Cayley - Hamilton Theorem:-

Def:- An Expression of the form $F(x) = A_0 + A_1 x + A_2 x^2 + \dots + A_m x^m$, $A_m \neq 0$ where $A_1, A_2, \dots, A_m$ are matrices each of order $n \times n$ over a field F, is called matrix polynomial of degree m.

The symbol x is called indeterminate and will be assumed that it is commutative with every matrix coefficient.

Def:- Equality of Matrix polynomials:-

Two matrix polynomials are Equal if and only if the coefficients of like powers of x are the same.

Addition and Multiplication of polynomials:-

Let $G(x) = A_0 + A_1 x + A_2 x^2 + \dots + A_m x^m$ and

$H(x) = B_0 + B_1 x + B_2 x^2 + \dots + B_k x^k$.

We define :- if $m > k$, then

$$G(x) + H(x) = (A_0 + B_0) + (A_1 + B_1) x + \dots + (A_k + B_k) x^k + A_{k+1} x^{k+1} + \dots + A_m x^m.$$

Also
$$G(x) \cdot H(x) = A_0 B_0 + (A_0 B_1 + A_1 B_0) x + (A_0 B_2 + A_1 B_1 + A_2 B_0) x^2 + \dots + A_m B_k x^{k+m}$$

* The Cayley - Hamilton Theorem :-

Statement :- Every square matrix satisfies its own characteristic equation.

i.e If the characteristic eqn of $A_{n\times n}$ is

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots + a_{n-1}\lambda + a_n = 0$$

$\because$ Ch. Eqn is $|A-\lambda I| = 0$

$\Rightarrow \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

By C-H theorem

i.e $A^3 - S_1A^2 + S_2A - S_3 = 0$

By Cayley - Hamilton theorem.

$$\boxed{A^n + a_1 . A^{n-1} + a_2 A^{n-2} + \dots + a_{n-1} A + a_n I = 0}$$

Application of Cayley - Hamilton Theorem :-

The important applications of C-H theorem are

1. To find the inverse of a matrix
2. To find higher powers of the matrix.

ie To find A^{-1} :-

By C-H theorem

$$A^n + a_1A^{n-1} + a_2A^{n-2} + \dots + a_{n-1}A + a_nI = 0$$

multiply A^{-1} on Both sides.

$$A^{-1}\left[A^n + a_1A^{n-1} + a_2A^{n-2} + \dots + a_{n-1}A + a_nI\right] = 0$$

$$\Rightarrow A^{n-1} + a_1A^{n-2} + a_2A^{n-3} + \dots + a_{n-1} . I + a_nA^{-1} = 0$$

$$\Rightarrow a_nA^{-1} = -\left[A^{n-1} + a_1A^{n-2} + \dots + a_{n-1}I\right]$$

$$\boxed{A^{-1} = \frac{-1}{a_n}\left[A^{n-1} + a_1 A^{n-2} + \cdots + a_{n-1} I\right].}$$

Similarly, To find A^{n+1}

By C-H Theorem

$$A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_{n-1} A + a_n I = 0$$

multiply A on B.S

$$A\left[A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_{n-1} A + a_n I\right] = 0$$

$$\Rightarrow A^{n+1} + a_1 A^n + a_2 A^{n-1} + \cdots + a_{n-1} A^2 + a_n I = 0$$

$$\Rightarrow \boxed{A^{n+1} = -\left[a_1 A^n + a_2 A^{n-1} + \cdots + a_{n-1} A^2 + a_n A\right]}$$

problems:-

1. Verify C-H theorem for $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ & hence find A^{-1} & A^4

Sol: Given $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$.

The ch. Eqn is $|A - \lambda I| = 0$

i.e $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

S_1 = Trace of $A = 1+1+1 = 3$

S_2 = Σ minors of diagonals of A $= 0 + (-2) + 1 = -1$

$S_3 = |A| = -9$.

$\therefore$ The ch. Eqn is $\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$

To verify C-H Theorem.

To prove $A^3 - 3A^2 - A + 9I = 0$.

<u>L.H.S :-</u>

$A^3 - 3A^2 - A + 9I$.

Here $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

<u>$A^2 = A * A$</u>

$= \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}$

<u>$A^3 = A^2 \cdot A$:-</u>

$= \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4 & -9 & 21 \\ 11 & -2 & 11 \\ 1 & -7 & 7 \end{bmatrix}$.

Take L.H.S $A^3 - 3A^2 - A + 9I$.

