EXERCISE - I

1. The probabilities that three students A, B and C will pass the common entrance test for engineering are $4/9$, $2/9$ and $1/3$ respectively. The probabilities that they will get admission in the same engineering college are $3/10$, $1/2$ and $4/5$ respectively.

Find the probability that they will get admission in the same engineering college.

[Ans. : $23/45$]

2. The chances that A, B and C will be the Education Minister of Government of India are in the ratio $4:1:2$. The probabilities that they will introduce reservations in professional colleges for backward classes are $0{\cdot}3$, $0{\cdot}8$ and $0{\cdot}5$ respectively.

Find the probability that the bill for reservation will be introduced.

[Ans. : $3/7$]

3. In a factory an article is produced on three machines. Their respective productions are 300 units by A, 250 units by B and 450 units by C. It is found that the percentages of defective articles for A, B, C are 1, 1·2 and 2 selected at random from a days production (which are mixed).

Find the probability that the selected article is defective.

[Ans. : 0·015]

We now state an important theorem known as Bayes' Theorem. It enables us to evaluate what may be called **reverse probabilities.** Suppose there are two boxes (I and II) which contain 2 white and 3 black balls; and 3 white and 4 black balls. If a box is chosen at random and a ball is drawn from it, what is the probability that the ball drawn is white ? We know how to calculate this probability. Now, consider the question : If the ball drawn is known to be white, what is the probability that it was drawn from the 1st box ? The question can be answered by Bayes' Theorem. If the 'result' is known, Bayes' Theorem enables us to find the probability of the 'cause'. For this reason it is also sometimes known as the formula for the **"probability of causes"**.

7. Bayes' Theorem

Let the events $A_1, A_2, \ldots, A_n$ represent a partition of the sample space S. Let B be any other event defined on S. If $P(A_i) \neq 0$, $i = 1, 2, \ldots, n$ and $P(B) \neq 0$ then

$$P(A_i/B) = \frac{P(A_i) \times P(B/A_i)}{\Sigma P(A_i) \times P(B/A_i)}$$

If we write $\quad p_1 = P(A_1),\ p_2 = P(A_2),\ p_3 = P(A_3)$ ...... etc.

and $\quad p'_1 = P(B/A_1),\ p'_2 = P(B/A_2),\ p'_3 = P(B/A_3)$ ...... etc.

then Bayes' theorem can be stated as

$$\boxed{P(A_i/B) = \frac{p_i\, p'_i}{p_1 p'_1 + p_2 p'_2 + \ldots + p_n p'_n}}$$

Proof : We have by conditional probability

$$P(A_i/B) = \frac{P(A_i \cap B)}{P(B)} \quad \text{......... (1)}$$

But $\quad P(B/A_i) = \dfrac{P(B \cap A_i)}{P(A_i)} \quad \therefore \quad P(A_i \cap B) = P(A_i) \cdot P(B/A_i) \quad \text{......... (2)}$

From (i) and (ii), we get

$$\therefore \quad P(A_i / B) = \frac{P(A_i) \cdot P(B/A_i)}{P(B)} = \frac{p_i \, p'_i}{P(B)} \qquad \text{.....................} \text{ (iii)}$$

But $\quad B = (B \cap A_1) \cup (B \cap A_2) \ldots\ldots (B \cap A_n)$

$$\therefore \quad P(B) = P(B \cap A_1) + P(B \cap A_2) + \ldots\ldots + P(B \cap A_n)$$

But $\quad P(B \cap A_i) = P(A_i) \cdot P(B/A_i)$ etc. $= p_1 \, p'_1$ etc.

Hence, from (iii) $\quad P(A_j / B) = \dfrac{p_i \, p'_i}{p_1 \, p'_1 + p_2 \, p'_2 + \ldots\ldots + p_n \, p'_n}$

Thomas Bayes (1701 - 1761)

Thomas Bayes was an English mathematician known for the theorem that bears his name. This theorem was published after his death by Richar Price. He published two works in his lifetime, one theological and one mathematical. He became a Fellow of the Royal Society in 1742. It is said that he learned mathematics and probability from a book by De Moivre. In his later years he took deep interest in probability.

Example 1 : There are in a bag three true coins and one false coin with head on both sides. A coin is chosen at random and tossed four times. If head occurs all the four times, what is the probability that the false coin was chosen and used ? **(M.U. 2003)**

Sol. : $\quad P(\text{selecting true coin}) = p_1 = \dfrac{3}{4}. \qquad P(\text{selecting false coin}) = p_2 = \dfrac{1}{4}.$

$$p'_1 = P(\text{getting all four heads with true coin}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}.$$

$$p'_2 = P(\text{getting all four heads with false coin}) = 1 \cdot 1 \cdot 1 \cdot 1 = 1.$$

$$\therefore \quad \text{Required Probability} = \frac{p_2 \, p'_2}{p_1 \, p'_1 + p_2 \, p'_2} = \frac{(1/4) \cdot 1}{(3/4) \cdot (1/16) + (1/4) \cdot 1} = \frac{16}{19}.$$

Example 2 : A coin is tossed. If it turns up heads two balls are drawn from urn A otherwise two balls are drawn from urn B. Urn A contains 3 black and 5 white balls. Urn B contains 7 black and one white ball. What is the probability that urn A was used, given that both balls drawn are black ? **(M.U. 2001, 04)**

Sol. : We have $p_1 = P(H) = \dfrac{1}{2}, \quad p_2 = P(T) = \dfrac{1}{2}$

$$p'_1 = P(\text{two black balls from } A) = \frac{{}^3C_2}{{}^8C_2}$$

$$p'_2 = P(\text{two black balls from } B) = \frac{{}^7C_2}{{}^8C_2}$$

$$\therefore \quad \text{Required Probability} = \frac{p_1 p'_1}{p_1 p'_1 + p_2 p'_2} = \frac{\frac{1}{2}\cdot\frac{{}^3C_2}{{}^8C_2}}{\frac{1}{2}\cdot\frac{{}^3C_2}{{}^8C_2}+\frac{1}{2}\cdot\frac{{}^7C_2}{{}^8C_2}}$$

$$= \frac{{}^3C_2}{{}^3C_2 + {}^7C_2} = \frac{3\cdot 2/2\cdot 1}{(3\cdot 2/2\cdot 1)+(7\cdot 6/2\cdot 1)}$$

$$= \frac{6}{6+42} = \frac{6}{48} = \frac{1}{8}.$$

Example 3 : In a certain test there are multiple choice questions. There are four possible answers to each question and one of them is correct. An intelligent student can solve 90% questions correctly by reasoning and for the remaining 10% questions he gives answers by guessing. A weak student can solve 20% questions correctly by reasoning and for the remaining 80% questions he gives answers by guessing. An intelligent students gets the correct answer, what is the probability that he was guessing ? **(M.U. 2004)**

Sol. : Consider the intelligent student.

