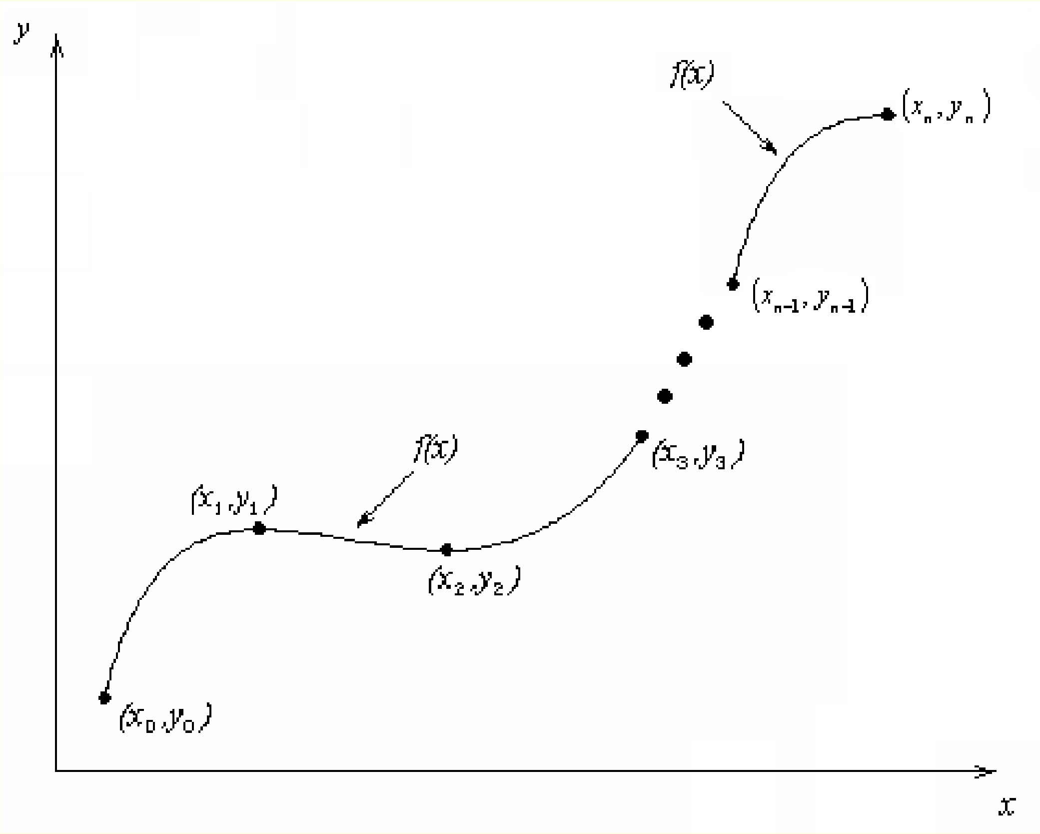
Unit 6

INTERPOLATION

WHAT IS INTERPOLATION?

Given (x_0,y_0), (x_1,y_1), …, (x_n,y_n), finding the value of 'y' at a value of 'x' in (x_0, x_n) is called **interpolation.**



INTERPOLANTS

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate,
- Differentiate, and
- Integrate.

LAGRANGIAN INTERPOLATION

Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where ' n ' in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ given at $(n+1)$ data points as $(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ is a weighting function that includes a product of $(n-1)$ terms with terms of $j = i$ omitted.

Example

A manufacturer of thermistors makes the following observations on a thermistor. Determine the temperature corresponding to 754.8 ohms using the Lagrangian method for interpolation.

