FACULTY OF ENGINEERING AND TECHNOLOGY
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PROBABILITY, STATISTICS AND NUMERICAL METHODS (303191251)
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UNIT: 2 Probability and Probability Distribution

Introduction:

The ‘Probability’ or ‘chance’ is very commonly used in day-to-day life. For example, statements like (1) probably it may rain today. (2) The chances of teams A and B winning a certain match is equal. (3) the Chance of making profits by investing in shares Company And so on.
The theory of probability has its origin in the games of chance related to gambling such as throwing a die, tossing a coin, drawing cards from a pack of cards, etc. Girolamo Cardano, an Italian mathematician, was the first man to write a book titled “Book on Games of Chance” which was published in 1663 after his death. Galileo, an Italian mathematician, was the first man to attempt quantitative measure of probability while dealing with some problems related to the theory of dice in gambling. Further systematic and scientific foundation to the mathematical theory of probability was laid by two French mathematicians B. Pascal and Pierre de Fermat. Swiss mathematician Jacques Bernoulli also contributed to the theory of probability.

Prerequisites:

Fundamental Principle of Counting:

If one job can be done in m ways and another job can be done in n ways, then the number of ways of performing the combination of the above two jobs is m * n. It can be extended to any finite number of operations

Permutation

If n objects are given and we have to arrange r $(r \leq n)$ out of them and the order in which these objects are arranged is important, such an arrangement is called a permutation of n objects taken r at a time.

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

Combination

If n objects are given and we have to choose $r(r \leq n)$ out of them and the order in which objects are arranged is not important, such a choice is called a combination of n objects taken at a time.

$$^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$$

Basic Concept in Probability:

Experiment: The term experiment refers to describe an act which can be repeated under some given conditions.

Random Experiment or Trial: If in an experiment all possible outcomes are known in advance and none of the outcomes can be predicted with certainty, then such an experiment is called random experiment.

E.g.: Given any triangle, without knowing the three angles, we can definitely say that the sum of measure of angles is 180°.

We also perform many experimental activities, where the result may not be same, when they are repeated under identical conditions. For example, when a coin is tossed it may turn up a head or a tail, but we are not sure which one of these results will actually be obtained.

Such experiments are called **random experiments**.

An experiment is called random experiment if it satisfies the following two conditions: (i) It has more than one possible outcome. (ii) It is not possible to predict the outcome in advance with certainty.

Sample space: The set consisting of all outcomes of a random experiment is called sample space.

E.g.: Sample space of tossing a coin once is {H, T}, Sample space of throwing a dice is {1, 2, 3, 4, 5, 6}.

Example: Two coins (a one-rupee coin and a two-rupee coin) are tossed once. Find a sample space.

Solution: Clearly the coins are distinguishable in the sense that we can speak of the first coin and the second coin. Since either coin can turn up Head (H) or Tail (T), the possible outcomes may be

Heads on both coins = (H, H) = HH
Head on first coin and Tail on the other = (H, T) = HT
Tail on first coin and Head on the other = (T, H) = TH
Tail on both coins = (T, T) = TT
Thus, the sample space is S = {HH, HT, TH, TT}

NOTE
The outcomes of this experiment are ordered pairs of H and T. For the sake of simplicity the commas are omitted from the ordered pairs.

Example
A coin is tossed. If it shows head, we draw a ball from a bag consisting of 3 blue and 4 white balls; if it shows tail we throw a die. Describe the sample space of this experiment.
Events: The outcomes of the random experiments are called the events.

For example,

For S = {H, T}, there are four events, ϕ , {H}, {T} and S.

Consider S = {1, 2, 3, 4, 5, 6}, which is the sample space of tossing a die. Some events connected with this sample space are given below.

A= Getting an odd number = {1, 3, 5}

B= Getting a number divisible by 2 = {2, 4, 6}

C= Getting a number greater than 2 = {3, 4, 5, 6}

D= Getting a number less than 7 = {1, 2,3,4,5, 6} = S

E= Getting a number greater than 6 = ϕ

The event S (or D) is called sure event and the event ϕ is called impossible event.

Occurrence of an event:

Consider the experiment of throwing a die.
Let E denotes the event "a number less than 4 appears". If actually '1' had appeared on the die then we say that event E has occurred.
As a matter of fact, if outcomes are 2 or 3 we say that event E has occurred

Thus, the event E of a sample space S is said to have occurred if the outcome ω of the experiment is such that ω ∈E. If the outcome ω is such that ω does not belong to E, we say that the event E has not occurred.

Types of events

Certain event: An event whose occurrence is inevitable (or certain) is called a certain event.

Impossible event: An event whose occurrence is impossible is called impossible event.

To understand these let us consider the experiment of rolling a die. The associated sample space is
S = {1, 2, 3, 4, 5, 6}
Let E be the event "the number appears on the die is a multiple of 7". Can you write the subset associated with the event E?

Clearly no outcome satisfies the condition given in the event, i.e., no element of the sample space ensures the occurrence of the event E. Thus, we say that the empty set only correspond to the event E.
In other words, we can say that it is impossible to have a multiple of 7 on the upper face of the die. Thus, **the event E = $\varphi$is an impossible event.**

Now let us take up another event F "the number turns up is odd or even". Clearly

F = {1, 2, 3, 4, 5, 6,} = S, i.e., all outcomes of the experiment ensure the occurrence of the event F. Thus, **the event F = S is a sure event.**

Intersection of two events: Intersection of two events means the set of all the sample points belonging to both the events.

Simple Event:

If an event E has only one sample point of a sample space, it iscalled a *simple* (or *elementary*) *event*.

In a sample space containing n distinct elements, there are exactly n simple events.
For example, in the experiment of tossing two coins, a sample space is
S= {HH, HT, TH, TT}
There are four simple events corresponding to this sample space. These are
E1= {HH}, E2= {HT}, E3= {TH} and E4={TT}.

Mutually exclusive events: Two events are said to be mutually exclusive or incompatible when both cannot happen simultaneously in a single trial.

Or

The occurrence of any one of the events prevents the occurrence of the other.

In the experiment of rolling a die, a sample space is S = {1, 2, 3, 4, 5, 6}.
Consider events, A 'an odd number appears' and B 'an even number appears'
Clearly the event A excludes the event B and vice versa.
In other words, there is no outcome which ensures the occurrence of events A and B simultaneously.
Here A = {1, 3, 5} and B = {2, 4, 6}
Clearly $A \cap B = \varphi$ i.e., A and B are disjoint sets.

In general,
two events A and B are called *mutually exclusive* events if the occurrence of any one of them excludes the occurrence of the other event, i.e., if they cannot occur simultaneously. In this case the sets A and B are disjoint.

Again, in the experiment of rolling a die, consider the events A 'an odd number appears' and event B 'a number less than 4 appears'
Obviously, A = {1, 3, 5} and B = {1, 2, 3}
Now $3 \in A$ as well as $3 \in B$
Therefore, A and B are not mutually exclusive events.

Remark: **Simple events of a sample space are always mutually exclusive.**

Independent events: Two or more events are said to be independent when the outcome of one does not affect and is not affected by the other.

E.g.: If a coin is tossed twice the second throw is not affected by the outcome of the first and vice-versa. Also, while throwing a dice the outcomes of none of the events are affected by the other.

Dependent events: Dependent events are those events in which the occurrence or non-occurrence of one event in any one trial affects the probability of other events in other trials.

E.g.: If a card is drawn from a pack of playing cards and is not replaced, this will alter the probability of next card drawn i. e. probability of drawing a queen from a pack of 52 cards is (4/52). But if the card drawn in first trial is queen and is not replaced then the probability of drawing queen in the next trial is (3/51).

Equally likely events: Events are said to be equally likely when one does not occur more often than the others.

E.g.: If we throw a dice each face may be expected to be observed approximately the same number.

Exhaustive events: Exhaustive events are a set of events in a sample space such that one of them compulsorily occurs while performing the experiment. In simple words, we can say that all the possible events in a sample space of an experiment constitute exhaustive events. For example, while tossing an unbiased coin, there are two possible outcomes - heads or tails. So, these two outcomes are exhaustive events as one of them will definitely occur while flipping the coin.

If E_1, E_2, E_3, …, E_n are n events of a sample space S, and if:

$$E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_n = S$$

then E_1, E_2, E_3, …, E_n are called exhaustive events.

In other words, events E_1, E_2, E_3, …, E_n are called **exhaustive events** if at least one of them necessarily occurs whenever the experiment is performed.

Consider the experiment of throwing a dice. We have S = {1, 2, 3, 4, 5, 6}. Let us define the following events

A: 'a number less than 4 appears',
B: 'a number greater than 2 but less than 5 appears'
C: 'a number greater than 4 appears'.
Then A = {1, 2, 3}, B = {3,4} and C = {5, 6}.
$A \cup B \cup C = \{1, 2, 3\} \cup \{3, 4\} \cup \{5, 6\} = S$.
Such events A, B and C are called **exhaustive events**.

Example: Two dice are thrown and the sum of the numbers which come up on the dice is noted. Let us consider the following events associated with this experiment
A: 'the sum is even'.
B: 'the sum is a multiple of 3'.
C: 'the sum is less than 4'.
D: 'the sum is greater than 11'.
Which pairs of these events are mutually exclusive?