Let p_1 = answering by reasoning $= \frac{90}{100} = \frac{9}{10}$.

p_2 = answering by guessing $= \frac{10}{100} = \frac{1}{10}$.

p_1' = answer is correct (by reasoning) = 1

p_2' = answering is correct (by guessing) $= \frac{1}{4}$.

$$\text{Required Probability} = \frac{p_2 p_2'}{p_1 p_1' + p_2 p_2'} = \frac{(1/10)\cdot(1/4)}{(9/10)\cdot 1 + (1/10)\cdot(1/4)}$$

$$= \frac{1/40}{37/40} = \frac{1}{37}.$$

Example 4 : A certain test for a particular cancer is known to be 95% accurate. A person submits to the test and the result is positive. Suppose that a person comes from a population of 100,000 where 2000 people suffer from that disease. What can we conclude about the probability that the person under test has that particular cancer ? **(M.U. 2006)**

Sol. : We have

p_1 = probability a person has the cancer $= \frac{2000}{100{,}000} = \frac{2}{100} = 0.02$

p_2 = probability that a person does not have the cancer = 1 − 0·02 = 0·98

p_1' = test is positive when a person has cancer $= \frac{95}{100} = 0.95$

p_2' = test is positive when person does not have a cancer $= \frac{5}{100} = 0.05$

$$\therefore \quad \text{Required probability} = \frac{p_1 p_1'}{p_1 p_1' + p_2 p_2'}$$

$$\therefore \quad \text{Required probability} = \frac{(2/100)\cdot(95/100)}{(2/100)\cdot(95/100)+(98/100)\cdot(5/100)}$$

$$= \frac{190}{680} = 0.279$$

Example 5 : A bag contains 7 red and 3 black balls and another bag contains 4 red and 5 black balls. One ball is transferred from the first bag to the second bag and then a ball is drawn from the second bag. If this ball happens to be red, find the probability that a black ball was transferred. **(M.U. 2002)**

Sol. : We have

p_1 = Probability of transferring black ball $= \frac{3}{10}$

p'_1 = Probability of now drawing a red ball $= \frac{4}{10}$

p_2 = Probability of transferring red ball $= \frac{7}{10}$

p'_2 = Probability of now drawing red ball $= \frac{5}{10}$

$$\therefore \quad \text{Required Probability} = \frac{p_1 p'_1}{p_1 p'_1 + p_2 p'_2} = \frac{(3/10)\cdot(4/10)}{(3/10)(4/10)+(7/10)(5/10)}$$

$$= \frac{12}{47}.$$

Example 6 : A man speaks truth 3 times out of 5. When a die is thrown, he states that it gave an ace. What is the probability that this event has actually happened ?

Sol. : We have

p_1 = Probability he speaks truth $= \frac{3}{5}$; p'_1 = Probability of an ace $= \frac{1}{6}$

p_2 = Probability he speaks a lie $= \frac{2}{5}$; p'_2 = Probability of not ace $= \frac{5}{6}$

$$P(\text{he speaks truth when ace has occured}) = \frac{p_1 p'_1}{p_1 p'_1 + p_2 p'_2}$$

$$= \frac{(3/5)(1/6)}{(3/5)(1/6)+(2/5)(5/6)} = \frac{3}{3+10} = \frac{3}{13}.$$

Example 7 : A lot of IC chips is known to contain 3% defective chips. Each chip is tested before delivery but the tester is not completely reliable. It is known that :

P (Tester says the chip is good / The chip is actually good) = 0·95 and P (Tester says the chip is defective / The chip is actually defective) = 0·96.

If a tested chip is declared defective by the tester. What is the probability that it is actually defective ?

Sol. : Let p_1 = Chip is defective = 0·03

p'_1 = Tester says the chip is defective / The chip is defective = 0·96

p_2 = Chip is good = 0·97

p'_2 = Tester says the chip is defective / The chip is good = 0·05.

By Bayes' Theorem,

P (Chip is defective / Tester says it is defective)

$$= \frac{p_1 p'_1}{p_1 p'_1 + p_2 p'_2} = \frac{0{\cdot}03 \times 0{\cdot}96}{0{\cdot}03 \times 0{\cdot}96 + 0{\cdot}97 \times 0{\cdot}05} = 0{\cdot}37.$$

Example 8 : A binary communication transmitter sends data as one of two types of signals denoted by 0 or 1. Due to noise, sometimes a transmitted 1 is received as 0 and vice versa.

If the probability that a transmitted 0 is correctly received as 0 is 0·9 and the probability that a transmitted 1 is correctly received as 1 is 0·8 and if the probability of transmitting 0 is 0·45, find the probability that (i) a 1 is received, (ii) a 0 is received, (iii) a 1 was transmitted given that 1 was received, (iv) a 0 was transmitted given that a 0 was received, (v) the error has occurred.

Sol. : We are given that

$P(T_0)$ = a 0 is transmitted = 0·45

$P(T_1)$ = a 1 is transmitted = $1 - P(T_0) = 1 - p_1 = 0{\cdot}55$

$P(R_0 / T_0)$ = a 0 is received when a 0 was transmitted = 0·9

$P(R_1 / T_0)$ = a 1 received when a 0 was transmitted $= 1 - 0{\cdot}9 = 0{\cdot}1$

$P(R_1 / T_1)$ = a 1 is received when a 1 was transmitted = 0·8

$P(R_0 / T_1)$ = a 0 is received when a 1 was transmitted $= 1 - 0{\cdot}8 = 0{\cdot}2$.

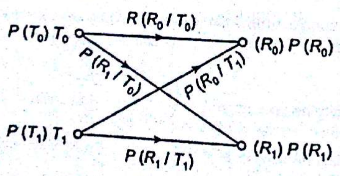


Fig. 5.11

Now, we calculate the required probabilities as follows :

(i) P(1 is received) = P(1 is received when 1 is transmitted) + P(1 is received when 0 is transmitted)

$$\therefore\ P(R_1) = P(R_1/T_1)\cdot P(T_1) + P(R_1/T_0)\cdot P(T_0)$$
$$= 0{\cdot}8 \times 0{\cdot}55 + 0{\cdot}1 \times 0{\cdot}45$$
$$= 0{\cdot}485$$

(ii) P(0 is received) = P(0 is received when 0 is transmitted) + P(0 is received when 1 is transmitted)

$$\therefore\ P(R_0) = P(R_0/T_0)\cdot P(T_0) + P(R_0/T_1)\cdot P(T_1)$$
$$= 0{\cdot}9 \times 0{\cdot}45 + 0{\cdot}2 \times 0{\cdot}55$$
$$= 0{\cdot}515$$

Now, by Bayes' Theorem

(iii) P(1 was transmitted given that 1 was received) *i.e.*

$$P(T_1/R_1) = \frac{P(R_1/T_1)\,P(T_1)}{P(R_1)} = \frac{0{\cdot}8 \times 0{\cdot}55}{0{\cdot}485} = 0{\cdot}907$$

(iv) P(0 was transmitted given that 0 was received) *i.e.*

$$P(T_0/R_0) = \frac{P(R_0/T_0)\cdot P(T_0)}{P(R_0)} = \frac{0{\cdot}9 \times 0{\cdot}45}{0{\cdot}515} = 0{\cdot}786$$

(v) P(Error) = P(0 was received when 1 is transmitted given that 1 was transmitted) + P(1 was received when 0 was transmitted given that 0 was transmitted)

$$= P(R_0/T_1)\cdot P(T_1) + P(R_1/T_0)\cdot P(T_0)$$
$$= 0{\cdot}2 \times 0{\cdot}55 + 0{\cdot}1 \times 0{\cdot}45 = 0{\cdot}155.$$

Example 9 : A box contains three biased coins A, B and C. The probability that a head will result when A is tossed is 1/3, when B is tossed, it is 2/3 and when C is tossed, it is 3/4.

