SOLUTIONS OF A SYSTEM OF LINEAR EQUATIONS

There are two methods to solve linear algebraic equations:

- (1) Direct methods
- (2) Iterative methods
- **<u>Notes:</u> A problem is called ill-posed if "small" changes in data cause "large" changes in solution and a problem is called well-posed if "small" changes in data cause "small" changes in solutions.**

(1) Direct methods:

- There are two direct methods, Guass-elimination and Guass-Jordan methods. Direct methods transform the original system of equations into equivalent system of equations that can be solved easily.
- The transformation of the original equations is carried out by applying elementary row transformations to the augmented matrix of the system of equations. In our syllabus we have to discuss about only one direct method, i.e. Gauss Elimination Method

Gauss Elimination Method:

- Consider the following general systems

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1 \\ . \\ . \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + + a_{nn}x_{nn} = b_n \end{cases}$$

- The principle is to eliminate at each step one unknown, starting from x_1 to x_{n-1}:
- We multiply the first equation by $\frac{a_{21}}{a_{11}}$ and we subtract it from the second equation.

Gauss Elimination Method:

$$\boxed{\left(a_{22} - \frac{a_{21}}{a_{11}} a_{12}\right)} x_2 + \ldots + \boxed{\left(a_{2n} - \frac{a_{21}}{a_{11}} a_{1n}\right)} x_n = \boxed{b_2 - \frac{a_{21}}{a_{11}} b_1}$$

Hence;

$$a'_2\, x_2 + \ldots + a'_n\, x_n = b'_2$$

Note that a_{11} has been removed from eq.2

Gauss Elimination Method:

The modified system is:

pivot coefficient or element

$$\begin{cases} \boxed{a_{11}} x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1 & \text{pivot equation} \\ \quad + a'_{22}\, x_2 + a'_{23}\, x_3 + \ldots + a'_{2n}\, x_n = b'_2 \\ . \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + + a_{nn}x_{nn} = b_n \end{cases}$$

You can notice that x_n can be found directly using the last equation. Then, a back-substitution is performed.

Gauss Elimination Method:

$$x_n = \frac{b^{(n-1)}}{a_{nn}^{(n-1)}}$$

and

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}} \qquad \text{for } i=(n-1), \ldots, 1$$

Partial Pivoting:

- Problems may arise when a pivot is zero or close to zero. The idea of pivoting is to look for the largest element in the same column below the zero pivot and then to switch the equation corresponding to this equation with the equation corresponding to the near zero pivot. This is called partial pivoting. If the column and rows are searched for the highest element and then switched, this is called complete pivoting.

Example:

- Solve the system of linear equations by Gauss-elimination method
- $x_1 + x_2 + 2x_3 = 8$,
- $-x_1 - 2x_2 + 3x_3 = 1$,
- $3x_1 - 7x_2 + 4x_3 = 10$
- **Solution:** Augmented matrix is
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-

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix}$$

Solution:

- **STEP 1.**
-
- $$\sim \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{bmatrix} \quad R_2 = R_1 + R_2, \quad R_3 = -3R_1 + R_3$$
-
-
-
- $$\sim \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & -52 & -104 \end{bmatrix} \quad -R_2, \quad R_3 = 10R_2 + R_3$$

Solution:

- $\sim \begin{bmatrix} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ $-R_3/52$
- Equivalent system of equations form is:
- $x_1 + x_2 + 2x_3 = 8$
- $x_2 - 5x_3 = -9$
- $x_3 = 2$

Solution:

- STEP 2. <u>**Back Substitution:**</u>
- $x_3 = 2$

  $x_2 = 5x_3 - 9 = 10 - 9 = 1$

  $x_1 = -x_2 - 2x_3 + 8 = -1 - 4 + 8 = 3$
-
- Solution is $x_1 = 3, \quad x_2 = 1, \quad x_3 = 2.$

Example:

- Solve the system of linear equations by Gauss-elimination method with partial pivoting.

$$x+y+z=7\ ,\ 3x+3y+4z=24\ ,\ 2x+y+3z=16$$

- **Solution:**
- The matrix form of the system is $Ax=B$

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 7 \\ 24 \\ 16 \end{bmatrix}$$

Solution:

- The augmented matrix of the system is

$$[A:B]=\left[\begin{array}{ccc|c}1 & 1 & 1 & 7\\ 3 & 3 & 4 & 24\\ 2 & 1 & 3 & 16\end{array}\right]$$

- In the left column, 3 is the pivot because it is the entry that has largest absolute value.

