Roots of algebraic and Transcendental

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- We all are aware about algebraic equation, so now we will discuss about transcendental functions and equations.
- **TRANSCENDENTAL FUNCTIONS AND EQUATION:**
- A **transcendental function** "transcends" algebra in that it cannot be expressed in terms of a finite sequence of the algebraic operations of addition, multiplication, and root extraction. Examples of transcendental functions include the exponential function, the logarithm, and the trigonometric functions. $x^{\pi}, c^{x}, x^{x}, \log_c x, \sin x$
- An equation which is includes transcendental functions is known as **Transcendental Equation.**

Roots of algebraic and Transcendental

- **To solve these types of equations we use the following methods:**
- 1) Bisection Method
- 2) Regula-Falsi Method(False-Position Method)
- 3) Newton-Raphson Method
- 4) Secant Method

1) Bisection Method:

- One of the first numerical methods developed to find the root of a nonlinear equation $f(x)=0$ was the bisection method (also called Binary-Search method). The method is based on the following theorem:
- **<u>Theorem:</u>** An equation $f(x)=0$, where $f(x)$ is a real continuous function, has at least one root between x_ℓ and x_u if $f(x_\ell)f(x_u)<0$ (Figure .1).

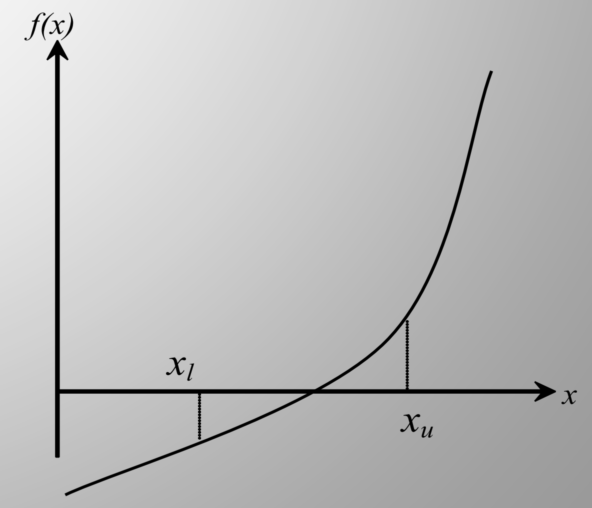


Fig.1 At least one root exists between two points if the function is real, continuous, and changes sign.

1) Bisection Method:

Note that if $f(x_\ell)f(x_u) > 0$, there may or may not be any root between x_ℓ and x_u (Figures 2 and 3). If $f(x_\ell)f(x_u) < 0$, then there may be more than one root between x_ℓ and x_u (Figure 4). So the theorem only guarantees one root between x_ℓ and x_u .

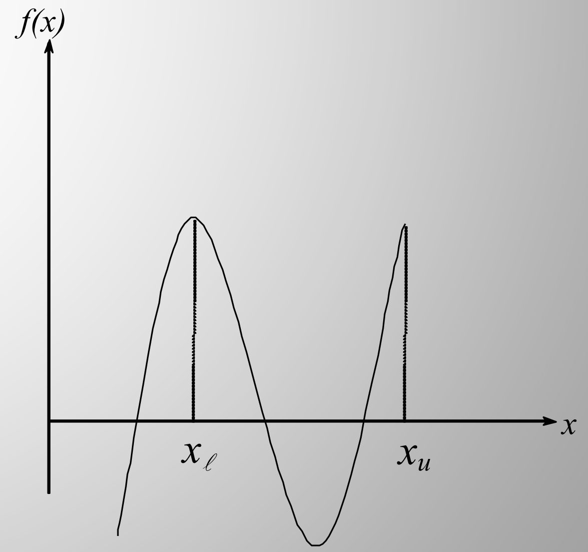


Fig.2 If function $f(x)$ does not change sign between two points, roots of $f(x) = 0$ may still exist between the two points.

1) Bisection Method:

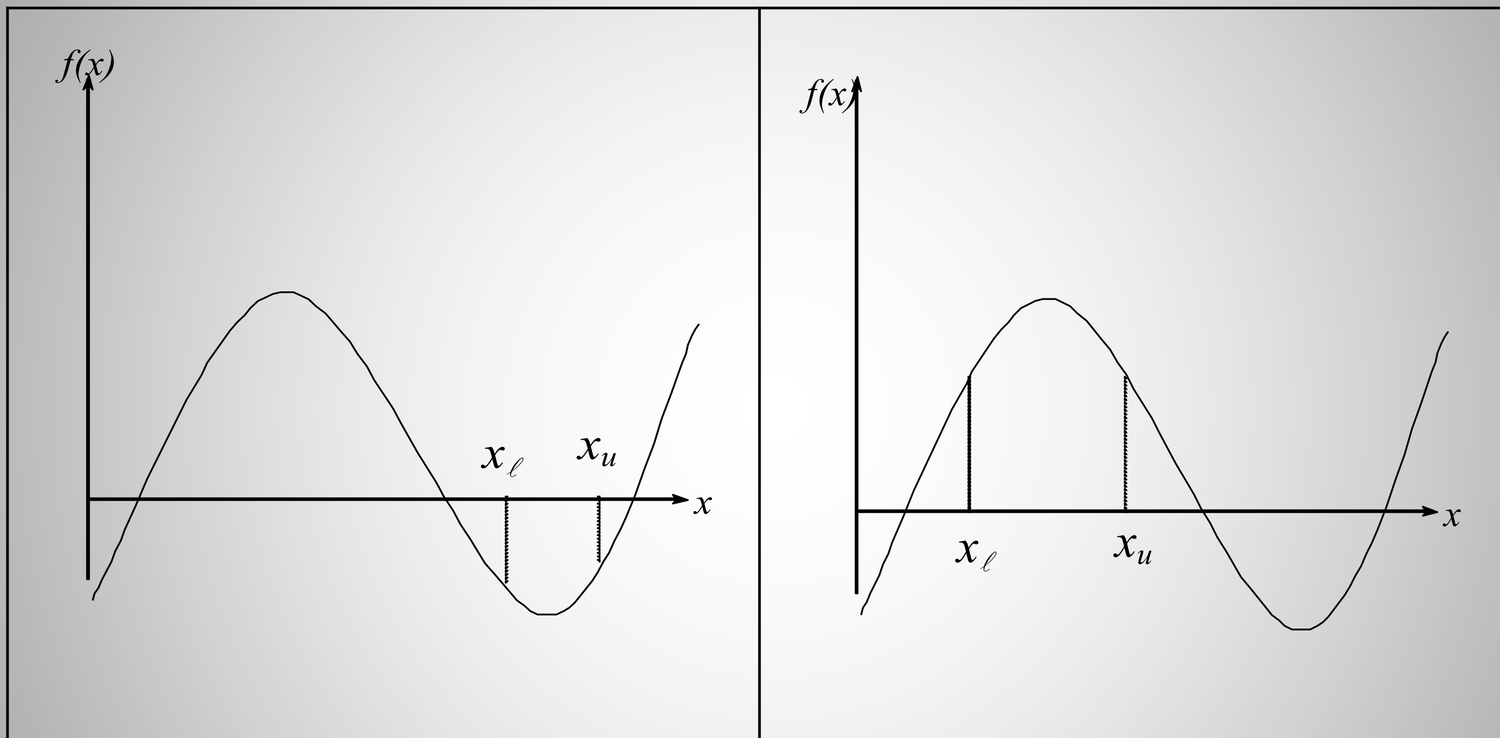


Fig.3 If the function $f(x)$ does not change sign between two points, there may not be any roots $f(x) = 0$ between the two points.

1) Bisection Method:

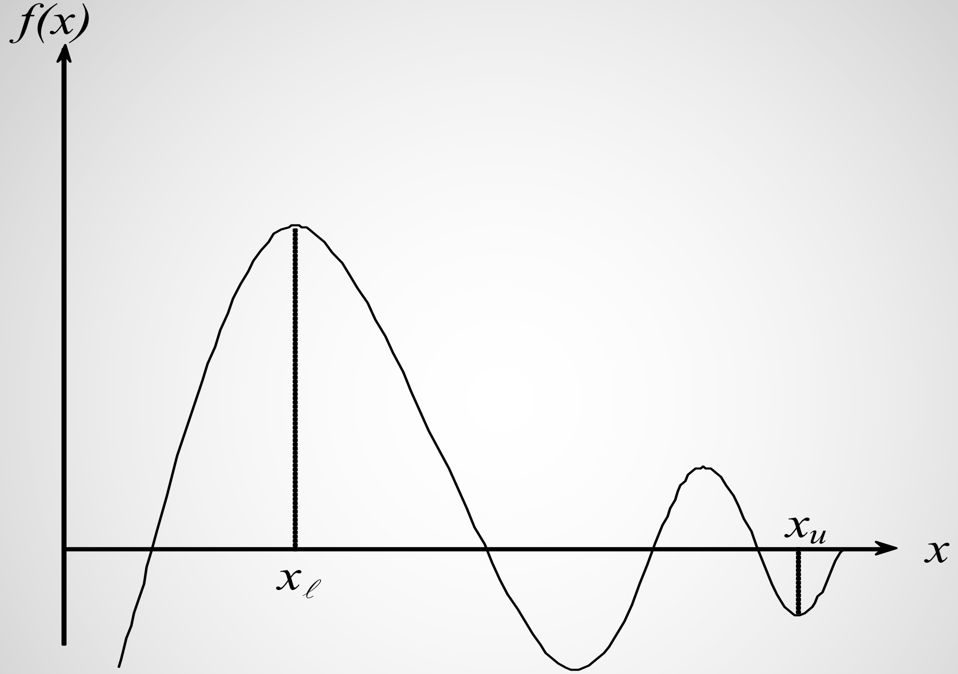


Fig .4 If the function $f(x)$ changes sign between two points, more than one root for $f(x) = 0$ may exist between the two points.

1) Bisection Method:

- A general rule is:
- - If the $f(x_l)$ and $f(x_u)$ have the same sign

$$f(x_\ell)f(x_u) > 0$$

- - There is no root between x_l and x_u.
- - There is an even number of roots between x_l and x_u.
- - If the $f(x_l)$ and $f(x_u)$ have different signs

$$f(x_\ell)f(x_u) < 0$$

- - The is an odd number of roots.

1) Bisection Method:

- **Advantages of Bisection Method:**
- The bisection method is always convergent. Since the method brackets the root, the method is guaranteed to converge.
- As iterations are conducted, the interval gets halved. So one can guarantee the decrease in the error in the solution of the equation.