R	T
Ohm	°C
1101.0	25.113
911.3	30.131
636.0	40.120
451.1	50.128

Solution

We are interpolating using a cubic polynomial

$$T(R) = \sum_{i=0}^{3} L_i(R)T(R_i)$$

$$= L_0(R)T(R_0) + L_1(R)T(R_1) + L_2(R)T(R_2) + L_3(R)T(R_3)$$

Solution

$$R_o = 1101.0,\ T(R_o) = 25.113 \qquad R_1 = 911.3,\ T(R_1) = 30.131$$

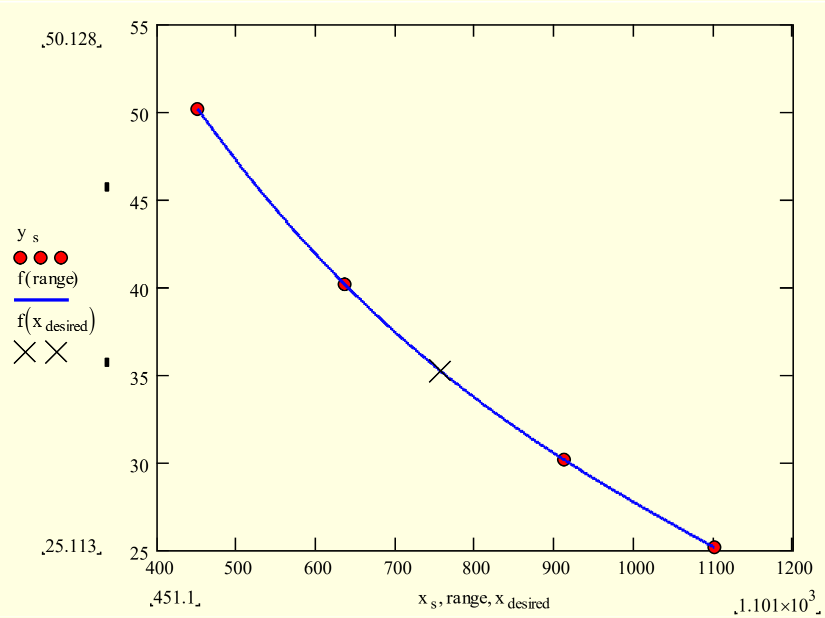
$$R_2 = 636.0,\ T(R_2) = 40.120 \qquad R_3 = 451.1,\ T(R_3) = 50.128$$

$$L_0(R) = \prod_{\substack{j=0\\ j\neq 0}}^{3} \frac{R-R_j}{R_0-R_j} = \left(\frac{R-R_1}{R_0-R_1}\right)\left(\frac{R-R_2}{R_0-R_2}\right)\left(\frac{R-R_3}{R_0-R_3}\right)$$

$$L_1(R) = \prod_{\substack{j=0\\ j\neq 1}}^{3} \frac{R-R_j}{R_1-R_j} = \left(\frac{R-R_0}{R_1-R_0}\right)\left(\frac{R-R_2}{R_1-R_2}\right)\left(\frac{R-R_3}{R_1-R_3}\right)$$

$$L_2(R) = \prod_{\substack{j=0\\ j\neq 2}}^{3} \frac{R-R_j}{R_2-R_j} = \left(\frac{R-R_0}{R_2-R_0}\right)\left(\frac{R-R_1}{R_2-R_1}\right)\left(\frac{R-R_3}{R_2-R_3}\right)$$

$$L_3(R) = \prod_{\substack{j=0\\ j\neq 3}}^{3} \frac{R-R_j}{R_3-R_j} = \left(\frac{R-R_0}{R_3-R_0}\right)\left(\frac{R-R_1}{R_3-R_1}\right)\left(\frac{R-R_2}{R_3-R_2}\right)$$



Solution

$$T(R)=\left(\frac{R-R_1}{R_0-R_1}\right)\left(\frac{R-R_2}{R_0-R_2}\right)\left(\frac{R-R_3}{R_0-R_3}\right)T(R_0)+\left(\frac{R-R_0}{R_1-R_0}\right)\left(\frac{R-R_2}{R_1-R_2}\right)\left(\frac{R-R_3}{R_1-R_3}\right)T(R_1)$$

$$+\left(\frac{R-R_0}{R_2-R_0}\right)\left(\frac{R-R_1}{R_2-R_1}\right)\left(\frac{R-R_3}{R_2-R_3}\right)T(R_2)+\left(\frac{R-R_0}{R_3-R_0}\right)\left(\frac{R-R_1}{R_3-R_1}\right)\left(\frac{R-R_2}{R_3-R_2}\right)T(R_3)$$

$$T(754.8)=\frac{(754.8-911.3)(754.8-636.0)(754.8-451.1)}{(1101.0-911.3)(1101.0-636.0)(1101.0-451.1)}(25.113)$$

$$+\frac{(754.8-1101.0)(754.8-636.0)(754.8-451.1)}{(911.3-1101.0)(911.3-636.0)(911.3-451.1)}(30.131)$$

$$+\frac{(754.8-1101.0)(754.8-911.3)(754.8-451.1)}{(636.0-1101.0)(636.0-911.3)(636.0-451.1)}(40.120)$$

$$+\frac{(754.8-1101.0)(754.8-911.3)(754.8-636.0)}{(451.1-1101.0)(451.1-911.3)(451.1-636.0)}(50.128)$$

$$=(-0.098494)(25.113)+(0.51972)(30.131)+(0.69517)(40.120)+(-0.11639)(50.128)$$

$$= 35.242°\mathrm{C}$$

NEWTONS DIVIDED DIFFERENCE

- What is divided difference?

$$f[x_0,x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$$

$$f[x_0,x_1,x_2] = \frac{f[x_1,x_2] - f[x_0,x_1]}{x_2 - x_1}$$

$$f[x_0, x_1, \ldots, x_{k-1}, x_k] = \frac{f[x_1, x_2 - x_k] - f[x_0, \ldots, x_{k-1}]}{x_k - x_0}$$

for k = 3, 4, ….. n.

