PARUL UNIVERSITY
FACULTY OF ENGINEERING AND TECHNOLOGY
DEPARTMENT OF APPLIED SCIENCE AND HUMANITIES
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PROBABILITY, STATISTICS AND NUMERICAL METHODS (303191251)
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Unit 4
Solution of a System of Linear Equations, Roots of Algebraic and Transcendental Equations

Iterative methods:

The direct methods lead to exact solutions in many cases but are subject to errors due to round off and other factors. In the iterative method, an approximation to the true solution is assumed initially to start the method. By applying the method repeatedly, better and better approximations are obtained. For large systems, iterative methods are faster than direct methods and round-off errors are also smaller. Any error made at any stage of computation gets automatically corrected in the subsequent steps.

We will discuss two iterative methods:

(i) Gauss-Jacobi method (ii) Gauss-Seidel method

DIAGONALLY DOMINANT PROPERTY (SUFFICIENT CONDITION FOR CONVEGENCE):

A matrix is said to be diagonally dominant if for every row of the matrix, the magnitude of the diagonal entry in a row is larger than or equal to the sum of the magnitudes of all the other (non-diagonal) entries in that row. More precisely, the matrix A is diagonally dominant if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}| \quad \textit{for all } i$$

where a_{ij} denotes the entry in the i^{th} row and j^{th} column.

Note that this definition uses a weak inequality, and is therefore sometimes called weak diagonal dominance. If a strict inequality (>) is used, this is called strict diagonal dominance. The unqualified term diagonal dominance can mean both strict and weak diagonal dominance.

(i) Gauss-Jacobi method:

Given a general set of n equations and n unknowns, we have

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = c_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = c_2$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = c_n$$

If the diagonal elements are non-zero, each equation is rewritten for the corresponding unknown, that is, the first equation is rewritten with x_1 on the left hand side, the second equation is rewritten with x_2 on the left hand side and so on as follows

$$x_1 = \frac{c_1 - a_{12}x_2 - a_{13}x_3 \ldots\ldots - a_{1n}x_n}{a_{11}}$$

$$x_2 = \frac{c_2 - a_{21}x_1 - a_{23}x_3 \ldots\ldots - a_{2n}x_n}{a_{22}}$$

$$\vdots$$

$$\vdots$$

$$x_{n-1} = \frac{c_{n-1} - a_{n-1,1}x_1 - a_{n-1,2}x_2 \ldots\ldots - a_{n-1,n-2}x_{n-2} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

$$x_n = \frac{c_n - a_{n1}x_1 - a_{n2}x_2 - \ldots\ldots - a_{n,n-1}x_{n-1}}{a_{nn}}$$

These equations can be rewritten in a summation form as

$$x_1 = \frac{c_1 - \sum_{\substack{j=1 \\ j\neq 1}}^{n} a_{1j}x_j}{a_{11}}$$

$$x_2 = \frac{c_2 - \sum_{\substack{j=1 \\ j\neq 2}}^{n} a_{2j}x_j}{a_{22}}$$

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$$x_{n-1} = \frac{c_{n-1} - \sum_{\substack{j=1 \\ j\neq n-1}}^{n} a_{n-1,j}x_j}{a_{n-1,n-1}}$$

$$x_n = \frac{c_n - \sum_{\substack{j=1 \\ j\neq n}}^{n} a_{nj}x_j}{a_{nn}}$$

Hence for any row i,

$$x_i = \frac{c_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j}{a_{ii}}, i = 1,2,\ldots,n.$$

Now to find x_i's, one assumes an initial guess for the x_i's and then uses the rewritten equations to calculate the new estimates. Remember, one always uses the most recent estimates to calculate the next estimates, x_i.

The above iteration process is continued until two successive approximations are nearly equal.

Working Rule:

(i) Arrange the equations in such a manner that the leading elements are large in magnitude in their respective rows satisfying the conditions

$$\begin{aligned} |a_{11}| &> |a_{12}| + |a_{13}| \\ |a_{22}| &> |a_{21}| + |a_{23}| \\ |a_{33}| &> |a_{31}| + |a_{32}| \end{aligned}$$

(ii) Express the variables having large coefficients in terms of other variables.

(iii) Start the iteration 1 by assuming the initial values of (x, y, z) *as* (x_0, y_0, z_0) and obtain

(x_1, y_1, z_1) using equations

$$x_1 = \frac{c_1 - a_{12}x_2 - a_{13}x_3 \ldots\ldots - a_{1n}x_n}{a_{11}}$$

$$x_2 = \frac{c_2 - a_{21}x_1 - a_{23}x_3 \ldots\ldots - a_{2n}x_n}{a_{22}}$$

$$\vdots$$

$$\vdots$$

$$x_{n-1} = \frac{c_{n-1} - a_{n-1,1}x_1 - a_{n-1,2}x_2 \ldots\ldots - a_{n-1,n-2}x_{n-2} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

$$x_n = \frac{c_n - a_{n1}x_1 - a_{n2}x_2 - \ldots\ldots - a_{n,n-1}x_{n-1}}{a_{nn}}$$

(iv) Start the iteration 2 by putting the values of (x, y, z) *as* (x_1, y_1, z_1) and obtain (x_2, y_2, z_2).