1) Bisection Method:

- **Drawbacks of Bisection Method:**
- The convergence of bisection method is slow as it is simply based on halving the interval.
- If one of the initial guesses is closer to the root, it will take larger number of iterations to reach the root.

1) Bisection Method:

The steps to apply the bisection method to find the root of the equation $f(x) \quad 0$ are:

1. Choose x_ℓ and x_u as two guesses for the root such that $f(x_\ell)f(x_u) < 0$, or in other words, $f(x)$ changes sign between x_ℓ and x_u.
2. Estimate the root, x_m of the equation $f(x) \quad 0$ as the mid-point between x_ℓ and x_u as

$$x_m = \frac{x_\ell + x_u}{2}$$

3. Now check the following
   a. If $f(x_\ell)f(x_m) < 0$, then the root lies between x_ℓ and x_m; then $x_\ell \quad x_\ell$ and $x_u \quad x_m$.
   b. If $f(x_\ell)f(x_m) > 0$, then the root lies between x_m and x_u; then $x_\ell \quad x_m$ and $x_u \quad x_u$.
   c. If $f(x_\ell)f(x_m) \quad 0$; then the root is x_m. Stop the algorithm if this is true.
4. Find the new estimate of the root

$$x_m = \frac{x_\ell + x_u}{2}$$

1) Bisection Method:

Example: Find a root of the equation $x^3 - 4x - 9 = 0$ using Bisection Method correct up to 3 decimal places.

Sol: Let $f(x) = x^3 - 4x - 9$

Since $f(-2)$ is –ve and $f(3)$ is +ve , a root lies between 2 and 3.

Therefore, the first approximation to the root is $x_1 = \frac{2+3}{2} = 2.5$

Thus $f(2.5) = (2.5)^3 - 4(2.5) - 9 = -3.375$ $i.e., -ve$

Therefore , the root lies between x_1 and 3.

Thus the second approximation to the root is $x_2 = \frac{2.5+3}{2} = 2.75$

Thus , $f(2.75) = (2.75)^3 - 4(2.75) - 9 = 0.7969$ $i.e., +ve$

$\therefore$ the root lies between x_1 and x_2 .

EXAMPLE:

Thus the third approximation to the root is $x_3 = \dfrac{2.5+2.75}{2} = 2.625$

Then $f(2.625) = (2.625)^3 - 4(2.625) - 9 = -1.4121$ *i.e.,* $-ve$

The root lies between x_2 *and* x_3 thus the fourth approximation to the root is $x_4 = \dfrac{2.625+2.75}{2} = 2.6875$

Repeating this process, the successive approximations are

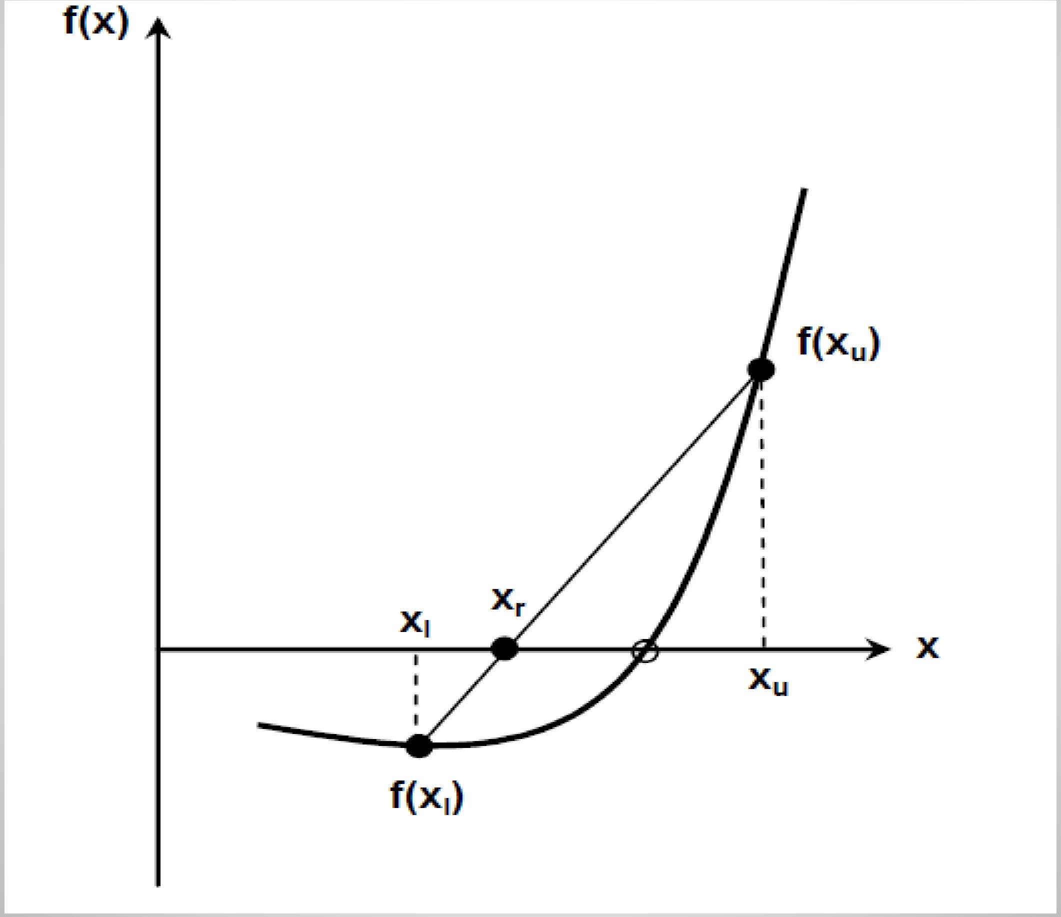
$x_5 = 2.71875 \quad x_6 = 2.70313 \quad x_7 = 2.71094$

$x_8 = 2.70703 \quad x_9 = 2.70508 \quad x_{10} = 2.70605$

$x_{11} = 2.70654 \quad x_{12} = 2.70642$

Hence the root is 2.70642.

