First Order Differential Equations

- An equation that defines a relationship between an unknown function and one or more of its derivatives is referred to as a *differential equation.*
- A first order differential equation:

$$\frac{dy}{dx} = f(x, y)$$

- Example:

$$\frac{dy}{dx} = 5x, \quad \text{with boundary condition } y = 2 \text{ at } x = 1.$$

Solving it, we get $y = \frac{5}{2}x^2 + c$

Substituting $y = 2$ and $x = 1$, we obtain $y = 2.5x^2 - 0.5$

- Example:

$$\frac{dy}{dx} = c(y - x)$$

- A second-order differential equation:

$$\frac{d^2y}{dx^2} = f(x, y, \frac{dy}{dx})$$

- Example:

$$y'' = 2x + xy + y'$$

Taylor Series Expansion

- Fundamental case, the first-order ordinary differential equation:

$$\frac{dy}{dx} = f(x) \qquad \text{subject to } y = y_0 \text{ at } x = x_0$$

Integrate both sides

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x)dx \qquad \text{or} \quad y = g(x) = y_0 + \int_{x_0}^{x} f(x)dx$$

- The solution based on Taylor series expansion:

$$y = g(x) = g(x_0) + (x - x_0)g'(x) + \frac{(x - x_0)^2}{2!} g''(x_0) + \ldots$$

where $y_0 = g(x_0)$ and $g'(x_0) = f(x_0)$

Example : First-order Differential Equation

Given the following differential equation:

$$\frac{dy}{dx} = 3x^2 \quad \text{such that } y = 1 \text{ at } x = 1$$

The higher-order derivatives:

$$\frac{d^2y}{dx^2} = 6x$$

$$\frac{d^3y}{dx^3} = 6$$

$$\frac{d^ny}{dx^n} = 0 \quad \text{for n} \geq 4$$

The final solution:

$$g(x) = 1 + (x-1)\frac{dy}{dx} + \frac{(x-1)^2}{2!}\frac{d^2y}{dx^2} + \frac{(x-1)^3}{3!}\frac{d^3y}{dx^3}$$

$$= 1 + (x-1)(3{x_0}^2) + \frac{(x-1)^2}{2!}(6x_0) + \frac{(x-1)^3}{3!}(6)$$

$$= 1 + 3(x-1) + 3(x-1)^2 + (x-1)^3$$

where $x_0 = 1$

Table: Taylor Series Solution

x	One Term	Two Terms	Three Terms	Four Terms
1	1	1	1	1
1.1	1	1.3	1.33	1.331
1.2	1	1.6	0.72	1.728
1.3	1	1.9	2.17	2.197
1.4	1	2.2	2.68	2.744
1.5	1	2.5	3.25	3.375
1.6	1	2.8	3.88	4.096
1.7	1	3.1	4.57	4.913
1.8	1	3.4	5.32	5.832
1.9	1	3.7	6.13	6.859
2	1	4	7	8

General Case

- The general form of the first-order ordinary differential equation:

$$\frac{dy}{dx} = f(x, y) \quad \text{subject to } y = y_0 \text{ at } x = x_0$$

- The solution based on Taylor series expansion:

$$y = g(x) = g(x_0, y_0) + (x - x_0)g'(x_0, y_0) + \frac{(x - x_0)}{2!}g''(x_0, y_0) + \ldots$$

Euler's Method

- Only the term with the first derivative is used:

$$g(x) = g(x_0) + (x - x_0)\frac{dy}{dx} + e$$

- This method is sometimes referred to as *the one-step Euler's method*, since it is performed one step at a time.

Example: One-step Euler's Method

- Consider the differential equation:

$$\frac{dy}{dx} = 4x^2 \text{ such that } y = 1 \text{ at } x = 1$$

- For x =1.1

$$\int_1^y dy = \int_1^{1.1} 4x^2 dx$$

$$y - 1 = \frac{4}{3} x^3 \Big|_1^{1.1} = 0.44133$$

Therefore, at x=1.1, y=1.44133 (true value).

With a step size of $\Delta x = (x - x_0) = 0.1$, we get

$$g(1.1) = 1 + 0.1[4(1)^2] = 1.4$$

The error $= 0.04133$ (in absolute value).

Use a step size of 0.05 and apply Euler's equation twice (at $x = 1$ and $x = 1.05$):

$$g(1.05) = g(1) + (1.05 - 1.00)[4(1)^2] = 1 + 0.2 = 1.2$$

$$g(1.10) = g(1.05) + (1.10 - 1.05)[4(1.05)^2] = 1.4205$$

The error is reduced to 0.020833.

For a step size of 0.02, after five steps, the estimated value

$$\mathrm{g}(1.10) = 1.43296$$

The error is 0.008373.

Errors with Euler`s Method

- *Local error*: over one step size.