(a) If one of the coins is chosen at random and is tossed 3 times, head resulted twice and tail once. What is the probability that the coin chosen was A ?

(b) What is the probability of getting head when a coin selected at random is tossed once ?

(c) What is the probability that we would get two heads in the first three tosses and a head again in the fourth toss with the same coin ?

Sol. : We have p_1 = Probability of choosing $A = \frac{1}{3}$.

p_2 = Probability of choosing $B = \frac{1}{3}$.

p_3 = Probability of choosing $C = \frac{1}{3}$.

p_1' = Probability of getting 2 heads in three tosses with the coin A

$$= {}^3C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right) = 3\cdot\frac{1}{9}\cdot\frac{2}{3} = \frac{2}{9}.$$

p_2' = Probability of getting 2 heads in three tosses with the coin B

$$= {}^3C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right) = 3\cdot\frac{4}{9}\cdot\frac{1}{3} = \frac{4}{9}.$$

p_3' = Probability of getting 2 heads in three tosses with the coin C

$$= {}^3C_2\left(\frac{3}{4}\right)^2\cdot\left(\frac{1}{4}\right) = 3\cdot\frac{9}{16}\cdot\frac{1}{4} = \frac{27}{64}.$$

$$\therefore \text{ Required Probability } = \frac{p_1 p'_1}{p_1 p'_1 + p_2 p'_2 + p_3 p_3'}$$
$$= \frac{(1/3)(2/9)}{(1/3)\cdot(2/9) + (1/3)\cdot(4/9) + (1/3)\cdot(27/64)}$$
$$= \frac{(2/9)}{(2/9) + (4/9) + (27/64)} = \frac{128}{627}.$$

(b) We do not know which coin was tossed.

If the coin was A, the probability that it will give head

$$= (\text{Prob. of choosing } A) \times (\text{Prob. of giving head})$$

$$= \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}.$$

If the coin was B, probability of getting head.

$$= (\text{Prob. of choosing } B) \times (\text{Prob. of giving head})$$

$$= \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}.$$

If the coin was C, probability of getting head

$$= (\text{Prob. of choosing } C) \times (\text{Prob. of giving head})$$

$$= \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$$

$\therefore$ The required probability $= \frac{1}{9} + \frac{2}{9} + \frac{1}{4} = \frac{7}{12}.$

(c) Now, getting two heads in the first three tosses and a head in the fourth toss with A

$$= P(\text{Choosing } A) \times P(\text{Getting two heads in three tosses with } A)$$
$$\times P(\text{Getting a head in the fourth toss with } A)$$

$$= p_1 \cdot p_1' \cdot \frac{1}{3} = \frac{1}{3} \cdot \frac{2}{9} \cdot \frac{1}{3} = \frac{2}{81}.$$

Similarly, getting two heads in the first three tosses and a head in the fourth toss with B

$$= p_2 \cdot p_2' \cdot \frac{2}{3} = \frac{1}{3} \cdot \frac{4}{9} \cdot \frac{2}{3} = \frac{8}{81}$$

And getting two heads in the first three tosses and a head in the fourth toss with C

$$= p_3\, p_3' \cdot \frac{3}{4} = \frac{1}{3} \cdot \frac{27}{64} \cdot \frac{3}{4} = \frac{27}{256}$$

$\therefore$ Required Probability $= \frac{2}{81} + \frac{8}{81} + \frac{27}{256} = 0.23.$

Example 10 : A bag contains five balls, the colours of which are not known. Two balls were drawn from the bag and they were found to be white. What is the probability that all balls are white? **(M.U. 2005)**

Sol. : Since two balls drawn are white, the bag may contain 2 white or 3 white or 4 white or 5 white balls.

Let these events be denoted by A_1, A_2, A_3, A_4 respectively. We can assume that the probabilities of these events are equal.

Let $p_1 = P(A_1),\ p_2 = P(A_2),\ p_3 = P(A_3),\ p_4 = P(A_4).$

$\therefore\ p_1 = p_2 = p_3 = p_4 = \frac{1}{4}$

Now, two balls out of 5 can be drawn in ${}^5C_2$ ways.

$\therefore\ p_1' = P(\text{drawing two balls when two balls are white}) = \frac{{}^2C_2}{{}^5C_2} = \frac{2 \cdot 1}{5 \cdot 4} = \frac{2}{20}.$

$p_2' = P(\text{drawing two white balls when 3 balls are white}) = \frac{{}^3C_2}{{}^5C_2} = \frac{3 \cdot 2}{5 \cdot 4} = \frac{6}{20}.$

$p_3' = P(\text{drawing 2 white balls when 4 balls are white}) = \frac{{}^4C_2}{{}^5C_2} = \frac{4 \times 3}{5 \times 4} = \frac{12}{20}.$

$p_4' = P(\text{drawing 2 white balls when 5 balls are white}) = \frac{{}^5C_2}{{}^5C_2} = \frac{5 \cdot 4}{5 \cdot 4} = \frac{20}{20}.$

$\therefore$ By Baye's theorem Required Probability

$$= \frac{p_4\, p_4'}{p_1\, p_1' + p_2\, p_2' + p_3 p_3' + p_4\, p_4'}$$

$$= \frac{(1/4) \cdot (20/20)}{(1/4) \cdot (2/20) + (1/4) \cdot (6/20) + (1/4) \cdot (12/20) + (1/4) \cdot (20/20)}$$

$$= \frac{20}{2 + 6 + 12 + 20} = \frac{20}{40} = \frac{1}{2}.$$

Example 11 : There are three boxes containing respectively 1 white 2 red, 3 black balls; 2 white, 3 red and 1 black ball; 3 white, 1 red and 2 black balls. A box is chosen at random and two balls are drawn from it. The two balls are found to be one red and one white. Find the probability that those have come from box 1, box 2 and box 3.

Sol. : Since there are three boxes, say, A_1, A_2, A_3, then

$$P(A_1) = P(A_2) = P(A_3) = \frac{1}{3} \qquad \therefore\ p_1 = p_2 = p_3 = \frac{1}{3}$$

Let B be the event that one ball is white and the other is red. Then,

$$p_1' = P(B/A_1) = \frac{{}^1C_1 \cdot {}^2C_1}{{}^6C_2} = \frac{1 \cdot 2 \cdot 2}{6 \cdot 5} = \frac{2}{15}$$

$$p_2' = P(B/A_2) = \frac{{}^2C_1 \cdot {}^3C_1}{{}^6C_2} = \frac{2 \cdot 3 \cdot 2}{6 \cdot 5} = \frac{6}{15}$$

$$p_3' = P(B/A_3) = \frac{{}^3C_1 \cdot {}^1C_1}{{}^6C_2} = \frac{3 \cdot 1 \cdot 2}{6 \cdot 5} = \frac{6}{15}$$

By Baye's Theorem,

$$P(A_1/B) = \frac{p_1\, p_1'}{p_1\, p_1' + p_2\, p_2' + p_3\, p_3'}$$

$$\therefore\ P(A_1/B) = \frac{(1/3)(2/15)}{(1/3)(2/15) + (1/3)(6/15) + (1/3)(3/15)} = \frac{2/45}{11/45} = \frac{2}{11}$$

$$P(A_2/B) = \frac{p_2\, p_2'}{p_1\, p_1' + p_2\, p_2' + p_3\, p_3'} = \frac{(1/3)(6/15)}{11/45} = \frac{6}{11}$$

$$P(A_3/B) = \frac{p_3\, p_3'}{p_1\, p_1' + p_2\, p_2' + p_3\, p_3'} = \frac{(1/3)(3/15)}{11/45} = \frac{3}{11}.$$

Example 12 : Three factories A, B, C produce 30 %, 50 % and 20 % of the total production of an item. Out of their production 80 %, 50 % and 10 % are defective. An item is chosen at random and found to be defective. Find the probability that it was produced by the factory A.