(v) The above iteration process is continued until two successive approximations are nearly equal.

Example: Solve the system of linear equations by Gauss- Jacobi method correct up to 2 decimal places. $6x+2y-z=4$, $x+5y+z=3$, $2x+y+4z=27$

Solution: Re-writing the equations,

$$\left.\begin{aligned} x &= \frac{1}{6}(4-2y+z) \\ y &= \frac{1}{5}(3-x-z) \\ z &= \frac{1}{4}(27-2x-y) \end{aligned}\right\} \text{.....................................(1)}$$

Iteration 1: Assuming $x_0 = 0$, $y_0 = 0$ *and* $z_0 = 0$ as initial approximation and putting in Eq.1

$$x_1 = \frac{2}{3} = 0.67$$
$$y_1 = \frac{1}{5}(3) = 0.6$$
$$z_1 = \frac{1}{4}(27) = 6.75$$

Iteration 2: Putting x_1, y_1 *and* z_1 in Eq.1

$$x_2 = \frac{1}{6}(4-2(0.6)+6.75) = 1.59$$
$$y_2 = \frac{1}{5}(3-0.67-6.75) = -0.884$$
$$z_2 = \frac{1}{4}(27-2(0.67)-0.6) = 6.265$$

Continuing in this way, we have the solution as

$$x_5 = 2.00$$
$$y_5 = -1.00$$
$$z_5 = 6.00$$

Which is the required solution.

(ii) Gauss-Seidel method:

The system of linear equations are same as given in Gauss-Jacobi method.

Working Rule:

(i) Arrange the equations in such a manner that the leading elements are large in magnitude in their respective rows satisfying the conditions

$$|a_{11}| > |a_{12}| + |a_{13}|$$
$$|a_{22}| > |a_{21}| + |a_{23}|$$
$$|a_{33}| > |a_{31}| + |a_{32}|$$

(ii) Express the variables having large coefficients in terms of other variables.

(iii) Start the iteration 1 by assuming the initial values of (x, y, z) *as* (x_0, y_0, z_0)

(iv) In the iteration 1, put $y = y_{0,} z = z_0$ in the equation of x to obtain x_1 ,put $x = x_1$ *and* $z = z_0$ in the eq. of y to obtain y_1, put $x = x_1$, $y = y_1$ in the eq. of z to obtain z_1 .

(v) The above iteration process is continued until two successive approximations are nearly equal.

Example: Find the solution to the following system of equations using the Gauss-Seidel method.

$12x_1 + 3x_2 - 5x_3 = 1$, $x_1 + 5x_2 + 3x_3 = 28$, $3x_1 + 7x_2 + 13x_3 = 76$

Use $x_1 = 1, x_2 = 0$ *and* $x_3 = 1$ as the initial guess .

Solution: The given system is diagonally dominant as

$$|a_{11}| = |12| = 12 \geq |a_{12}| + |a_{13}| = |3| + |-5| = 8$$

$$|a_{22}| = |5| = 5 \geq |a_{21}| + |a_{23}| = |1| + |3| = 4$$

$$|a_{33}| = |13| = 13 \geq |a_{31}| + |a_{32}| = |3| + |7| = 10$$

and the inequality is strictly greater than for at least one row. Hence, the solution should converge using the Gauss-Seidel method.

Rewriting the equations, we get

$$x_1 = \frac{1 - 3x_2 + 5x_3}{12}$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5}$$

$$x_3 = \frac{76 - 3x_1 - 7x_2}{13}$$

Assuming an initial guess of $x_1 = 1, x_2 = 0 \; and \; x_3 = 1$

Iteration 1

$$x_1^{\,1} = \frac{1-3(0)+5(1)}{12}$$

$$= 0.50000$$

$$x_2^{\,1} = \frac{28-(0.50000)-3(1)}{5}$$

$$= 4.9000$$

$$x_3^{\,1} = \frac{76-3(0.50000)-7(4.9000)}{13}$$

$$= 3.0923$$

Iteration 2

$$x_1^{\,2} = \frac{1-3(4.9000)+5(3.0923)}{12}$$

$$= 0.14679$$

$$x_2^{\,2} = \frac{28-(0.14679)-3(3.0923)}{5}$$

$$= 3.7153$$

$$x_3^{\,2} = \frac{76-3(0.14679)-7(3.7153)}{13}$$

$$= 3.8118$$

as you conduct more iterations, the solution converges as follows.

