{3} [(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)] \\ &= \frac{1}{3} [(1.02 + 0.55) + 4(0.94 + 0.79 + 0.62) + 2(0.89 + 0.71)] \\ &= 4.7233\end{aligned}$$

**Example-2:** A rocket is launched from the ground. Its acceleration is registered during the first 80 seconds and is given as follows:

$t(s)$	0	10	20	30	40	50	60	70	80
$a(m/s^2)$	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67

By Simpson's 1/3 rule, find the velocity at  $t = 80$  s.

**Solution:**  $a = 0, b = 80, h = 10$

By Simpson's 1/3 rule,

$$\begin{aligned}\text{Velocity} &= \int_0^{80} a dt = \frac{h}{3} [(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)] \\ &= \frac{10}{3} [(30 + 50.67) + 4(31.63 + 35.47 + 40.33 + 46.69) + 2(33.34 + 37.75 + 43.25)] = 3086.1 \text{ m/s}\end{aligned}$$

### 3) Simpson's $\frac{3}{8}$ Rule:

By the Newton-Cotes quadrature formula,

$$\therefore \int_{x_0}^{x_0+nh} f(x) dx = hn \left[ y_0 + \frac{n}{2} \Delta y_0 + \frac{n(2n-3)}{12} \Delta^2 y_0 + \frac{n(n-2)^2}{24} \Delta^3 y_0 + \dots \right]$$

Putting  $n = 3$  in above formula and ignoring the differences of order higher than three,

$$\begin{aligned}\therefore \int_{x_0}^{x_0+3h} f(x) dx &= 3h \left[ y_0 + \frac{3}{2} \Delta y_0 + \frac{3}{4} \Delta^2 y_0 + \frac{1}{8} \Delta^3 y_0 \right] \\ &= \frac{3h}{8} (y_0 + 3y_1 + 3y_2 + y_3)\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{x_0+3h}^{x_0+6h} f(x) dx &= \frac{3h}{8} (y_3 + 3y_4 + 3y_5 + y_6) \\ &\vdots \\ &\vdots \\ &\vdots \\ \int_{x_0+(n-3)h}^{x_0+nh} f(x) dx &= \frac{3h}{8} (y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)\end{aligned}$$

Adding all these integrals,

$$\therefore \int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{8} [(y_0 + y_n) + 2(y_3 + y_6 \dots + y_{n-3}) + 3(y_1 + y_2 + y_4 + y_5 \dots + y_{n-1})]$$

$$= \frac{3h}{8} [X + 2T + 3R]$$

Where  $X$  = extreme terms,  $T$  = multiple of three terms,  $R$  = remaining terms

This is known as Simpson's  $\frac{3}{8}$  Rule.

**Note:** To apply this rule, the number of sub-intervals must be a multiple of 3.

**Example 1:** Evaluate  $\int_0^3 \frac{1}{1+x} dx$ , with  $n = 6$  by using Simpson's  $\frac{3}{8}$  rule and hence calculate  $\log 2$ .  
(Winter 22-23)

**Solution:**  $a = 0, b = 3, n = 6, h = \frac{b-a}{n} = \frac{3-0}{6} = 0.5, y = f(x) = \frac{1}{1+x}$

$x$	0	0.5	1.0	1.5	2.0	2.5	3.0
$f(x)$	1	0.6667	0.5	0.4	0.3333	0.2857	0.25

By Simpson's  $\frac{3}{8}$  rule,

$$\int_0^3 \frac{1}{1+x} dx = \frac{3h}{8} [(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)]$$

$$= \frac{3(0.5)}{8} [(1 + 0.25) + 2(0.4) + 3(0.6667 + 0.5 + 0.3333 + 0.2875)]$$

$$= 1.3888 \rightarrow (1)$$

By direct integration,  $\int_0^3 \frac{1}{1+x} dx = |\log(1+x)|_0^3 = \log 4 = \log(2)^2 = 2 \log 2 \rightarrow (2)$

From Eqs (1) and (2),  $2 \log 2 = 1.3888 \therefore \log 2 = 0.6944$

**Example 2:** Find the volume of a solid of revolution formed by rotating about the  $x$ -axis the area bounded by the lines  $x = 0, x = 1.5, y = 0$  and the curve passing through the following points:

$x$	0.00	0.25	0.50	0.75	1.00	1.25	1.50
$y$	1.00	0.9826	0.9589	0.9089	0.8415	0.7624	0.7589

**Solution:** Volume is given by  $V = \int \pi y^2 dx, h = 0.5$

$x$	0.00	0.25	0.50	0.75	1.00	1.25	1.50
$y^2$	1.00	0.9826	0.9589	0.9089	0.8415	0.7624	0.7589

By Simpson's  $\frac{3}{8}$  rule,

$$\therefore \int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{8} [(y_0 + y_n) + 2(y_3 + y_6 \dots + y_{n-3}) + 3(y_1 + y_2 + y_4 + y_5 \dots + y_{n-1})]$$

$$\therefore \int y^2 dx = \frac{3h}{8} [(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)]$$

$$= \frac{3(0.25)}{8} [(1.00 + 0.5759) + 2(0.8261) + 3(0.9655)] = 1.1954$$

$$\text{Volume} = \pi \int y^2 dx = \pi(1.1954) = 3.7555$$

## Exercise

- 1) Use Trapezoidal rule to compute  $\int_1^2 \frac{1}{x} dx$  using three subintervals. **(Winter 23-24)**
- 2) Evaluate  $\int_0^6 \frac{dx}{1+x^2}$  by using (1) Trapezoidal rule (2) Simpson's  $\frac{1}{3}$  Rule (3) Simpson's 3/8 rule.
- 3) A river is 80 meters wide. The depth 'd' in meters at a distance x meter from one bank is given by the following table. Calculate the area of cross-section of the river using Simpson's  $\frac{1}{3}$  Rule.

x	0	10	20	30	40	50	60	70	80
d	0	4	7	9	12	15	14	8	3

### Error formulae for Trapezoidal and Simpson's formulas:

#### 1. Error in Trapezoidal rule:

$$E = -\frac{(b-a)h^2}{12} y''(\bar{x})$$

#### 2. Error in Simpson's $\frac{1}{3}$ rule:

$$E = -\frac{(b-a)h^4}{180} y^{iv}(\bar{x})$$

#### 3. Error in Simpson's $\frac{3}{8}$ rule:

$$E = -\frac{3h^5}{80} y^{iv}(\bar{x})$$

### Gaussian Quadrature Formulae:

In Newton-Cotes quadrature formula, we require values at equally spaced points of the interval. The Gaussian quadrature formula uses the values of  $x$  which are not all equidistant points. Consider

$I = \int_a^b f(t)dt$ . The substitution  $t = \frac{1}{2}[(b-a)x + (b+a)]$  transforms the above integral into  $\int_{-1}^1 f(x)dx$ .

An n-point Gaussian quadrature formula is a quadrature formula constructed to give an exact result for polynomials of degree  $2n-1$  or less by a suitable choice of the point  $x_i$  and weights  $w_i$  for  $i = 1, 2, 3, \dots, n$ .

Gauss quadrature formula can be expressed as  $\int_{-1}^1 f(x)dx = \sum_{i=1}^n w_i f(x_i)$

#### 1) One-point Gaussian Quadrature Formula:

Consider a function  $f(x)$  over the interval  $[-1, 1]$  with sampling point  $x_1$  and weight  $w_1$ .

$\int_{-1}^1 f(x)dx = 2f(0)$ . This is known as one-point Gaussian quadrature formula. This formula is exact for polynomials up to degree one.

#### 2) Two-point Gaussian Quadrature Formula:

Consider a function  $f(x)$  over the interval  $[-1, 1]$  with sampling point  $x_1, x_2$  and weight  $w_1, w_2$  respectively.

$\int_{-1}^1 f(x)dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$ . This is known as two-point Gaussian quadrature formula. This formula is exact for polynomials up to degree three.

### 3) Three-point Gaussian Quadrature Formula:

Consider a function  $f(x)$  over the interval  $[-1, 1]$  with sampling point  $x_1, x_2, x_3$  and weight  $w_1, w_2, w_3$

respectively.  $\int_{-1}^1 f(x) dx = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$

This is known as three-point Gaussian quadrature formula. This formula is exact for polynomials up to degree five.