Solution: There are 36 elements in the sample space S = {(x, y): x, y = 1, 2, 3, 4, 5, 6}.
Then
A = {(1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (2, 6), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4), (4, 6), (5, 1), (5, 3), (5, 5), (6, 2), (6, 4), (6, 6)}

B = {(1, 2), (2, 1), (1, 5), (5, 1), (3, 3), (2, 4), (4, 2), (3, 6), (6, 3), (4, 5), (5, 4), (6, 6)}
C = {(1, 1), (2, 1), (1, 2)} and D = {(6, 6)}

We find that
A ∩B = {(1, 5), (2, 4), (3, 3), (4, 2), (5, 1), (6, 6)} $\neq \emptyset$
Therefore, A and B are not mutually exclusive events.

Similarly, A ∩C$\neq \emptyset$,A ∩D$\neq \emptyset$B ∩C$\neq \emptyset$and B ∩D$\neq \emptyset$.
Thus, the pairs, (A, C), (A, D), (B, C), (B, D) are not mutually exclusive events.
Also, C ∩D =$\emptyset$ and so C and D are mutually exclusive events.
Example: A coin is tossed three times, consider the following events.
A: 'No head appears', B: 'Exactly one head appears' and C: 'At least two heads appear'.
Do they form a set of mutually exclusive and exhaustive events?

Complementary events: For two events A and B, A is said to be complementary event of B, if A and B are mutually exclusive and exhaustive events.

E.g.: When a dice is thrown, occurrence of even numbers (2, 4, 6) and odd numbers (1, 3, 5) are complementary events.

Favorable cases: The number of sample points favorable to the happening of an event A is known as favorable cases.

E.g.: In drawing a card from a pack of cards, the favorable cases for getting a club is 13.

Probability:

$$p = probability\ of\ sucess$$

$$q = probability\ of\ failure$$

Mathematical or Classical approach: If an experiment results in 'n' equally likely ways, and out of which 'm' is favorable to the happening of the event A, then the probability of happening of event A is defined as the ration of m: n.

$$P(A) = \frac{Number of favorable cases}{Total number of equally likely cases} = \frac{m}{n}$$

The number of favorable cases is always less than or equal to the total number of equally likely cases.

$$0 \leq m \leq n$$

$$0 \leq \frac{m}{n} \leq 1$$

$$0 \leq P(A) \leq 1$$

Thus, the probability of an event is always between 0 and 1.

The probability of a certain event is always 1 and that of impossible event is 0.

If 'm' cases are favorable to the happening of event A, then 'n-m' are the cases not favorable to the happening of A i.e. favorable to the happening of complement of event A.

$$P(A') = \frac{n-m}{n}$$

$$P(A') = \frac{n}{n} - \frac{m}{n}$$

$$P(A') = 1 - \frac{m}{n}$$

$$P(A') = 1 - P(A)$$

$$P(A) + P(A') = 1$$

Thus, the probability of happening and non-happening of an event is 1.

Shortcomings of the classical approach:

1. It cannot be applied whenever it is not possible to find the no. of equally likely cases.
2. It cannot be applied if exhaustive cases are infinite.
3. It cannot be applied if the events are not equally likely.

Statistical or Empirical or A posteriori definition of Probability: If an experiment is repeated under same conditions for a large number of times, then the limit of the ratio of number of times the event happens to the total number of trials is Statistical Probability.

$$\text{i.e.,} P(A) = \lim_{n\to\infty} \left(\frac{m}{n}\right)$$

Modern or Axiomatic definition of Probability: The axiomatic approach to probability was introduced by the Russian mathematician A. N. Kolmogorov in the year 1993 in the book *Foundations of Probability*.

If A is any event from the sample space S, then $P(A)$is called the probability of A if it satisfies the following axioms:

i. $0 \le P(A) \le 1$
ii. $P(S) = 1$
iii. If A and B are mutually exclusive events then the probability of occurrence of either A or B is denoted by $P(A \cup B)$

$$P(A \cup B) = P(A) + P(B)$$

EXAMPLE: Three unbiased coins are tossed. Find the probability of getting (i) exactly 2 heads, (ii) at least one tail, (iii) at most 2 heads, (iv) a head on the second coin and, (v) exactly 2 heads in succession.

Solution:

When three coins are tossed, then sample space S given by

S= {HHH, HTH, THH, HHT, TTT, THT, TTH, HTT}

n(s)=8

(i) Let A be the event of getting exactly 2 heads.

A= {HTH, THH, HHT}
n(A)=3

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{8}$$

(ii) Let B be the events of getting at least one tail.
B= {HTH, THH, HHT, TTT, THT, TTH, HTT}
n(B)=7

$$P(A) = \frac{n(B)}{n(S)} = \frac{7}{8}$$

(iii) Let C be the event of getting at most 2 heads.
C= {HTH, THH, HHT, TTT, THT, TTH, HTT}
n(C)=7

$$P(A) = \frac{n(C)}{n(S)} = \frac{7}{8}$$

(iv) Let D be the event of getting a head on the second coin.
D= {HHH, THH, HHT, THT}
n(D)=4

$$P(A) = \frac{n(D)}{n(S)} = \frac{4}{8}$$

(v) Let E be the event of getting 2 heads in succession.
E= {HHH, THH, HHT}
n(E)=3

$$P(A) = \frac{n(E)}{n(S)} = \frac{3}{8}$$

EXAMPLE: Find the probability that an odd number appears in a single toss of a fair die.

Solution: The sample for the experiment of tossing a die is S= {1,2,3, 4, 5,6}. Let A be the event that an odd number appears in a single toss of the die. Hence A= {1, 3,5}

$$P(A) = \frac{N(A)}{N(S)} = \frac{3}{6} = \frac{1}{2}$$

EXAMPLE: A card is drawn from a pack of well- shuffled cards. Find the probability of the following events.

(i) The card drawn is a spade.
(ii) The card drawn is a king.
(iii) The card drawn is a face card.
(iv) The card drawn is not a club.
(v) The card drawn is either a heart or a diamond.

Solution: In the pack there are 52 cards. One card can be drawn from the pack in 52 ways. Hence the sample space for the experiment of drawing a card has 52 elements i.e. N(S)=52.

(i) Let A= The card drawn is spade.
There are 13 spade cards in the pack. Hence N(A)=13

$$\therefore P(A)=\frac{N(A)}{N(S)}=\frac{13}{52}=\frac{1}{4}$$

(ii) Let B= The card drawn is king.
There are 4 kings in the pack. Hence N(B)=4

$$\therefore P(B)=\frac{N(B)}{N(S)}=\frac{4}{52}=\frac{1}{13}$$

(iii) Let C= The card drawn is face card.
There are 12face cards in the pack (4 king, 4 queen, 4 jacks). Hence N(C)=12

$$\therefore P(C)=\frac{N(A)}{N(S)}=\frac{12}{52}=\frac{3}{13}$$

(iv) Let D= The card drawn is club.
There are 13 spade cards in the pack. Hence N(D)=13

$$\therefore P(D)=\frac{N(A)}{N(S)}=\frac{13}{52}=\frac{1}{4}$$

(v) Let E= The card drawn is heart and F= The card drawn is diamond.

$$\therefore P(E)=\frac{N(E)}{N(S)}=\frac{13}{52}=\frac{1}{4},\ \therefore P(F)=\frac{N(F)}{N(S)}=\frac{13}{52}=\frac{1}{4}$$

Now, $E\cup F$ =The card is a heart or diamond.

EXAMPLE 4. What is the probability that a leap year selected at random will have 53 Sundays?

Solution: -A leap year has 366 days, that is 52 weeks and 2 days. These 2 days can occur in the following manner:

(i) Monday and Tuesday
(ii) Tuesday and Wednesday
(iii) Wednesday and Thursday
(iv) Thursday and Friday
(v) Friday and Saturday
(vi) Saturday and Sunday
(vii) Sunday and Monday

Number of exhaustive cases n=7

Number of favorable cases m=2

Let A be the event of getting 53 Sundays in a leap year $P(A)=\frac{m}{n}$ =2/7.

EXAMPLE 5: Find the probability of getting at least one head in two throws of unbiased coin.

Solution: The sample space associated with the random experiment of throwing a coin twice is S= {HH, HT, TH, TT}

Let A be the event of getting at least one head.

$\therefore A = \{HH, HT, TH\}$

$$\therefore P(A) = \frac{N(A)}{N(S)} = \frac{3}{4}$$

EXERCISE: -

1)A committee of two persons is selected from two men and two women.
What is the probability that the committee will have (a) no man? (b) one man? (c) two men?
[Ans: - (i) 1/6, (ii) 2/3, (iii) 1/6]

2.From a collection of 10 bulbs, of which 4 are defective, 3 bulbs are selected at random and fitted into lamps. Find the probability that (i) all three bulbs glow, and (ii) the rooms is lit. [Ans: - (i) 1/6, (ii) 29/30]

3.Two students Anil and Ashima appeared in an examination. The probability that Anil will qualify the examination is 0.05 and that Ashima will qualify the examination is 0.10. The probability that both will qualify the examination is 0.02. Find the probability that

(a) Both Anil and Ashima will not qualify the examination. [Ans: 0.87]
(b) At least one of them will not qualify the examination. [Ans: 0.98]
(c) Only one of them will qualify the examination. [Ans: 0.11]

Addition Theorem: If A and B are two events, the probability of occurrence of either A or B is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

i. If A and B are mutually exclusive events then $A \cap B = \varphi$ hence$P(A \cap B) = 0$

$$P(A \cup B) = P(A) + P(B)$$

ii. For 3 events A, B and C, the probability that at least one will occur is

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

iii. $P(A \cup B) = 1 - P(A' \cap B')$

Multiplication Theorem: If two events A and B are independent, the probability of occurrence of both is equal to the product of the individual probabilities of both the events.

$$P(A \cap B) = P(A) * P(B)$$

Example: There are 5 red ad 7 black balls in an urn. Two balls are drawn at random one after the other. If they are drawn (i) with replacement (ii) without replacement, find the probability that both the balls are red. (Summer 22-23)

Solution: There are 5 red and 7 black balls in urn. So, Total balls are 12.