Sol. :

$$p_1 = P\,(\text{item is produced by } A) = 0{\cdot}3$$

$$p_2 = P\,(\text{item is produced by } B) = 0{\cdot}5$$

$$p_3 = P\,(\text{item is produced by } C) = 0{\cdot}2$$

Let D be the event that the item is defective then

$$p_1' = P\,(D\,/\,A) = 0{\cdot}8, \quad p_2' = P\,(D\,/\,B) = 0{\cdot}5, \quad p_3' = P\,(D\,/\,C) = 0{\cdot}1$$

Now, the required event is $A\,/\,D$.

$$\therefore \quad P\,(A\,/\,D) = \frac{p_1\,p_1'}{p_1\,p_1' + p_2\,p_2' + p_3\,p_3'} = \frac{0{\cdot}3 \times 0{\cdot}8}{0{\cdot}3 \times 0{\cdot}8 + 0{\cdot}5 \times 0{\cdot}5 + 0{\cdot}2 \times 0{\cdot}1}$$

$$= \frac{0{\cdot}24}{0{\cdot}24 + 0{\cdot}25 + 0{\cdot}02} = \frac{0{\cdot}24}{0{\cdot}51} = 0{\cdot}47.$$

17. A coin is tossed. If it turns up head two balls are drawn from urn A, otherwise two balls are drawn from urn B. Urn A contains 3 black and 5 white balls. Urn B contains 7 black and one white ball. In both cases selection is made with replacement. What is the probability that A was chosen given that both the balls drawn are black. (M.U. 2002) [Ans. : 9 / 58]

18. An urn A contains ten red and three black balls. Another urn B contains three red and five black balls. Two balls are transferred from urn A to the urn B without noticing their colour. One ball now is drawn from the urn B and it is found to be red. What is the probability that one red and one black ball have been transferred ? [Ans. : 20 / 59]

19. There are three boxes A, B and C. The probability of getting a white ball from the box A is 1/3, from the box B is 2 / 3 and from the box C is 3 / 4.

A box is chosen at random and three balls are drawn from it (without replacement) and it was found that two of them were white.

(a) What is the probability that the box A was chosen ?

(b) If a ball is drawn from a box selected at random, what is the probability that it will be white ?

(c) What is the probability that we would get two white balls in the first three draws and a white ball again in the fourth draw ?

(Hint : See Solved Ex. 9) [Ans. : (a) 128 / 627, (b) 7 / 12, (c) 0·23]

20. For a certain binary communication channel, the probability that a transmitted '0' is received as a '0' is 0·95 while the probability that a transmitted '1' is received as '1' is 0.90. If the probability of transmitting a '0' is 0·4, find the probability that (i) a '1' is received, (ii) a '1' was transmitted given that '1' was received. [Ans. : (i) 0·56, (ii) 27 / 28]

8. Random Variable

In the previous chapter we learnt how to find the probability of an outcome and the laws of probability. We saw that the outcome of an experiment can be anything : it may be colour (black-white-red .....) of a ball, a gender (male-female) of a child, a suit (club-diamond-spade-heart) of a card or a number (1 – 2 – 3 – 4 – 5 – 6) of a die or a logical answer (yes - no) of a question or result of a toss (head - tail) of a coin. In most of the problems the outcome of an experiment is a number *e.g.* the salary of a person, the height of a student, the temperature at a place, the rainfall on a particular day. However, when the outcome is not a number we can express these outcomes in numbers by agreeing to denote,

(a) head by 1 and tail by 0 (b) boy by 1 and girl by 0

(c) yes by 1 and no by 0 (d) club by 1, diamond by 2, spade by 3 and heart by 4

(e) red by 1, white by 2 and black by 3, etc.

In probability problems it is found convenient to think of a variable and consider the values of the variable which describe the outcomes of the experiment. In the toss of a coin the variable takes values 1 and 0; in the selection of a child it takes values 1 and 0, in the answers of the question it takes the values 1 and 0, in drawing a card it takes values 1, 2, 3, 4, in drawing a ball it takes values 1, 2, 3 etc. This variable may take discrete values or may take any value in a range continuously *e.g.* the arrival time of a bus at a stop, say, between 9 a.m. and 9.10 a.m. may be any number between the range. Such a variable is called a **random variable**. Actually random variable is a misnomer. It is a function which assigns a real number to the outcome of an experiment. A random variable is denoted by X and a particular value of X is denoted by x. In the toss of a coin X assigns the value 1 to H and 0 to T, in the selection of a ball X assigns value 1 to red, 2 to white and 3 to black.

In these cases X takes discrete values and is called a discrete random variable. But in the case of arrival time of a bus X takes continuously any value between 9 am. and 9.10 am. and hence X is a continuous random variable. In this way, we consider X as a function from sample space S to the set of real numbers R. Thus we get the following definition.

(a) Definition

Let E be an experiment and S be the sample space associated with it. A function X assigning to every element s of S one and only one real number $x = X(s)$ of R is called a *random variable*.

Since X is a function whose domain is the set of outcomes of an experiment and whose range is a part or the whole of real line $(-\infty < x < \infty)$, it can be shown pictorially as a mapping from the sample space to the real line.

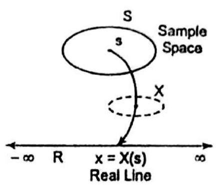


Fig. 5.12

The random variable X can be discrete or continuous depending upon the nature of its domain.

Remarks ....

1. In simple words a variable used to denote the numerical value of the outcome of an experiment is called the random variable, abbreviated as r.v.
2. X is a *function* and still we call it a variable.
3. We are not interested in the *functional nature* of X but in the *values of X*.
4. X must be single valued *i.e.* for every s of S there corresponds *exactly one value* of X. Different elements of S may lead to the same value of X (See Example 2). But two values of X cannot be assigned to the same sample point.
5. We shall denote *random variables* by capital letters X, Y, Z, ... and shall denote the *unknown values* of these random variables by small letters $x, y, z, \ldots, x_1, x_2, \ldots, y_1, y_2, \ldots, z_1, z_2, \ldots$ etc. This is an important distinction and students should note it carefully. With this notation it is meaningless to write $P(x \geq 10)$ say since x being a value of X either is or is not ≥ 10. Instead we should write $P(X \geq 10)$.

Example 1 : Suppose the experiment E is to toss a fair coin.

Then $S = \{H, T\}$. If X is the random variable denoting the number of heads then we have $X(H) = 1$ and $X(T) = 0$. [See Fig. 5.13]

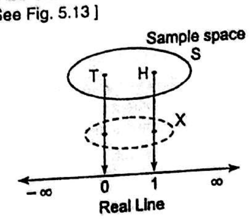


Fig. 5.13

Example 2 : Suppose the experiment E is to toss two fair coins.