Iteration	x_1	x_2	x_3
1	0.50000	4.9000	3.0923
2	0.14679	3.7153	3.8118
3	0.74275	3.1644	3.9708
4	0.94675	3.0281	3.9971
5	0.99177	3.0034	4.0001
6	0.99919	3.0001	4.0001

Examples:

(1) $5x+y-z=10; 2x+4y+z=14; x+y+8z=20$ by using Gauss-Seidel method correct upto three decimal places. $(x=2; y=2; z=2)$

(2) $6x+y+z=105; 4x+8y+3z=155; 5x+4y-10z=65$ by using Gauss-Seidel method correct upto three decimal places. $(x=15; y=10; z=5)$

(3) $20x+y-2z=17; 3x+20y-z=-18; 2x-3y+20z=25$ by using Gauss-Jacobi method correct upto three decimal places. $(x=1; y=-1; z=1)$

(4) Using Gauss – Seidel method, solve the system of linear equation, $27x+6y-z=85$ $6x+10y+2z=72$ $x+y+54z=110$ Correct upto three decimal places.**(winter 23-24)**

(5) Solve the following system of equations by Gauss Jacobi method $6x+2y-z=4$,$x+5y+z=3$, $2x+y+4z=2$**(Summer 23-24)**

ROOTS OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

We all are aware about algebraic equation, so now we will discuss about transcendental functions and equations.

TRANSCENDENTAL FUNCTIONS AND EQUATION:

A **transcendental function** "transcends" algebra in that it cannot be expressed in terms of a finite sequence of the algebraic operations of addition, multiplication, and root extraction. Examples of transcendental functions include the exponential function, the logarithm, and the trigonometric functions.(x^{π}, c^{x} , x^{x}, $\log_c x$, $\sin x$ etc.)

An equation which is includes transcendental functions is known as **Transcendental Equation.To solve these types of equations we use the following methods:**

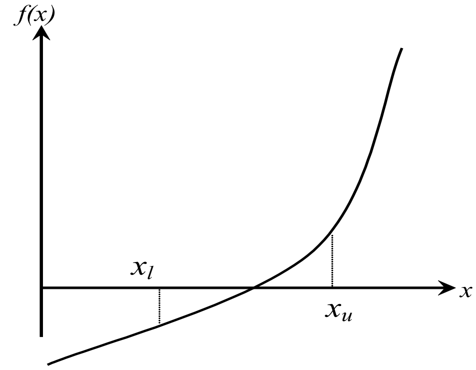
1) Bisection Method

2) Regula-Falsi Method (False-Position Method)

3) Newton-Raphson Method

1) Bisection Method: One of the first numerical methods developed to find the root of a nonlinear equation $f(x)=0$ was the bisection method (also called Binary-Search method). The method is based on the following theorem:

Theorem : An equation $f(x)=0$, where $f(x)$ is a real continuous function, has at least one root between x_ℓ and x_u if $f(x_\ell)f(x_u)<0$ (Figure .1).Note that if $f(x_\ell)f(x_u)>0$, there may or may not be any root between x_ℓ and x_u (Figures 2 and 3). If $f(x_\ell)f(x_u)<0$, then there may be more than one root between x_ℓ and x_u (Figure 4). So the theorem only guarantees one root between x_ℓ and x_u.



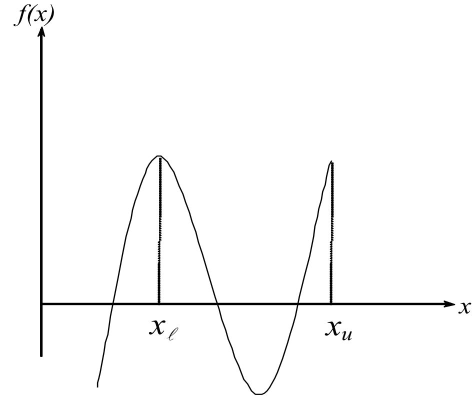


Fig 1 **Fig 2**

Fig.1 At least one root exists between two points if the function is real, continuous, and changes sign.
Fig.2 If function) (x f does not change sign between two points, roots of 0) (=x f may still exist between the two points.

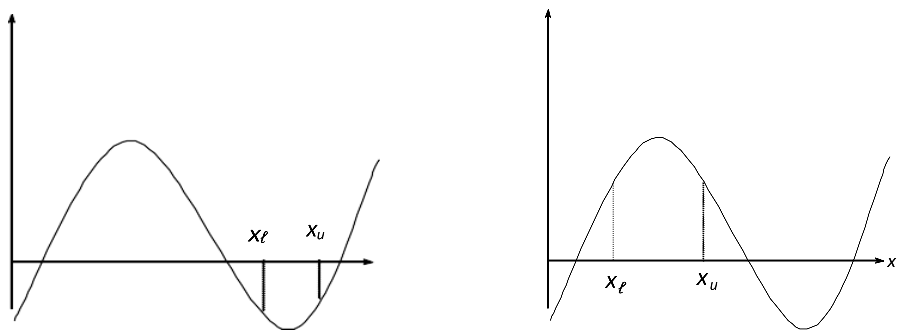


Fig.3 If the function $f(x)$ does not change sign between two points, there may not be any roots $f(x)=0$ between the two points.

