Interpolation

NUMERICAL METHODS

Introduction:

The study of finite difference calculus has become very important due to its wide variety of application in everyday life. It has ben of great use for mathematicians and was originated by Sir Issac Newton.

1.7 Separation of symbols

The relationship between the operators E and Δ can be used to prove a number of useful identities. Recall that the method is known as separation of symbols.

Example : Construct difference table for the following data

X	45	50	55	60	65
F(x)	2.871	2.404	2.083	1.862	1.712

Table 1.4 (Difference Table)

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
45	2.871				
		-0.467			
50	2.404		0.146		
		-0.321		-0.046	
55	2.083		0.100		0.017
		-0.221		-0.029	
60	1.862		0.071		
		-0.150			
65	1.712				

Interpolation

Argument Given
Equal Intervals

Argument Given
At Unequal intervals

Newton' Forward & Backward Difference

Central Differences Gauss Difference

Divided Differences

Lagrange's Method

- **In Equal Interval the following situations are experienced:**
- Given n observations we want to find some intermediate term.
- Knowing .... Observations out of n the missing observation is to be calculated.
- Knowing n-m observations out of n. the unknown m observations are to be determined.

CASE I EQUAL INTERVALS

NEWTON – GREGORY FORMULA

Given X a   a+h   a+2h   ……   a+nh

f(x)   f(a)   f(a+h)   f(a+2h) ….. f(a+nh)

To find f(x) for x=a+mh.

The using ordinary difference and assuming that the given set of values can be expressed by a polynomial of degree n.

$f(a+mh) = f(a)+{}^{m}C_1\Delta f(a)+ {}^{n}C_2\Delta^2 f(a) + \ldots + \Delta^n f(a)$

Example

In an examination the number of candidates who obtained marks between certain limits were as follows:

Marks	No. of Candidates
00-19	41
20-39	62
40-59	65
60-79	50
80-99	17

Estimate the number of candidates who obtained fewer than 70 marks.

Sol. Construct a cumulative frequency table to solve this problem.

Marks	f	lessthan	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0-19	41	19	41				
				62			
20-39	62	39	103		3		
				65		-18	
40-59	65	59	168		-15		0
				50		-18	
60-79	50	79	218		-33		
				17			
80-99	77	99	235				

We want to find the number of candidates getting less than 70 marks

here h=39-19 = 59-38 = 20

$\therefore$ x0 = 19 , x = 70 i.e. p = 2.55

By Newton Gregory formula

$$f(x) = y_0 + p\,\Delta y_0 + \frac{p\,(p-1)}{2!}\,\Delta^2 y_0 + \frac{p\,(p-1)(p-2)}{3!}\,\Delta^3 y_0 + \ \dots + \frac{p\,(p-1)(p-2)\dots(p-(n-1))}{n!}\,\Delta^n y_0$$

where $p = \frac{x-x_0}{h}$

$$= 41 + 2.55(62) + \frac{2.55(2.55-1)}{2}(3) + \frac{2.55(2.55-1)(2.55-2)}{6}(-18) + 0$$

$=198.405$

Case of Missing terms

Ex. Given $\mu_0 = 580$, $\mu_1 = 556$, $\mu_2 = 520$ and $\mu_4 = 580$ find $\mu_{3.}$

It is a typical case of one missing observation. There can be five observations μ_0 , μ_1 , μ_2 , μ_3 , μ_4 at equal interval of width 1 but one of them i.e. μ_3 is missing. Only four observations are known so we can fit a polynomial of degree three and 4^{th} degree difference will be zero.

In Equal Intervals the following situations are experienced:

(i) Given n observations we want to some intermediate term.

(ii) Knowing $\overline{n-1}$ observations out of n the missing observation is to be calculated.

(iii) Knowing n-m observations out of n, the unknown m observations are to be determined.

Some Typical Examples:

I. Find the Missing Term in the following table and explain why the resulting value differ from 3^3?

X :	0	1	2	3	4
Y :	1	3	9	?	81

II. The following table gives

X :	0	1	2	3	4
Y :	3	6	11	18	27

What is the form of the function f(x) ?

NEWTONS BACKWARD DIFFERENCES

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$
a	f(a)		
a+h	f(a+h)	f(a+h) - f(a) = ∇ f(a+h)	∇ f(a+2h) - ∇ f(a+h) = ∇^2 f(a+2h)
a+2h	f(a+2h)	f(a+2h)-f(a+h) = ∇ f(a+2h)	∇ f(a+3h) - ∇ f(a+2h) = ∇^2 f(a+3h)
a+3h	f(a+3h)	f(a+3h)-f(a+2h) = ∇f(a+3h)	=∇^2 f(a+3h) - ∇^2 f(a+2h) = ∇^3 f(a+3h)

The Newton backward interpolation formula is given by

$$f(x) = y_n + p\,\nabla y_n + \frac{p\,(p+1)}{2!}\,\nabla^2 y_n + \frac{p\,(p+1)(p+2)}{3!}\,\nabla^3 y_0 + \,...+\, \frac{p\,(p+1)(p+2)...(p+(n-1))}{n!}\,\nabla^n y_0$$

where $p = \frac{x-x_0}{h}$

The formula is suitable if one wants to interpolate near the bottom of the of the table.