Solution:

$$[A:B]=\left[\begin{array}{ccc|c}1 & 1 & 1 & 7\\ 3 & 3 & 4 & 24\\ 2 & 1 & 3 & 16\end{array}\right]$$

$$[A:B]=\left[\begin{array}{ccc|c}1 & 1 & 1 & 7\\ 3 & 3 & 4 & 24\\ 2 & 1 & 3 & 16\end{array}\right] R_{12}$$

Solution:

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & \frac{4}{3} & 8 \\ 1 & 1 & 1 & 7 \\ 2 & 1 & 3 & 16 \end{array}\right] \quad R_1 \rightarrow \left(\frac{1}{3}\right) R_1$$

Solution:

$$R_2 = R_2 - R_1, R_3 = R_3 - 2R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & \frac{4}{3} & \frac{8}{} \\ 0 & 0 & -\frac{1}{3} & -1 \\ 0 & -1 & \frac{1}{3} & \frac{}{0} \end{array}\right]$$

This completes the first pass. For the second pass, the pivot is -1 in the submatrix formed by deleting the first row and first column.

Solution:

$$\sim\left[\begin{array}{ccc|c} 1 & 1 & \frac{4}{3} & \frac{8}{} \\ 0 & -1 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} & \frac{}{-1} \end{array}\right] \quad R_{23}$$

$$\sim\left[\begin{array}{ccc|c} 1 & 1 & \frac{4}{3} & \frac{8}{} \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} & \frac{}{-1} \end{array}\right] \quad R_2 \to (-1)R_2$$

Solution:

- The corresponding system of equations is

$$x + y + \frac{4}{3}z = 8$$

$$y - \frac{1}{3}z = 0$$

$$-\frac{1}{3}z = -1$$

- Solving these equations by back substitution,
- The solution is $x = 3 \quad , \; y = 1 \quad , \; z = 3$

(2) Iterative methods:

- The direct methods lead to exact solutions in many cases but are subject to errors due to round off and other factors. In the iterative method, an approximation to the true solution is assumed initially to start the method. By applying the method repeatedly, better and better approximations are obtained. For large systems, iterative methods are faster than direct methods and round-off errors are also smaller. Any error made at any stage of computation gets automatically corrected in the subsequent steps.
- We will discuss two iterative methods:
- (i) Gauss-Jacobi method
- (ii) Gauss-Seidel method

DIAGONALLY DOMINANT PROPERTY(CONDITION FOR CONVEGENCE):

- A matrix is said to be diagonally dominant if for every row of the matrix, the magnitude of the diagonal entry in a row is larger than or equal to the sum of the magnitudes of all the other (non-diagonal) entries in that row. More precisely, the matrix A is diagonally dominant if

$$\left|a_{ii}\right| \geq \sum_{j \neq i} \left|a_{ij}\right| \quad for\ all\ i$$

- where denotes the entry in the i^{th} row and j^{th} column.
- Note that this definition uses a weak inequality, and is therefore sometimes called weak diagonal dominance. If a strict inequality (>) is used, this is called strict diagonal dominance. The unqualified term diagonal dominance can mean both strict and weak diagonal dominance.

(i) Gauss-Jacobi method:

- Given a general set of n equations and n unknowns, we have

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = c_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = c_2$$

  - ....
  - ....
  - ....
  - ....

