Numerical Integration

Parul Institute of Engineering & Technology

For CSE , IT , Mech, Civil Students

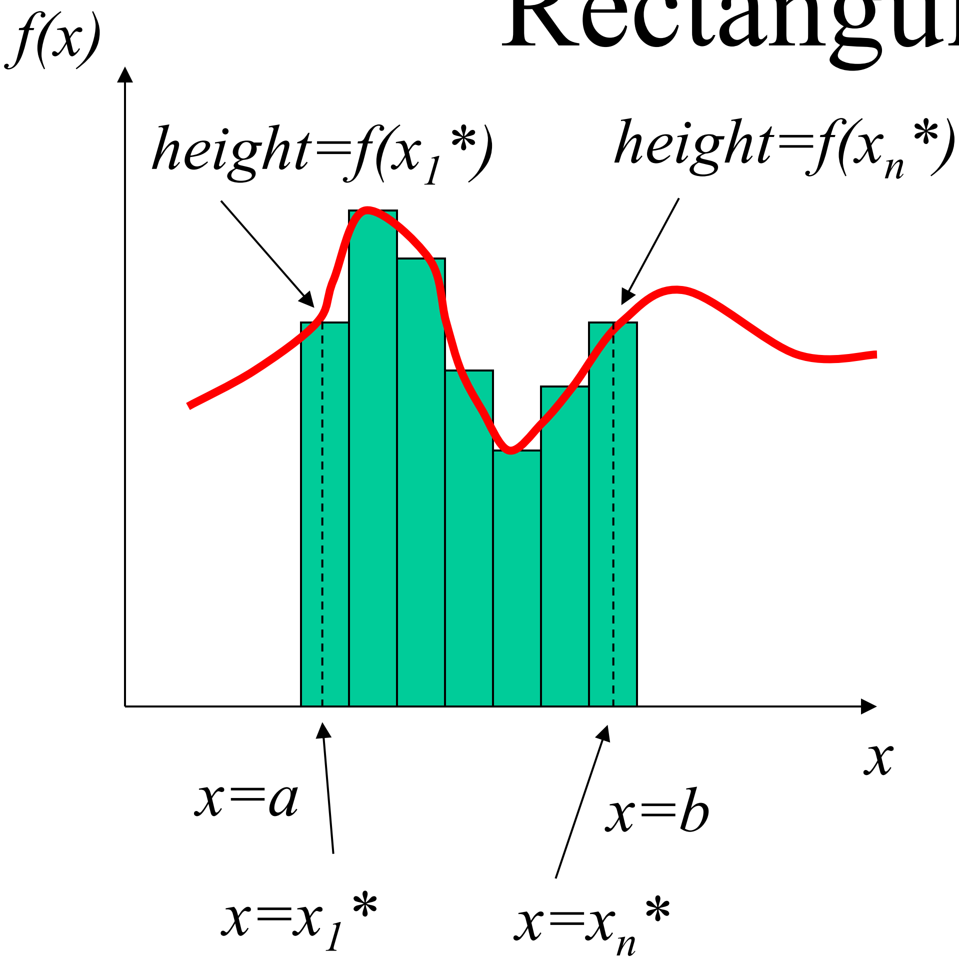
Numerical Integration

- In general, a numerical integration is the approximation of a definite integration by a "weighted" sum of function values at discretized points within the interval of integration.

$$\int_a^b f(x)dx \approx \sum_{i=0}^{N} w_i f(x_i)$$

where w_i is the weighted factor depending on the integration schemes used, and $f(x_i)$ is the function value evaluated at the given point x_i

