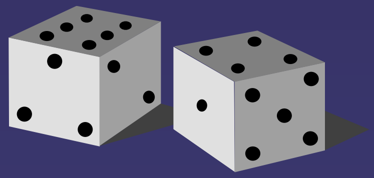
Random Variables And Probability Distributions

DEPARTMENT OF APPLIED SCIENCES
PARUL UNIVERSITY

Topics

- Probability Distributions
- Expected Value and Variance of a RV



Topics

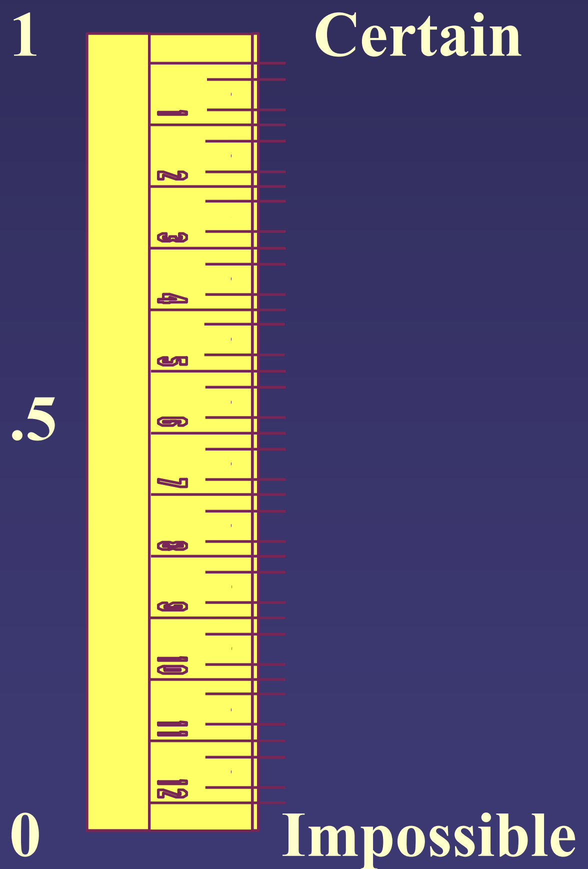
- Discrete Probability Distributions
  - Bernoulli and Binomial Distributions
  - Poisson Distributions
- Continuous Probability Distributions
  - Uniform
  - Normal

Topics

- **Random Sampling and Probability Distributions**
  - **Random Numbers**
  - **Sampling from Probability Distributions**

Probability

- Probability is the likelihood that the event will occur.
- Two Conditions:
  - Value is between 0 and 1.
  - Sum of the probabilities of all events must be 1.

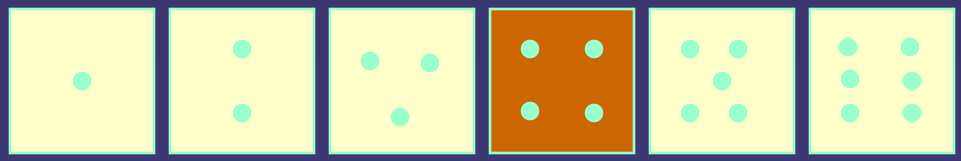


Random Variable

• A numerical description of the outcome of an experiment

Example: **Discrete RV: countable # of outcomes**

Throw a die twice: Count the number of times 4 comes up (0, 1, or 2 times)



Discrete Random Variable

- **Discrete Random Variable:**
  - **Obtained by Counting (0, 1, 2, 3, etc.)**
  - **Usually finite by number of different values**

e.g.

Toss a coin 5 times. Count the number of tails. (0, 1, 2, 3, 4, or 5 times)

Random Variable

- A numerical description of the outcome of an experiment

  Example: Continuous RV:

  - The Value of the DJIA
  - Time to repair a failed machine
- RandomVariable: Given by Capital Letters X & Y.
- Specific Values Given by lower case

Probability Distribution

Characterization of the possible values that a RV may assume along with the probability of assuming these values.

Discrete Probability Distribution

- List of all possible [x_i, $p(x_i)$] pairs

  X_i = value of random variable

  $P(x_i)$ = probability associated with value

- Mutually exclusive (nothing in common)
- Collectively exhaustive (nothing left out)

$$0 \leq p(x_i) \leq 1$$

$$\Sigma P(x_i) = 1$$

Weekly Demand of a Slow-Moving Product

Probability Mass Function

Demand, x	Probability, p(x)
0	0.1
1	0.2
2	0.4
3	0.3
4 or more	0

Weekly Demand of a Slow-Moving Product

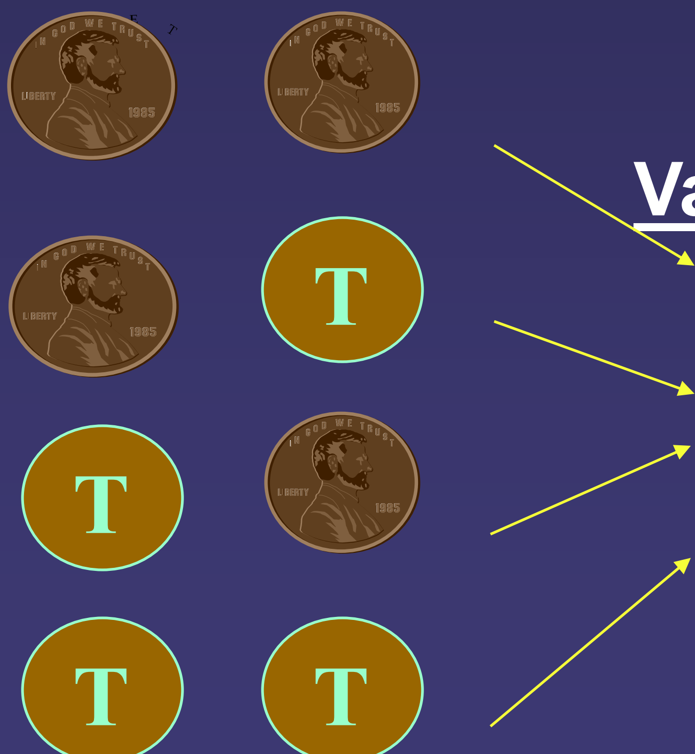
A Cumulative Distribution Function:

Probability that RV assume a value <= a given value, x

Demand, x	Cumulative Probability, P(x)
0	0.1
1	0.3
2	0.7
3	1

Discrete Probability Distribution Example

Event: Toss 2 Coins. Count # Tails.