Then $S = \{(H, H), (H, T), (T, H), (T, T)\}$. If X is the random variable denoting the number of heads then $X(H, H) = 2$, $X(H, T) = 1$, $X(T, H) = 1$, $X(T, T) = 0$.

[See Fig. 5.14]

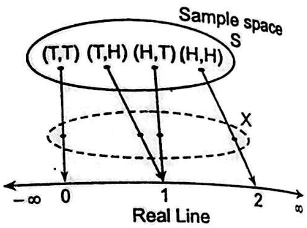


Fig. 5.14

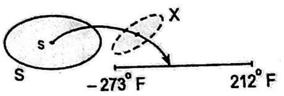


Fig. 5.15

Example 3 : Suppose the experiment is to record the temperature at a place.

If X denoted the temperature then X can take any value from -273° F to 212° F say. [See Fig. 5.15]

Example 4 : Suppose the experiment is to record the time required to complete a software project.

If X denotes the time then X can take any value (theoretically) from 0 to ∞. [See Fig. 5.16]

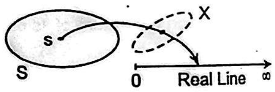


Fig. 5.16

In examples 1 and 2, X takes discrete values and in examples 3 and 4, X takes continuously all values between a specified interval. In the first case X is called a ***discrete random variable*** in the second case it is called ***continuous random variable.***

(b) Definition

Let X be a random variable. If X takes finite or countably infinite values $x_0, x_1, x_2, \ldots$ then X is called a ***discrete random variable.***

(c) Definition

Let X be a random variable. If X takes uncountably infinite values in a given interval then X is called a ***continuous random variable.***

9. Probability Distribution of a Discrete Random Variable

As we know already, with every possible outcome of an experiment there will be associated its probability. We shall be interested in the values of the random variable X along with their probabilities. If x_i is the value of X and $P(x_i)$ is the probability of x_i then set of pairs $\{x_i, P(x_i)\}$ is called the probability distribution.

Definition : Let X be a discrete random variable. Let $x_1, x_2, \ldots, x_n, \ldots$ be the possible values of X. With each possible outcome x_i we associate a number $p(x_i) = P(X = x_i)$ called the probability of x_i. The numbers $p(x_i)$, $i = 1, 2, \ldots, n \ldots$ must satisfy the following conditions :

1. $p(x_i) \geq 0$ for all i
2. $\sum_{i=1} p(x_i) = 1$

The function p is called the **probability function** or **probability mass function (p.m.f)** or **probability density function (p.d.f.)** of the random variable X and the set of pairs (x_i, p_i) is called the probability distribution of X.

The probability distribution of a discrete random variable X taking values $x_1, x_2, x_3 \ldots x_n \ldots$ with probabilities $p_1, p_2, p_3 \ldots p_n \ldots$ where $p_i \geq 0$ and $\Sigma p_i = 1$ can be given in tabular form as

X	x_1	x_2	x_3	.....	x_n	.....
$P(x_i)$	p_1	p_2	p_3	.....	p_n	.....

Example 1 : State true or false with justification :

(a) A random variable X takes values 0, 1, 2 and 3 then $p(X = x) = \dfrac{x-1}{2}$ can be its probability distribution.

(b) A random variable takes values 0, 1, 2 and $p(x) = \dfrac{x+1}{3}$ is its probability distribution.

Sol. : As seen above for a probability distribution, two conditions must be satisfied.

(i) Each probability must be equal to or greater than zero but less than one.

(ii) The sum of all probabilities must be equal to unity.

Putting $x = 0, 1, 2, 3$ in (a), we get

$$P(0) = -\frac{1}{2},\ P(1) = 0,\ P(2) = \frac{1}{2},\ P(3) = 1$$

Since, the probability cannot be negative, $P(X = x) = \dfrac{x-1}{2}$ cannot be a probability distribution.

Putting $x = 0, 1, 2, 3$, in (b), we get

$$P(0) = \frac{1}{3},\ P(1) = \frac{2}{3},\ P(2) = 1.$$

Although all probabilities are positive, the sum of all the probabilities is 2, (greater than 1). Hence, $P(X = x) = \dfrac{x+1}{3}$ also cannot be a probability distribution.

Example 2 : From the past experience it was found that the daily demand at an autogarage was as under.

Daily Demand	:	5	6	7
Probability	:	0·25	0·65	0·10

Check if this is a probability distribution. Find also the probability that over a period of two days the number of demands would be 11 or 12.

Sol. : Since the sum of all probabilities = 0·25 + 0·65 + 0·10 = 1, it is a probability distribution.

P(11 requests over two days)

$= P$(5 requests on the first day and 6 on the second)

$+ P$(6 request on the first day and 5 on the second)

$= (0{\cdot}25 \times 0{\cdot}65) + (0{\cdot}65 \times 0{\cdot}25)$

$= 0{\cdot}1625 + 0{\cdot}1625 = 0{\cdot}325$

P(12 requests over two days)

$= P$(5 requests on the first day and 7 on the second)

$+ P$(6 requests on the first day and 6 on the second)

$+ P$(7 requests on the first day and 5 on the second)

$= (0{\cdot}25 \times 0{\cdot}10) + (0{\cdot}65 \times 0{\cdot}65) + (0{\cdot}10 \times 0{\cdot}25)$

$= 0{\cdot}025 + 0{\cdot}4224 + 0{\cdot}025 = 0{\cdot}7975.$

Example 3 : Find the probability distribution of number of heads (X) obtained when a fair coin is tossed 4 times.

Sol. : When a coin is tossed 4 times, there are $2^4 = 16$ outcomes which are listed below.

HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT,

THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT.

(Write first *H, T, H, T* alternately for 16 times. Then 2 *H*'s and 2 *T*'s alternately, then 4 *H*'s and 4 *T*'s alternatively and lastly 8 *H*'s and 8 *T*'s alternately). Hence, the probability distribution of X is

X	0	1	2	3	4
$P(X=x)$	1/16	4/16	6/16	4/16	1/16

Example 4 : Write down the probability distribution of the sum of numbers appearing on the toss of two unbiased dice. **(M.U. 2004, 06)**

Sol. : When two dice are thrown, we get the sum of numbers as shown below by slant dotted lines.

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

It is easy to see that the sum 2 appears once, 3 twice, 4 thrice etc. The probability of each single event is 1/36.

The probability distribution obtained is shown below.

X	2	3	4	5	6	7	8	9	10	11	12
$P(X=x)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Example 5 : For the above distribution, (i) find the probability that X is an odd number, (ii) find the probability that X lies between 3 and 9. **(M.U. 1997)**

Sol. : $P(X \text{ is odd}) = P(X = 3, 5, 7, 9 \text{ or } 11)$

$= P(X=3) + P(X=5) + P(X=7) + P(X=9) + P(X=11)$

$\therefore\ P(X \text{ is odd}) = \dfrac{2}{36} + \dfrac{4}{36} + \dfrac{6}{36} + \dfrac{4}{36} + \dfrac{2}{36} = \dfrac{18}{36} = \dfrac{1}{2}$

$P(3 \le X \le 9) = P(X = 3, 4, 5, 6, 7, 8 \text{ or } 9)$

$= P(X=3) + P(X=4) + \ldots\ldots + P(X=9)$

$= \dfrac{2}{36} + \dfrac{3}{36} + \ldots\ldots + \dfrac{4}{36} = \dfrac{29}{36}.$

Example 6 : Two unbiased dice are thrown. Let x_1 and x_2 denote the scores obtained and y denote the maximum of them *i.e.* $y = \max.\ (x_1, x_2)$. Find the probability distribution of Y.