Let A= event of taking first ball red from urn

Let B= event of taking second ball red from urn

We are drawing **two balls** and want the probability that both are **red**. That can occur in two cases:

(i) With replacement

After drawing the first ball, it is replaced, so the total number of balls remains the same for both draws. The probability of drawing a red ball in each draw is:

P(A)=$\frac{5}{12}$, P(B) = $\frac{5}{12}$

Probability to get both are the red $P(A \cap B) = P(A) * P(B)$= $\frac{5}{12} * \frac{5}{12} = \frac{25}{144}$

(ii) Without replacement

After drawing the first ball, it is not replaced, so the total number of balls decreases by 1, and the number of red balls also changes if a red ball is drawn.

Then, P(A)=$\frac{5}{12}$, $P(B) = \frac{4}{11}$

Probability to get both are the red $P(A \cap B) = P(A) * P(B)$= $\frac{5}{12} * \frac{4}{11} = \frac{20}{132} = \frac{5}{33}$

Example: A card is drawn from a well-shuffled pack of cards. What is the probability that it is either a spade or an ace? (Summer 22-23)

Solution: -Let A and B be the events of getting a spade and an ace card respectively.

$$\text{P(A)} = \frac{{}^{13}C_1}{{}^{52}C_1} = \frac{13}{52}$$

$$\text{P(B)} = \frac{{}^{4}C_1}{{}^{52}C_1} = \frac{4}{52}$$

$$P(A \cap B) = \frac{{}^{1}C_1}{{}^{52}C_1} = \frac{1}{52}$$

Probability of getting either a spade or an ace card Now, using formula

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

Example: The probability that a contractor will get a plumbing contract is 2/3 and the probability that he will not get an electric contract is 5/9. If the probability of getting any one contract is 4/5, what is the probability that he will get both the contract?

Solution: - Let A and B be the events that the contractor will get plumbing and electric contracts respectively. $P(A) = \frac{2}{3}, P(\bar{B}) = \frac{5}{9}, P(A \cup B) = \frac{4}{5}$

$$P(B) = 1 - P(\bar{B}) = 1 - \frac{5}{9} = 4/9$$

Probability that the contractor will get any one contract

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Probability that the contractor will get both the contracts

$P(A \cap B) = P(A) + P(B) - P(A \cup B) = \frac{2}{3} + \frac{4}{9} - \frac{4}{5} = \frac{14}{45}$.

EXERCISE: -

1) Two cards are drawn from a pack of cards. Find the probability that they will be both red and both pictures. [**Ans: -188/663**]
2) A box contains 4 white, 6 red, 5 black balls and 5 balls of other colors. Two balls are drawn from the box at random. Find the probability that (i) both are white or both are red, and (ii) both are red or both are black. [**Ans: - (i) 21/190, (ii) 5/38**]
3) If $P(A) = \frac{1}{3}, P(B') = \frac{1}{4}$, and $P(A \cap B) = \frac{1}{6}$, find $P(A \cup B), P(A' \cap B')$ and $P(\frac{A'}{B'})$.
4) If an unbiased dice is rolled. Find the probability of getting: (i) Even Number (ii) A perfect square (iii) A number divisible by 3**. (Summer 22-23)**
5) A card is drawn at random from a pack of well shuffled 52 cards. Find the probability that the card drawn is (i) a king (ii) not a diamond (iii) An ace of red hearts or diamonds
6) Two students X and Y work independently on a problem. The probability that X will solve it is $\frac{3}{4}$ and probability that Y will solve it is $\frac{2}{3}$. What is the probability that the problem will be solved?

Conditional Probability:

Up till now in probability, we have discussed the methods of finding the probability of events. If we have two events from the same sample space, does the information about the occurrence of one of the events affect the probability of the other event? Let us try to answer this question by taking up a random experiment in which the outcomes are equally likely to occur.
Consider the experiment of tossing three fair coins. The sample space of the experiment is
S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Since the coins are fair, we can assign the probability 1/8 to each sample point. LetE be the event 'at least two heads appear' and F be the event 'first coin shows tail'.
Then
E = {HHH, HHT, HTH, THH}
and F = {THH, THT, TTH, TTT}
Therefore P(E) = P ({HHH}) + P ({HHT}) + P ({HTH}) + P ({THH})
=1/8+ 1/8 +1/8+ 1/8 =1/2

and $P(F) = P(\{THH\}) + P(\{THT\}) + P(\{TTH\}) + P(\{TTT\})$
$=1/8 + 1/8 + 1/8 + 1/8 = 1/2$

Also, $E \cap F = \{THH\}$
with $P(E \cap F) = P(\{THH\}) = 1/8$

Now, suppose we are given that the first coin shows tail, i.e. F occurs, then what is the probability of occurrence of E?
With the information of occurrence of F, we are sure that the cases in which first coin does not result into a tail should not be considered while finding the probability of E. This information reduces our sample space from the set S to its subset F for the event E.

In other words, the additional information really amounts to telling us that the situation may be considered as being that of a new random experiment for which the sample space consists of all those outcomes only which are favorable to the occurrence of the event F.
Now, the sample point of F which is favorable to event E is THH.

Thus, Probability of E considering F as the sample space = 1/4,
or Probability of E given that the event F has occurred = 1/4

This probability of the event E is called the *conditional probability of E given that F has already occurred*, and is denoted by P (E|F). Thus P(E|F) = 1/4

Note that the elements of F which favor the event E are the common elements of E and F, i.e. the sample points of $E \cap F$.

Thus, we can also write the conditional probability of E given that F has occurred as

$$P(E|F) = \frac{\text{Number of elementary events favorable to } E \cap F}{\text{Number of elementary events which are favorable to } F}$$

$$= \frac{n(E \cap F)}{n(F)}$$

Dividing the numerator and the denominator by total number of elementary events of the sample space, we see that P(E|F) can also be written as

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

Thus, we can define the conditional probability as follows:

Definition: If E and F are two events associated with the same sample space of a random experiment, the conditional probability of the event E given that F has occurred, i.e. P (E|F) is given by

$$\mathbf{P(E|F) = \frac{P(E \cap F)}{P(F)}}$$

Provided P(F)$\neq$ 0

Properties of conditional probability

Let E and F be events of a sample space S of an experiment, then we have

Property 1 P(S|F) = P(F|F) = 1

Property 2 If A and B are any two events of a sample space S and F is an event of S such that P(F) $\neq$ 0,then P ((A ∪ B) |F) = P(A|F) + P(B|F) – P ((A ∩ B) |F)
Property 3 P(E′|F) = 1-P (E|F)

Example: A family has two children. What is the probability that both the children are boys given that at least one of them is a boy?
Solution Let *b* stand for boy and *g* for girl. The sample space of the experiment is
S = {(*b, b*), (*g, b*), (*b, g*), (*g, g*)}
Let E and F denote the following events:
E: ‘both the children are boys’
F: ‘at least one of the child is a boy’
Then E = {(*b*,*b*)} and F = {(*b*,*b*), (*g*,*b*), (*b*,*g*)}
Now E ∩F = {(*b*,*b*)}
Thus P(F) =3/4 and P (E ∩F)=1/4
Therefore P(E|F) =(1/4) / (3/4)=1/3

Independent Events

Consider the experiment of drawing a card from a deck of 52 playing cards, in which the elementary events are assumed to be equally likely. If E and F denote the events the card drawn is a spade' and 'the card drawn is an ace' respectively, then
P(E) =13/52 =1/4 and P(F)= 4/52 =1/3,
Also, E and F is the event the card drawn is the ace of spades' so that P(E∩F) =1/52.
Hence P(E|F) = $\frac{P(E\cap F)}{P(F)} = \frac{1/_{52}}{4/_{52}}$ = 1/4
Since P(E) =1/4 = P (E|F), We can say that the occurrence of event F has not affected the probability of occurrence of the event E.
We also have P(F|E) = 1/13 =P(F)
Which shows that occurrence of event E has not affected the probability of occurrence of the event F.

Thus, E and F are two events such that the probability of occurrence of one of them is not affected by occurrence of the other. Such events are called *independent events*.

<u>**Definition**</u> **Two events E and F are said to be independent, if**

P(F|E) = P (F) provided P (E)$\neq$ 0 and

P (E|F) = P (E) provided P (F) $\neq$ 0

Thus, in this definition we need to have P (E) $\neq$ 0and P(F) $\neq$ 0
Now, by the multiplication rule of probability, we have

P(E ∩F) = P(E) . P (F|E) ... (1)
If E and F are independent, then (1) becomes
P(E ∩F) = P(E) . P(F) ... (2)
Thus, event E and F are independent if P (E ∩F) = P(E). P(F)
Note: If A and B are independent events, then $P(A \cup B) = P(A) + P(B) - P(A)P(B)$.

Example: A die is thrown. If E is the event 'the number appearing is a multiple of 3' and F be the event 'the number appearing is even' then find whether E and F are independent?

Solution We know that the sample space is S = {1, 2, 3, 4, 5, 6}
Now E = {3, 6}, F = {2, 4, 6} and E ∩ F = {6}
Then P(E) =2/6 =1/3
P(F)= 3/6=1/2
and P (E∩ F) =1/6

Clearly P(E∩F) = P(E). P (F)
Hence E and F are independent events.