Probability distribution

Values	probability
0	1/4 = .25
1	2/4 = .50
2	1/4 = .25

Discrete Random Variable Summary Measures

Expected value (The mean)

Weighted average of the probability distribution

$$\mu = E(X) = \sum x_i \, p(x_i)$$

In slow-moving product demand example,

the expected value is :

$$E(X) = 0 \times 0.1 + 1 \times .2 + 2 \times .4 + 3 \times .3 = 1.9$$

The average demand on the long run is 1.9

Discrete Random Variable Summary Measures

Variance

Weighted average squared deviation about mean

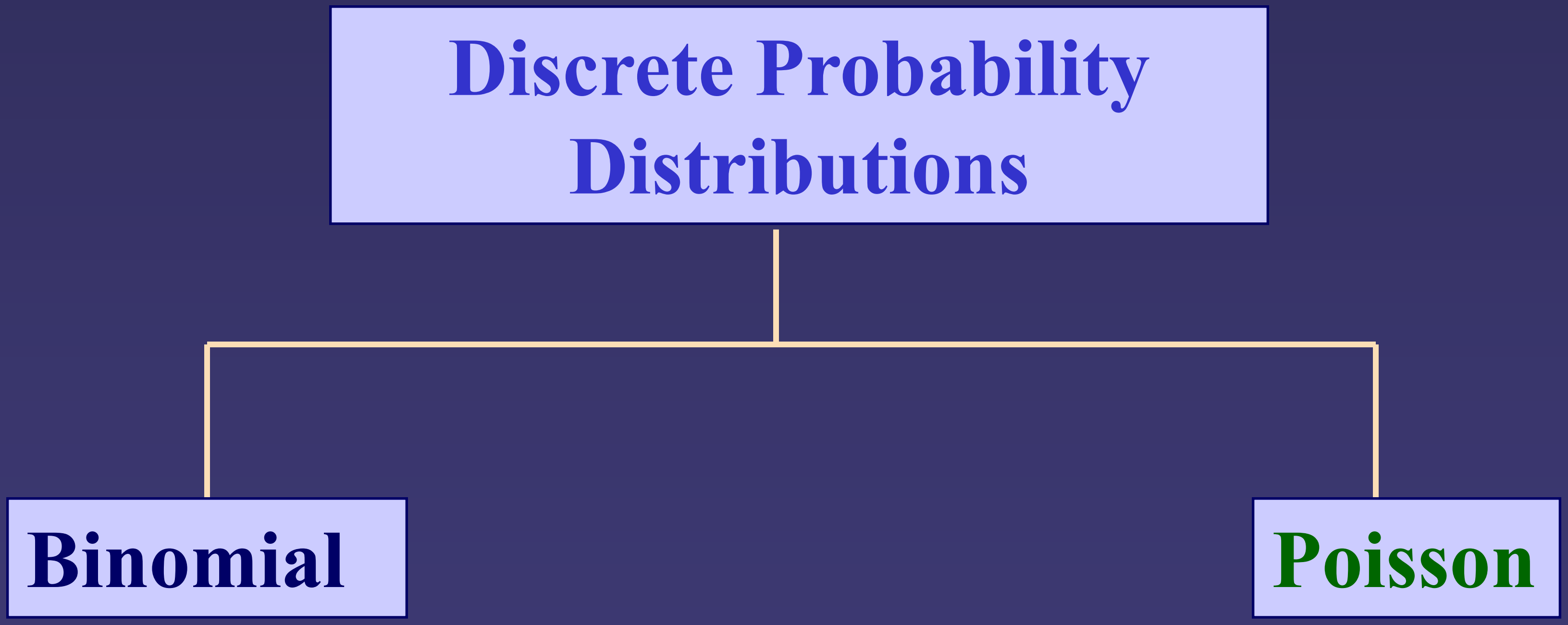
$$\text{Var}[X] = \sigma^2 = E\,[\,(x_i - E(X)\,)^2] = \Sigma\,(x_i - E(X)\,)^2 p(x_i)$$

For the Product demand example, the variance is:

$$\text{Var}[X] = \sigma^2 = (0 - 1.9)^2(.1) + (1 - 1.9)^2(.2) +$$

$$(2 - 1.9)^2(.4) + (3 - 1.9)^2(.3) = .89$$

Important Discrete Probability Distribution Models



Binomial Distribution

- ‘N’ identical trials
  - Example: 15 tosses of a coin, 10 light bulbs taken from a warehouse
- 2 mutually exclusive outcomes on each trial
  - Example: Heads or tails in each toss of a coin, defective or not defective light bulbs

Binomial Distributions

- Constant Probability for each Trial
  - Example: Probability of getting a tail is the same each time we toss the coin and each light bulb has the same probability of being defective
- 2 Sampling Methods:
  - Infinite Population Without Replacement
  - Finite Population With Replacement
- Trials are Independent:
  - The Outcome of One Trial Does Not Affect the Outcome of Another

Binomial Probability Distribution Function

$$P(X) = \frac{n!}{X!(n-X)!} p^X (1-p)^{n-X}$$

$P(X)$ = probability that X successes given a knowledge of n and p

X = number of 'successes' in sample, ($X = 0, 1, 2, ..., n$)

p = probability of each 'success'