**Example-1:** Evaluate  $\int_{-1}^1 \frac{dx}{1+x^2}$  by one-point, two-point and three-point Gaussian formulae. (Winter 22-23)

**Solution:**  $f(x) = \frac{1}{1+x^2}$

By the one-point Gaussian formula,

$$\begin{aligned}\int_{-1}^1 \frac{1}{1+x^2} dx &= 2f(0) \\ &= 2\left(\frac{1}{1+0}\right) \\ &= 2\end{aligned}$$

By the two-point Gaussian formula,

$$\begin{aligned}\int_{-1}^1 \frac{1}{1+x^2} dx &= f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \\ &= \frac{1}{1+\frac{1}{3}} + \frac{1}{1+\frac{1}{3}} \\ &= 1.5\end{aligned}$$

By the three-point Gaussian formula,

$$\begin{aligned}\int_{-1}^1 \frac{1}{1+x^2} dx &= \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right) \\ &= \frac{5}{9} \left(\frac{1}{1+\frac{3}{5}}\right) + \frac{8}{9} (1) + \frac{5}{9} \left(\frac{1}{1+\frac{3}{5}}\right) \\ &= 1.5833\end{aligned}$$

**Example-2:** Evaluate the integral  $\int_{-2}^6 (1+x^2)^{3/2} dx$  by the Gaussian formula for  $n=3$

**Solution:** Here  $a=-2$ ,  $b=6$ . The original limits of integration have changed to -1 to 1 hence change the variable  $x$  to  $t$  is needed such that  $x = \frac{b-a}{2}t + \frac{b+a}{2} = 4t + 2$ ,  $x = 4t + 2$ ,  $dx = 4dt$

$$\int_{-2}^6 (1+x^2)^{3/2} dx = \int_{-1}^1 (1+(4t+2)^2)^{3/2} 4 dt = \int_{-1}^1 (5+16t+16t^2)^{3/2} 4 dt$$

$$\therefore f(t) = 4(5 + 16t + 16t^2)^{3/2}$$

By the three-point Gaussian formula,

$$\int_{-2}^6 (1 + x^2)^{3/2} dx = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right) = 7.2833 + 39.7523 + 311.6573 = 358.70$$

**Example-3:** Evaluate  $\int_0^1 \frac{dt}{1+t}$  by three-point Gaussian formula

**Solution:** Here, the original limits 0 to 1 of the integration have to be changed to -1 to 1 for this we use the substitution,  $t = \frac{1}{2}[(b-a)x + (b+a)]$  where  $a = 0$  and  $b = 1$

$$\begin{aligned}\therefore t &= \frac{1}{2}[(1-0)x + (1+0)] \\ &= \frac{1}{2}(x+1)\end{aligned}$$

$$\therefore dt = \frac{1}{2} dx \quad \text{and} \quad x = 2t - 1$$

$$t = 0 \Rightarrow x = -1$$

$$t = 1 \Rightarrow x = 1$$

$$\begin{aligned}\text{Now, } I &= \int_0^1 \frac{dt}{1+t} \\ &= \int_{-1}^1 \frac{dx}{3+x}\end{aligned}$$

$$\text{Here, } f(x) = \frac{1}{3+x}$$

By the one-point Gaussian formula,

$$\int_{-1}^1 \frac{dx}{3+x} = 2f(0) = 2\left(\frac{1}{3}\right) = 0.6667$$

By the two-point Gaussian formula,

$$\int_{-1}^1 \frac{dx}{3+x} = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = 0.6923$$

By the three-point Gaussian formula,

$$\int_{-1}^1 \frac{dx}{3+x} = \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right) = 0.6931$$

**Exercise:**

1) Using two-point Gaussian quadrature Formula compute  $\int_{-2}^2 e^{\frac{-x}{2}} dx$

2) Using three-point Gaussian Quadrature formula evaluate  $\int_2^4 (1 + x^4) dx$ .