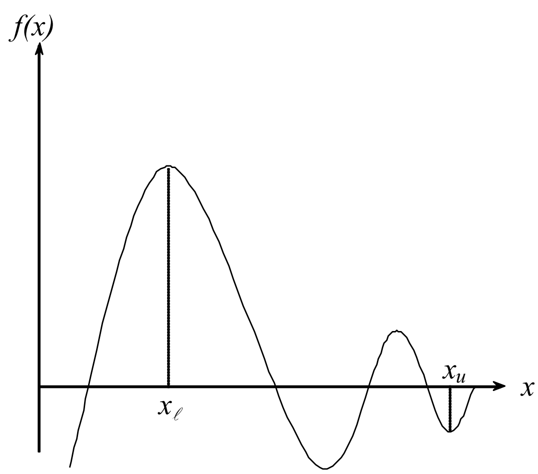


Fig .4 If the function $f(x)$ changes sign between two points, more than one root for $f(x)=0$ may exist between the two points.

A general rule is:

- If the $f(x_l)$ and $f(x_u)$ have the same sign ($f(x_\ell)f(x_u) > 0$):

  - There is no root between x_l and x_u.

  - There is an even number of roots between x_l and x_u.

- If the $f(x_l)$ and $f(x_u)$ have different signs ($f(x_\ell)f(x_u) < 0$):

  - The is an odd number of roots.

Advantages of Bisection Method:

a) The bisection method is always convergent. Since the method brackets the root, the method is guaranteed to converge.
b) As iterations are conducted, the interval gets halved. So one can guarantee the decrease in the error in the solution of the equation.

Drawbacks of Bisection Method:

a) The convergence of bisection method is slow as it is simply based on halving the interval.
b) If one of the initial guesses is closer to the root, it will take larger number of iterations to reach the root.
c) If a function $f(x)$ is such that it just touches the x-axis such as $f(x) = x^2 = 0$ it will be unable to find the lower guess, x_ℓ, and upper guess, x_u, such that $f(x_\ell)f(x_u) < 0$

Working rule for Bisection Method :

The steps to apply the bisection method to find the root of the equation $f(x) = 0$ are:

1. Choose x_ℓ and x_u as two guesses for the root such that $f(x_\ell)f(x_u) < 0$, or in other words, $f(x)$ changes sign between x_ℓ and x_u.
2. Estimate the root, x_m of the equation $f(x) = 0$ as the mid-point between x_ℓ and x_u as
$$x_m = \frac{x_\ell + x_u}{2}$$
3. Now check the following

a. If $f(x_\ell)f(x_m) < 0$, then the root lies between x_ℓ and x_m; then $x_\ell = x_\ell$ and $x_u = x_m$.
b. If $f(x_\ell)f(x_m) > 0$, then the root lies between x_m and x_u; then $x_\ell = x_m$ and $x_u = x_u$.
c. If $f(x_\ell)f(x_m) = 0$; then the root is x_m. Stop the algorithm if this is true.

4. Find the new estimate of the root $x_m = \frac{x_\ell + x_u}{2}$

   Find the absolute approximate relative error as $|\varepsilon_a| = \left|\frac{x_m^{new} - x_m^{old}}{x_m^{new}}\right| \text{x}\, 100$

   Where x_m^{new} = estimated root from present iteration x_m^{old} = estimated root from previous iteration

5. Compare the absolute relative approximate error $|\varepsilon_a|$ with the pre-specified relative error tolerance ε_s. If $|\varepsilon_a| > \varepsilon_s$, then go to Step 3, else stop the algorithm. Note one should also check whether the number of iterations is more than the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user about it.

Example: Find a root of the equation $x^3 - 4x - 9 = 0$ using Bisection Method correct up to 3 decimal places.(summer 23-24)

Sol: Let $f(x) = x^3 - 4x - 9$

Since $f(-2)$ is –ve and $f(3)$ is +ve , a root lies between 2 and 3.

Therefore, the first approximation to the root is $x_1 = \frac{2+3}{2} = 2.5$

Thus $f(2.5) = (2.5)^3 - 4(2.5) - 9 = -3.375$ *i.e.,* $-ve$

Therefore , the root lies between x_1 and 3.

Thus the second approximation to the root is $x_2 = \frac{2.5+3}{2} = 2.75$

Thus , $f(2.75) = (2.75)^3 - 4(2.75) - 9 = 0.7969$ *i.e.,* $+ve$

$\therefore$ the root lies between x_1 *and* x_2.

Thus the third approximation to the root is $x_3 = \frac{2.5+2.75}{2} = 2.625$

Then $f(2.625) = (2.625)^3 - 4(2.625) - 9 = -1.4121$ *i.e.,* $-ve$

The root lies between x_2 *and* x_3 thus the fourth approximation to the root is

$$x_4 = \frac{2.625+2.75}{2} = 2.6875$$

Repeating this process, the successive approximations are

$x_5 = 2.71875$ $x_6 = 2.70313$ $x_7 = 2.71094$
$x_8 = 2.70703$ $x_9 = 2.70508$ $x_{10} = 2.70605$
$x_{11} = 2.70654$ $x_{12} = 2.70642$

Hence the root is 2.70642.

Example: Find the real root of $\cos x - xe^x = 0$ by bisection method correct upto three decimal places.