n = sample size

Tails in 2 Tosses of Coin

X	$P(X)$
0	1/4 = .25
1	2/4 = .50
2	1/4 = .25

Binomial Distribution Characteristics

Mean

$$\mu = E(X) = np$$

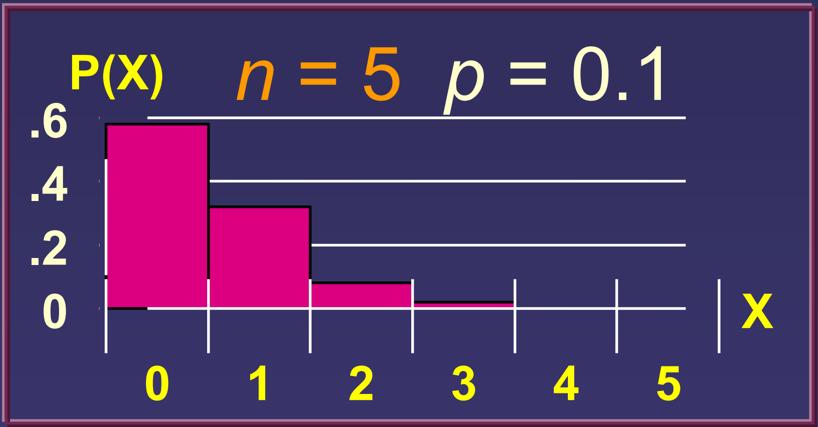
e.g. $\mu = 5\,(.1) = .5$

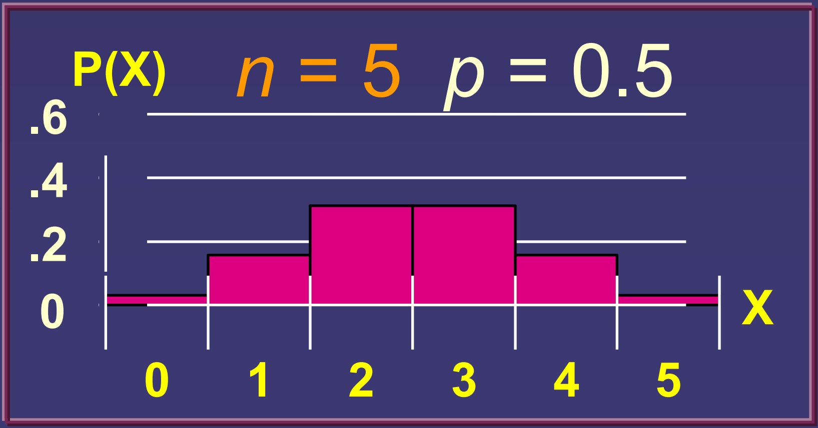
Standard Deviation

$$\sigma = \sqrt{np\,(1 - p)}$$

e.g. $\sigma = \sqrt{5(.5)(1 - .5)}$

$= 1.118$





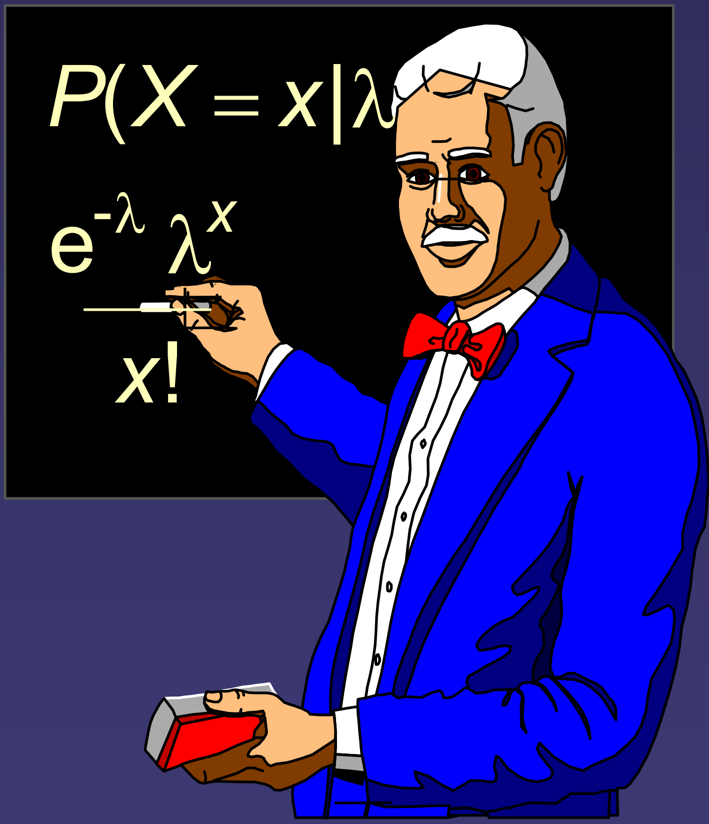
Computing Binomial Probabilities using Excel Function BINOMDIST

Binomial Probabilities	
n	10
p	0.8
x	p(x)
0	0.000000
1	0.000004
2	0.000074
3	0.000786
4	0.005505
5	0.026424
6	0.088080
7	0.201327
8	0.301990
9	0.268435
10	0.107374

Poisson Distribution

Poisson process:

- Discrete events in an 'interval'
  - The probability of one success in an interval is stable
  - The probability of more than one success in this interval is 0
- Probability of success is Independent from interval to Interval



Examples of Poisson Distribution

Poisson Distribution Function

$$P(X) = \frac{e^{-\lambda} \lambda^{X}}{X!}$$

P*(*X*) = probability** of ***X successes given λ

λ = expected (**mean**) number of 'successes'

e = 2.71828 (base of natural logs)

X = number of **'successes' per unit**

e.g. Find the probability of 4 customers arriving in 3 minutes when the mean is 3.6