These I^{st}, II^{nd}... and k^{th} order differences are denoted by Δf, $\Delta^2 f$, …, $\Delta^k f$.

INTERPOLATION USING DIVIDED DIFFERENCE

- The *divided difference interpolation polynomial* is:

$$P(x) = f(x_0) + (x - x_0)\, f[x_0, x_1] + \cdots + (x - x_0) \cdots (x - x_{n-1})\, f[x_0, x_1, \ldots, x_n]$$

Example

- For the data

x:	–1	0	2	5
f(x) :	7	10	22	235

- Find the divided difference polynomial and estimate f(1).

Solution

X	f	Δf	$\Delta^2 f$	$\Delta^3 f$
-1	7			
		3		
0	10		1	
		6		2
			13	
2	22			
		71		
5	235			

$P(x) = f(x_0) + (x - x_0)\, f[x_0, x_1] + (x - x_0)(x - x_1)\, f[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)\, f[x_0, x_1, x_2, x_3]$

$= 7 + (x+1) \times 3 + (x+1)(x-0) \times 1 + (x+1)(x-0)(x-2) \times 2$

$= 2x^3 - x^2 + 10$

$P(1) = 11$

INTERPOLATION FOR EQUALLY SPACED POINTS

Let $(X_{0,},Y_0)$, $(X_{1,},Y_1)$, …, $(X_{n,},Y_n)$ be the given points with $X_{i+1} = X_i + h, i = 0,1,2,\ldots,(n-1)$.

- Finite Difference Operators
  - Forward difference operator
    $\Delta f(x_i) = f(x_i + h) - f(x_i)$
  - Backward difference operator
    $\nabla f(x_i) = f(x_i) - f(x_i - h)$
  - Central difference operator
    - $\delta f(x_i) = f\left(x_i + \frac{h}{2}\right) - f\left(x_i - \frac{h}{2}\right)$

INTERPOLATION FOR EQUALLY SPACED POINTS

- Shift operators

$$E\, f(x_i) = f(x_i + h)$$
$$E^r\, f(x_i) = f\,(x_i + rh)$$

- Averaging operator

$$\mu\, f(x_i) = \frac{1}{2}\left[\, f\left(x_i + \frac{h}{2}\right) + f\left(x_i - \frac{h}{2}\right)\right]$$

RELATION BETWEEN OPERATORS

- $\Delta f_i = \nabla f_{i+1} = \delta f_{i+\frac{1}{2}}$
- $\Delta = E - 1, \nabla = 1 - E^{-1}$
- $\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$
- $\mu = (E^{\frac{1}{2}} + E^{-\frac{1}{2}})$

NEWTON GREGORY FORWARD INTERPOLATION

For convenience we put $p = \dfrac{x - x_0}{h}$ and $f_0 = y_0$. Then we have

$$P(x_0 + ph) = y_o + pDy_0 + \frac{p(p-1)}{2!}D^2y_0 + \frac{p(p-1)(p-2)}{3!}D^3y_0 + \cdots +$$

$$\frac{p(p-1)(p-2)L(p-n+1)}{n!}D^ny_0$$

Example

Estimate f (3.17)from the data using Newton Forward Interpolation.

x:	3.1	3.2	3.3	3.4	3.5
f(x):	0	0.6	1.0	1.2	1.3

Solution

First let us form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3.1	**0**				
		0.6			
3.2	0.6		- **0.2**		
		0.4		**0**	
3.3	1.0		- 0.2		**0.1**
		0.2		0.1	
3.4	1.2		-0.1		
		0.1			
3.5	1.3				

Here $x_0 = 3.1$, $x = 3.17$, $h = 0.1$.

Solution

$$p = \frac{x - x_0}{h} = \frac{0.07}{0.1} = 0.7$$

Newton forward formula is:

$$P(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$P(3.17) = 0 + 0.7\times 0.6 + \frac{0.7(0.7-1)}{2}\times(-0.2) + \frac{0.7(0.7-1)(0.7-2)}{6}\times 0 + \frac{0.7(0.7-1)(0.7-2)(0.7-3)}{24}\times 0.1$$

$$= 0.4384$$

Thus $f(3.17) = 0.4384$.