$\begin{bmatrix} 4 & -9 & 21 \\ 11 & -2 & 11 \\ 1 & -7 & 7 \end{bmatrix} - 3\begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 4-12-1+9 & -9+9 & 21-18-3 \\ 11-9-2 & -2-6-1+9 & 11-12+1 \\ 1-1 & -7+6+1 & 7-15-1+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$\therefore A^3 - 3A^2 - A + 9I = 0$

$\therefore$ C-H Theorem is verified.

<u>To find A^{-1}</u> :-

By C-H Theorem

$$A^3 - 3A^2 - A + 9I = 0$$

Multiply with A^{-1} on b.s

$$A^{-1}[A^3 - 3A^2 - A + 9I] = 0$$

$$\Rightarrow A^2 - 3A - I + 9A^{-1} = 0$$

$$\Rightarrow 9A^{-1} = -A^2 + 3A + I$$

$$\Rightarrow 9A^{-1} = -\begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow 9A^{-1} = \begin{bmatrix} -4+3+1 & 3 & -6+9 \\ -3+6 & -2+3+1 & -4-3 \\ 3 & 2-3 & -5+3+1 \end{bmatrix}$$

$$9A^{-1} = \begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$$

To find A^4 :-

By C-H Theorem, $A^3 - 3A^2 - A + 9I = 0$

Multiply with A on b.s, we get

$$A^4 - 3A^3 - A^2 + 9A = 0$$

$$\Rightarrow A^4 = 3A^3 + A^2 - 9A.$$

$$= 3\begin{bmatrix} 4 & -9 & 21 \\ 11 & -2 & 11 \\ 1 & -7 & 7 \end{bmatrix} + \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} - 9\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 12+4-9 & -27-3 & 63+6-27 \\ 33+3-18 & -6+2-9 & 33+4+9 \\ 3+0-9 & -21-2+9 & 21+5-9 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -30 & 42 \\ 18 & -13 & 46 \\ -6 & -14 & 17 \end{bmatrix} //.$$

2. Verify C-H theorem & use it to find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$

Sol:- Ch. Eqn is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$

Here $S_1 = 1 - 1 - 1 = -1$

$S_2 = -3 - 10 - 5 = -18$

$S_3 = 1(-3) - 2(-14) + 3(5) = 40.$

$\therefore$ The ch. Eqn is $\lambda^3 + \lambda^2 - 18\lambda - 40 = 0.$

To verify C-H Theorem, To prove

$$A^3 + A^2 - 18A - 40I = 0$$

L.H.S: $A^3 + A^2 - 18A - 40I$.

To find A^2:-

$$A^2 = A \cdot A.$$

$$A^2 = \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}$$

To find A^3:-

$$A^3 = A^2 \cdot A$$

$$= \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 2. \\ 52 & 14 & 8 \end{bmatrix}$$

Take L.H.S is $A^3 + A^2 - 18A - 40I$.

$$\begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 2 \\ 52 & 14 & 8 \end{bmatrix} + \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} - 18 \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} - 40 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 \text{ R.H.S}$$

C-H theorem verified //.

To find A^{-1} :-

By C-H theorem $A^3+A^2-18A-40I=0$

Multiply A^{-1} on b.s, we get

$$A^{-1}\left[A^3+A^2-18A-40I\right]=0$$

$$\Rightarrow A^2+A-18I-40A^{-1}=0$$

$$\Rightarrow 40A^{-1}=A^2+A-18I.$$

$$\Rightarrow 40A^{-1}=\begin{bmatrix}14 & 3 & 8\\ 12 & 9 & -2\\ 2 & 4 & 14\end{bmatrix}+\begin{bmatrix}1 & 2 & 3\\ 2 & -1 & 4\\ 3 & 1 & -1\end{bmatrix}-\begin{bmatrix}18 & 0 & 0\\ 0 & 18 & 0\\ 0 & 0 & 18\end{bmatrix}$$

$$40A^{-1}=\begin{bmatrix}-3 & 5 & 11\\ 14 & -10 & 2\\ 5 & 5 & -5\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{40}\begin{bmatrix}-3 & 5 & 11\\ 14 & -10 & 2\\ 5 & 5 & -5\end{bmatrix}$$ //.

H.W.

1. Verify C-H theorem for $A=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$ find A^{-1} & A^4.

2. verify C-H theorem $A=\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 5\\ 3 & 5 & 6\end{bmatrix}$ & find A^{-1}.

3. Using Cayley - Hamilton theorem, find A^8 if $A=\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}$.