Exercise Three coins are tossed simultaneously. Consider the event E 'three heads or three tails', F 'at least two heads and G 'at most two heads. Of the pairs (E, F), (E, G) and (F, G), which are independent? Which are dependent?

Example: The personal department of a company has records which show the following analysis of its 200 engineers.

Age (Year)	Bachelor's Degree only	master's Degree	Total
Under 30	90	10	100
30 to 40	20	30	50
Over 40	40	10	50
	150	50	200

If one engineer is selected at random from the company, find
(i) The probability that he has only a bachelor's degree
(ii) The probability that he has a master's degree given that he is over 40
(iii) The probability that he is under 30 given that he has only a bachelor's degree.

Solution:
(i) Probability that the engineer has only a bachelor's degree

The total number of engineers with only a bachelor's degree is 150
The total number of engineers is 200.

$$P(Bachelor's\ Degree\ Only) = \frac{150}{200} = \frac{3}{4}$$

(ii) Probability that the engineer has a master's degree given that they are over 40

To find the conditional probability P (Master's Degree |Over 40), we use the formula:
$P(A/B) = \frac{P(A \cap B)}{P(B)}$
Where, P(A∩B) = Number of engineers with a master's degree and over 40 = 10
P(B) = Total number of engineers over 40 = 50
$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{10}{50} = \frac{1}{5}$
(iii) Probability that the engineer is under 30 given that they have only a bachelor's degree

To find the conditional probability P (Under 30|Bachelor's Degree Only) we use:
$P(A/B) = \frac{P(A \cap B)}{P(B)}$
Where, P(A∩B) = Number of engineers who are under 30 and have only a bachelor's degree = 90, P(B) = Total number of engineers with only a bachelor's degree = 150.
$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{90}{150} = \frac{3}{5}$

Example: In a certain college, 25% students failed in mathematics, 15% students failed in statistics and 30% students failed in at least one of the subjects selected at random. Find the probability that

(a)he failed in both mathematics and statistics Ans = 0.1

(b)he failed in mathematics if he also failed in statistics Ans =0.4

(c)he failed in statistics if he also failed in mathematics Ans=0.67

(d)he failed in mathematics given that he passed in statistics.

Solution:
Let Probability of failing in Mathematics P(M)=0. 25
Let Probability of failing in Statistics. P(S)=0.15
Therefore, Probability of failing in at least one of the subjects P(M∪S) = 0.30

From the formula for the union of two events:

P(M∪S) =P(M)+P(S)−P(M∩S)

Therefore, P(M∩S) = 0.25+0.15-0.30 = 0.10

(a) Probability of failing both Mathematics and Statistics is P(M∩S) = 0.10
(b) Probability of failed in mathematics if he also failed in statistics is
$$P(M \mid S) = \frac{P(M \cap S)}{P(S)} = \frac{0.10}{0.15} = 0.666$$
(c) Probability of failed in statistics if he also failed in mathematics is
$P(S \mid M) = \frac{P(M \cap S)}{P(M)} = \frac{0.10}{0.25} = 0.4$
(d) Probability of failed in mathematics given that he passed in statistics

The probability of passing in Statistics is $P(S^c) = 1-P(S) = 1-0.15 = 0.85$

$$so, P(M \cap S^c) = P(M) - P(M \cap S) = 0.25 - 0.10 = 0.15$$

$$P(M \mid S^c) = \frac{P(M \cap S^c)}{P(S^c)} = \frac{0.15}{0.85} = 0.1764$$

Example: If A and B are 2 events such that p(A)=2/3, $P(\bar{A} \cap B) = \frac{1}{6} and P(A \cap B) = \frac{1}{3}$, find P(B), P(A$\cup B$), $P(\bar{A} \cup B)$, P(A/B), P(B/A) and $P(\bar{B})$. Also examine whether the events A and B are (i) equally likely, (ii) exhaustive, (iii) mutually exclusive, and (iv) independent

Solution: -

P(B)=$P(\bar{A} \cap B) + P(A \cap B)$=1/6+1/3=1/2.

P(A$\cup B$)=P(A)+P(B)-$P(A \cap B)$=2/3+1/2-1/3=5/6

P(A/B) = $\frac{P(A \cap B)}{P(B)} = \frac{1/3}{1/2} = 2/3$

P(B/A) = $\frac{P(A \cap B)}{P(A)} = \frac{1/3}{2/3} = 1/2$

$P(\bar{A} \cup B)$=$P(\bar{A}) + P(B) - P(\bar{A} \cap B)$=1/3+1/2-1/6=2/3

P($\bar{B}$)=1-P(B)=1-1/2=1/2

(i) Since $P(A) \neq P(B)$, A and B are not equally like events.

(ii) Since P (A$\cup B$) $\neq 1$, A and B are not exhaustive events.

(iii) Since $P(A \cap B) \neq 0$, A and B are not mutually exclusive events.

(iv) Since $P(A \cap B) = P(A)P(B)$, A and B are independent events.

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EXERCISE: -

1) A department store has been the target of many shoplifters during the past month,

but owing to increased security precautions, 250 shoplifters have been caught. Each shoplifter's sex is noted, also noted is whether he/she was a first-time or repeat offender. The data are summarized in the table below

Sex	First-Time Offender	Repeat Offender	Total
Males	60	70	130
Females	44	76	120
	104	146	250

Assuming that an apprehended shoplifter is chosen at random, find:

a) The Probability that the shoplifter is male. **Ans=130/250**

b) The Probability that the shoplifter is a first-time offender, given that the shoplifter is male. **Ans=60/130**

c) The Probability that the shoplifter is female, given that the shoplifter is a repeat offender. **Ans=76/146**

d) The Probability that the shoplifter is female, given that the shoplifter is a first-time offender. Ans= **44/104**

2) If a and B are 2 events with p(A)=1/3, P(B)=1/4, $P(A \cap B) = 1/12$. Find (i) P(A/B), (ii) P(B/A), (iii) P (B/$\bar{A}$), and (iv) $P(A \cap \bar{B})$. **[Ans: - (i) 1/3, (ii) ¼, (iii) ¼, (iv) ¼]**

3) Find the probability of drawing a queen and a king from a pack of cards in 2 consecutive draws, the cards drawn not being replaced. **[Ans: - 4/663]**

4) A bag contains 8 red and 5 white balls. Two successive draws of 3 balls each are made such that (i) the balls are replaced before the Second trial, and (ii) the balls are not replaced before the second trial. Find the probability that the first draw will give 3 white and the second 3 red balls. **[Ans: - (i) 140/20449, (ii) 7/429]**

5) From a bag containing 4 white and 6 black balls, 2 balls are drawn at random. If the balls are drawn one after the other without replacements, find the probability that the first ball is white and the second ball is black. **[Ans: - 4/15]**

Bayes' theorem:

Consider that there are two bags I and II. Bag I contains 2 white and 3 red balls and Bag II contains 4 white and 5 red balls. One ball is drawn at random from one of the bags. We can find the probability of selecting any of the bags (i.e.1/2) or probability of drawing a ball of a particular color (say white) from a particular bag (say Bag I). In other words, we can find the probability that the ball drawn is of a particular color, if we are given the bag from which the ball is drawn. But, can we find the probability that the ball drawn is from a particular bag (say Bag II), if the color of the ball drawn is given? Here, we have to find the reverse probability of Bag II to be selected when an event occurred after it is known. Famous mathematician, John Bayes' solved the problem of finding reverse probability by using conditional probability. The formula developed by him is known as *'Bayes theorem'* which was published posthumously in 1763.

In conditional probability we consider the probability of an event when we have information about the occurrence of an earlier event. Bayes theorem determines the probability of an earlier event based on the information about the occurrence of a later event.

Statement of Baye's Theorem: If an event A corresponds to a number of mutually exclusive events $B_1, B_2, \ldots\ldots B_n$ *and if* $P(B_i)$ *and* $P(A|B_i)$ are known and $P(A) > 0,$ *then*

$$P(B_i|A) = \frac{P(B_i)P(A|B_i)}{\sum_{i=1}^{n} P(B_i)P(A|B_i)}, \quad i = 1,2,\ldots.n$$

Particular Case: If n=2, the Bayes' theorem can be stated as follows.

Statement: If an event A corresponds to two mutually exclusive events B_1 *and* B_2 *and* if $P(B_1), P(B_2), P(A|B_1), P(A|B_2)$ are known and P(A)>0, then

$$P(B_1|A) = \frac{P(B_1)P(A|B_1)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2)} \; and \quad P(B_2|A) = \frac{P(B_2)P(A|B_2)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2)}$$

Example: In a pharmaceutical factory, machines A and B manufacture 40% and 60% of the total output. Of this production of tablets, machines A and B produce 5% and 10% defective tablets. A tablet is picked at random and is found to be defective. What is the probability that the tablet was produced by the machine A? (Summer 23-24)

Solution: Let B_1=the tablet is produced by machine A

B_2=the tablet is produced by machine B.

$$\therefore P(B_1) = \frac{40}{100} = 0.4, P(B_2) = \frac{60}{100} = 0.6$$

Let A = the tablet is defective.

$$P(A|B_1) = \frac{5}{100} = 0.05\,, P(A|B_2) = \frac{10}{100} = 0.10$$

We wish to find $P(B_1|A)$. By Bayes' theorem,

$$P(B_1|A) = \frac{P(B_1)P(D|B_1)}{P(B_1)P(D|B_1) + P(B_2)P(D|B_2)}$$

$$= \frac{0.4*0.05}{0.4*0.05 + 0.6*0.10} = \frac{0.02}{0.08} = 0.25$$

Example: In a bolt factory machines A, B and C manufacture 25%, 35% and 40% of the total. Of their output 5%, 4% and 2% are defective bolts. A bolt is drawn at random from the product and is found to be defective. What is the probability that it was manufactured by machine A? (winter 22-23)

Solution: Let A, B, C be the event that bolts are manufactured by machines A, B and C respectively. Let D be the event that the bolt drawn is defective.