$$P(X) = \frac{e^{-3.6} \, 3.6^{4}}{4!} = .1912$$

Poisson Distribution Characteristics

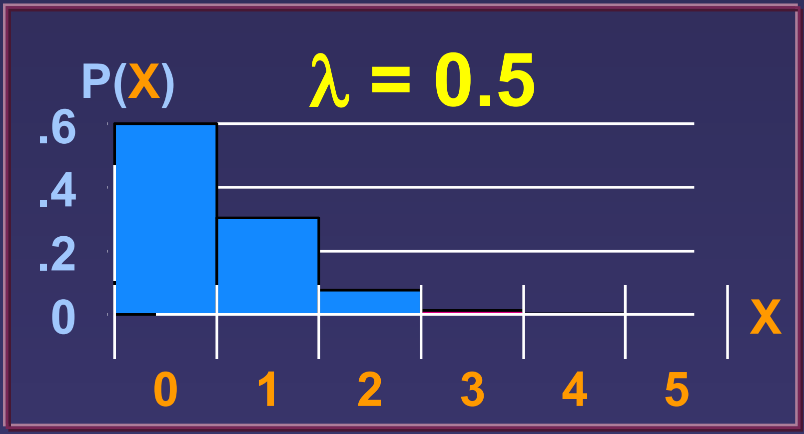
Mean

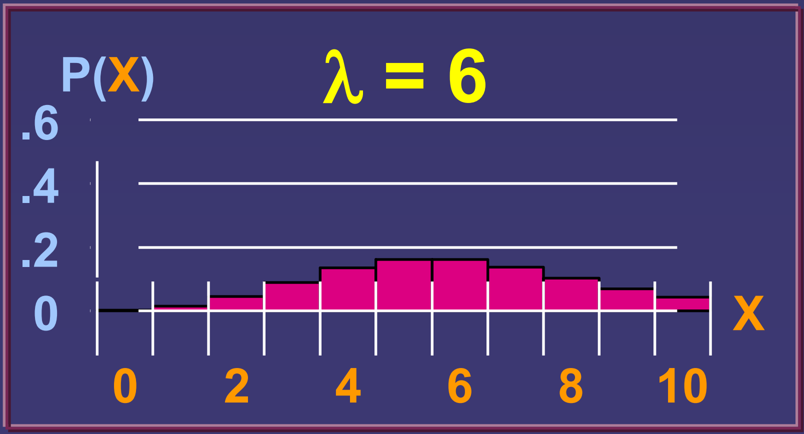
$$\mu = E(X) = \lambda$$

$$= \sum_{i=1}^{N} X_i P(X_i)$$

Standard Deviation

$$\sigma = \sqrt{\lambda}$$





Computing Poisson Probabilities using Excel Function POISSON

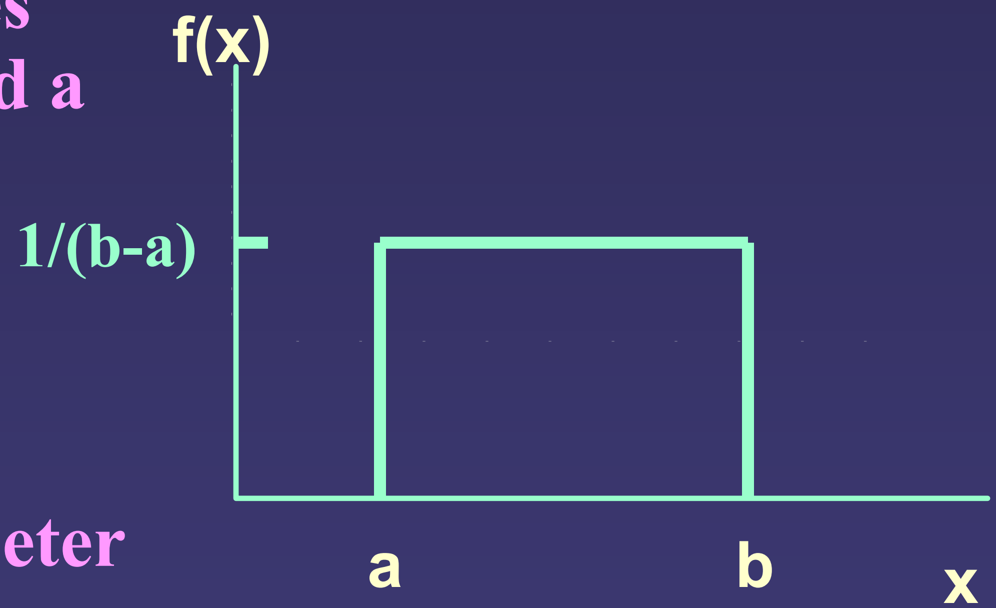
Poisson Distribution	
Mean	12
x	p(x)
1	0.00007
2	0.00044
3	0.00177
4	0.00531
5	0.01274
6	0.02548
7	0.04368
8	0.06552
9	0.08736
10	0.10484
11	0.11437
12	0.11437

Continuous Probability Distributions

- Uniform
- Normal

The Uniform Distribution

- Equally Likely chances of occurrences of RV values between a maximum and a minimum
- Mean = (b+a)/2
- Variance = $(b-a)^2/12$
- 'a' is a location parameter
- 'b-a' is a scale parameter
- no shape parameter



The Uniform Distribution

Probability Density Function

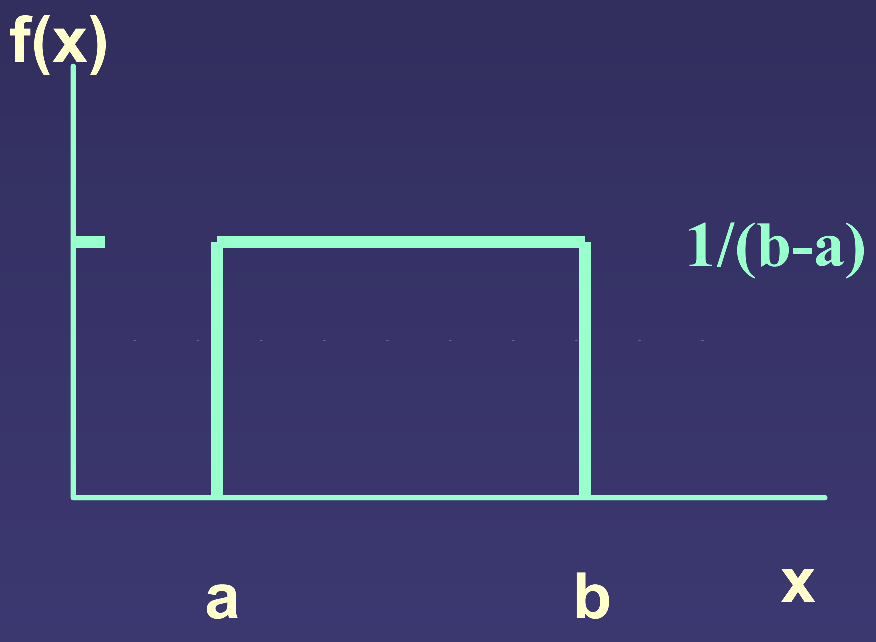
$$f(x) = \frac{1}{b-a} \qquad if \;\; a \le x \le \mathrm{b}$$

Distribution Function

$$F(x) = 0 \qquad if \;\; x < \mathrm{a}$$

$$F(x) = \frac{x-a}{b-a} \qquad if \;\; a \le x \le \mathrm{b}$$

$$F(x) = 1 \qquad if \;\; b < x$$



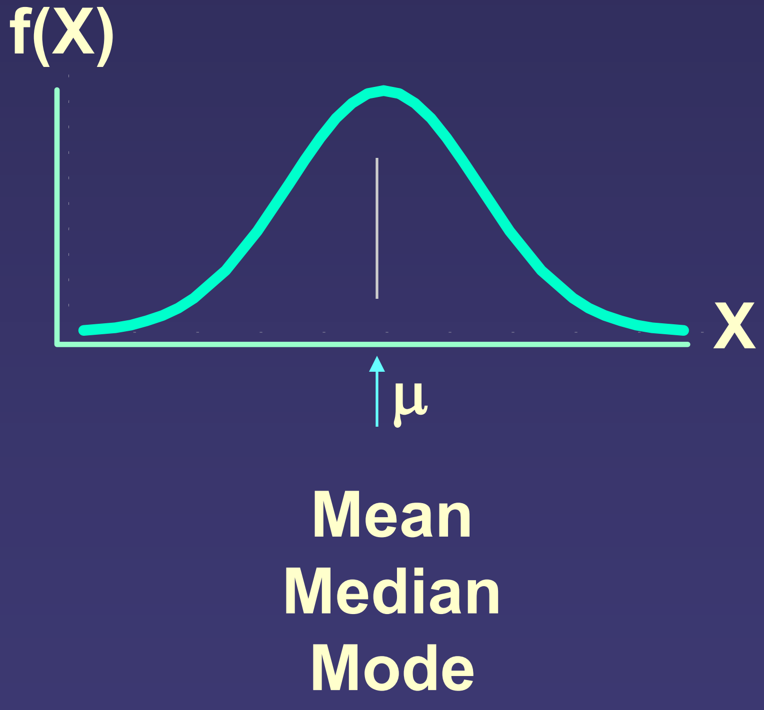
The Normal Distribution

- 'Bell Shaped'
- Symmetrical
- Mean, Median and

  Mode are Equal
- 'Middle Spread'