Solution: Given equation $f(x) = \cos x - xe^x = 0$

$\therefore f(0) = 1$ *and* $f(1) = -2.17$

Therefore, the root lies between 0 and 1. Hence by Bisection method:

n	x_n	$f(x_n)$	$interval$
	$a = 0$	$1 > 0$	
	$b = 1$	-2.17<0	[0,1]
1	0.5	0.053	$[x_1, b]$
2	0.75	-0.85	$[x_1, x_2]$
3	0.625	-0.3567	$[x_1, x_3]$
4	0.5625	-0.14	$[x_1, x_4]$
5	0.5313	-0.04	$[x_1, x_5]$
6	0.5157	0.0064	$[x_6, x_5]$
7	0.5235	-0.017	$[x_6, x_7]$
8	0.5196	-0.0056	$[x_6, x_8]$
9	0.5177	0.0003	$[x_9, x_8]$
10	0.5187	-0.0027	$[x_9, x_{10}]$
11	0.5182		

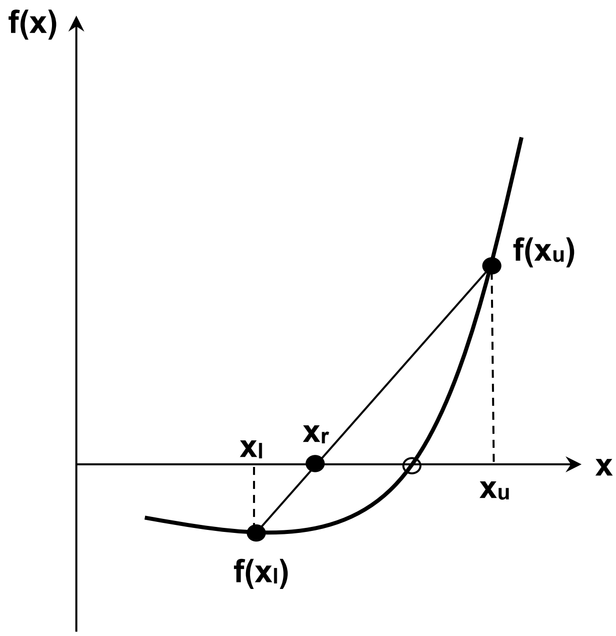
∴ The root correct to three decimal places is 0.5182

Example:

(1) Find the real root of the equation $3x = \cos x + 1$, correct up to four decimal places. $(x = 0.6071)$

(2) Find the real root of the equation $xe^x = 1$, correct up to four decimal places. $(x = 0.5671)$

2) Regula-Falsi Method (False-Position Method):



A shortcoming of the bisection method is that in dividing the interval from x_l to x_u into equal halves, no account is taken of the magnitude of $f(x_i)$ and $f(x_u)$. Indeed, if $f(x_i)$ is close to zero, the root is more close to x_l than x_0.

The false position method uses this property:

A straight line joins $f(x_i)$ and $f(x_u)$. The intersection of this line with the x-axis represents an improvement estimate of the root. This new root can be computed as:

$$\frac{f(x_l)}{x_r - x_l} = \frac{f(x_u)}{x_r - x_u}$$

Giving $x_r = x_u - \dfrac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$ or $x_r = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$ This is called the false-position formula

Then, x_r replaces the initial guess for which the function value has the same sign as $f(x_r)$
The difference between bisection method and false-position method is that in bisection method, both limits of the interval have to change. This is not the case for false position method, where one limit may stay fixed throughout the computation while the other guess converges on the root.

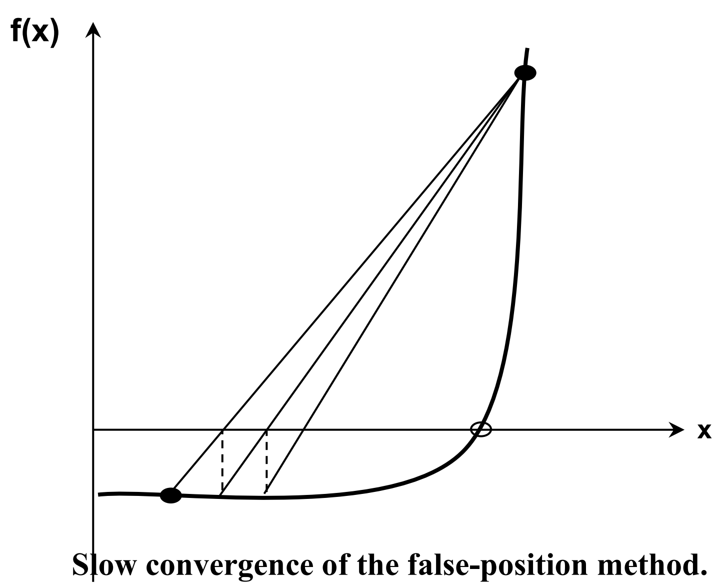
Working rule for False-Position Method (Regula-Falsi Method):

1. Define the first interval (a,b) such that solution exists between them. Check $f(a)f(b)<0$.
2. Compute the first estimate of the numerical solution x_r using the above equation.
3. Find out whether the actual solution is between a and x_r or x_r and b .This is accomplished by checking the sign of the product $f(a)f(x_r)$

- If $f(a)f(x_r)<0$, the solution is between a and x_r.
- If $f(a)f(x_r)>0$, the solution is between x_r and b

4. Select the subinterval that contains the solution (a,x_r) or (x_r,b) is the new interval (a,b) and go to step no.2 and repeat the process. The method of False Position always converges to an answer, provided a root is initially bracketed in the interval (a,b).