P(A)=0.25: Probability that a bolt is manufactured by machine A.
P(B)=0.35 Probability that a bolt is manufactured by machine B.
P(C)=0.40 Probability that a bolt is manufactured by machine C.
P(D|A) =0.05 Probability that a bolt from machine A is defective.
P(D|B) =0.04 Probability that a bolt from machine B is defective.
P(D|C) =0.02 Probability that a bolt from machine C is defective.

Using Baye's Theorem we have to find P(A|D)

$$P(A|D) = \frac{P(A)P(D|A)}{P(A)P(D|A) + P(B)P(D|B) + P(C)P(D|C)}$$

$$P(A|D) = \frac{0.25 * 0.05}{(0.25 * 0.05) + (0.35 * 0.04) + (0.40 * 0.02)} = 0.3623$$

Example: In a railway reservation office, two clerks are engaged in checking reservation forms. On an average the clerk A checks 55% of the forms, while the clerk B does the remaining work. The clerk A has an error rate of 0.03 and the clerk B has an error rate of 0.02. A reservation form is selected at random from the total number of forms checked during a day and discovered to have an error. Find the probabilities that it was checked by the clerks A and B respectively.

Solution: Let B_1=the form is checked by the clerk A

B_2= the form is checked by the clerk B.

$$\therefore P(B_1) = \frac{55}{100} = 0.55, P(B_2) = \frac{45}{100} = 0.45$$

Let A = the form selected at random has an error.

$$P(A|B_1) = 0.03, P(A|B_2) = 0.02$$

We wish to find $P(B_1|A)$ and $P(B_1|A)$. By Bayes' theorem,

$$P(B_1|A) = \frac{P(B_1)P(A|B_1)}{P(B_1)P(A|B_1) + P(B_2)P(A|B_2)}$$

$$= \frac{0.55 * 0.03}{0.55 * 0.03 + 0.45 * 0.02} = \frac{165}{255} = 0.647$$

$$Now, P(B_2|A) = 1 - P(B_1|A) = 1 - 0.647 = 0.353$$

Example: Suppose that 5 men out of 100 and 25 women out of 10,000 are color blind. A color blind person is chosen at random. What is the probability of his being male? (Assume males and females to be in equal numbers.)

Solution: Let B_1=the person is a male

B_2= the person is a female.

$$\therefore P(B_1) = \frac{1}{2}, P(B_2) = \frac{1}{2}$$

Let A = the person is color blind.

$$P(A|B_1) = \frac{5}{100}, P(A|B_2) = \frac{25}{10000}$$

We wish to find $P(B_1|A)$. By Bayes' theorem,

$$P(B_1|A)=\frac{P(B_1)P(A|B_1)}{P(B_1)P(A|B_1)+P(B_2)P(A|B_2)}$$

$$=\frac{\frac{1}{2}*\frac{5}{100}}{\frac{1}{2}*\frac{5}{100}+\frac{1}{2}*\frac{25}{10000}}=\frac{500}{500+25}=\frac{500}{525}=\frac{20}{21}$$

EXERCISE: -

1) A company has 2 plants to manufacture hydraulic machines. Plant I manufactures 70% of the hydraulic machines, and plant II manufactures 30%. At plant I, 80% of hydraulic machines are rated standard quality; and at plant II, 90% of hydraulic machines are rated as standard quality. A machine is picked at random and is found to be of standard quality. What is the chance that it has come from plant I? **[Ans: - 0.6747]**
2) A bag A contains 2 white and 3 red balls, and a bag B contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and it is found to be red. Find the probability that the red ball is drawn from the bag B. **[Ans: - 25/52]**

Probability Distributions

Overview:

The theory of probability owes its origins to the study of games of chance or gambling. For example, the chance of winning a cricket match, the chance of getting pass in examination, the chance of getting railway ticket booking confirmed etc. Probability theory is designed to deal with uncertainties regarding happening of given phenomena. Thus, when we throw a coin, a head is likely to occur but may not occur. When a product is manufactured may or may not defective. The cricket board refers numbers of names in the list who are likely to play for the country but are not certain to be included in the team.

RANDOM VARIABLE: -

A **random variable** is a numerical description of the outcome of an experiment.
In effect, a random variable associates a numerical value with each possible experimental outcome. The particular numerical value of the random variable depends on the outcome of the experiment. A random variable can be classified as being either discrete or continuous depending on the numerical values it assumes.

Discrete Random Variables: -

A random variable that may assume either a finite number of values or an infinite sequence of values such as 0, 1, 2, . . . is referred to as a **discrete random variable**. For example, consider the experiment of an accountant taking the certified public accountant (CPA) examination. The examination has four parts. We can define a random variable as x = the number of parts of the CPA examination passed. It is a discrete random variable because it may assume the finite number of values 0, 1, 2, 3, or 4.

As another example of a discrete random variable, consider the experiment of cars arriving at a tollbooth. The random variable of interest is x _ the number of cars arriving during a one-day period. The possible values for x come from the sequence of integers 0, 1, 2, and so on. Hence, x is a discrete random variable assuming one of the values in this infinite sequence.

TABLE 1 EXAMPLES OF DISCRETE RANDOM VARIABLES

Experiment	Random Variable (x)	Possible Values for the Random Variable
Contact five customers	Number of customers who place an order	0, 1, 2, 3, 4, 5
Inspect a shipment of 50 radios	Number of defective radios	0, 1, 2, · · · , 49, 50
Operate a restaurant for one day	Number of customers	0, 1, 2, 3, · · ·
Sell an automobile	Gender of the customer	0 if male; 1 if female

Discrete Probability Distributions: -

The **probability distribution** for a random variable describes how probabilities are distributed over the values of the random variable. For a discrete random variable x, the probability distribution is defined by a **probability function,** denoted by f (x). The probability function provides the probability for each value of the random variable.

REQUIRED CONDITIONS FOR A DISCRETE PROBABILITY FUNCTION

$$f(x) \geq 0$$
$$\sum f(x) = 1$$

DISCRETE UNIFORM PROBABILITY FUNCTION

$$f(x) = 1/n$$

Where, n = the number of values the random variable may assume

For example, suppose that for the experiment of rolling a die we define the random variable x to be the number of dots on the upward face. For this experiment, n= 6 values are possible for the random variable; x = 1, 2, 3, 4, 5, 6. Thus, the probability function for this discrete uniform random variable is $f(x) = 1/6 \qquad x = 1, 2, 3, 4, 5, 6$

The possible values of the random variable and the associated probabilities are shown.

x	*1*	*2*	*3*	*4*	*5*	*6*
f(x)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Expected Value and Variance: -

Expected Value: - The **expected value**, or mean, of a random variable is a measure of the central location for the random variable. The formula for the expected value of a discrete random variable x follows.

EXPECTED VALUE OF A DISCRETE RANDOM VARIABLE

If $x_1, x_2, x_3 ...., x_n$ are different values of a random variable x with their respective probabilities Then the expected value of the random variable x is defined as follows:

$$E(x) = \mu = x_1 f(x_1) + x_2 f(x_2) + ... + x_n f(x_n)$$
$$= \sum x_i f(x_i)$$

The expected value of x is the mean of x and is denoted by μ.

$E(x) = \mu$

Some properties of expected value:

(i) The expected value of a constant is a constant itself $i.e.\ E(k) = k$

(ii) If k is a constant, E(kx)=k E(x)

(iii) $E(ax+b) = aE(x) + b$

(iv) If x and y are two Random Variables E(x+y) =E(x)+E(y)

(v) If x and y are two independent random variables E(xy)= E(x)E(y)

$(vi)\ E(x-\mu) = E(x) - E(\mu)$
$$= \mu - \mu$$
$$= 0$$

(vii) If g(x) is a function of x, E[g(x)] =∑g(x). f(x)

Variance: -

Even though the expected value provides the mean value for the random variable, we often need a measure of variability, or dispersion. We now use **variance** to summarize the variability in the values of a random variable. The formula for the variance of a discrete random variable follows.

VARIANCE OF A DISCRETE RANDOM VARIABLE

$$Var(x) = \sigma^2 = \sum (x-\mu)^2 f(x)$$

The notations $Var(x)$ and σ^2 are both used to denote the variance of a random variable. The **standard deviation,** σ, is defined as the positive square root of the variance.

Variance of a random variable:

The average of the squares of the deviations from the mean of a random variable x is said to Be its variance.

$Var(x) = E(x-\mu)^2$

$= E(x^2 - 2x\mu + \mu^2)$

$= E(x^2) - 2\mu E(x) + E(\mu^2)$

$= E(x^2) - 2\mu\mu + \mu^2$

$= E(x^2) - \mu^2$

$= E(x^2) - [E(x)]^2$

If mean of a random Variable x is $E(x) = \mu$ the variance of then variance of the x can be given as follows:

Some properties of variance:

(1) If k is a constant, $V(k) = 0$

(2) $V(kx) = k^2 V(x)$

(3) $V(ax+b) = a^2 V(x)$

(4) If x *and* y are two independent random variables $V(ax+by) = a^2 V(x) + b^2 V(x)$

Example: 1. A tray of electronics components contains nine good components and three defective components. If two components are selected at random, what is the expected number of defective components?