  Equals 1.33 σ
- Random Variable has

  Infinite Range

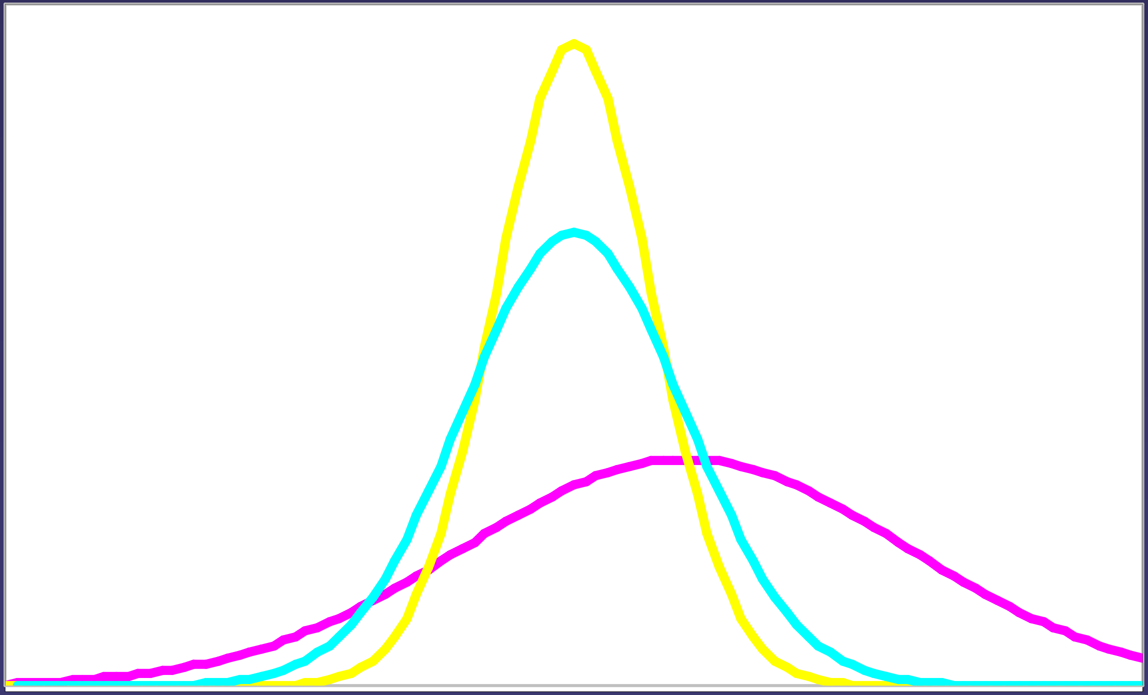


The Mathematical Model

$$f(X) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(X-\mu)^2}$$

$f(X)$	=	**frequency of random variable** X
π	=	**3.14159;** e **= 2.71828**
σ	=	**population standard deviation**
X	=	**value of random variable** $(-\infty < X < \infty)$
μ	=	**population mean**

Many Normal Distributions

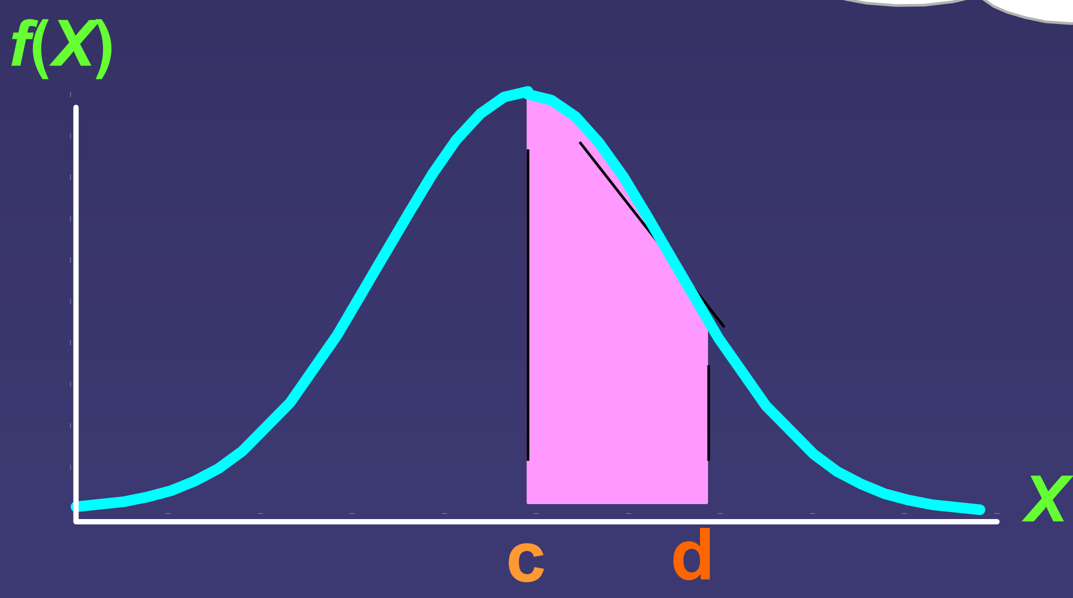


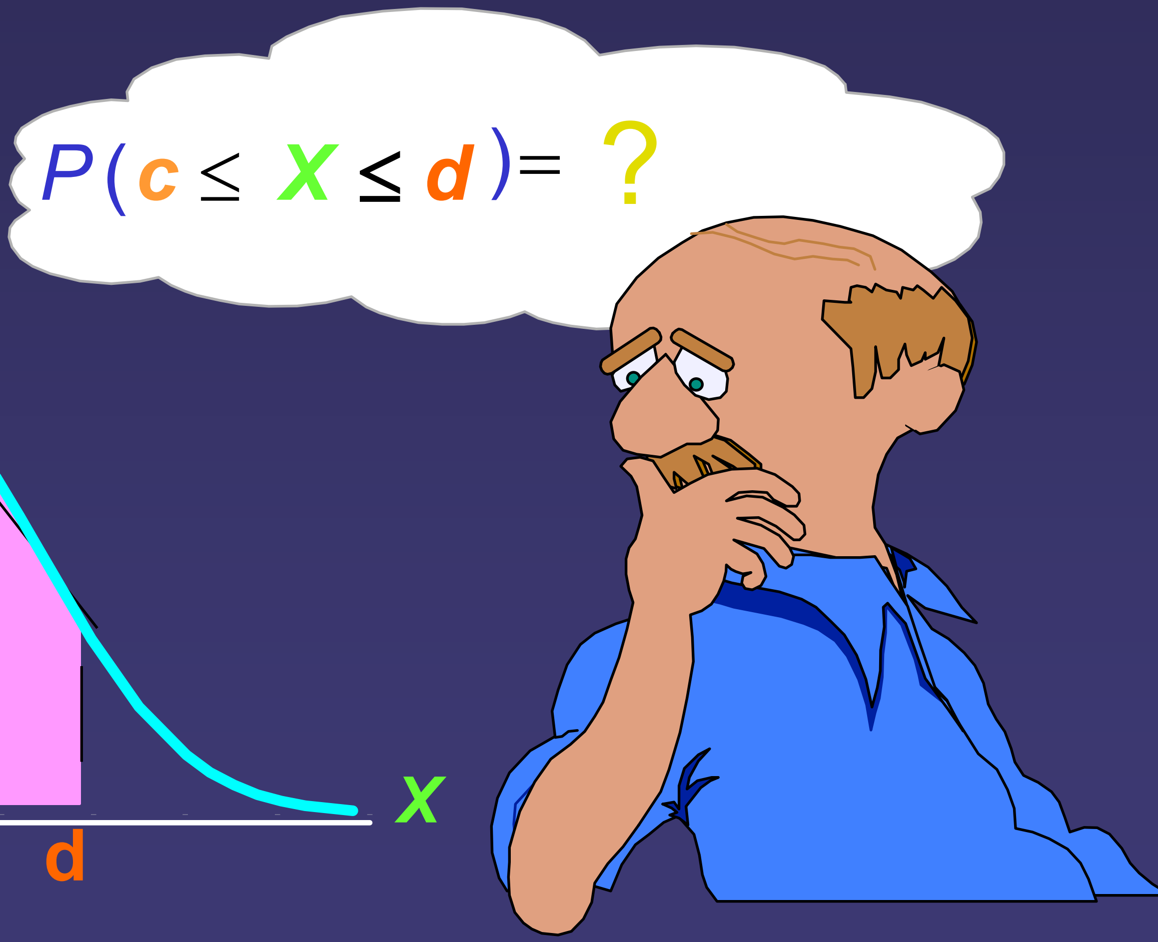
There are
an Infinite
Number

Varying the Parameters σ and μ, we obtain
Different Normal Distributions.