Rectangular Rule

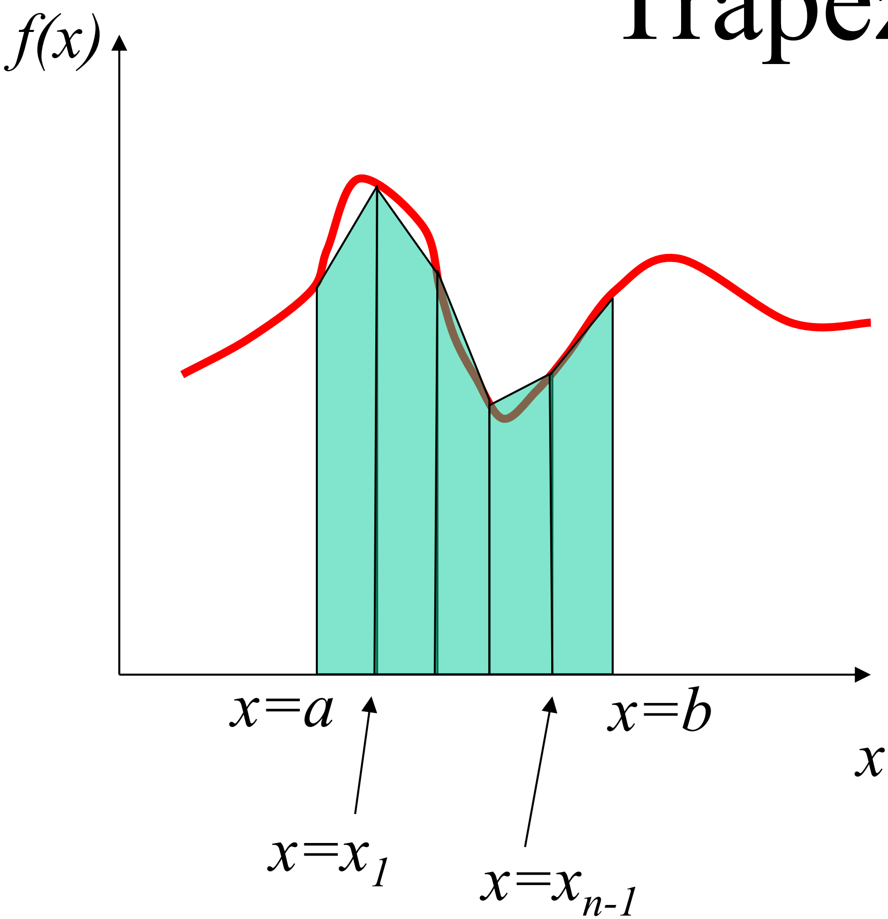


Approximate the integration, $\int_a^b f(x)dx$, that is the area under the curve by a series of rectangles as shown.
The base of each of these rectangles is $\Delta x=(b-a)/n$ and its height can be expressed as $f(x_i^*)$ where x_i^* is the midpoint of each rectangle

$$\int_a^b f(x)dx = f(x_1^*)\Delta x + f(x_2^*)\Delta x + .. f(x_n^*)\Delta x$$

$$= \Delta x[f(x_1^*) + f(x_2^*) + .. f(x_n^*)]$$

Trapezoidal Rule



The rectangular rule can be made more accurate by using trapezoids to replace the rectangles as shown. A linear approximation of the function locally sometimes work much better than using the averaged value like the rectangular rule does.

$$\int_a^b f(x)dx = \frac{\Delta x}{2}[f(a)+f(x_1)] + \frac{\Delta x}{2}[f(x_1)+f(x_2)] + .. + \frac{\Delta x}{2}[f(x_{n-1})+f(b)]$$

$$= \Delta x[\frac{1}{2}f(a)+f(x_1)+..f(x_{n-1})+\frac{1}{2}f(b)]$$

Simpson's Rule

Still, the more accurate integration formula can be achieved by approximating the local curve by a higher order function, such as a quadratic polynomial. This leads to the Simpson's rule and the formula is given as:

$$\int_a^b f(x)dx = \frac{\Delta x}{3}[f(a) + 4f(x_1) + 2f(x_2) + 4f(x_3) + ..$$

$$..2f(x_{2m-2}) + 4f(x_{2m-1}) + f(b)]$$

It is to be noted that the total number of subdivisions has to be an even number in order for the Simpson's formula to work properly.

Examples

Integrate $f(x) = x^3$ between $x = 1$ and $x = 2$.

$$\int_1^2 \mathrm{x}^3 \mathrm{dx} = \frac{1}{4} x^4 \Big|_1^2 = \frac{1}{4}(2^4 - 1^4) = 3.75$$

Using 4 subdivisions for the numerical integration: $\Delta \mathrm{x} = \dfrac{2\text{-}1}{4} = 0.25$

Rectangular rule:

i	x_i*	$f(x_i$*$)$
1	*1.125*	*1.42*
2	*1.375*	*2.60*
3	*1.625*	*4.29*
4	*1.875*	*6.59*

$$\int_1^2 x^3 dx$$
$$= \Delta x[f(1.125) + f(1.375) + f(1.625) + f(1.875)]$$
$$= 0.25(14.9) = 3.725$$

Trapezoidal Rule

i	x_i	$f(x_i)$
	1	*1*
1	*1.25*	*1.95*
2	*1.5*	*3.38*
3	*1.75*	*5.36*
	2	*8*

$$\int_1^2 x^3 dx$$

$$= \Delta x[\frac{1}{2} f(1) + f(1.25) + f(1.5) + f(1.75) + \frac{1}{2} f(2)]$$

$$= 0.25(15.19) = 3.80$$

Simpson's Rule

$$\int_1^2 x^3 dx = \frac{\Delta x}{3}[f(1) + 4f(1.25) + 2f(1.5) + 4f(1.75) + f(2)]$$

$$= \frac{0.25}{3}(45) = 3.75 \;\Rightarrow \text{perfect estimation}$$