Pitfalls of the false-position method:
Although, the false position method is an improvement of the bisection method. In some cases, the bisection method will converge faster and yields to better results



Slow convergence of the false-position method.

Example Using the False Position method, find a root of the function $f(x)=e^x-3x^2$ to an accuracy of 5 digits. The root is known to lie between 0.5 and 1.0.
Sol: We apply the method of False Position with a = 0.5 and b = 1.0. and the equation is

$$x_r=\frac{af(b)-bf(a)}{f(b)-f(a)}$$

The calculations based on the method of False Position are shown in the Table

n	A	b	$f(a)$	$f(b)$	x_r	$f(x_r)$
1	0.5	1	0.89872	-0.28172	0.88067	0.08577
2	0.88067	1	0.08577	-0.28172	0.90852	0.00441
3	0.90852	1	0.00441	-0.28172	0.90993	0.00022
4	0.90993	1	0.00022	-0.28172	0.91000	0.00001
5	0.91000	1	0.00001	-0.28172	0.91001	0

The root is 0.91 accurate to five digits.

Example: Use False-position method to find a root of the function $x^2 - x - 2 = 0$ in th range $1 < x < 3$, correct upto three decimal places.

Solution: We have $f(x) = x^2 - x - 2 = 0$. Given that $x_0 = 1, x_1 = 3$. $\therefore f(x_0) = -2, f(x_1) = 4$

Using false position method, we have $x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$

n	x_n	$f(x_n)$	a	b
0	1	-2		
1	3	4	1	3
2	1.6667	-0.8889	1.6667	3
3	1.9091	-0.2644	1.9091	3
4	1.9767	-0.0692	19767	3
5	1.9941	-0.0177	1.9941	3
6	1.9985	-0.0044	1.9985	3
7	1.9996	-0.0012	1.9996	3
8	1.9999	-0.0012		

Example: (1) Find a positive root of $x^3 - 4x + 1$ correct up to three decimal places. ($x = 0.254$)

(2) Find a positive root of $x \log_{10} x = 1.2$ correct up to three decimal places. ($x = 2.740$)

(3) Find the real root of the equation xex−1=0 lies between 0.3 and 0.7 using Regula Falsi method. (Perform only three steps).**(Winter 23-24)**

3) Newton-Raphson method:

Newton-Raphson method is based on the principle that if the initial guess of the root of f(x)=0 is at x_i, then if one draws the tangent to the curve at $f(x_i)$, the point x_{i+1} where the tangent crosses the x-axis is an improved estimate of the root (Figure 3.12).

Using the definition of the slope of a function, at $x = x_i$

$$f'(x_i) = = \frac{f(x_i) - 0}{x_i - x_{i+1}} \quad \text{which gives} \quad x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

This equation is called the Newton-Raphson formula for solving nonlinear equations of the form $f(x) = 0$. So starting with an initial guess, x_i, one can find the next guess, x_{i+1}, by using the above equation. One can repeat this process until one finds the root within a desirable tolerance.

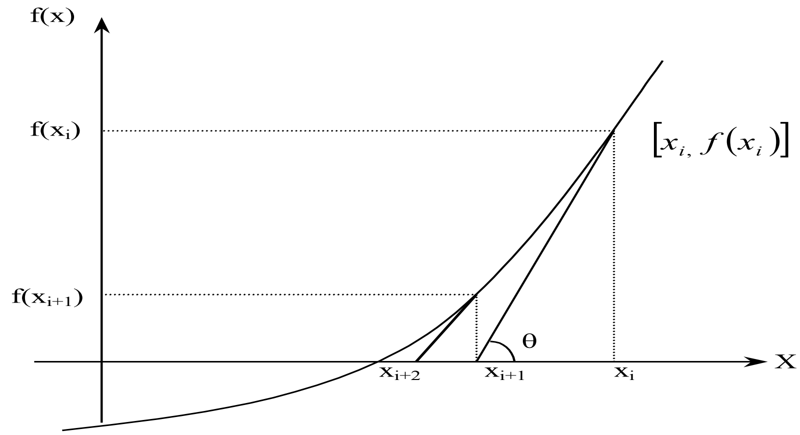


Figure. Geometrical illustration of the Newton-Raphson method.

Working rule for Newton-Raphson method:

The steps to apply using Newton-Raphson method to find the root of an equation f(x) = 0 are

1. Evaluate f′(x) symbolically
2. Use an initial guess of the root, x_i, to estimate the new value of the root x_{i+1} as

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

3. Find the absolute relative approximate error, $|\varepsilon_a|$ as

$$|\varepsilon_a| = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| \times 100$$

4. Compare the absolute relative approximate error, $|\varepsilon_a|$ with the pre-specified relative error tolerance, ε_s. If $|\varepsilon_a| > \varepsilon_s$, then go to step 2, else stop the algorithm. Also, check if the number of iterations has exceeded the maximum number of iterations.