Solution: Let a random variable X be the number of defective components selected.

X can have the value 0,1, or 2. We need the probability of each of those numbers.

P(0) = Probability of no. of defective

$= 9C_2 / 12C_2$

$= 36 / 66 = 12 / 22$

P(1) = Probability of 1good and 1 defective

$= 9C_1 . 3C_1 / 12C_2$

$= 27 / 66 = 9 / 22$

P(2) = Probability of two defective

$= 3C_2 / 12C_2$

$= 3 / 66 = 1 / 22$

The expected value is

E(X) = 12/22 (0) + 9/22 (1) +1/22 (2)

= 11/22 = ½.

So, the expected number of components is ½.

Example: 2Which of the following functions are probability density function?

(i) **$F(x) = (1/2)^x (1/2)^{1-x}$; x =0,1**

Solution:

x	0	1
F(x)	0.5	0.5

Here $\sum f(x) = 1$ so, it is probability function.

(ii)

X	**-1**	**0**	**1**
F(x)	**0.5**	**0.8**	**-3**

Solution: Here the last value of f(x) is negative which is not possible. Therefore, given distribution is not possible.

Example: 3 The Probability distribution of random variable X is given below.

Find (i) E(x) (ii) V(x) (iii) E(2x-3) (iv) V(2x-3)

X	-2	-1	0	1	2
P(x)	0.2	0.1	0.3	0.3	0.1

Solution:

x	$P(x)$	$xP(x)$	x^2	$x^2P(x)$
-2	0.2	-0.4	4	0.8
-1	0.1	-0.1	1	0.1
0	0.3	0.0	0	0.0
1	0.3	0.3	1	0.3
2	0.1	0.2	4	0.4
		$\sum xP(x) = 0$		$\sum x^2P(x) = 1.6$

From the table, we have

(i) $E(x) = \sum xP(x) = 0$

(ii) $E(x^2) = \sum x^2P(x) = 1.6$

Now, $V(x) = E(x^2) - [E(x)]^2 = 1.6 - 0 = 1.6$

(iii) Now, $E(ax-b) = a\,E(x) - b$

$E(2x-3) = 2E(x) - 3 = 2(0) - 3 = -3$

(iv) Also, $V(aX+b) = a^2V(X)$

$V(2x-3) = (2)^2 V(x) = 4(1.6) = 6.4$

Exercise:

Example: 1 Two unbiased coins are tossed. Find expected values of number of heads.

Example: 2 In a lottery of 10,000 tickets, only one ticket bears a prize of Rs. 5,000. The price of a ticket is Rs. 100. Rakesh has one ticket of this lottery. Find his expectation.

Example:3 The probability distribution of demand of a commodity is given below:

Demand x	5	6	7	8	9	10
Probability p(x)	0.05	0.1	0.3	0.4	0.1	0.05

Find the expected demand and its variance.

Continuous Random variable

If X is a continuous random variable having the probability density function f(x) then the function,

$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx, \quad -\infty < x < \infty$ is called the distribution function or cumulation function of the random variable X.

Properties of Probability Density Function

i. $f(x) \geq 0, \quad -\infty < x < \infty$
ii. $\int_{-\infty}^{\infty} f(x)dx = 1$
iii. $P(a < x < b) = \int_{a}^{b} f(x)dx$

Properties of Cumulative Distribution Function

i. $F(-\infty) = 0$
ii. $F(\infty) = 1$
iii. $0 \leq F(X) \leq 1, \ -\infty < x < \infty$
iv. $P(a < X < b) = F(b) - F(a)$
v. $F'(x) = \frac{d}{dx}F(x) = f(x), f(x) \geq 0$

Example: Find the constant k such that the function

$$f(x) = kx^2 \qquad 0 < x < 3$$
$$= 0 \qquad otherwise$$

Is a probability density function and compute (i) P(1<x<2), (ii)P(X<2), and (iii)P(X≥2).

Solution: Since f(x) is a probability density function $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_{-\infty}^{0} f(x)dx + \int_{0}^{3} f(x)dx + \int_{3}^{\infty} f(x)dx = 1$$

$$0+\int_{0}^{3} kx^2dx+0=1$$

$$k\left|\frac{x^3}{3}\right|_0^3=1$$

$$\frac{k}{3}(27-0)=1$$

$$9k=1$$

$$k=\frac{1}{9}$$

Hence, $$f(x)=kx^2 \qquad 0<x<3$$

$$=0 \qquad otherwise$$

(i) $P(1<X<2)=\int_1^2 f(x)dx=\int_1^2 \frac{1}{9}x^2dx=\frac{1}{9}\left|\frac{x^3}{3}\right|_1^2=\frac{1}{27}(8-1)=\frac{7}{27}$

(ii) $P(X<2)=\int_{-\infty}^{2} f(x)dx=\int_{-\infty}^{0} f(x)dx+\int_0^2 f(x)dx=0+\int_0^2 \frac{1}{9}x^2dx$

$$=\frac{1}{9}\left|\frac{x^3}{3}\right|_0^2=\frac{1}{27}(8-0)=\frac{8}{27}$$

(iii) $P(X\geq 2)=1-P(X<2)=1-\frac{8}{27}=\frac{19}{27}$

Example: If the probability density function of a random variable is given by

$$f(x)=k(1-x^2) \qquad 0<x<1$$

$$=0 \qquad otherwise$$

Find the value of k and the probabilities that a random variable having this probability density will take on a value (i) between 0.1 and 0.2 and (ii) greater than 0.5

Solution: Since f(x) is a probability density function, $\int_{-\infty}^{\infty} f(x)dx=1$

$$\int_{-\infty}^{0} f(x)dx+\int_{0}^{1} f(x)dx+\int_{1}^{\infty} f(x)dx=1$$

$$0+\int_{0}^{1} k(1-x^2)dx+0=1$$

$$k\left|x-\frac{x^3}{3}\right|_0^1=1$$

$$k\left(1-\frac{1}{3}\right)=1$$

$$k=\frac{3}{2}$$

Hence,

$$f(x) = k(1 - x^2) \qquad 0 < x < 1$$
$$= 0 \qquad otherwise$$

(i) Probability that the variable will take on a value between 0.1 to 0.2

$$P(0.1 < X < 0.2) = \int_{0.1}^{0.2} f(x)dx = \int_{0.1}^{0.2} \frac{3}{2}(1 - x^2)dx = \frac{3}{2}\left| x - \frac{x^3}{3} \right|_{0.1}^{0.2} = 0.1465$$

(ii) Probability that the variable will take on a value greater than 0.5

$$P(X > 0.5) = \int_{0.5}^{\infty} f(x)dx$$
$$= \int_{0.5}^{1} f(x)dx + \int_{1}^{\infty} f(x)dx = \int_{0.5}^{1} \frac{3}{2}(1 - x^2)dx + 0 = \frac{3}{2}\left| x - \frac{x^3}{3} \right|_{0.5}^{1}$$
$$= \frac{3}{2}\left[\left(1 - \frac{1}{3}\right) - (0.5 - \frac{0.125}{3})\right] = 0.3125$$

Example: Let the Continuous random variable X have the probability density function

$f(x) = \frac{2}{x^3} \qquad 1 < x < \infty$

$= 0 \qquad otherwise$

Find F(x).

Solution: $F(x) = \int_{-\infty}^{x} f(x) = \int_{-\infty}^{1} f(x)dx + \int_{1}^{\infty} f(x)dx = 0 + \int_{1}^{x} \frac{2}{x^3}dx = 2\frac{3}{2}\left|\frac{x^{-2}}{-2}\right|_{1}^{x} = 1 - \frac{1}{x^2}$

Hence, $F(x) = 1 - \frac{1}{x^2} \qquad 1 < x < \infty$

$= 0 \qquad otherwise$

Exercise:

1) Let X be a continuous random variable with probability distribution

$f(x) = \begin{cases} \frac{x}{6} + k & 0 \le x \le 3 \\ 0 & otherwise \end{cases}$ $find\ k, and\ P(1 \le X \le 2)$ $\qquad$ $\boldsymbol{Ans}: \mathbf{1, 1/3}$

2) If a random Variable has the Probability density function

$$f(x) = k(x^2 - 1) \qquad -1 \le x \le 3$$
$$= 0 \qquad otherwise$$

$Find\ (i) the\ value\ of\ k\ and\ (ii) P\left(\frac{1}{2} \le X \le \frac{5}{2}\right).$ $\qquad$ $\boldsymbol{Ans}: (\boldsymbol{i})\frac{\mathbf{3}}{\mathbf{28}} (\boldsymbol{ii})\frac{\mathbf{19}}{\mathbf{56}}$

Binomial Probability Distribution: -

The binomial probability distribution is a discrete probability distribution that provides many applications. It is associated with a multiple-step experiment that we call the binomial experiment.

A **binomial experiment** exhibits the following four properties.

PROPERTIES OF A BINOMIAL EXPERIMENT

1. The experiment consists of a sequence of n identical trials.

2. Two outcomes are possible on each trial. We refer to one outcome as a success and the other outcome as a failure.

3. The probability of a success, denoted by p, does not change from trial to trial. Consequently, the probability of a failure, denoted by $1-p$, does not change from trial to trial.

4. The trials are independent If properties 2, 3, and 4 are present, we say the trials are generated by a Bernoulli process. If, in addition, property 1 is present, we say we have a binomial experiment. Figure 1 depicts one possible sequence of successes and failures for a binomial experiment involving eight trials.