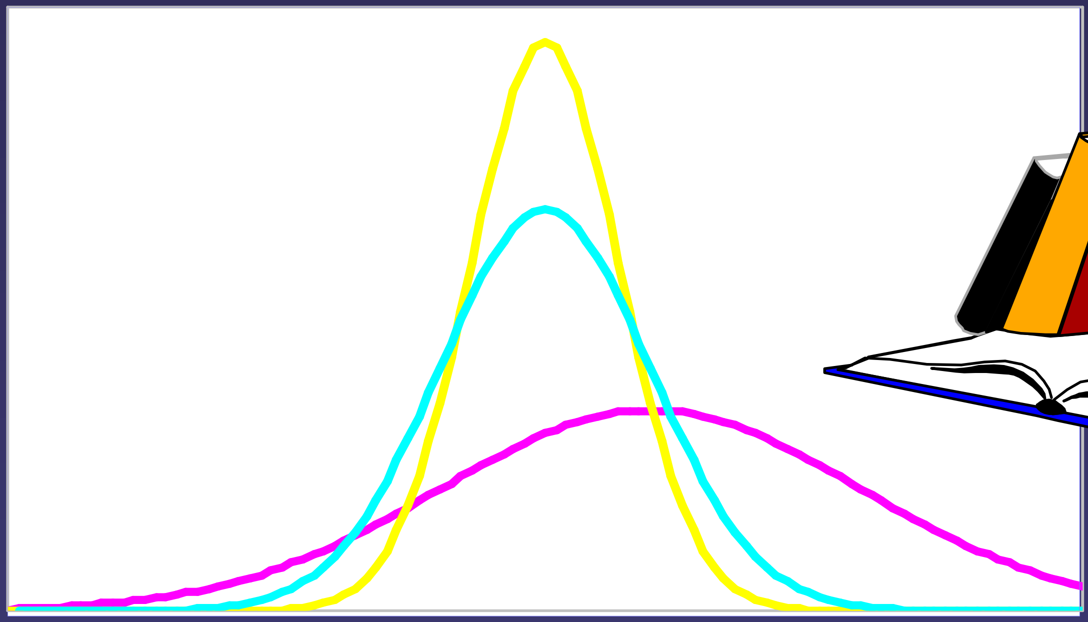
Normal Distribution: Finding Probabilities

Probability is the area under the curve!





Which Table?



Each distribution has its own table?

Infinitely Many Normal Distributions Means Infinitely Many Tables to Look Up!

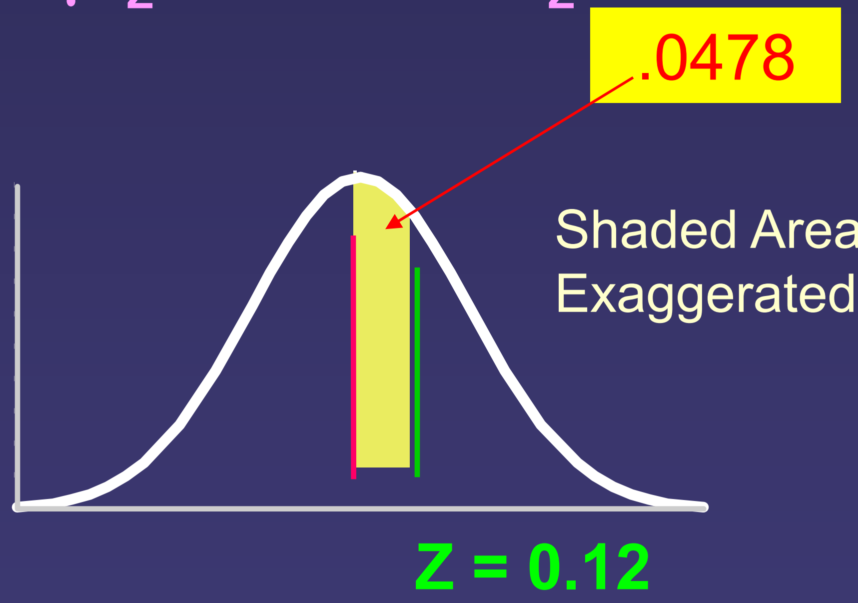
Solution (I): The Standardized Normal Distribution

Standardized Normal Distribution Table (Portion)

$\mu_Z = 0$ and $\sigma_Z = 1$

Z	.00	.01	**.02**
0.0	.0000	.0040	.0080
0.1	.0398	.0438	**.0478**
0.2	.0793	.0832	.0871
0.3	.0179	.0217	.0255

Probabilities



Only One Table is Needed

Solution (II): The Cumulative Standardized Normal Distribution

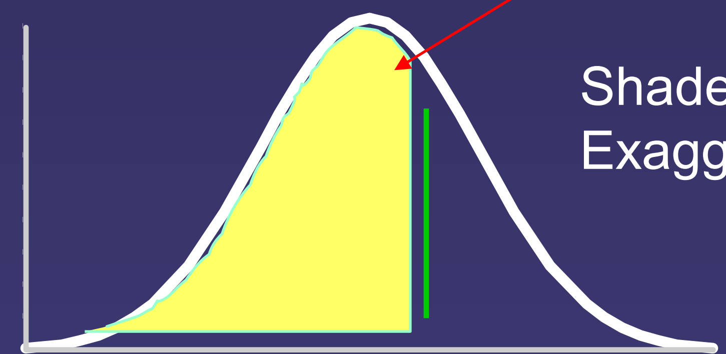
Cumulative Standardized Normal Distribution Table (Portion)