## Numerical Solution of Ordinary Differential Equations: -

The fundamental laws of physics, mechanics, electricity and thermodynamics are usually based on empirical observations that explain variations in physical properties and states of the systems. Rather than describing the state of physical system directly, the laws are usually couched in terms of spatial and temporal changes. The following table gives a few examples of such fundamental laws that are written in terms of the rate of change of variables ( $t$  = time and  $x$  = position)

Physical Law	Mathematical Expression	Variables and Parameters
Newton's second law of motion	$\frac{dv}{dt} = \frac{F}{m}$	Velocity( $v$ ), force ( $F$ ) and mass ( $M$ )
Fourier's Law of Heat Conduction	$q = -k \frac{dT}{dx}$	Heat flux ( $q$ ), thermal conductivity ( $k$ ) and temperature ( $T$ )
Faraday's Law (Voltage drop across an inductor)	$\Delta V_L = L \frac{di}{dt}$	Voltage drops ( $\Delta V_L$ ), inductance( $L$ ) and current ( $i$ )

The above laws define mechanism of change. When combined with continuity laws for energy, mass or momentum, differential equation arises. The mathematical expression in the above table is an example of the Conversion of a Fundamental law to an Ordinary Differential Equation. Subsequent integration of these differential equations results in mathematical functions that describe the spatial and temporal state of a system in terms of energy, mass or velocity variations. In fact, such mathematical relationships are the basis of the solution for a great number of engineering problems. But many ordinary differential equations arising in real-world applications and having lot of practical significance cannot be solved exactly using the classical analytical methods. These odes can be analyzed qualitatively. However, qualitative analysis may not be able to give accurate answers. A numerical method can be used to get an accurate approximate solution to a differential equation. There are many programs and packages available for solving these differential equations. With today's computer, an accurate solution can be obtained rapidly. In this chapter we focus on basic numerical methods for solving initial value problems.

Analytical methods, when available, generally enable to find the value of  $y$  for all values of  $x$ . Numerical methods, on the other hand, lead to the values of  $y$  corresponding only to some finite set of values of  $x$ . That is the solution is obtained as a table of values, rather than as continuous function. Moreover, analytical solution, if it can be found, is exact, whereas a numerical solution inevitably involves an error which should be small but may, if it is not controlled, swamp the true solution. Therefore, we must be concerned with two aspects of numerical solutions of ODEs: both the method itself and its accuracy.

The general form of first order differential equation, in implicit form, is  $F(x, y, y')=0$  and in the explicit form is  $\frac{dy}{dx} = f(x, y)$ . An Initial Value Problem (IVP) consists of a differential equation and a condition which the solution must satisfy (or several conditions referring to the same value of  $x$  if the differential equation is of higher order). In this chapter we shall consider IVPs of the form

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0. \quad (1)$$

Assuming  $f$  to be such that the problem has a unique solution in some interval containing  $x_0$ , we shall discuss the methods for computing numerical values of the solution. These methods are step-by-step methods. That is, we start from  $y_0 = y(x_0)$  and proceed stepwise. In the first step, we compute an approximate value  $y_1$  of the solution  $y$  of (1) at  $x = x_1 = x_0 + h$ . In the second step we compute an approximate value  $y_2$  of the solution  $y$  at  $x = x_2 = x_0 + 2h$ , etc. Here  $h$  is fixed number for example 0.1 or 0.001 or 0.5 depends on the requirement of the problem. In each step the computations are done by the same formula. The following methods are used to solve the IVP (1).

1. Taylor's Series Method
2. Euler and Modified Euler Method
3. Runge – Kutta Method

## 1. Taylor's Series Method:

Consider an IVP  $\frac{dy}{dx} = f(x, y)$ ,  $y(x_0) = y_0$ . Let us approximate the exact solution  $y(x)$  to a power series in  $(x - x_0)$  using Taylor's series. The Taylor's series expansion of  $y(x)$  about the point  $x = x_0$  is

$$y(x) = y(x_0) + \frac{(x - x_0)}{1!} y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \frac{(x - x_0)^4}{4!} y^{iv}(x_0) + \dots \quad (1)$$

From the differential equation, we have  $y'(x) = \frac{dy}{dx} = f(x, y)$ . Differentiating this successively, we can get  $y''(x), y'''(x), y^{iv}(x)$ , etc. Putting  $x = x_0$  and  $y = y_0$ , the values of  $y''(x_0), y'''(x_0), y^{iv}(x_0)$ , etc. can be obtained. Hence the Taylor's series (1) gives the values of  $y$  for every value of  $x$  for which (1) converges. On finding the value of  $y_1$  for  $x = x_1$  from (1),  $y''(x), y'''(x), y^{iv}(x)$ , etc. can be evaluated at  $x = x_1$  from the differential equation

**Example-1:** Find by Taylor's series method the value of  $y$  at  $x = 0.1$  and  $0.2$  five places of decimals for the IVP  $\frac{dy}{dx} = x^2 y - 1$ ,  $y(0) = 1$