In a binomial experiment, our interest is in the number of successes occurring in the n trials. If we let x denote the number of successes occurring in the n trials, we see that x can assume the values of 0, 1, 2, 3,..., n. Because the number of values is finite, x is a discrete random variable. The probability distribution associated with this random variable is called the **binomial probability distribution.**

For example, consider the experiment of tossing a coin five times and on each toss observing whether the coin lands with a head or a tail on its upward face. Suppose we want to count the number of heads appearing over the five tosses. Does this experiment show the properties of a binomial experiment? What is the random variable of interest? Note that:

1. The experiment consists of five identical trials; each trial involves the tossing of one coin.
2. Two outcomes are possible for each trial: a head or a tail. We can designate head a success and tail a failure.
3. The probability of a head and the probability of a tail are the same for each trial, with $p=0.5$ and $1-p=0.5$.
4. The trials or tosses are independent because the outcome on any one trial is not affected by what happens on other trials or tosses.

In applications involving binomial experiments, a special mathematical formula, called the binomial probability function, can be used to compute the probability of x successes in the n trials. The number of experimental outcomes resulting in exactly x successes in n trials can be computed using the following formula.

Uses of Binomial distribution:

1. For finding out probability of number of successes out of n independent Bernoulli trials.
2. For finding control limits of p and np charts in Statistical Quality Control.
3. In large sample tests for attributes.

NUMBER OF EXPERIMENTAL OUTCOMES PROVIDING EXACTLY x SUCCESSES IN n TRIALS

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

where $n! = n(n-1)(n-2)....(2)(1)$

$0! = 1$

BINOMIAL PROBABILITY FUNCTION

$$f(x) = \binom{n}{x} p^x (1-p)^{(n-x)}$$

where

$f(x) = \text{the probability of } x \text{ successes in } n \text{ trials}$

$n = \text{the number of trials}$

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

$p = \text{the probability of a success on any one trial}$

$1 - p = \text{the probability of a failure on any one trial}$

Expected Value and Variance for the Binomial Distribution

we provided formulas for computing the expected value and variance of a discrete random variable. In the special case where the random variable has a binomial distribution with a known number of trials n and a known probability of success p ,the general formulas for the expected value and variance can be simplified. The results follow.

Mean and Variance of Binomial distribution:

Mean=

$$\begin{aligned}
&= E(x) \\
&= \sum x \cdot f(x) \\
&= \sum x \cdot {}^nC_x p^x q^{n-x} \\
&= \sum \frac{xn!}{x!(n-x)!} p^x q^{n-x} \\
&= \sum \frac{xn(n-1)!}{x(x-1)!(n-x)!} p^x q^{n-x} \\
&= np \sum \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} q^{n-x} \\
&= np \sum {}^{(n-1)}C_{(x-1)} p^{x-1} q^{n-x} \\
&= np(p+q)^{n-1} \\
&= np(1)^{n-1} \\
&= np
\end{aligned}$$

$$\boldsymbol{Mean\ of\ the\ Binomial\ Distribution\ is\ E(x) = \mu = np}$$

Variance $= E(x)^2 - [E(x)]^2$

$$E(x^2) = E\,x[(x-1)+x]$$
$$= E\,x(x-1)f(x) + \sum xf(x)$$
$$E\,x(x-1)f(x) = \sum x(x-1)\,{}^nC_x p^x q^{n-x}$$
$$= \sum \frac{x(x-1)n!}{x!(n-x)!} p^x q^{n-x}$$
$$= \sum \frac{x(x-1)n(n-1)(n-2)!}{x(x-1)(x-2)!(n-x)!} p^x q^{n-x}$$
$$= n(n-1)p^2 \sum \frac{(n-2)!}{(x-2)!(n-x)!} p^{x-2} q^{n-x}$$
$$= n(n-1)p^2 \sum {}^{(n-2)}C_{(x-2)} p^{x-2} q^{n-x}$$
$$= n(n-1)p^2 (p+q)^{n-2}$$
$$= n(n-1)p^2 (1)$$
$$= n(n-1)p^2$$
$$E(x^2) = E\,x[(x-1)+x]$$
$$= E\,x(x-1)f(x) + \sum xf(x)$$
$$= n(n-1)p^2 + np$$
$$V(x) = E(x^2) - [E(x)]^2$$
$$= n(n-1)p^2 + np - (np)^2$$
$$= n^2p^2 - np^2 + np - n^2p^2$$
$$= np - np^2$$
$$= np(1-p)$$
$$= npq$$

$$\boldsymbol{Variance\ of\ binomial\ distribution\ is\ V(x) = \sigma^2 = npq}$$

EXPECTED VALUE AND VARIANCE FOR THE BINOMIAL DISTRIBUTION

$$E(x) = \mu = np$$
$$Var(x) = \sigma^2 = np(1-p)$$

Example 1 For the binomial distribution with n = 20, p = 0.35. Find mean and variance of binomial distribution. (Winter 22-23)

Solution: n = 20 p = 0.35 q = 1-p =0.65

Mean = μ = np = 20(0.35) = 7

Variance = σ^2 = npq. = (20) (0.35) (0.65) = 4.55

$\sigma = \sqrt{npq}$= 2.133.

Example 2An unbiased coin is tossed 6 times. Find the probability of getting (i) exactly 4 heads (ii) at least 4 heads.

Solution: Probability of getting head in each trial p = ½, q = 1-p = ½.
Now the coin is tossed 6 times So, n = 6.
P(x) = $nC_x p^x q^{n-x}$;
= $6C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{6-x}$;
= $6C_x \left(\frac{1}{2}\right)^6$;
(i) probability of getting 4 heads
X = 4
P(4) = $6C_4 \left(\frac{1}{2}\right)^6$ = 15/64.
(ii) Probability of getting at least 4 heads
x≥ 4 x = 4,5,6.
P(4) = 15/64
P(5) = $6C_5 \left(\frac{1}{2}\right)^6$ = 6/64.
P(6) = $6C_6 \left(\frac{1}{2}\right)^6$ = 1/64.
P (x≥ 4) = P (4) + P (5) +P (6) = 15/64 + 6/64 + 1/64. = 11/32.
Exercise:

Example: 1 For the binomial distribution with n = 160, p = 0.21.

Example: 2 The probability that a man aged 60 will live up to 70 is 0.65. What is the probability that out of 10 such men now at 60 at least 7 will live up to 70?

Example:3 Mean and Variance of a binomial distribution are 4 and $\frac{4}{3}$ $\boldsymbol{respectively\ find\ P(x \geq 1)}$.(Winter-23)

Poisson Probability Distribution: -

In this section we consider a discrete random variable that is often useful in estimating the number of occurrences over a specified interval of time or space. For example, the random variable of interest might be the number of arrivals at a car wash in one hour, the number of repairs needed in 10 miles of highway, or the number of leaks in 100 miles of pipeline. If the following two properties are satisfied, the number of occurrences is a random variable described by the **Poisson probability distribution.**

PROPERTIES OF A POISSON EXPERIMENT
1.The probability of an occurrence is the same for any two intervals of equal length.
2. The occurrence or nonoccurrence in any interval is independent of the occurrence or nonoccurrence in any other interval.

The **Poisson probability function** is defined by equation

$$f(x)=\frac{\mu^x e^{-\mu}}{x!}$$

Where, $f(x)=$ the probability of x occurrences in an interval, $\mu=$ expected value or mean number of occurrences in an interval $e=2.71828$.Before we consider a specific example to see how the Poisson distribution can be applied, note that the number of occurrences, x, has no upper limit. It is a discrete random variable that may assume an infinite sequence of values (x = 0,1, 2, …)

Mean of Poisson distribution:

$$= E(x)$$
$$= \sum x\cdot f(x)$$
$$= \sum x\cdot\frac{\mu^x e^{-\mu}}{x!}$$
$$= \sum x\frac{\mu^x e^{-\mu}}{x(x-1)!}$$
$$= \mu\sum\frac{\mu^{x-1}e^{-\mu}}{(x-1)!}$$
$$= \mu(1)$$
$$= \mu$$

Variance of Poisson distribution:

$$\text{Variance} = E(x)^2-[E(x)]^2$$
$$E(x^2)=\sum x^2\cdot f(x)$$
$$=\sum[x(x-1)+x]\cdot f(x)$$
$$=\sum x(x-1)f(x)+\sum x.f(x)$$
$$Now,\ \sum x(x-1)\frac{\mu^x e^{-\mu}}{x!}$$
$$=\sum x(x-1)\cdot\frac{\mu^x e^{-\mu}}{x(x-1)(x-2)!}$$
$$=\mu^2\sum\frac{\mu^{x-2}e^{-\mu}}{(x-2)!}$$
$$=\mu^2(1)$$
$$=\mu^2$$
$$E(x^2)=E[x(x-1)+x]$$
$$=E(x(x-1))+E(x)$$
$$=\mu^2+\mu$$
$$V(x)=E(x^2)-[E(x)]^2$$
$$=\mu^2+\mu-\mu^2$$
$$=\mu$$

Mean and Variance of Poisson distribution are equal.

Application of Poisson distribution:
The following are some of the instances where Poisson distribution can be applied:

1. Number of accidents on a road.
2. Number of misses prints per page of a book.
3. Number of defects in a radio set.
4. Number of air bubbles in a glass bottles.

Example 1 The variate X has a Poisson distribution and is given that P (X = 2) = 0.25 and P (X =3) = 0.125 Find P (X = 0), P (X = 1) and P (X< 3). (Summer 23-24)

Solution: From the recurrence relation

P(x+1) = $\frac{\lambda}{x+1}p(x)$

P (3) = $\frac{\lambda}{3}p(2)$

0.125 = $\frac{\lambda}{3}(0.25)$

So, λ = 1.5.