Sol. : Refer to the chart of Ex. 4 above. It is easy to see that y takes values, 2, 3, 4, 5, 6 with probabilities 1/36, 3/36, 5/36, 7/36, 9/36, 11/36 respectively as shown by thick arrows.

Y	1	2	3	4	5	6
$P(Y=y)$	1/36	3/36	5/36	7/36	9/36	11/36

Example 7 : For the above distribution, (i) find the probability that X is an odd number, (ii) find the probability that X lies between 3 and 9.

Sol. : $P(X \text{ is odd}) = P(X = 3, 5, 7, 9 \text{ or } 11)$

$= P(X=3) + P(X=5) + P(X=7) + P(X=9) + P(X=11)$

$= \dfrac{2}{36} + \dfrac{4}{36} + \dfrac{6}{36} + \dfrac{4}{36} + \dfrac{2}{36} = \dfrac{18}{36} = \dfrac{1}{2}$

$P(3 \le X \le 9) = P(X = 3, 4, 5, 6, 7, 8 \text{ or } 9)$

$= P(X=3) + P(X=4) + \ldots\ldots + P(X=9)$

$= \dfrac{2}{36} + \dfrac{3}{36} + \ldots\ldots + \dfrac{4}{36} = \dfrac{29}{36}.$

Example 8 : The probability mass function of a random variable X is zero except at the points $X = 0, 1, 2$. At these points it has the values $P(0) = 3c^3$, $P(1) = 4c - 10c^2$, $P(2) = 5c - 1$.

(i) Determine c, (ii) Find $P(X < 1)$, $P(1 < X \le 2)$, $P(0 < X \le 2)$. **(M.U. 2001)**

Sol. : Since $\Sigma\, p_i = 1$, we have, $P(0) + P(1) + P(2) = 1$.

$\therefore\ 3c^3 - 10c^2 + 4c + 5c - 1 = 1 \qquad \therefore\ 3c^3 - 10c^2 + 9c - 2 = 0$

$(3c - 1)(c - 2)(c - 1) \qquad \therefore\ c = 1/3$

(The other values are not admissible. Why?)

$\therefore$ The probability distribution is

X	0	1	2
$P(X=x)$	1/9	2/9	2/3

$\therefore\ P(X < 1) = P(X = 0) = \dfrac{1}{9}; \quad P(1 < X \le 2) = P(X = 2) = \dfrac{2}{3};$

$P(0 < X \le 2) = P(X = 1) + P(X = 2) = \dfrac{2}{9} + \dfrac{2}{3} = \dfrac{8}{9}.$

Example 9 : A random variable X has the following probability distribution

X	0	1	2	3	4	5	6	7
$P(X=x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

(i) Find k, (ii) $P\left(\frac{1.5 < X < 4.5}{X > 2}\right)$, (iii) The smallest value of λ for which $P(X \le \lambda) > \frac{1}{2}$.

Sol. : (i) Since $\Sigma\, p(x_i) = 1$, we get

$10k^2 + 9k = 1$ $\therefore\ 10k^2 + 9k - 1 = 0$

$\therefore\ 10k^2 + 10k - k - 1 = 0$ $\therefore\ 10k(k+1) - 1\,(k+1) = 0$

$\therefore\ (10k-1)(k+1) = 0$ $\therefore\ k = 1/10$. k cannot be -1.

$\therefore$ The probability distribution of X is

X	:	0	1	2	3	4	5	6	7
$P(X=x)$	:	0	1/10	2/10	2/10	3/10	1/100	2/100	17/100

(ii) Now, $P(A/B) = \frac{P(A \cap B)}{P(B)}$

$$\therefore\ P\left(\frac{1.5 < X < 4.5}{X > 2}\right) = \frac{P[(1.5 < X < 4.5) \cap (X > 2)]}{P(X>2)} = \frac{P(2 < X < 4.5)}{P(X > 2)}$$

$$= \frac{P(X = 3, 4)}{P(X = 3, 4, 5, 6, 7)} = \frac{5/10}{70/100} = \frac{5}{7}.$$

Now from the table we find that

$$P(X \le 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= 0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} = \frac{5}{10} = \frac{1}{2}.$$

Hence, $P(X \le 4) = \frac{8}{10} > \frac{1}{2}$. Hence, $\lambda = 4$.

Example 10 : An urn contains 4 white and 3 red balls. Find the probability distribution of the number of red balls in three draws made successively with replacement from the urn. **(M.U. 2007)**

Sol. : We get the following probabilities.

No. of red balls	Probability
0	$\frac{4}{7}\cdot\frac{4}{7}\cdot\frac{4}{7}$
1	$\frac{3}{7}\cdot\frac{4}{7}\cdot\frac{4}{7}$, $\frac{4}{7}\cdot\frac{3}{7}\cdot\frac{4}{7}$, $\frac{4}{7}\cdot\frac{4}{7}\cdot\frac{3}{7}$
2	$\frac{3}{7}\cdot\frac{3}{7}\cdot\frac{4}{7}$, $\frac{3}{7}\cdot\frac{4}{7}\cdot\frac{3}{7}$, $\frac{4}{7}\cdot\frac{3}{7}\cdot\frac{3}{7}$
3	$\frac{3}{7}\cdot\frac{3}{7}\cdot\frac{3}{7}$

$\therefore$ Probability distribution is

x	0	1	2	3	Total
$p(x)$	$\frac{64}{343}$	$\frac{144}{343}$	$\frac{108}{343}$	$\frac{27}{343}$	1

Example 11 : A random variable X has the following probability function :

X	:	1	2	3	4	5	6	7
$P(X=x)$	:	k	$2k$	$3k$	k^2	k^2+k	$2k^2$	$4k^2$

Find (i) k, (ii) $P(X < 5)$, (iii) $P(X > 5)$, (iv) $P\left(\frac{X<5}{2 < X \le 6}\right)$, (v) $P\left(\frac{X=4}{3 \le X \le 5}\right)$

Sol. : Since $\Sigma\, p(x_i) = 1$,

$$k + 2k + 3k + k^2 + k^2 + k + 2k^2 + 4k^2 = 1$$

$\therefore\ 8k^2 + 7k - 1 = 0$ $\therefore\ (8k-1)(k+1) = 0$

$\therefore\ k = 1/8$ or $k = -1$ which is impossible (why ?)

Thus, we have the following probability distribution.