Z	.00	.01	.02
0.0	.5000	.5040	.5080
0.1	.5398	.5438	.5478
0.2	.5793	.5832	.5871
0.3	.5179	.5217	.5255

Probabilities

$\mu_Z = 0$ and $\sigma_Z = 1$

.5478



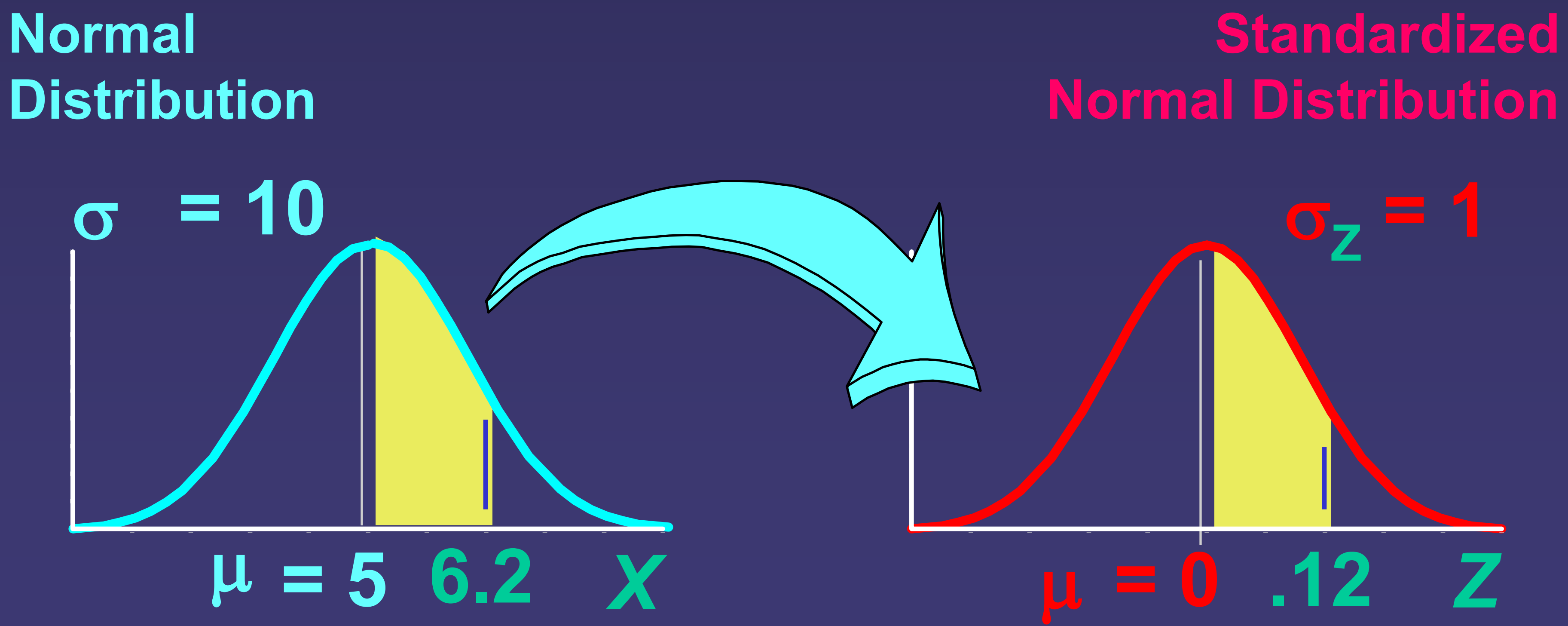
Shaded Area Exaggerated

Z = 0.12

Only One Table is Needed

Standardizing Example

$$Z = \frac{X - \mu}{\sigma} = \frac{6.2 - 5}{10} = 0.12$$



Shaded Area Exaggerated

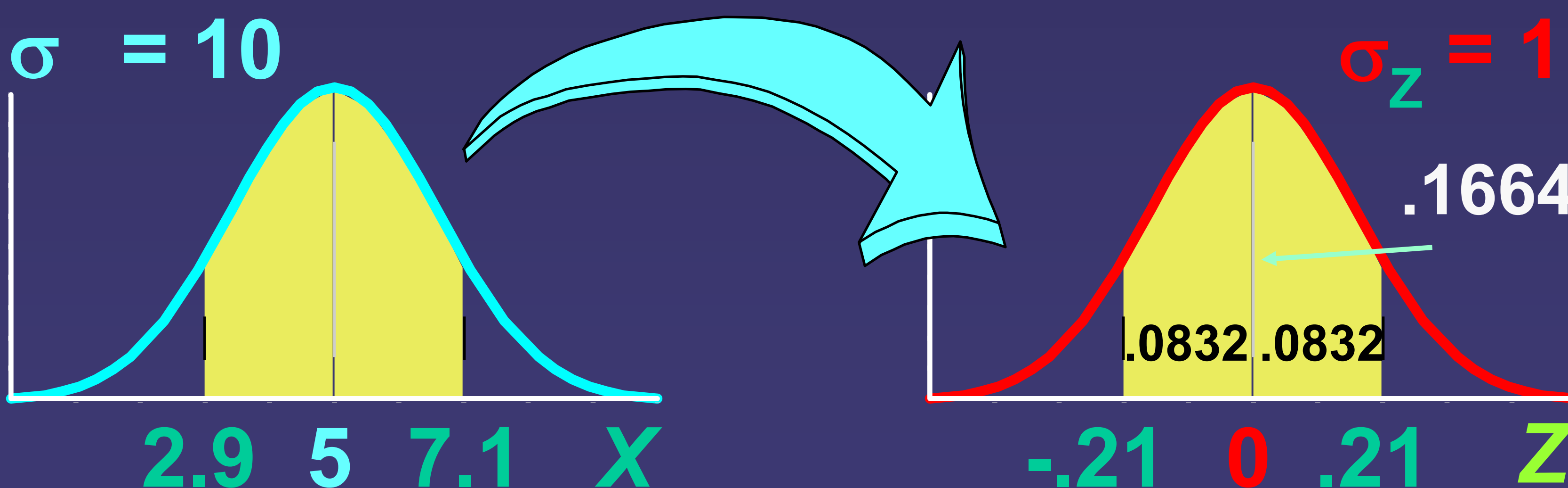
Example:
$P(2.9 < X < 7.1) = .1664$

$$z = \frac{x - \mu}{\sigma} = \frac{2.9 - 5}{10} = -.21$$

$$z = \frac{x - \mu}{\sigma} = \frac{7.1 - 5}{10} = .21$$

Normal Distribution

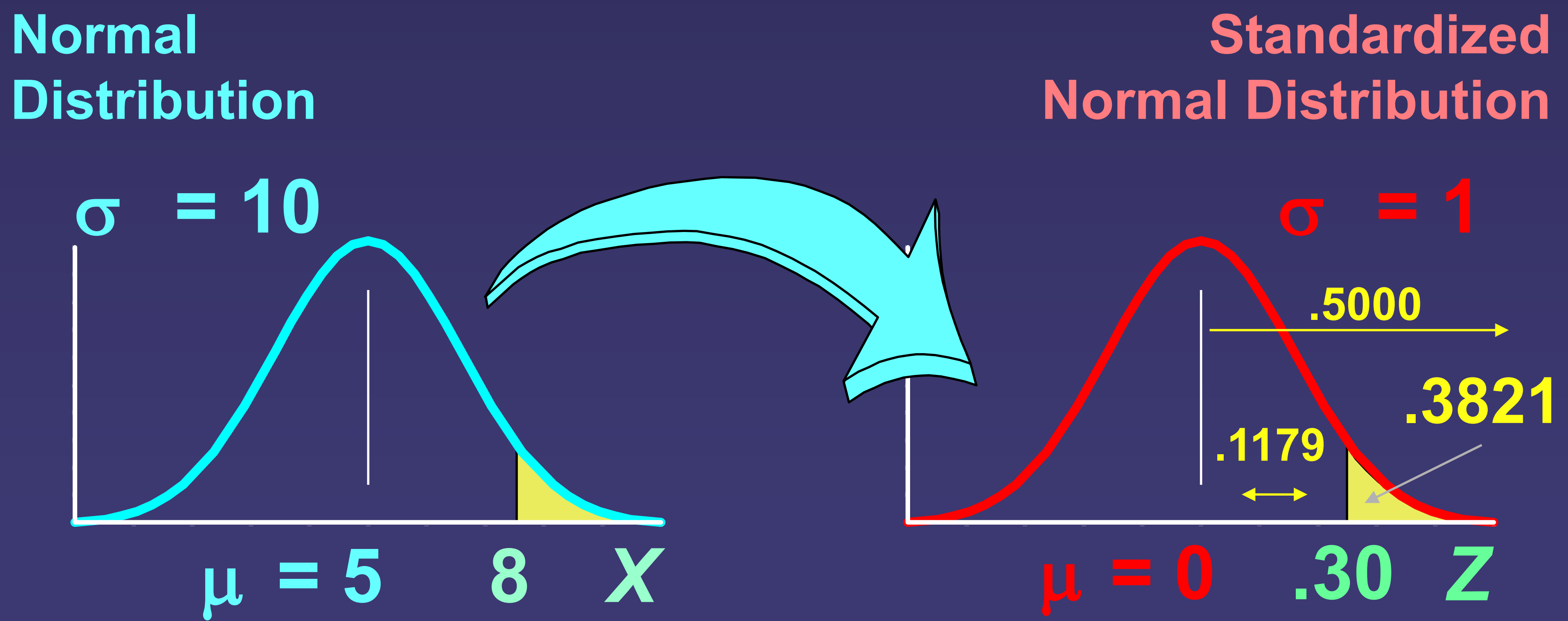