NEWTON GREGORY BACKWARD INTERPOLATION FORMULA

Taking $p = \frac{x - x_n}{h}$, we get the interpolation formula as:

$$P(x_n+ph) = y_0 + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots + \frac{p\,(p+1)\,(p+2)\ldots(p+n-1)}{n!}\,\nabla^n y_n$$

Example

Estimate f(42) from the following data using newton backward interpolation.

x:	20	25	30	35	40	45
f(x):	354	332	291	260	231	204

Solution

The difference table is:

x	f	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$	$\nabla^5 f$
20	354					
		- 22				
25	332		- 19			
		- 41		29		
30	291		10		-37	
		- 31		- 8		45
35	260		2		**8**	
		- 29		**0**		
40	231		**2**			
		- 27				
45	**204**					

Here $x_n = 45$, $h = 5$, $x = 42$

and $p = -0.6$

Solution

Newton backward formula is:

$$P(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \frac{p(p+1)(p+2)(p+3)(p+4)}{5!}\nabla^5 y_n$$

$$P(42) = 204 + (-0.6)(-27) + \frac{(-0.6)(0.4)}{2}\times 2 + \frac{(-0.6)(0.40(1.4)}{6}\times 0 + \frac{(-0.6)(0.4)(1.4)(2.4)}{24}\times 8 + \frac{(-0.6)(0.4)(1.4)(2.4)(3.4)}{120}\times 45 = 219.1430$$

Thus, f(42) = 219.143

INTERPOLATION USING CENTRAL DIFFERENCES

- Suppose the values of the function f (x) are known at the points a -3h, a – 2h, a – h, a, a +h, a + 2h, a + 3h, ... etc. Let these values be y_{-3}, y_{-2}, y_{-1}, y_0, y_1, y_2, y_3 ..., and so on. Then we can form the central difference table as:

x	f(x)	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$	$\Delta^6 f$
a-3h	y_{-3}						
		Δy_{-3}					
a-2h	y_{-2}		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
a-h	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
a	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
a+h	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		
		Δy_1		$\Delta^3 y_0$			
a+2h	y_2		$\Delta^2 y_1$				
		Δy_2					
a+3h	y_3						

We can relate the central difference operator δ with Δ and E using the operator relation $\delta = \Delta E^{½}$.

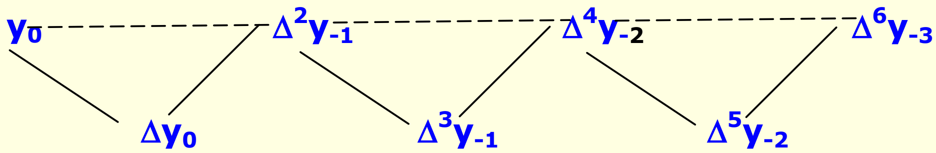
GAUSS FORWARD INTERPOLATION FORMULA

$$P(x) = y_0 + \binom{p}{1}\Delta y_0 + \binom{p}{2}\Delta^2 y_{-1} + \binom{p+1}{3}\Delta^3 y_{-1} + \binom{p+1}{4}\Delta^4 y_{-2} + \binom{p+2}{5}\Delta^5 y_{-2} + \cdots \text{ where}$$

$$p = \frac{x - x_0}{h} \text{ and } \binom{p}{r} = \frac{p(p-1)(p-2)\cdots(p-r+1)}{r!}$$

GAUSS FORWARD INTERPOLATION FORMULA

- The value p is measured forwardly from the origin and $0<p<1$.
- The above formula involves odd differences below the central horizontal line and even differences on the line. This is explained in the following figure.