**Solution:** Given  $x_0 = 0$ ,  $y_0 = 1$  and  $f(x, y) = x^2 y - 1$

Taylor's series expansion about the point  $x = 0 (= x_0)$  is

$$y(x) = y(0) + \frac{(x-0)}{1!} y'(0) + \frac{(x-0)^2}{2!} y''(0) + \frac{(x-0)^3}{3!} y'''(0) + \frac{(x-0)^4}{4!} y^{iv}(0) + \dots$$

$$\text{i.e. } y(x) = y(0) + xy'(0) + \frac{x^2}{2} y''(0) + \frac{x^3}{6} y'''(0) + \frac{x^4}{24} y^{iv}(0) + \dots \quad (1)$$

It is given that  $y(0) = 1$

$$\frac{dy}{dx} = y'(x) = x^2 y - 1 \quad \Rightarrow \Rightarrow \Rightarrow \quad y'(0) = -1$$

Differentiating  $y'(x) = x^2 y - 1$  successively three times and putting  $x = 0$  &  $y = 1$ , we get

$$y''(0) = 0, y'''(0) = 2, y^{iv}(0) = -6$$

Putting the values of  $y(0), y'(0), y''(0), y'''(0), y^{iv}(0)$  in equation (1), we get

$$y(x) = 1 + x(-1) + \frac{x^2}{2}(0) + \frac{x^3}{6}(2) + \frac{x^4}{24}(-6) \Rightarrow y(x) = 1 - x + \frac{x^3}{3} - \frac{x^4}{4}$$

Hence,  $y(0.1) = 0.90033$  and  $y(0.2) = 0.80227$ .

**Example-2:** Using Taylor's series method solve  $\frac{dy}{dx} = x^2 - y$ ,  $y(0) = 1$  at  $0.1 \leq x \leq 0.4$ . Compare the values with the exact solution.

**Solution:** Given  $x_0 = 0$ ,  $y_0 = 1$  and  $f(x, y) = x^2 - y$

Taylor's series expansion about the point  $x = 0$  is  $y(x) = y(0) + xy'(0) + \frac{x^2}{2} y''(0) + \frac{x^3}{6} y'''(0) + \frac{x^4}{24} y^{iv}(0) + \dots$

It is given that  $y(0) = 1$

$$\frac{dy}{dx} = y'(x) = x^2 - y \quad \Rightarrow \quad y'(0) = (0)^2 - 1 = -1$$

$$y''(x) = 2x - y' \quad \Rightarrow \quad y''(0) = 0 - y'(0) = 0 - (-1) = 1$$

$$y'''(x) = 2 - y'' \quad \Rightarrow \quad y'''(0) = 2 - y''(0) = 1$$

$$y^{iv}(x) = -y''' \quad \Rightarrow \quad y^{iv}(0) = -y'''(0) = -1$$

Putting the values of  $y(0), y'(0), y''(0), y'''(0), y^{iv}(0)$  in (4), we get  $y(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24}$

$$\text{Hence, } y(0.1) = 1 - (0.1) + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{6} - \frac{(0.1)^4}{24} = 0.9051625$$

$$y(0.2) = 1 - (0.2) + \frac{(0.2)^2}{2} + \frac{(0.2)^3}{6} - \frac{(0.2)^4}{24} = 0.8212667$$

$$y(0.3) = 1 - (0.3) + \frac{(0.3)^2}{2} + \frac{(0.3)^3}{6} - \frac{(0.3)^4}{24} = 0.7491625$$

$$y(0.4) = 1 - (0.4) + \frac{(0.4)^2}{2} + \frac{(0.4)^3}{6} - \frac{(0.4)^4}{24} = 0.6896.$$

### Exercise:

1) Use the Taylor's series method to solve  $\frac{dy}{dx} = 1 + y^2, y(0) = 0$ . Also find  $y(0.2)$ .