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = c_n$$

(i) Gauss-Jacobi method:

- If the diagonal elements are non-zero, each equation is rewritten for the corresponding unknown, that is, the first equation is rewritten with x_1 on the left hand side, the second equation is rewritten with x_2 on the left hand side and so on as follows

$$x_1 = \frac{c_1 - a_{12}x_2 - a_{13}x_3 \ldots\ldots - a_{1n}x_n}{a_{11}}$$

$$x_2 = \frac{c_2 - a_{21}x_1 - a_{23}x_3 \ldots\ldots - a_{2n}x_n}{a_{22}}$$

$$\vdots$$

$$\vdots$$

$$x_{n-1} = \frac{c_{n-1} - a_{n-1,1}x_1 - a_{n-1,2}x_2 \ldots\ldots - a_{n-1,n-2}x_{n-2} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

$$x_n = \frac{c_n - a_{n1}x_1 - a_{n2}x_2 - \ldots\ldots - a_{n,n-1}x_{n-1}}{a_{nn}}$$

(i) Gauss-Jacobi method:

- These equations can be rewritten in a summation form as

$$x_1 = \frac{c_1 - \sum_{\substack{j=1 \\ j\neq 1}}^{n} a_{1j} x_j}{a_{11}}$$

$$x_2 = \frac{c_2 - \sum_{\substack{j=1 \\ j\neq 2}}^{n} a_{2j} x_j}{a_{22}}$$

- ....
- ....

$$x_n = \frac{c_n - \sum_{\substack{j=1 \\ j\neq n}}^{n} a_{nj} x_j}{a_{nn}}$$

(i) Gauss-Jacobi method:

- Hence for any row i

$$x_i = \frac{c_i - \sum\limits_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j}{a_{ii}}, i = 1, 2, \ldots, n.$$

- The above iteration process is continued until two successive approximations are nearly equal.

Working Rule:

- **(i)** Arrange the equations in such a manner that the leading elements are large in magnitude in their respective rows satisfying the conditions

$$|a_{11}| > |a_{12}| + |a_{13}|$$
$$|a_{22}| > |a_{21}| + |a_{23}|$$
$$|a_{33}| > |a_{31}| + |a_{32}|$$

- **(ii)** Express the variables having large coefficients in terms of other variables.
- **(iii)** Start the iteration 1 by assuming the initial values of (x, y, z) *as* (x_0, y_0, z_0) and obtain (x_1, y_1, z_1) using equations

Working Rule:

$$x_1 = \frac{c_1 - a_{12}x_2 - a_{13}x_3 \ldots\ldots - a_{1n}x_n}{a_{11}}$$

$$x_2 = \frac{c_2 - a_{21}x_1 - a_{23}x_3 \ldots\ldots - a_{2n}x_n}{a_{22}}$$

$$\vdots$$

$$\vdots$$

$$x_{n-1} = \frac{c_{n-1} - a_{n-1,1}x_1 - a_{n-1,2}x_2 \ldots\ldots - a_{n-1,n-2}x_{n-2} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

$$x_n = \frac{c_n - a_{n1}x_1 - a_{n2}x_2 - \ldots\ldots - a_{n,n-1}x_{n-1}}{a_{nn}}$$

(iv) Start the iteration 2 by putting the values of (x, y, z) *as* (x_1, y_1, z_1) and obtain (x_2, y_2, z_2)

(v) The above iteration process is continued until two successive approximations are nearly equal.

Example:

- Solve the system of linear equations by Gauss- Jacobi method correct up to 2-decimal places.

$$6x + 2y - z = 4\ ,\ x + 5y + z = 3\ ,\ 2x + y + 4z = 27$$

- **Solution:** Re-writing the equations,

$$\left.\begin{aligned} x &= \frac{1}{6}(4 - 2y + z) \\ y &= \frac{1}{5}(3 - x - z) \\ z &= \frac{1}{4}(27 - 2x - y) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

- Iteration 1: Assuming $x_0 = 0\ ,\ y_0 = 0\ and\ z_0 = 0$ as initial approximation and putting in Eq.1