2) Regula-Falsi Method (False-Position Method):



2) Regula-Falsi Method (False-Position Method):

A shortcoming of the bisection method is that in dividing the interval from x_l to x_u into equal halves, no account is taken of the magnitude of $f(x_i)$ and $f(x_u)$. Indeed, if $f(x_i)$ is close to zero, the root is more close to x_l than x_0.

The false position method uses this property:

A straight line joins $f(x_i)$ and $f(x_u)$. The intersection of this line with the x-axis represents an improvement estimate of the root. This new root can be computed as:

$$\frac{f(x_l)}{x_r - x_l} \quad \frac{f(x_u)}{x_r - x_u}$$

Giving $x_r \quad x_u - \dfrac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$ or $x_r \quad \dfrac{af(b) - bf(a)}{f(b) - f(a)}$ This is called the false-position formula

Then, x_r replaces the initial guess for which the function value has the same sign as $f(x_r)$

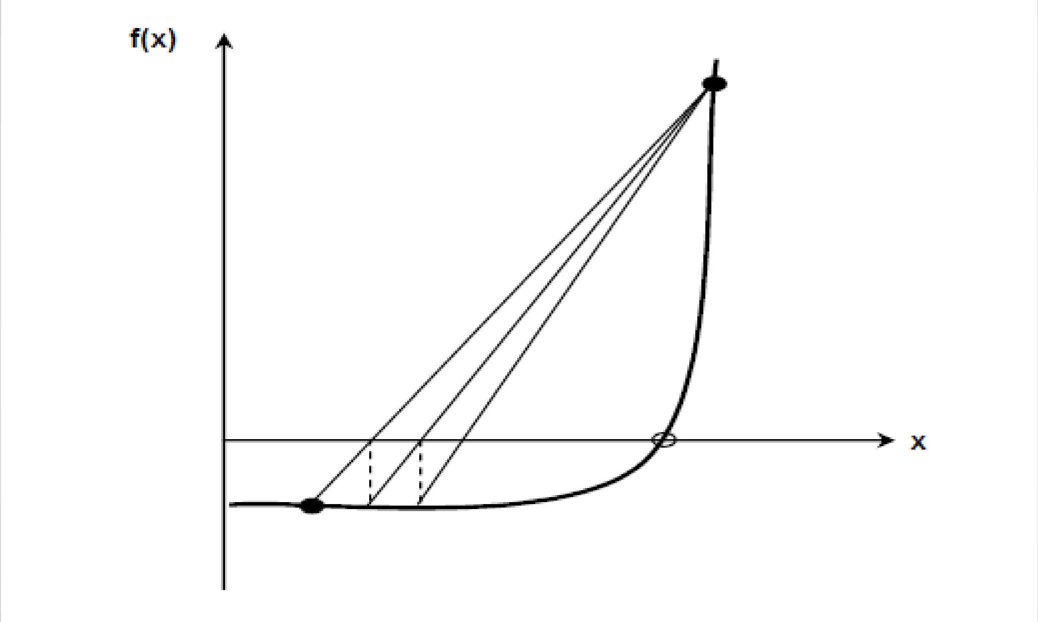
The difference between bisection method and false-position method is that in bisection method, both limits of the interval have to change. This is not the case for false position method, where one limit may stay fixed throughout the computation while the other guess converges on the root.

2) Regula-Falsi Method (False-Position Method):

Working rule for False-Position Method (Regula-Falsi Method):

1. Define the first interval (a,b) such that solution exists between them. Check $f(a)f(b)<0$.
2. Compute the first estimate of the numerical solution x_r using the above equation.
3. Find out whether the actual solution is between a and x_r or x_r and b. This is accomplished by checking the sign of the product $f(a)f(x_r)$
   - If $f(a)f(x_r)<0$, the solution is between a and x_r.
   - If $f(a)f(x_r)>0$, the solution is between x_r and b.
4. Select the subinterval that contains the solution (a,x_r) or (x_r,b) is the new interval (a,b) and go to step no.2 and repeat the process. The method of False Position always converges to an answer, provided a root is initially bracketed in the interval (a,b).

2) Regula-Falsi Method (False-Position Method):



Modified false position method:

As the problem with the original false-position method is the fact that one limit may stay fixed during the computation, we can introduce a modified method in which if one limit is detected to be stuck, the function value of the stagnant point is divided by 2.

Incremental searches and determining initial guesses:

To determine the number of roots within an interval, it is possible to use an incremental search. The function is scanned from one side to the other using a small increment. When the function changes sign, it is assumed that a root falls within the increment.

2) Regula-Falsi Method (False-Position Method):

- **Drawbacks of Bisection Method:**

The problem is the choice of the increment length:

Too small	very time consuming
Too large	some root may be missed

The solution is to use plotting or information from the physical problem.