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$	$\nabla^5 f(x)$	$\nabla^6 f(x)$
100	10.63						
		2.40					
150	13.03		-0.39				
		2.01		0.15			
200	15.04		-0.24		-0.07		
		1.77		0.08		0.02	
250	16.81		-0.16		-0.05		0.02
		1.61		0.03		0.04	
300	18.42		-0.13		-0.01		
		1.48		0.01			
350	19.90		-0.11				
		1.37					
400	21.27						

$$x0 = 400$$

$$h = 50$$

$$x = 410$$

$$\therefore 410 - 400 = p(50)$$

$$\therefore p = \frac{10}{50} = 0.2$$

Now

$$f(x) = y_n + p\,\nabla y_n + \frac{p\,(p+1)}{2!}\,\nabla^2 y_n + \frac{p\,(p+1)(p+2)}{3!}\,\nabla^3 y_0 + \ ... + \frac{p\,(p+1)(p+2)...(p+(n-1))}{n!}\,\nabla^n y_0$$

where $p = \frac{x - x_0}{h}$

$$= 21.27 + 0.2(1.37) + \frac{(0.2)(1.2)}{2!}(-0.11) + ...$$

Find the unique Polynomial of degree 2 such that

P(1)=1, P(3)=27, P(4)=64

Use Lagrange's method of Interpolation [2002-2003]

Sol: Here x = 1 3 4

Here x	=	1	3	4
P(x)	=	1	27	64

Now by Lagrange's formula

$$P(x)=\frac{P(1)(x-3)(x-4)}{(1-3)(1-4)}+\frac{P(3)(x-1)(x-4)}{(3-1)(3-4)}+\frac{P(4)(x-1)(x-3)}{(4-1)(4-3)}$$

$$P(x)=\frac{1(x-3)(x-4)}{(-2)(-3)}+\frac{27(x-1)(x-4)}{(2)(-1)}+\frac{64(x-1)(x-3)}{(3)(1)}$$

$$P(x)=\frac{x^2-7x+12}{6}+\frac{27(x^2-5x+4)}{-2}+\frac{64(x^2-4x+3)}{3}$$

$$=\frac{1}{6}(x^2-7x+12-81x^2+405x-108+128x^2-512x+384)$$

$$=\frac{1}{6}(48x^2-114x+288)$$

$$=8x^2-19x+48$$

Ex. The following table gives the distance in nautical miles of the visible horizon for the given heights in feet above the earths surface.

X : Height	:100	150	200	250	300	350	400
Y : Distance	:10.63	13.03	15.04	16.81	18.42	19.90	21.27

Use Newtons Gregory's forward formula and backward interpolation formula to find the values of Y when X=218 and 410 [2001-2001]

Sol:

Since x = 218 is nearer is 200 and lies between the given set of values, we prepare forward difference table is :

X	Y	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$	$\Delta^6 f(x)$
100	10.63						
		2.40					
150	13.03		-0.39				
		2.01		0.15			
200	15.04		-0.24		-0.07		
		1.77		0.08		0.02	
250	16.81		-0.16		-0.05		0.02
		1.61		0.03		0.04	
300	18.42		-0.13		-0.01		
		1.48		0.01			
350	19.90		-0.11				
		1.37					
400	21.27						

Here x0 = 100, h = 50, x = 218

$$\therefore 218 - 100 = m(50)$$

$$\therefore m = \frac{118}{50} = 2.36$$

Now Newtons Gregory formula is give by

f(x)

$$= f(a) + m\Delta f(a) + \frac{m(m-1)}{2!}\Delta^2 f(a) + \ldots + \frac{m(m-1)\ldots(m-5)}{6!}\Delta^6 f(a)$$

Ex. Given the following data

X	2.5	3.0	4.5	4.75	6.0
f(x)	8.85	11.45	20.66	22.85	38.60

Find f(3.5)

Sol:

Here the argument is given at unequal intervals. So we construct divided difference table.

X	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
2.5	8.85				
		5.20			
3.0	11.45		0.470		
		6.14		0.456	
4.5	20.66		1.497		-0.0.29
		8.76		0.354	
4.75	22.85		2.560		
		12.60			
6.0	38.60				

Here x = 3.5 lies between 3.0 and 4.5 so we take a = 3.

So we take a =3

Using Newtons Divided difference formula

$$f(x) = f(a) + (x-a)\Delta f(a) + (x-a)(x-b)\Delta^2 f(a)$$

$$+ (x-a)(x-b)(x-c)\Delta^3 f(a) + ...$$

$$= 11.45 + (3.5-3)(6.14) + (3.5-3)(3.5-4.5)(1.497)$$

$$= 13.993 + (3.5-3)(3.5-4.5)(3.5-4.75)(0.354)$$

Ex. Use Gauss forward formula to evaluate y_{30}

$y_{21} = 18.4708$, $y_{25} = 17.8144$, $y_{29} = 17.1070$, $y_{33} = 16.3432$ and $Y_{37} = 15.5154$

Sol. y_{30} lies in the neighbourhood of y_{29} so we can use Gauss forward formula using origin at x_{29} .

The following differences are calculated.

X	Y	ΔY	$\Delta^2 Y$	$\Delta^3 Y$	$\Delta^4 Y$
21	18.4708				
		-0.6564			
25	17.8144		-0.0510		
		-0.7074		.0946	
29	17.1070		-0.0564		.1022
		-0.7638		-0.0076	
33	16.3432		-0.0640		
		-0.8278			
37	15.5154				

The Gauss forward formula is given by

$$f(x) = f(29) + (x-29)\Delta f(30) + (x-29)(x-33)\Delta^2 f(25)+\ldots$$