Hence P(X=0) = $e^{-1.5}$= 0.223.

P (X = 1) = 0.335.

P (X< 3) = P(X$\leq$ 2) = $e^{-1.5}\sum_{x=0}^{2}\frac{1.5^2}{x!}$= 0.808

Example 2: 100 electric bulbs are found to be defective in a lot of 5000 bulbs. Find the probability that at the most 3 bulbs are defective in a box of 100 bulbs ($e^{-2} = 0.1353$) (Winter 23-24)

Solution: Probability that a bulb is defective = p = 100/5000. = 0.02

In a box there are 100 bulbs

n = 100

np = λ = 100(0.02) = 2

By Poisson distribution, the Probability that x bulbs are defective is P(x).

Where P(x) = $\frac{e^{-\lambda}\lambda^x}{x!}$, $\lambda = 2$

For, at the most 3 bulbs defective x = 0 or 1 or 2 or 3.

Required probability is P (x$\leq$ 3)

Now, P (x $\leq$ 3) = P (0) + P (1) + P (2) + P (3)

= $\frac{e^{-2}2^0}{0!} + \frac{e^{-2}2^1}{1!} + \frac{e^{-2}2^2}{2!} + \frac{e^{-2}2^3}{3!}$

= $e^{-2}(1 + 2 + 2 + 1.33)$

= 0.1353 * 6.33

= 0.8566.

Example: The following mistake per Page observed in a book. Fit a Poisson Distribution.

No. Of Mistake per page	**0**	**1**	**2**	**3**	**4**
No. of pages	**211**	**90**	**19**	**5**	**0**

(Summer 22-23)

Solution: Mean $= \frac{\sum fx}{\sum f} = \frac{211(0)+90(1)+19(2)+5(3)+0(4)}{211+90+19+5+0} = \frac{143}{325} = 0.44$

For a Poisson distribution, $\lambda = 0.44$

P(x) = $\frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-0.44}0.44^x}{x!}, \qquad x = 0,1,2,3,4$

N=$\sum f = 325$

Expected frequency $f(x) = N\,P(X = x)$

$$f(x) = \frac{325e^{-0.44}0.44^x}{x!}$$

$$f(0) = \frac{325e^{-0.44}0.44^0}{0!} = 209.311 \approx 209$$

$$f(1) = \frac{325e^{-0.44}0.44^1}{1!} = 92.09 \approx 92$$

$$f(2) = \frac{325e^{-0.44}0.44^2}{2!} = 20.26 \approx 20$$

$$f(3) = \frac{325e^{-0.44}0.44^3}{3!} = 2.971 \approx 3$$

$$f(4) = \frac{325e^{-0.44}0.44^4}{4!} = 0.326 \approx 1$$

Poisson Distribution,

No. Of mistake per page	0	1	2	3	4
Expected Poisson Distribution	209	92	20	3	1

Exercise:

Example 1: For Poisson variate X; if P (X = 3) = P (X = 4) then, find P (X = 0).

Example2: There are 100 misprints in a book of 100 pages. If a page is selected at random, find the probabilities that (i) there will be no misprint in the page (ii) there will be 1 misprint (iii) there will be at the most 2 misprints.

Example 3: If mean of Poisson Distribution is 1.8. find $P(x > 1) and\ P(x = 5)$ **(Summer 23)**

Example 4: The probability that patient will get reaction of a particular injection is 0.001. 2000 patients are given that injection. Find the probabilities that (i) 3 patients will get reaction (ii)More than 2 patients will get reaction. $(e^{-2} = 0.135)$

Normal Distribution:

The most important probability distribution for describing a continuous random variable is the **normal probability distribution**. The normal distribution has been used in a wide variety of practical applications in which the random variables are heights and weights of people, test scores, scientific measurements, amounts of rainfall, and other similar values.
It is also widely used in statistical inference, which is the major topic of the remainder of this book. In such applications, the normal distribution provides a description of the likely results obtained through sampling.

Normal Curve
The form, or shape, of the normal distribution is bell-shaped normal curve. The probability density function that defines the bell-shaped curve of the normal distribution follows.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\mu = mean, \sigma = standarddeviation, \pi = 3.14159, e = 2.71828$$

The characteristics of the normal distribution.

1. The entire family of normal distributions is differentiated by two parameters: the mean $\mu$and the standard deviation.
2. The highest point on the normal curve is at the mean, which is also the median and mode of the distribution.
3. The mean of the distribution can be any numerical value: negative, zero, or positive.
4. The normal distribution is symmetric, with the shape of the normal curve to the left of the mean a mirror image of the shape of the normal curve to the right of the mean. The tails of the normal curve extend to infinity in both directions and theoretically never touch the horizontal axis. Because it is symmetric, the normal distribution is not skewed; its skewness measure is zero.
5. The standard deviation determines how flat and wide the normal curve is. Larger values of the standard deviation result in wider, flatter curves, showing more variability in the data.
6. Probabilities for the normal random variable are given by areas under the normal curve. The total area under the curve for the normal distribution is 1. Because the distribution is symmetric, the area under the curve to the left of the mean is 0.50 and the area under the curve to the right of the mean is 0.50
7. The percentage of values in some commonly used intervals are:
a. 68.3% of the values of a normal random variable are within plus or minus one standard deviation of its mean.
b. 95.4% of the values of a normal random variable are within plus or minus two standard deviations of its mean.
c. 99.7% of the values of a normal random variable are within plus or minus three standard deviations of its mean.

Standard Normal Probability Distribution
A random variable that has a normal distribution with a mean of zero and a standard deviation of one is said to have a **standard normal probability distribution**. The letter *z* is commonly used to designate this particular normal random variable.

Standard Normal Density Function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

The three types of probabilities we need to compute include (1) the probability that the standard normal random variable z will be less than or equal to a given value; (2) the probability that z will be between two given values; and (3) the probability that z will be greater than or equal to a given value.

Example :1 What is the probability that a standard normal variate Z will be (i) greater than 1.09? (ii) less than -1.65? (iii) lying between -1 and 1.96? (iv)lying between 1.25 And 2.75?

Solution: (i) Z > 1.09

$$P(Z > 1.09) = 0.5 - P(0 \leq Z \leq 1.09)$$

$$= 0.5 - 0.3621 \quad = 0.1379$$

(ii) $Z \leq -1.65$

$$P(Z \leq -1.65) = 1 - P(Z > -1.65)$$

$$= 1 - P[0.5 + P(-1.65 < Z < 0)]$$

$$= 1 - P[0.5 + P(0 < Z < 1.65)] \text{ (By symmetry)}$$

$$= 0.5 - P(0 < Z < 1.65 = 0.5 - 0.45 = 0.0495$$

(iii) $P(-1 < Z < 1.96) = P(-1 < Z < 0) + P(0 < Z < 1.96)$

$$= P(0 < Z < 1) + P(0 < Z < 1.96) = 0.3413 + 0.4750 = 0.8163$$

(iv) $P(1.25 < Z < 2.75) = P(0 < Z < 2.75) - P(0 < Z < 1.25) = 0.4970 - 0.3944 = 0.1026$

Example2: Students at Nirma Institute of Technology spend average of 24.3 hours per week on homework, with a standard deviation of 1.4 hours. (a) What percentage of students spend more than 28 hours per week on homework? (b) What is the probability that a student spends more than 28 hours per week on homework

Solution:(a) The value of z corresponding to 28 hours is

$$Z = \frac{x-\mu}{\sigma} = \frac{28-24.3}{1.4} = \frac{3.7}{1.4} = 2.64$$

From the table A = 0.4959 when z = 2.64

All of the area under the curve to the right of the mean is one half of the total area.

The area to the right of z = 2.64 is 0.5 – 0.4959 = 0.0041.

Therefore 0.41% of the students study more than 28 hours.

(b) The probability that a student studies more than 28 hours is the fraction of the area that is to the right of 28, that is, 0.0041

Example 3: A sample of 100 dry battery cells tested and found that average life of 12 hours and a standard deviation 3 hours. Assuming the data to be normally distributed, What percentage of battery cells are expected to have life (i)more than 15 hours (ii) less than 6 hours (iii) between10 to 14 hours?

Solution: $Here \quad \mu$ = 12 hours, σ = 3 hours

$Z = \frac{x-\mu}{\sigma} = \frac{x-12}{3}$

(i) When x = 15, z = 1

P(X> 15) = P (Z > 1)

= P (0 < z < ∞) – P (0 < z < 1) = 0.5 – 0.3413= 0.1587 = 15.87%

(ii) When x = 6, z = -2

P (X< 6) = P (Z < -2)

= P (z > 2) = P (0 < z < ∞) – P (0 < z < 2)= 0.5 – 0.4772.= 0.0228 = 2.28%

(iii) When x = 10 , z = -2/3. = - 0.67, When x = 14 , z = 2/3 = 0.67

P(10 < X < 14) = P(-0.67 < X < 0.67) = 2P(0 < X < 0.67)= 2(0.2487) = 0.4974 = 49.74%

Exercise:

Example 1: In AEC Company, the amount of light bills follows normal distribution with standard deviation 60. 11.31% of customers pay light bill less than Rs 260. Find average amount of light bill.

Example 2: A manufacture knows from his experience that the resistance of resistor he produces is normal with μ=100 and SD=σ=2 ohms. What percentage of resistor will have resistance between 98 ohms and 102 ohms?

Example 3: The daily profit of a business man is Rs. 120 and the standard deviation of the profit is Rs. 15. Find the number of days out of 365 days on which his profit will be less than Rs. 100.