2) Regula-Falsi Method (False-Position Method):

Example Using the False Position method, find a root of the function $f(x) = e^x - 3x^2$ **to an accuracy of 5 digits. The root is known to lie between 0.5 and 1.0.**

Sol:

We apply the method of False Position with $a = 0.5$ and $b = 1.0$. and the equation is

$$x_r \quad \frac{af(b) - bf(a)}{f(b) - f(a)}$$

The calculations based on the method of False Position are shown in the Table

n	a	b	$f(a)$	$f(b)$	x_r	$f(x_r)$
1	0.5	1	0.89872	-0.28172	0.88067	0.08577
2	0.88067	1	0.08577	-0.28172	0.90852	0.00441
3	0.90852	1	0.00441	-0.28172	0.90993	0.00022
4	0.90993	1	0.00022	-0.28172	0.91000	0.00001
5	0.91000	1	0.00001	-0.28172	0.91001	0

The root is 0.91 accurate to five digits.

3) Newton-Raphson method:

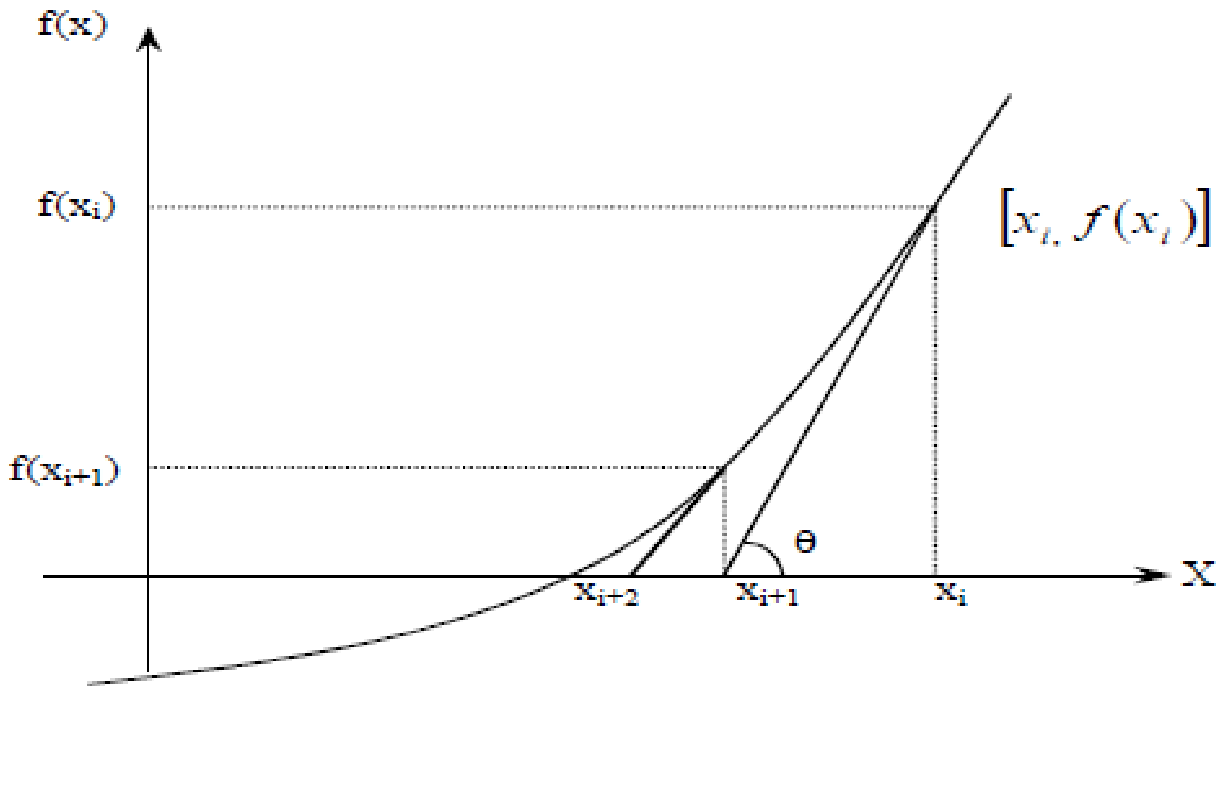


Figure. Geometrical illustration of the Newton-Raphson method.

3) Newton-Raphson method:

The steps to apply using Newton-Raphson method to find the root of an equation f(x) = 0 are

1. Evaluate f'(x) symbolically
2. Use an initial guess of the root, x_i, to estimate the new value of the root x_{i+1} as

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Drawbacks of Newton-Raphson Method:

Divergence at inflection points: If the selection of a guess or an iterated value turns out to be close to the inflection point of f(x), that is, near where f''(x)=0, the roots start to diverge away from the true root.

$$x_{i+1} \quad x_i - \frac{f(x_i)}{f'(x_i)}$$

Consequently if an iteration value, x_i is such that $f'(x_i) \cong 0$, then one can face division by zero or a near-zero number. This will give a large magnitude for the next value, x_{i+1}.

3) Newton-Raphson method:

Example: Find the root of the function $x^3 + x - 1 = 0$ correct up to four decimal places.

Sol:

Let $$f(x) = x^3 + x - 1$$

$$f(0) = -1 \text{ and } f(1) = 1$$

Since, $f(0) < 0$ and $f(1) > 0$, the root lies between 0 and 1.