X	:	1	2	3	4	5	6	7
$P(X=x)$	:	1/8	2/8	3/8	1/64	9/64	2/64	4/64

(i) $P(X < 5) = P(X = 1, 2, 3, 4)$

$$= P(X=1) + P(X=2) + P(X=3) + P(X=4)$$

$$= \frac{1}{8} + \frac{2}{8} + \frac{3}{8} + \frac{1}{64} = \frac{49}{64}.$$

(ii) $P(X > 5) = P(X = 6, 7) = P(X=6) + P(X=7)$

$$= \frac{2}{64} + \frac{4}{64} = \frac{6}{64} = \frac{3}{32}.$$

(iii) $$P\left(\frac{X<5}{2<X\le 6}\right) = \frac{P(X<5 \cap 2<X\le 6)}{P(2<X\le 6)} = \frac{P(2<X<5)}{P(2<X\le 6)}$$

$$= \frac{P(X=3,4)}{P(X=3,4,5,6)} = \frac{25/64}{36/64} = \frac{25}{36}.$$

(iv) $$P\left(\frac{X=4}{3\le X\le 5}\right) = \frac{P(X=4 \cap 3 \le X \le 5)}{P(3 \le X \le 5)}$$

$$= \frac{P(X=4)}{P(X=3,4,5)} = \frac{1/64}{34/64} = \frac{1}{64}.$$

Example 12 : The probability of a man hitting the target is 1/4. How many times must he fire so that the probability of his hitting the target at least once is greater than 2/3? **(M.U. 2016)**

Sol. : We are given that the probability of hitting the target $p = 1/4$.

Probability of not hitting the target, $q = 1 - \frac{1}{4} = \frac{3}{4}$.

$\therefore$ Probability of not hitting the target in n trials

$$= \left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\ldots\ldots(n \text{ times}) = \left(\frac{3}{4}\right)^n$$

$\therefore$ Probability of hitting the target at least once in n trials

$$= 1 - \left(\frac{3}{4}\right)^n$$

We want this probability to be greater 2/3.

$$\therefore \quad 1 - \left(\frac{3}{4}\right)^n > \frac{2}{3} \qquad \therefore \quad 1 - \left(\frac{2}{3}\right) > \left(\frac{3}{4}\right)^n$$

$$\therefore \quad \frac{1}{3} > \left(\frac{3}{4}\right)^n \qquad \therefore \quad \left(\frac{3}{4}\right)^n < \frac{1}{3}$$

Taking logarithms of both sides

$$n\,(\log 3 - \log 4) < -\log 3$$

$$\therefore \quad n\,(0{\cdot}4771 - 0{\cdot}6021) < -\log 3 \qquad \therefore \quad -n\,(0{\cdot}1250) < -0{\cdot}4771$$

$$\therefore \quad n > \frac{0{\cdot}4771}{0{\cdot}1250} = 3{\cdot}81 \qquad \therefore \quad n > 4$$

$\therefore$ He must fire at least 4 times, so that he will hit the target at least once, with probability of success greater 2/3.

15. The amount of bread X (in hundred kgs) that a certain bakery is able to sell in a day is a random variable with probability density function is given by ,

$$f_X(x) = \begin{cases} Ax, & 0 < x < 5 \\ A(10 - x), & 5 < x < 10 \\ 0, & \text{elsewhere} \end{cases}$$

Find (i) A, (ii) the probabilities of the events : B, the amount of bread sold in a day is more than 500 kgs, C : the amount of bread sold in a day is less than 500 kgs, D : the amount of bread sold in a day is between 250 kgs and 750 kgs. (iii) Are the events B and C exclusive ? (iv) Are the events B and D exclusive. (M.U. 2005) [Ans. : (i) $A = 1/25$, (ii) 1/2, 1/2, 0.75, (iii) Yes, (iv) No]

10. Distribution Function of a Discrete Random Variable X

Probability distribution of X gives us the probability $p(x)$ that X will take a particular value x. Sometimes we need to know the probability that X will take a value less than or equal to a given value x. This probability is obtained by adding the probabilities of all values less than or equal to x.

Suppose, X is a discrete random variable taking values $x_1, x_2, \ldots, x_n$ with probabilities $p(x_i)$, $i = 1, 2, \ldots, n, \ldots$ such that

(i) $p(x_i) \geq 0$ for all i,

(ii) $\Sigma\, p(x_i) = 1$ and consider the following table.

X	x_1	x_2	x_3	.....	x_n	.....
$F(x_i) = F(X = x_i)$	$p(x_1)$	$\sum_1^2 p(x_i)$	$\sum_1^3 p(x_i)$	.....	$\sum_1^n p(x_i)$	.....

The table states that

$$F(x_1) = P(a \leq X < x_1) = p(x_1)$$
$$F(x_2) = P(X \leq x_2) = p(a \leq X \leq x_1) + p(x_1 \leq X \leq x_2)$$
$$= p(x_1) + p(x_2)$$
$$F(x_3) = P(X \leq x_3) = p(a \leq X \leq x_1) + p(x_1 \leq X \leq x_2) + p(x_2 \leq X \leq x_3)$$
$$= p(x_1) + p(x_2) + p(x_3),$$

And so on.

The graph shown in the Fig. 5.17 shows this diagrammatically.

The function F is called the **distribution function**. We have more precise definition as follows.

(a) Definition

Let X be a discrete random variable taking values $x_1, x_2, \ldots$ such that $x_1 < x_2 < x_3 \ldots$ with probabilities $p(x_1), p(x_2), \ldots$ such that $p(x_i) \geq 0$ for all i and $\Sigma\, p(x_i) = 1$.

Consider F defined by $\quad F(x_i) = p(X \leq x_i),\ i = 1, 2, 3, \ldots$

i.e. $\quad F(x_i) = p(x_1) + p(x_2) + \ldots + p(x_i)$

then the function F is called the **cumulative distribution function** or simply **distribution function** and the set of pairs $\{x_i, F(x_i)\}$ is called the **cumulative probability distribution.**

Note ....

The distribution function is to the probability mass function as cumulative frequency distribution is to frequency distribution.

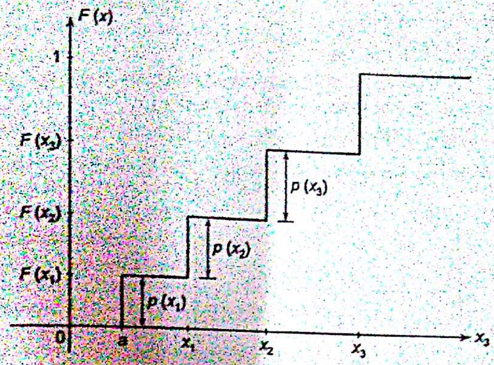


Fig. 5.17 : Graph of Distribution Function.

(b) Important Properties of Distribution function

The distribution function F of a random variable X has the following important properties.

1. $0 \leq F(x) \leq 1$

Proof : Since, $F(x_1) = p(x_1)$, $F(x_2) = p(x_1) + p(x_2)$

.... $F(x_n) = p(x_1) + p(x_2) + \ldots + p(x_n)$

and $0 \leq p(x_i) \leq 1$ for every i and $\Sigma\, p(x_i) = 1$, it is clear that $0 \leq F(x) \leq 1$.

2. $F(x) = 0$ for $x < a$ and $F(x) = 1$ for $x > b$ where $a < x_1 < x_2 < \ldots\ x_n < b$.

Proof : If $x < a$ where $a < x_1$, $p(x) = 0$ and hence, $F(x) = 0$ for $x < a$.

If $x > b$ then $F(x) = p(x_1) + p(x_2) + \ldots + p(x_n) = 1$.