2) Using the Taylor's series method, find correct up to four decimal places, the value of  $y(0.1)$ , given  $\frac{dy}{dx} = x^2 + y^2$  and  $y(0) = 1$

## 2. Euler's Method:

Consider the IVP  $\frac{dy}{dx} = f(x, y)$ ,  $y(x_0) = y_0$ . The following two

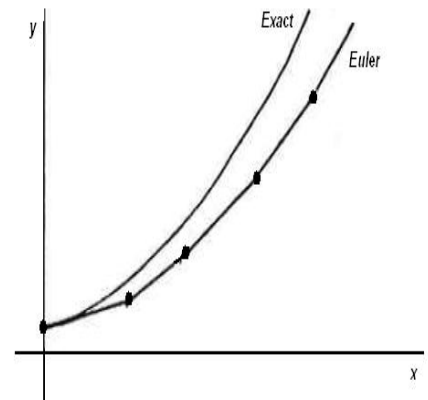
methods can be used to determine the solution at a point  $x = x_n = x_0 + nh$ . The solution of the differential equation is represented by the curve as shown in the fig. The point  $(x_0, y_0)$  lies on the curve. The equation of tangent to the curve at the point is

$$\text{given by } y - y_0 = \left( \frac{dy}{dx} \Big|_{x=x_0} \right) (x - x_0) = f(x_0, y_0)(x - x_0)$$

$$y = y_0 + f(x_0, y_0)(x - x_0)$$

But we know that  $h = x_n - x_{n-1}$ ,

$$\text{Hence } y_{n+1} = y_n + h[f(x_n, y_n)], \quad n = 0, 1, 2, 3, \dots$$



Euler's Method:

$$y_{n+1} = y_n + h[f(x_n, y_n)], \quad n = 0, 1, 2, 3, \dots$$

Modified Euler's Method:

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})], \quad n = 0, 1, 2, 3, \dots$$

**Remark:** When Modified Euler's method is applied to find the solution at a given point, we first find the solution at that point by using Euler's method and the same will be used in the calculation of Modified Euler's method. Also Modified Euler's method has to be applied repeatedly until the solution is stationary.

**Example-1:** Using Euler's method, find  $y(0.2)$  given  $\frac{dy}{dx} = y - \frac{2x}{y}$ ,  $y(0) = 1$  with  $h=0.1$ .

**Solution:**  $\frac{dy}{dx} = y - \frac{2x}{y}$ ,  $x_0 = 0, y_0 = 1, h = 0.1, x = 0.2, n = \frac{x-x_0}{h} = 2$

for  $x_1 = 0.1$

$$y_1 = y_0 + h[f(x_0, y_0)] = 1 + 0.1f(0, 1) = 1.1,$$

$$y_2 = y_1 + h[f(x_1, y_1)] = 1.1 + 0.1f(0.1, 1.1) = 1.1918$$

Hence,  $y_2 = y(0.2) = 1.1918$

**Example-2:** Solve the initial-value problem  $\frac{dy}{dx} = x\sqrt{y}$ ,  $y(1) = 1$  and hence, find  $y(1.5)$  by taking  $h = 0.1$  using Euler's method.

**Solution:** Here,  $\frac{dy}{dx} = f(x, y) = x\sqrt{y}$ ,  $x_0 = 1, y_0 = 1, h = 0.1, x = 1.5, n = \frac{x-x_0}{h} = 5$

$$x_1 = 1.1, x_2 = 1.2, x_3 = 1.3, x_4 = 1.4$$

$$y_1 = y_0 + h[f(x_0, y_0)] = 1 + 0.1f(1, 1) = 1.1,$$

$$y_2 = y_1 + h[f(x_1, y_1)] = 1.1 + 0.1f(1.1, 1.1) = 1.2154$$

$$y_3 = y_2 + h[f(x_2, y_2)] = 1.2154 + 0.1f(1.2, 1.2154) = 1.3477$$

$$y_4 = y_3 + h[f(x_3, y_3)] = 1.3477 + 0.1f(1.3, 1.3477) = 1.4986$$

$$y_5 = y_4 + h[f(x_4, y_4)] = 1.4986 + 0.1f(1.4, 1.4986) = 1.67$$

Hence,  $y_5 = y(1.5) = 1.67$

**Example-3:** Determine the value of  $y$  when  $x = 0.1$  correct up to four decimal places by taking  $h=0.05$ .

Given that  $y(0) = 1$  and  $\frac{dy}{dx} = x^2 + y$  using modified Euler's method.