Let $x_0 = 1$

$$f'(x) = 3x^2 + 1$$

By the Newton-Raphson method, $$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

3) Newton-Raphson method:

$$f(x_0) = f(1) = 1$$
$$f'(x_0) = f'(1) = 4$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1}{4} = 0.75$$

$$\therefore \quad f(x_1) = f(0.75) = 0.171875$$
$$\therefore \quad f'(x_1) = f'(0.75) = 2.6875$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.75 - \frac{0.171875}{2.6875} = 0.68605$$

$$\therefore \quad f(x_2) = f(0.68605) = 0.00894$$
$$\therefore \quad f'(x_2) = f'(0.68605) = 2.41198$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.68605 - \frac{0.00894}{2.41198} = 0.68234$$

$$\therefore \quad f(x_3) = f(0.68234) = 0.000028$$
$$\therefore \quad f'(x_3) = f'(0.68234) = 2.39676$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.68234 - \frac{0.000028}{2.39676} = 0.68233$$

Since x_3 *and* x_4 are same up to four decimal places, the root is 0.6823.

Some deduction from Newton-Raphson's Formula:

We can derive the following useful results from the Newton-Raphson's Formula

1) Iteration formula for $\frac{1}{N}$:

Let $x = \frac{1}{N} \Rightarrow N = \frac{1}{x} \Rightarrow \frac{1}{x} - N = 0$

Also let $f(x) = \frac{1}{x} - N$ *and* $f'(x) = -\frac{1}{x^2}$

By the Newton-Raphson method ,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{\left(\frac{1}{x_i} - N\right)}{\left(-\frac{1}{x_i^2}\right)}$$

$$= x_i + x_i^2\left(\frac{1}{x_i} - N\right)$$

$$= x_i(2 - Nx_i)$$

Some deduction from Newton-Raphson's Formula:

2) Iteration formula for $\sqrt{N}$:

Let $x=\sqrt{N} \Rightarrow N=x^2 \Rightarrow x^2-N=0$

Also let $f(x)=x^2-N$ *and* $f'(x)=2x$

By the Newton-Raphson method ,

$$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)}=x_i-\frac{\left(x_i^{\,2}-N\right)}{\left(2x_i\right)}$$

$$=x_i-\frac{1}{2}\left(x_i-\frac{N}{x_i}\right)$$

$$=\frac{1}{2}\left(x_i+\frac{N}{x_i}\right)$$

Some deduction from Newton-Raphson's Formula:

3) Iteration formula for $\frac{1}{\sqrt{N}}$:

Let $x = \frac{1}{\sqrt{N}} \Rightarrow x^2 = \frac{1}{N} \Rightarrow x^2 - \frac{1}{N} = 0$

Also let $f(x) = x^2 - \frac{1}{N}$ *and* $f'(x) = 2x$

By the Newton-Raphson method ,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{\left(x_i^2 - \frac{1}{N}\right)}{(2x_i)}$$

$$\left(\frac{2x_i^2 - x_i^2 + \frac{1}{N}}{2x_i}\right)$$

$$= \frac{1}{2}\left(x_i + \frac{1}{x_i N}\right)$$

Some deduction from Newton-Raphson's Formula:

4) Iteration formula for $\sqrt[k]{N}$:

Let $x = \sqrt[k]{N} \Rightarrow N = x^k \Rightarrow x^k - N = 0$

Also let $f(x) = x^k - N$ *and* $f'(x) = kx^{k-1}$

By the Newton-Raphson method ,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{\left(x_i^{\,k} - N\right)}{\left(kx_i^{\,k-1}\right)}$$

$$\left(\frac{(x_i)(kx_i^{\,k-1}) - \left(x_i^{\,k} - N\right)}{kx_i^{\,k-1}} \right)$$

$$= \frac{1}{k}\left((k-1)x_i + \frac{N}{x_i^{\,k-1}} \right)$$

Some deduction from Newton-Raphson's Formula:

Example: Evaluate the following (correct to four decimal places) by Newton's iteration method:

$$(i)\ \frac{1}{31} \quad (ii)\ \sqrt{24} \quad (iii)\ 1/\sqrt{14} \quad (iv)\ \sqrt[3]{24}$$

Sol: $(i)\ \frac{1}{31}$

Taking $N \quad 31$ in the equation $x_{i+1} = x_i(2 - Nx_i)$

We have $x_{i+1} = x_i(2 - 31x_i)$, Since an approximate value of 1/31=0.03, we take $x_0 \quad 0.03$.

Then

$$x_1 = x_0(2 - 31x_0) = 0.03(2 - 31(0.03)) = 0.0321$$
$$x_2 = x_1(2 - 31x_1) = 0.0321(2 - 31(0.0321)) = 0.032257$$
$$x_3 = x_2(2 - 31x_2) = 0.032257(2 - 31(0.032257)) = 0.03226$$

Since $x_2 \quad x_3$ up to 4 decimal places, we have $1/31 \quad 0.0323$

Some deduction from Newton-Raphson's Formula:

(ii) $\sqrt{28}$

Taking $N \quad 24$ in the equation $x_{i+1} = \frac{1}{2}\left(x_i + \frac{N}{x_i}\right)$

We have $x_{i+1} = \frac{1}{2}\left(x_i + \frac{28}{x_i}\right)$, Since the value of $\sqrt{25} \quad 5$, we take $x_0 \quad 5$.