Example

Find f(30) from the following table values using Gauss forward difference formula:

x:	21	25	29	33	37
F(x):	18.4708	17.8144	17.1070	16.3432	15.5154

Solution

The difference table is

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$
21	18.4708				
		-0.6564			
25	17.8144		- 0.0510		
		-0.7074		- 0.0054	
29	17.1070		0.0564		- 0.0022
		-0.7638		- 0.0076	
33	16.3432		- 0.0640		
		-0.8278			
37	15.5154				

$$P(x) = y_0 + \binom{p}{1}\Delta y_0 + \binom{p}{2}\Delta^2 y_{-1} + \binom{p+1}{3}\Delta^3 y_{-1} + \binom{p+1}{4}\Delta^4 y_{-2} + \cdots$$

f (30) = 16.9217

GAUSS BACKWARD INTERPOLATION FORMULA

This formula is used to interpolate the value of the function for a negative value of p and $-1<p<0$. By substituting for Δy_0, $\Delta^2 y_0$, $\Delta^3 y_0$, ... in terms of central difference in the Newton Forward Difference formula, we get

$$P(x) = y_0 + \binom{p}{1}\Delta y_{-1} + \binom{p+1}{2}\Delta^2 y_{-1} + \binom{p+1}{3}\Delta^3 y_{-2} + \binom{p+2}{4}\Delta^4 y_{-2} + \ldots$$

GAUSS BACKWARD INTERPOLATION FORMULA

- This formula involves odd differences above the central horizontal line and even differences on the central line.

	Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$

Example

Estimate cos 51° 42' by Gauss backward interpolation from the following data:

x:	**50°**	**51°**	**52°**	**53°**	**54°**
cos x:	**0.6428**	**0.6293**	**0.6157**	**0.6018**	**0.5878**

Solution

$x_0 = 52$, $x = 51^\circ\ 42' = 51.7^\circ$, $h = 1$

$$P = \frac{x - x_0}{h} = \frac{51.7 - 52}{1} = -0.3$$

The difference table is

X	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
50	0.6428				
		- 0.0135			
51	0.6293		- 0.0001		
		- 0.0136		- 0.0002	
x_0 = 52	0.6157		- 0.0003		0.0004
		- 0.0139		- 0.0002	
53	0.6018		- 0.0001		
		- 0.0140			
54	0.5878				

Solution

The Gauss backward formula is:

$$P(x) = y_0 + \binom{p}{1}\Delta y_{-1} + \binom{p+1}{2}\Delta^2 y_{-1} + \binom{p+1}{3}\Delta^3 y_{-2} + \binom{p+2}{3}\Delta^4 y_{-2}$$

P(51.7) = 0.6198

STIRLING'S FORMULA

- This formula gives the average of the values obtained by Gauss forward and backward interpolation formulae. For using this formula we should have $-\frac{1}{2} < p < \frac{1}{2}$.

We can get very good estimates if $-\frac{1}{4} < p < \frac{1}{4}$.
The formula is:

$$P(x) = y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p-1)}{4!}\Delta^4 y_{-2} + \ldots$$

Example

- Using **Sterling Formula** estimate f (1.63) from the following table:

x:	1.50	1.60	1.70	1.80	1.90
f(x):	17.609	20.412	23.045	25.527	27.875

Solution

$X_0 = 1.60,\ x = 1.63,\ h = 0.1$

$$p = \frac{1.63 - 1.60}{0.1} = 0.3$$

Solution

- Difference table

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^2 y$
	1.50	17.609				
			2.803			
$x_0=$	1.60	20.412		-0.170		
			2.633		0.019	
	1.70	23.045		-0.151		-0.02
			2.482		0.017	
	1.80	25.527		-0.134		
			2.348			
	1.90	27.875				

BESSELS' INTERPOLATION FORMULA

- This formula involves means of even difference on and below the central line and odd difference below the line.
- The formula is

$$P(x) = \frac{y_0 + y_1}{2} + \left(p - \frac{1}{2}\right)\Delta y_0 + \frac{p(p-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right)$$

$$+ \frac{\left(p - \frac{1}{2}\right) p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)(p-1)(p-2)}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-}}{2}\right)$$

BESSELS' INTERPOLATION FORMULA

Note

- When p = 1/2 , the terms containing odd differences vanish.
- Then we get the formula in a more simple form:

$$P(x) = \frac{y_0 + y_1}{2} + \frac{p\,(p-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{(p+1)\,p(p-1)(p-2)}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \cdots$$

Example

Use Bessels' Formula to find $(46.24)^{1/3}$ from the following table of $x^{1/3}$.

X:	41	45	49	53
$X^{1/3}$:	3.4482	3.5569	3.6593	3.7563

Solution

$x_0 = 45$, $x = 46.24$, $h = 4$ $p = 0.31$

Difference table is:

	x	Y	Δy	$\Delta^2 y$	$\Delta^3 y$
	41	3.4482			
			0.1087		
$x_0 =$	45	3.5569		-0.0063	
			0.1024		-0.00091
	49	3.6593		-0.0054	
			0.0970		
	53	3.7563			

Applying Bessels' formula, we get

$$(46.24)^{1/3} = 3.5893$$