**Solution:**  $\frac{dy}{dx} = f(x, y) = x^2 + y$

(i) Given:  $x_0 = 0, y_0 = 1, h = 0.05, x_1 = 0.05$

$$f(x_0, y_0) = 0 + 1 = 1$$

$$y_1^0 = y_0 + hf(x_0, y_0) = 1 + 0.05(1) = 1.05$$

First approximation to  $y_1$

$$\begin{aligned}
y_1^1 &= y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^0)] \\
&= 1 + \frac{0.05}{2} [1 + f(0.05, 1.05)] \\
&= 1.0513
\end{aligned}$$

Second approximation to  $y_1$

$$\begin{aligned}
y_1^2 &= y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^1)] \\
&= 1 + \frac{0.05}{2} [1 + f(0.05, 1.0513)] \\
&= 1.0513
\end{aligned}$$

Since the values of  $y_1^1$  and  $y_1^2$  are equal.

$$y_1 = y(0.05) = 1.0513$$

(ii) Now,  $x_1 = 0.05$ ,  $y_1 = 1.0513$ ,  $h = 0.05$ ,  $x_2 = 0.1$

$$f(x_1, y_1) = f(0.05, 1.0513) = 1.0538$$

$$y_2^0 = y_1 + hf(x_1, y_1) = 1.0513 + 0.05(1.0538) = 1.1040$$

First approximation to  $y_2$

$$\begin{aligned}
y_2^1 &= y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^0)] \\
&= 1.0513 + \frac{0.05}{2} [1.0538 + f(0.1, 1.1040)] \\
&= 1.1055
\end{aligned}$$

Second approximation to  $y_2$

$$\begin{aligned}
y_2^2 &= y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^1)] \\
&= 1.0513 + \frac{0.05}{2} [1.0538 + f(0.1, 1.1055)] \\
&= 1.1055
\end{aligned}$$

Since the values of  $y_2^1$  and  $y_2^2$  are equal.

$$y_2 = y(0.1) = 1.1055$$

### Exercise:

- 1) Apply the modified Euler's method to solve the initial value problem  $\frac{dy}{dx} = x + y$  with  $y(0)=0$  choosing  $h=0.2$  and compute  $y$  for  $x=0.2, x=0.4$
- 2) Using Euler's method, find the approximate value of  $y$  at  $x=1$  taking  $h=0.2$ . Given  $\frac{dy}{dx} = x^2 + y^2$  and  $y(0) = 1$ .

### 3. Runge – Kutta Method:

The Taylor's series method to solve IVPs is restricted by the difficulty in finding the higher order derivatives. However, Runge – Kutta method do not require the calculations of higher order derivatives. Euler's method and modified Euler's method are Runge – Kutta methods of first and second order

respectively. Consider the IVP  $\frac{dy}{dx} = f(x, y)$ ,  $y(x_0) = y_0$ . Let us find the approximate value of  $y$  at  $x = x_{n+1}$ ,  $n = 0, 1, 2, 3, \dots$  of this numerically, using Runge – Kutta method, as follows:

#### Second Order method:

First let us calculate the quantities  $k_1$  and  $k_2$  using the following formulae.

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf(x_n + h, y_n + k_1)$$

$$k = \frac{1}{2}(k_1 + k_2)$$

Finally, the required solution  $y$  is given by  $y_{n+1} = y_n + k$

#### Fourth Order method:

First let us calculate the quantities  $k_1, k_2, k_3$  and  $k_4$  using the following formulae.

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

Finally, the required solution  $y$  is given by  $y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

**Example-1:** Given that  $y = 1.3$  when  $x = 1$  and  $\frac{dy}{dx} = 3x + y$ . Use the second order R-K method to approximate  $y$  when  $x = 1.2$ . Use a step size of 0.1. **(Winter 22-23)**

**Solution:**  $\frac{dy}{dx} = f(x, y) = 3x + y$ .

**(i)** Given:  $x_0 = 1, y_0 = 1.3, h = 0.1$  and  $n = 0$

$$k_1 = hf(x_0, y_0)$$

$$= 0.1f(1, 1.3)$$

$$= 0.43$$

$$k_2 = hf(x_0 + h, y_0 + k_1)$$

$$= 0.1f(1 + 0.1, 1.3 + 0.43)$$

$$= 0.503$$

$$k = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(0.43 + 0.503) = 0.4665$$

$$y_1 = y_0 + k = 1.3 + 0.4665 = 1.7665$$

**(ii)** Now,  $x_1 = 1.1, y_1 = 1.7665, h = 0.1$  and  $n = 1$

$$\begin{aligned}
k_1 &= hf(x_1, y_1) \\
&= 0.1f(1.1, 1.7665) \\
&= 0.5067 \\
k_2 &= hf(x_1 + h, y_1 + k_1) \\
&= 0.1f(1.1 + 0.1, 1.7665 + 0.5067) \\
&= 0.5873 \\
k &= \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(0.5067 + 0.5873) = 0.5470 \\
y_2 &= y_1 + k = 1.7665 + 0.5470 = 2.3135
\end{aligned}$$