Then

$$x_1 = \frac{1}{2}\left(x_0 + \frac{28}{x_0}\right) = \frac{1}{2}\left(5 + \frac{28}{5}\right) = 5.3$$

$$x_2 = \frac{1}{2}\left(x_1 + \frac{28}{x_1}\right) = \frac{1}{2}\left(5.3 + \frac{28}{5.3}\right) = 5.29151$$

$$x_3 = \frac{1}{2}\left(x_2 + \frac{28}{x_2}\right) = \frac{1}{2}\left(5.29151 + \frac{28}{5.29151}\right) = 5.29150$$

Since $x_2 \quad x_3$ up to 4 decimal places, we have $\sqrt{28} \quad 5.2915$.

Some deduction from Newton-Raphson's Formula:

$(iii)\ 1/\sqrt{14}$

Taking $N \quad 14$in the equation $x_{i+1} = \frac{1}{2}\left(x_i + \frac{1}{x_i N}\right)$

We have $x_{i+1} = \frac{1}{2}\left(x_i + \frac{1}{x_i 14}\right)$, Since the value of $1/\sqrt{16} = 1/4 = 0.25$, we take $x_0 \quad 0.25$.

Then

$$x_1 = \frac{1}{2}\left(x_0 + \frac{1}{x_0 14}\right) = \frac{1}{2}\left((0.25) + \frac{1}{(0.25)14}\right) = 0.26785$$

$$x_2 = \frac{1}{2}\left(x_1 + \frac{1}{x_1 14}\right) = \frac{1}{2}\left((0.26785) + \frac{1}{(0.26785)14}\right) = 0.2672618$$

$$x_3 = \frac{1}{2}\left(x_2 + \frac{1}{x_2 14}\right) = \frac{1}{2}\left((0.2672618) + \frac{1}{(0.2672618)14}\right) = 0.2672612$$

Since $x_2 \quad x_3$ up to 4 decimal places, we have $1/\sqrt{14} \quad 0.2673$.

Some deduction from Newton-Raphson's Formula:

(iv) $\sqrt[3]{24}$

Taking $N = 24$ *and* $k = 3$ in the equation $x_{i+1} = \frac{1}{k}\left((k-1)x_i + \frac{N}{x_i^{k-1}}\right)$

We have $x_{i+1} = \frac{1}{3}\left((2)x_i + \frac{24}{x_i^2}\right)$, Since the value of $27^{\frac{1}{3}} \quad 3$, we take $x_0 \quad 3$.

Then

$$x_1 = \frac{1}{3}\left((2)x_0 + \frac{24}{x_0^2}\right) = \frac{1}{3}\left((2)(3) + \frac{24}{(3)^2}\right) = 2.8889$$

$$x_2 = \frac{1}{3}\left((2)x_1 + \frac{24}{x_1^2}\right) = \frac{1}{3}\left((2)(2.8889) + \frac{24}{(2.8889)^2}\right) = 2.88451$$

$$x_3 = \frac{1}{3}\left((2)x_2 + \frac{24}{x_2^2}\right) = \frac{1}{3}\left((2)(2.88451) + \frac{24}{(2.88451)^2}\right) = 2.8845$$

Since $x_2 \quad x_3$ up to 4 decimal places, we have $\sqrt[3]{24} \quad 2.8845$

Since $x_2 \quad x_3$ up to 4 decimal places, we have $\sqrt[3]{24} \quad 2.8845$

4) The Secant method:

The Newton-Raphson method of solving the nonlinear equation f(x) =0 is given by the recursive formula

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \qquad (1)$$

From the above equation, one of the drawbacks of the Newton-Raphson method is that you have to evaluate the derivative of the function.

To overcome this drawback, the derivative, f'(x) of the function, f(x) is approximated as

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \qquad (2)$$

Substituting Equation (2) in (1), gives

4) The Secant method:

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})} \quad \text{OR} \quad x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})} \qquad (3)$$

The above equation is called the Secant method. This method now requires two initial guesses, but unlike the bisection method, the two initial guesses do not need to bracket the root of the equation. The Secant method may or may not converge, but when it converges, it converges faster than the bisection method. However, since the derivative is approximated, it converges slower then Newton-Raphson method.

The Secant method can be also derived from geometry shown in Figure. Taking two initial guesses, xi and xi-1, one draws a straight line between f(xi) and f(xi-1) passing through the x-axis at xi+1. ABE and DCE are similar triangles. Hence

$$\frac{AB}{AE} = \frac{DC}{DE}$$

$$\frac{f(x_i)}{x_i - x_{i+1}} = \frac{f(x_{i-1})}{x_{i-1} - x_{i+1}}$$

Rearranging, it gives the secant method as

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

4) The Secant method:

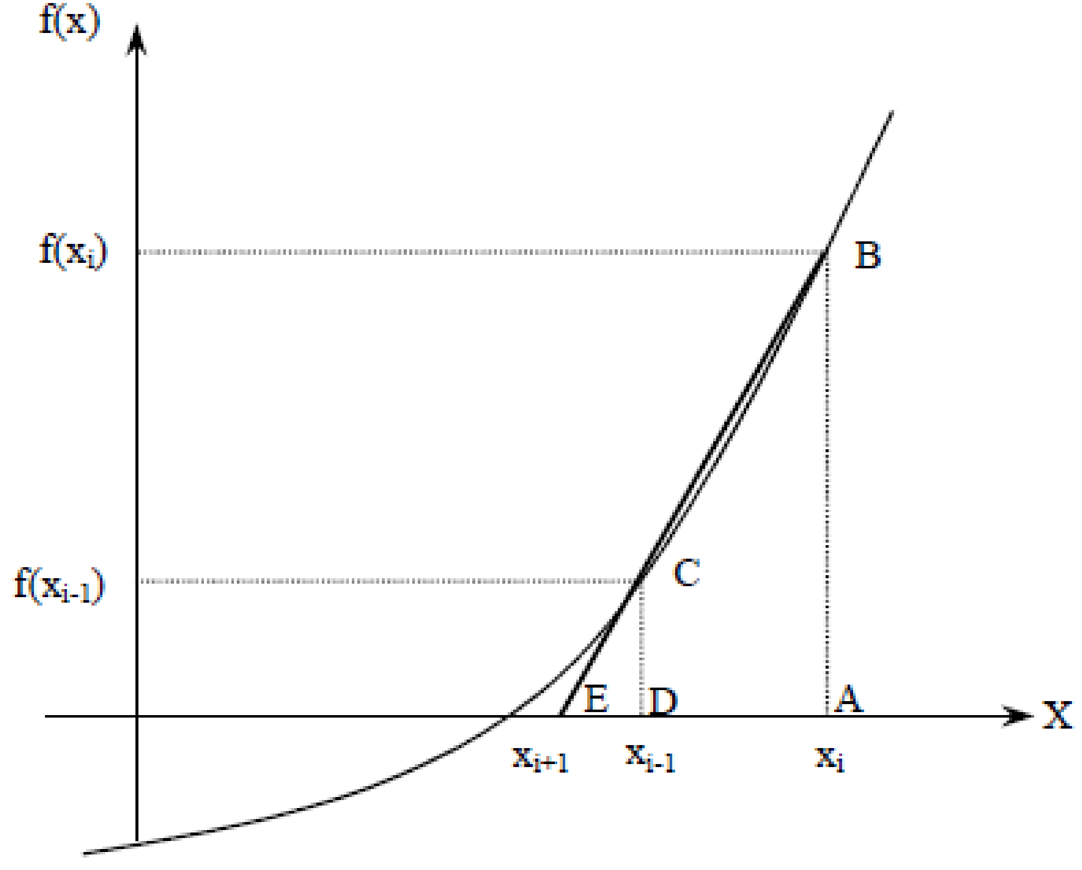


Figure. Geometrical representation of the Secant method.

4) The Secant method:

Example: Find a root of the equation $x^3 - 2x - 5 = 0$ using Secant Method correct up to 3 decimal places.

Sol: Let $f(x) = x^3 - 2x - 5$

Since $f(2) = -1$ is –ve and $f(3)\quad 16$ is +ve , a root lies between 2 and 3. Therefore $x_0 = 2$ *and* $x_1 = 3$.

Therefore, the second approximation to the root is $x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = 2.058823$

Thus $f(x_2) = f(2.058823) = -0.390799$

The third approximation to the root is $x_3 = \dfrac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = 2.081263$

4) The Secant method:

Thus , $f(x_3) = f(2.081263) = -0.147204$

The fourth approximation to the root is $x_4 = \dfrac{x_2 f(x_3) - x_3 f(x_2)}{f(x_3) - f(x_2)} = 2.094824$

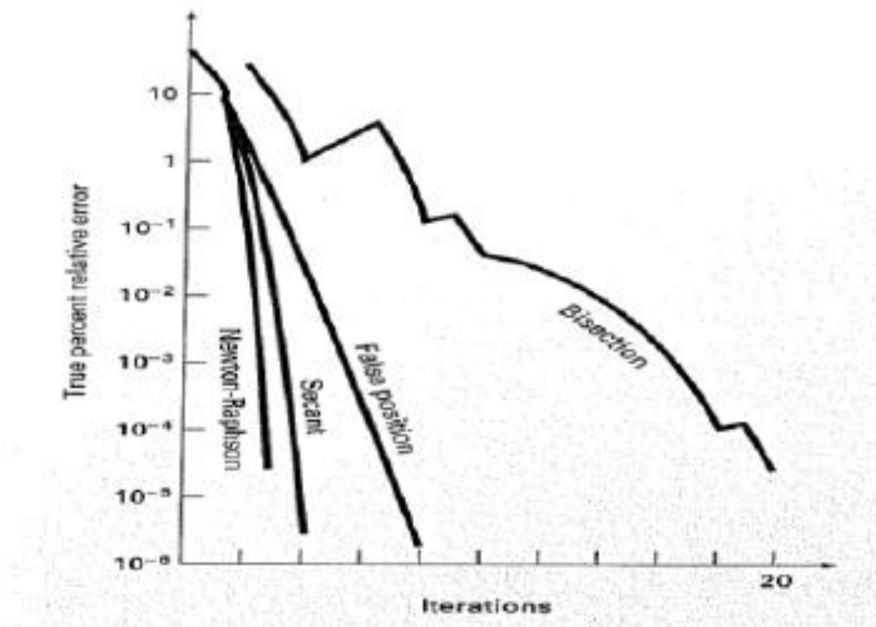
Thus , $f(x_4) = f(2.094824) = 0.003042$

The fifth approximation to the root is $x_5 = \dfrac{x_3 f(x_4) - x_4 f(x_3)}{f(x_4) - f(x_3)} = 2.094549$

Hence the root is 2.094 correct up to 3 decimal places.

Rate of convergence:

- Figure compares the convergence of all the methods.



Rate of convergence of all the methods.