Standardized Normal Distribution



Shaded Area Exaggerated

Example: $P(X \geq 8) = .3821$

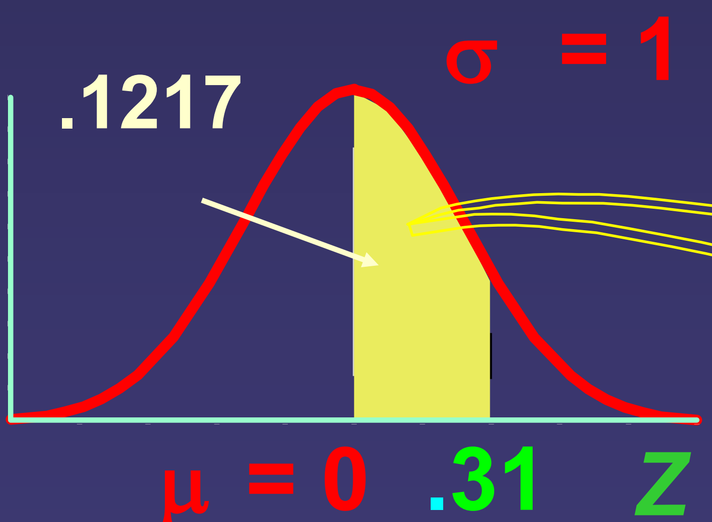
$$z = \frac{x - \mu}{\sigma} = \frac{8 - 5}{10} = .30$$



Shaded Area Exaggerated

Finding *Z* Values for Known Probabilities

What Is *Z* Given Probability = 0.1217?

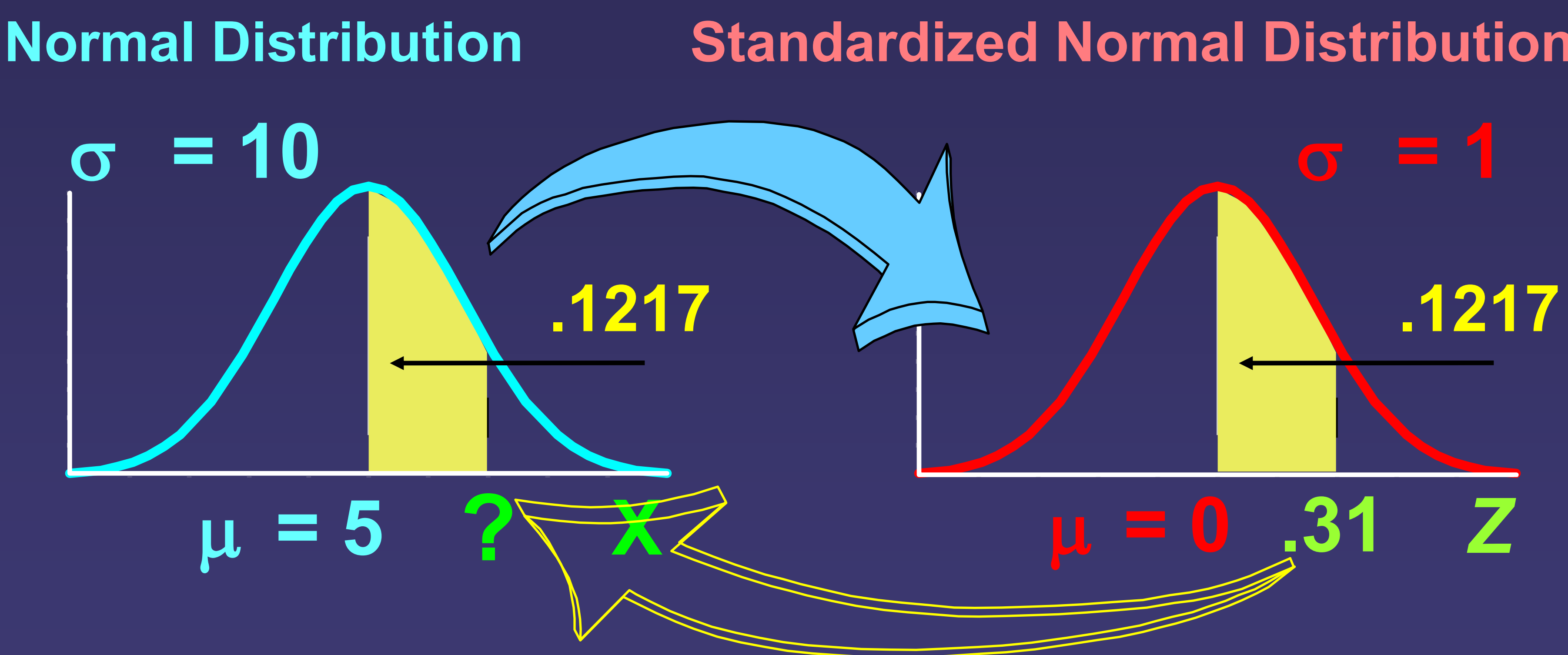


Shaded Area Exaggerated

Standardized Normal Probability Table (Portion)

Z	.00	.01	0.2
0.0	.0000	.0040	.0080
0.1	.0398	.0438	.0478
0.2	.0793	.0832	.0871
0.3	.1179	.1217	.1255

Recovering *X* Values for Known Probabilities



$$X = \mu + Z\sigma = 5 + (0.31)(10) = 8.1$$

Shaded Area Exaggerated

THANK YOU