Solution:

$$x_1 = \frac{2}{3} = 0.67$$

$$y_1 = \frac{1}{5}(3) = 0.6$$

$$z_1 = \frac{1}{4}(27) = 6.75$$

- Iteration 2: Putting $x_1,\ y_1\ and\ z_1$ in Eq.1

$$x_2 = \frac{1}{6}(4 - 2(0.6) + 6.75) = 1.59$$

$$y_2 = \frac{1}{5}(3 - 0.67 - 6.75) = -0.884$$

$$z_2 = \frac{1}{4}(27 - 2(0.67) - 0.6) = 6.265$$

Solution:

- Continuing in this way, we have the solution as

$$x_5 = 2.00$$

$$y_5 = -1.00$$

$$z_5 = 6.00$$

Which is the required solution.

(ii) Gauss-Seidel method:

- The system of linear equations are same as given in Gauss-Jacobi method.
- ***Working Rule:***
- **(i)** Arrange the equations in such a manner that the leading elements are large in magnitude in their respective rows satisfying the conditions

$$|a_{11}| > |a_{12}| + |a_{13}|$$

$$|a_{22}| > |a_{21}| + |a_{23}|$$

$$|a_{33}| > |a_{31}| + |a_{32}|$$

(ii) Express the variables having large coefficients in terms of other variables.

Working Rule:

- **(iii)** Start the iteration 1 by assuming the initial values of (x, y, z) *as* (x_0, y_0, z_0)
- **(iv)** In the iteration 1, put $y = y_{0,} z = z_0$ in the equation of
- x to $x_1$obtain ,put $x = x_1$ *and* $z = z_0$ in the eq. of y to obtain y_1 , put $x = x_1, y = y_1$ in the eq. of z to obtain z_1.
- **(v)** The above iteration process is continued until two successive approximations are nearly equal.

Example:

- Find the solution to the following system of equations using the Gauss-Seidel method.

$$12x_1 + 3x_2 - 5x_3 = 1$$

$$x_1 + 5x_2 + 3x_3 = 28$$

$$3x_1 + 7x_2 + 13x_3 = 76$$

- Use $x_1 = 1, x_2 = 0 \; and \; x_3 = 1$ as the initial guess.
- **Solution:**
- The given system is diagonally dominant as

$$|a_{11}| = |12| = 12 \geq |a_{12}| + |a_{13}| = |3| + |-5| = 8$$

$$|a_{22}| = |5| = 5 \geq |a_{21}| + |a_{23}| = |1| + |3| = 4$$

$$|a_{33}| = |13| = 13 \geq |a_{31}| + |a_{32}| = |3| + |7| = 10$$

Solution:

- and the inequality is strictly greater than for at least one row. Hence, the solution should converge using the Gauss-Seidel method.
- Rewriting the equations, we get

$$x_1 = \frac{1 - 3x_2 + 5x_3}{12}$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$

$$x_3 = \frac{76 - 3x_1 - 7x_2}{13}$$

- Assuming an initial guess of $x_1 = 1, x_2 = 0 \; and \; x_3 = 1$

Solution:

- **Iteration 1**

$$x_1^{\,1} = \frac{1-3(0)+5(1)}{12} = 0.50000$$

$$x_2^{\,1} = \frac{28-(0.50000)-3(1)}{5} = 4.9000$$

$$x_3^{\,1} = \frac{76-3(0.50000)-7(4.9000)}{13} = 3.0923$$

- **Iteration 2**

- $$x_1^{\,2} = \frac{1-3(4.9000)+5(3.0923)}{12} = 0.14679$$

$$x_2^{\,2} = \frac{28-(0.14679)-3(3.0923)}{5} = 3.7153$$

Solution:

$$x_3^{\,2} = \frac{76 - 3(0.14679) - 7(3.7153)}{13} = 3.8118$$

as you conduct more iterations, the solution converges as follows.

Iteration			
1	0.50000	4.9000	3.0923
2	0.14679	3.7153	3.8118
3	0.74275	3.1644	3.9708
4	0.94675	3.0281	3.9971
5	0.99177	3.0034	4.0001
6	0.99919	3.0001	4.0001