Hence,  $y_2 = y(1.2) = 2.3135$

**Example-2:** Apply Runge – Kutta fourth order method, to find an approximate value of y when x = 0.2 given that  $\frac{dy}{dx} = x + y$ ,  $y(0) = 1$ .

**Solution:**  $x_0 = 0, y_0 = 1, h = 0.2$  and  $f(x, y) = x + y$

By R – K method (for n = 0) is:  $y_1 = y(0.2) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$  ----- (1)

Now  $k_1 = hf(x_0, y_0) = 0.2 \times [0 + 1] = 0.2$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2 \times \left[\left(0 + \frac{0.2}{2}\right) + \left(1 + \frac{0.2}{2}\right)\right] = 0.2400$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2 \times \left[\left(0 + \frac{0.2}{2}\right) + \left(1 + \frac{0.24}{2}\right)\right] = 0.2440$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2 \times [(0 + 0.2) + (1 + 0.2440)] = 0.2888$$

Using the values of  $k_1, k_2, k_3$  and  $k_4$  in (1), we get

$$y_1 = y(0.2) = 1 + \frac{1}{6}(0.2 + 0.24 + 0.244 + 0.2888) = 1.2468$$

Hence the required approximate value of y is 1.2468.

**Example-3:** Using Runge – Kutta method of fourth order, solve  $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ ,  $y(0) = 1$  at x = 0.2 & 0.4.

**Solution:** Given:  $x_0 = 0, y_0 = 1, h = 0.2$  and  $f(x, y) = \frac{y^2 - x^2}{y^2 + x^2}$

For  $y_1 = y(0.2)$

By R – K method (for n = 0) is:  $y_1 = y(0.2) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$  ----- (2)

$$k_1 = hf(x_0, y_0) = 0.2 \times f(0,1) = 0.2$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2 \times f(0.1, 1.1) = 0.19672$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2 \times f(0.1, 1.0936) = 0.1967$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2 \times f(0.2, 1.1967) = 0.1891$$

Using the values of  $k_1, k_2, k_3$  and  $k_4$  in (2), we get

$$y_1 = y(0.2) = 1 + \frac{1}{6}(0.2 + 2(0.19672) + 2(0.1967) + 0.1891) = 1 + 0.19599 = 1.19599$$

Hence the required approximate value of y is 1.19599.

For  $y_2 = y(0.4)$

We have  $x_1 = 0.1, y_1 = 1.19599$  and  $h = 0.2$

By R – K method (for n = 1) is:  $y_2 = y(0.4) = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$  ----- (3)

$$k_1 = hf(x_1, y_1) = 0.2 \times f(0.2, 1.19599) = 0.1891$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.2 \times f(0.3, 1.2906) = 0.1795$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.2 \times f(0.3, 1.2858) = 0.1793$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2 \times f(0.4, 1.3753) = 0.1688$$

Using the values of  $k_1, k_2, k_3$  and  $k_4$  in (3), we get

$$y_2 = y(0.4) = 1.19599 + \frac{1}{6}(0.1891 + 2(0.1795) + 2(0.1793) + 0.1688) = 1.19599 + 0.1792 = 1.37519$$

Hence the required approximate value of y is 1.37519.

### Exercise:

1) Solve the differential equation  $\frac{dy}{dx} = x + y$ , with the 4<sup>th</sup> order Runge-Kutta method, where  $y(0)=1$ , with  $x=0$  to  $x=0.2$  with  $h=0.1$  (**Summer 22-23**)

2) Use the second -order Runge-Kutta method to find an approximate value of y given that  $\frac{dy}{dx} = x - y^2$  and  $y(0)=1$  at  $x=0.2$  taking  $h=0.1$