  Global error: cumulative over the range of the solution.
- The error ε using Euler`s method can be approximated using the second term of the Taylor series expansion as

$$\varepsilon = \frac{(x - x_0)^2}{2!} \frac{d^2 y}{dx^2}$$

where $\frac{d^2 y}{dx^2}$ is the maximum in $[x_0, x]$.

- If the range is divided into n increments, then the error at the end of range for x would be $n\varepsilon$.

Modified Euler's Method

Use an average slope, rather than the slope at the start of the interval :

a. Evaluate the slope at the start of the interval
b. Estimate the value of the dependent variable y at the end of the interval using the Euler's metod.
c. Evaluate the slope at the end of the interval.
d. Find the average slope using the slopes in a and c.
e. Compute a revised value of the dependent variable y at the end of the interval using the average slope of step d with Euler's method.

Example : Modified Euler's Method

$$\frac{dy}{dx} = x\sqrt{y} \qquad \text{such that } y = 1 \text{ at } x = 1$$

The five steps of the first iteration for $\Delta x = 0.1$:

1a. $\left.\dfrac{dy}{dx}\right|_1 = 1\sqrt{1} = 1$

1b. $g(1.1) = g(1.0) + (1.1 - 1.0)\left.\dfrac{dy}{dx}\right|_1 = 1 + 0.1(1) = 1.1$

1c. $\left.\dfrac{dy}{dx}\right|_{1.1} = 1.1\sqrt{1.1} = 1.15369$

1d. $\left.\dfrac{dy}{dx}\right|_a = \dfrac{1}{2}(1 + 1.15369) = 1.07684$

1e. $g(1.1) = g(1.0) + (1.1 - 1.0)\left.\dfrac{dy}{dx}\right|_a = 1 + 0.1(1.07684) = 1.10768$

The steps for the second interval :

2a. $\left.\frac{dy}{dx}\right|_{1.1} = x\sqrt{y} = 1.1\sqrt{1.10768} = 1.15771$

2b. $g(1.2) = g(1.1) + (1.2 - 1.1)\left.\frac{dy}{dx}\right|_{1.1} = 1.10768 + 0.1(1.15771) = 1.22345$

2c. $\left.\frac{dy}{dx}\right|_{1.2} = 1.2\sqrt{1.22345} = 1.32732$

2d. $\left.\frac{dx}{dy}\right|_{a} = \frac{1}{2}(\left.\frac{dx}{dy}\right|_{1.1} + \left.\frac{dx}{dy}\right|_{1.2}) = 1.24251$

2e. $g(1.2) = g(1.1) + (1.2 - 1.1)\left.\frac{dy}{dx}\right|_{a} = 1.23193$

Second-order Runge-Kutta Methods

- The modified Euler's method is a case of the second-order Runge-Kutta methods. It can be expressed as

$$y_{i+1} = y_i + 0.5[f(x_i, y_i) + f(x_i + h, y_i + hf(x_i, y_i))]h$$

where $y_i = g(x_i), \quad y_{i+1} = g(x_i + \Delta x),$

$$x_{i+1} = x_i + \Delta x, \quad h = \Delta x$$

- The computations according to Euler's method:

1. Evaluate the slope at the start of an interval, that is, at (x_i, y_i) .

$$S_1 = f(x_i, y_i)$$

2. Evaluate the slope at the end of the interval (x_{i+1}, y_{i+1}) :

$$S_2 = f(x_i + h, y_i + hS_1)$$

3. Evaluate y_{i+1} using the average slope S_1 of and S_2 :

$$y_{i+1} = y_i + 0.5(S_1 + S_2)h$$

Fourth-order Runge-Kutta Methods

$$\frac{dy}{dx} = f(x, y) \quad \text{such that } y = y_0 \text{ at } x = x_0 \qquad \Delta x = h.$$

1. Compute the slope S_1 at (x_i, y_i).

$$S_1 = f(x_i, y_i)$$

2. Estimate y at the mid-point of the interval.

$$y_{i+1/2} = y_i + \frac{h}{2} f(x_i, y_i)$$

3. Estimate the slope S_2 at mid-interval.

$$S_2 = f(x_i + 0.5h, y_i + 0.5hS_1)$$

4. Revise the estimate of y at mid-interval

$$y_{i+1/2} = y_i + \frac{h}{2} S_2$$

5. Compute a revised estimate of the slope S_3 at mid-interval.

$$S_3 = f(x_i + 0.5h, y_i + 0.5hS_2)$$

6. Estimate y at the end of the interval.

$$y_{i+1} = y_i + hS_3$$

7. Estimate the slope S_4 at the end of the interval

$$S_4 = f(x_i + h, y_i + hS_3)$$

8. Estimate y_{i+1} again.

$$y_{i+1} = y_i + \frac{h}{6}(S_1 + 2S_2 + 2S_3 + S_4)$$