Drawbacks of Newton-Raphson Method:

Divergence at inflection points: If the selection of a guess or an iterated value turns out to be close to the inflection point of f(x), that is, near where f'(x)=0, the roots start to diverge away from the true root.

- Division by zero or near zero: The formula of Newton-Raphson method is

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Consequently if an iteration value, x_i is such that $f'(x_i) \cong 0$, then one can face division by zero or a near-zero number. This will give a large magnitude for the next value, x_{i+1}.

- **Root jumping:** In some case where the function f(x) is oscillating and has a number of roots, one may choose an initial guess close to a root. However, the guesses may jump and converge to some other root.

Example: Find the root of the function $x^3 + x - 1 = 0$ **correct up to four decimal places.**

Sol : Let, $f(x) = x^3 + x - 1$, $f(0) = -1$ and $f(1) = 1$

Since, $f(0) < 0$ and $f(1) > 0$,the root lies between 0 and 1. Let $x_0 = 1$

$$f'(x) = 3x^2 + 1$$

By the Newton-Raphson method , $x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$

$$f(x_0) = f(1) = 1$$
$$f'(x_0) = f'(1) = 4$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1}{4} = 0.75$$

$$\therefore \quad f(x_1) = f(0.75) = 0.171875$$
$$\therefore \quad f'(x_1) = f'(0.75) = 2.6875$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.75 - \frac{0.171875}{2.6875} = 0.68605$$

$$\therefore \quad f(x_2) = f(0.68605) = 0.00894$$
$$\therefore \quad f'(x_2) = f'(0.68605) = 2.41198$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.68605 - \frac{0.00894}{2.41198} = 0.68234$$

$$\therefore \quad f(x_3) = f(0.68234) = 0.000028$$
$$\therefore \quad f'(x_3) = f'(0.68234) = 2.39676$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.68234 - \frac{0.000028}{2.39676} = 0.68233$$

Since x_3 *and* x_4 are same up to four decimal places, the root is 0.6823.

Example :Find a real root of $xe^x = 2$, correct upto three decimal places, by using Newton-Raphson's method.

Solution: Let $f(x) = xe^x - 2 = 0$

$\therefore f(0) = -2, f(1) = 0.71$

$\therefore$ The root lies between 0 and 1.

Using Newton-Raphson's method, we have $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, n = 0,1,2, \ldots$

Choose the initial approximation $x_0 = 0.5$

Here, $f'(x) = e^x + xe^x = e^x(1 + x)$

n	x_n	$f(x_n)$	$f'(x_n)$
0	0.5	-1.1756	2.4731
1	0.9754	0.5867	5.2392
2	0.8634	0.0473	4.4185
3	0.8527	0.0004	4.3464
4	0.8526		

$\therefore$ The approximate value to the root = 0.8526

Example:

(1) Find the root of the function $x^4 - x - 10 = 0$ correct up to three decimal places. $(x = 2.7406)$

(2) Find the root of the function $e^x = 3x$ correct up to three decimal places. $(x = 1.512)$

Some deduction from Newton-Raphson's Formula:

We can derive the following useful results from the Newton-Raphson's Formula

1) Iteration formula for $\frac{1}{N}$:

Let $x = \frac{1}{N} \Rightarrow N = \frac{1}{x} \Rightarrow \frac{1}{x} - N = 0$,

Also let $f(x) = \frac{1}{x} - N$ *and* $f'(x) = -\frac{1}{x^2}$

By the Newton-Raphson method ,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{\left(\frac{1}{x_i} - N\right)}{\left(-\frac{1}{x_i^2}\right)}$$

$$= x_i + x_i^2\left(\frac{1}{x_i} - N\right)$$

$$= x_i(2 - Nx_i)$$

2) Iteration formula for $\sqrt{N}$:

Let $x = \sqrt{N} \Rightarrow N = x^2 \Rightarrow x^2 - N = 0$ Also let $f(x) = x^2 - N$ *and* $f'(x) = 2x$ By the Newton-Raphson method ,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{\left(x_i^2 - N\right)}{\left(2x_i\right)}$$

$$= x_i - \frac{1}{2}\left(x_i - \frac{N}{x_i}\right)$$

$$= \frac{1}{2}\left(x_i + \frac{N}{x_i}\right)$$

3) Iteration formula for $\frac{1}{\sqrt{N}}$:

Let $x = \frac{1}{\sqrt{N}} \Rightarrow x^2 = \frac{1}{N} \Rightarrow x^2 - \frac{1}{N} = 0$, Also let $f(x) = x^2 - \frac{1}{N}$ *and* $f'(x) = 2x$

By the Newton-Raphson method ,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{\left(x_i^2 - \frac{1}{N}\right)}{\left(2x_i\right)}$$

$$= \left(\frac{2x_i^2 - x_i^2 + \frac{1}{N}}{2x_i}\right)$$

$$= \frac{1}{2}\left(x_i + \frac{1}{x_i N}\right)$$

4) Iteration formula for $\sqrt[k]{N}$:

Let $x = \sqrt[k]{N} \Rightarrow N = x^k \Rightarrow x^k - N = 0$, Also let $f(x) = x^k - N$ *and* $f'(x) = kx^{k-1}$ By the Newton-Raphson method ,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{\left(x_i^{\,k} - N\right)}{\left(kx_i^{\,k-1}\right)}$$

$$= \left(\frac{(x_i)(kx_i^{\,k-1}) - \left(x_i^{\,k} - N\right)}{kx_i^{\,k-1}} \right)$$

$$= \frac{1}{k}\left((k-1)x_i + \frac{N}{x_i^{\,k-1}} \right)$$

Example: Evaluate the following (correct to four decimal places) by Newton's iteration method:

$$(i)\ \frac{1}{31} \quad (ii)\ \sqrt{24} \quad (iii)\ 1/\sqrt{14} \quad (iv)\ \sqrt[3]{24}$$

Sol: $(i)\ \frac{1}{31}$ Taking $N = 31$ in the equation $x_{i+1} = x_i(2 - Nx_i)$

We have $x_{i+1} = x_i(2 - 31x_i)$, Since an approximate value of 1/30=0.03, we ta $x_0 = 0.03$

Then

$$x_1 = x_0(2 - 31x_0) = 0.03(2 - 31(0.03)) = 0.0321$$
$$x_2 = x_1(2 - 31x_1) = 0.0321(2 - 31(0.0321)) = 0.032257$$
$$x_3 = x_2(2 - 31x_2) = 0.032257(2 - 31(0.032257)) = 0.03226$$

Since $x_2 = x_3$ up to 4 decimal places, we have $1/31 = 0.0323$

$(ii)\ \sqrt{28}$

Taking $N = 24$ in the equation $x_{i+1} = \frac{1}{2}\left(x_i + \frac{N}{x_i} \right)$

We have $x_{i+1} = \frac{1}{2}\left(x_i + \frac{28}{x_i} \right)$, Since the value of $\sqrt{25} = 5$, we take $x_0 = 5$.

Then

$$x_1 = \frac{1}{2}\left(x_0 + \frac{28}{x_0} \right) = \frac{1}{2}\left(5 + \frac{28}{5} \right) = 5.3$$
$$x_2 = \frac{1}{2}\left(x_1 + \frac{28}{x_1} \right) = \frac{1}{2}\left(5.3 + \frac{28}{5.3} \right) = 5.29151$$
$$x_3 = \frac{1}{2}\left(x_2 + \frac{28}{x_2} \right) = \frac{1}{2}\left(5.29151 + \frac{28}{5.29151} \right) = 5.29150$$

Since $x_2 = x_3$ up to 4 decimal places, we have $\sqrt{28} = 5.2915$.

$(iii)\ 1/\sqrt{14}$

Taking $N = 14$ in the equation $x_{i+1} = \frac{1}{2}\left(x_i + \frac{1}{x_i N}\right)$

We have $x_{i+1} = \frac{1}{2}\left(x_i + \frac{1}{x_i 14}\right)$, Since the value of $1/\sqrt{16} = 1/4 = 0.25$, we take $x_0 = 0.25$.

Then

$$x_1 = \frac{1}{2}\left(x_0 + \frac{1}{x_0 14}\right) = \frac{1}{2}\left((0.25) + \frac{1}{(0.25)14}\right) = 0.26785$$

$$x_2 = \frac{1}{2}\left(x_1 + \frac{1}{x_1 14}\right) = \frac{1}{2}\left((0.26785) + \frac{1}{(0.26785)14}\right) = 0.2672618$$

$$x_3 = \frac{1}{2}\left(x_2 + \frac{1}{x_2 14}\right) = \frac{1}{2}\left((0.2672618) + \frac{1}{(0.2672618)14}\right) = 0.2672612$$

Since $x_2 = x_3$ up to 4 decimal places, we have $1/\sqrt{14} = 0.2673$.

$(iv)\ \sqrt[3]{24}$

Taking $N = 24$ *and* $k = 3$ in the equation $x_{i+1} = \frac{1}{k}\left((k-1)x_i + \frac{N}{x_i^{k-1}}\right)$

We have $x_{i+1} = \frac{1}{3}\left((2)x_i + \frac{24}{x_i^2}\right)$, Since the value of $27^{\frac{1}{3}} = 3$, we take $x_0 = 3$.

Then

$$x_1 = \frac{1}{3}\left((2)x_0 + \frac{24}{x_0^2}\right) = \frac{1}{3}\left((2)(3) + \frac{24}{(3)^2}\right) = 2.8889$$

$$x_2 = \frac{1}{3}\left((2)x_1 + \frac{24}{x_1^2}\right) = \frac{1}{3}\left((2)(2.8889) + \frac{24}{(2.8889)^2}\right) = 2.88451$$

$$x_3 = \frac{1}{3}\left((2)x_2 + \frac{24}{x_2^2}\right) = \frac{1}{3}\left((2)(2.88451) + \frac{24}{(2.88451)^2}\right) = 2.8845$$

Since $x_2 = x_3$ up to 4 decimal places, we have $\sqrt[3]{24} = 2.8845$

Since $x_2 = x_3$ up to 4 decimal places, we have $\sqrt[3]{24} = 2.8845$