3. $F(x)$ is a step function

Proof : By definition, $\quad F(x_1) = F(X < x_1) = p(x_1)$

$$F(x_2) = F(X < x_2) = p(x_1) + p(x_2)$$

i.e. $F(X)$ has the same value $p(x_1)$ for $x_1 \leq x \leq x_2$,

and the same value $p(x_1) + p(x_2)$ for $x_2 \leq x \leq x_3$ .....

Hence, the graph of $F(x)$ is made up of horizontal line segments taking "jumps" at the possible values x_i of X. The jump is of magnitude $p(x_i) = P(X = x_i)$.

Hence, $F(x)$ is a step functions.

Example 1 : A random variable X has the probability function given below ;

$f(x) = k$ if $x = 0$; $f(x) = 2k$ if $x = 1$;

$f(x) = 3k$ if $x = 2$; $f(x) = 0$ otherwise.

(i) Determine the value of k, (ii) Evaluate $P(X < 2)$, $P(X \leq 2)$, $P(0 < X < 2)$, (iii) Obtain the distribution function.

Sol. : The probability distribution can be tabulated as

X	0	1	2
$p(x)$	k	$2k$	$3k$

(i) Since $\Sigma p_i = 1$, $k + 2k + 3k = 1$ $\quad \therefore \; k = 1/6$.

(ii) $P(X < 2) = P(X = 0) + P(X = 1) = k + 2k = 3/6 = 1/2$

$P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2) = 6k = 1$.

$P(0 < X < 2) = P(X = 1) = 2k = 1/3$.

(iii) Distribution function of X is

X	0	1	2
$F(x)$	1/6	1/2	1

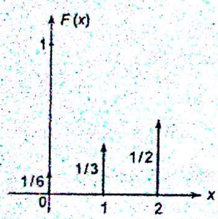


Fig. 5.18 (a)
Probability Density Function.

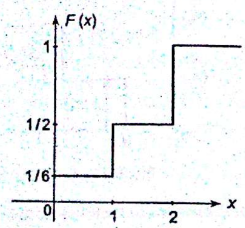


Fig. 5.18 (b)
Distribution Function.

EXERCISE - III

1. A random variable takes values 1, 2, 3, 4 such that $2P(X = 1) = 3P(X = 2) = P(X = 3) = 5P(X = 4)$. Find the probability distribution and the cumulative distribution function. **(M.U. 2004)**

[Ans. :

x	1	2	3	4
$P(x)$	15/61	10/61	30/61	6/61
$F(x)$	15/61	25/61	55/61	1

2. A shipment of 8 computers contains 3 that are defective. If a college makes a random purchase of 2 of these computers, find the probability distribution of the defective computers. Find also distribution function. **(M.U. 2005)**

[Ans. :

x	0	1	2
$P(x)$	10/28	15/28	3/28
$F(x)$	10/28	25/28	1

3. The probability density function of a random variable X is

X	0	1	2	3	4	5	6
$P(X = x)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

Find $P(X < 4)$, $P(3 < X \le 6)$.

[Ans. : $k = 1/49$, 16/49, 33/49]

11. Continuous Random Variable

Definition : A random variable is called a **continuous random variable** if it takes all values between an interval (a, b).

For example, age, height, weight are continuous random variables.

12. Probability Density Function of A Continuous Random Variable

Let $y = f(x)$ be a continuous function of x such that the area $f(x)\,\delta x$ represents the probability that X will lie in the interval $(x, x + \delta x)$. Symbolically,

$$P(x \le X \le x + \delta x) = f_x(x)\,\delta x$$

where, $f_x(x)$ denotes the value of $f(x)$ at x.

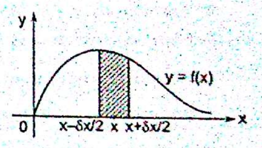


Fig. 5.19

The adjoining figure denotes the curve $y = f(x)$ and the area under the curve in the interval $(x, x + \delta x)$. The function satisfying certain conditions giving the probability that x will lie between certain limits is called **probability density function**, or simply **density function** of a continuous random variable X and is abbreviated as p.d.f. The curve given by $y = f(x)$ is called the **probability density curve** or simply **probability curve**. The expression $f(x)\,dx$ is usually denoted by $df(x)$ and is known as **probability differential**.

Definition : A continuous function $y = f(x)$ such that

(i) $f(x)$ is integrable. (ii) $f(x) \ge 0$

(iii) $\int_a^b f(x)\,dx = 1$ if X lies in $[a, b]$ and

(iv) $\int_\alpha^\beta f(x)\,dx = P(\alpha \le X \le \beta)$ where $a < \alpha < \beta < b$

is called **probability desnity function** of a continuous random variable X.

Thus, for a continuous random variale X,

$$\boxed{P(\alpha \le X \le \beta) = \int_\alpha^\beta f(x)\,dx}$$

Clearly $\int_\alpha^\beta f(x)\,dx$ represents the area under the curve $y = f(x)$, the x-axis and the ordinates at $x = \alpha$ and $x = \beta$. Further since, the total probability is one, if X lies in the interval $[a, b]$ then $\int_a^b f(x)\,dx = 1$. For a continuous random variable X the range may be finite $[a, b]$ or infinite $[-\infty, \infty]$.

Properties of Probability Density Function

The probability density function $f(x)$ has the following properties.

(i) $f(x) \ge 0$, $-\infty < X < \infty$ (*i.e.* the curve $y = f(x)$ lies above the x-axis in the first and second quadrants only)

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$ (*i.e.* the total area under the curve and the x-axis is one.)

(iii) The probability that $\alpha \le X \le \beta$ is given by $P(\alpha \le X \le \beta) = \int_\alpha^\beta f(x)\,dx$.

Notes ...

1. The property (1) and the property (2) can be used to verify whether a given function $f(x)$ can be a probability density function.

2. You know that for A discrete random variable the probability at $X = c$ may not be zero. But in a continuous random variable $P(X = c)$ is always zero because $P(X = c) = \int_c^c f_X(x)\,dx$ and this definite integral is zero. Hence, for a continuous random variable X,

$$P(\alpha \le X \le \beta) = P(\alpha < X < \beta) = P(\alpha < X \le \beta) = P(\alpha \le X < \beta)$$

In other words we may include or may not include the end-points in the interval.

3. Any function $f(x)$ of a real variable x can be a probability density function if it satisfies the first two properties given above viz. $f(x)$ is non-negative for all value of x and $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Sometimes $\int_{-\infty}^{\infty} f(x)\,dx$ is not equal to 1 but $\int_{-\infty}^{\infty} kf(x) = 1$ for some value of k.

In such cases k is called the ***normalising factor*** or ***normalisation constant***.

Example 1 : Find the normalising factor, k if the following function is a probability density function

$$f(x) = \begin{cases} k(1 - x^2) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Also find $P(0.1 < X < 0.2)$ and $P(X > 0.5)$.

Sol. : Since $0 < x < 1$, $f(x) \ge 0$ for all x.

Now, $\int_0^1 f(x)\,dx = k\int_0^1 (1 - x^2)\,dx = k\left[x - \frac{x^3}{3}\right]_0^1 = k\frac{2}{3}$

But this must be equal to 1.

$$\therefore \frac{2k}{3} = 1 \quad \therefore k = \frac{3}{2}$$

(i) $P(0.1 < X < 0.2) = \int_{0.1}^{0.2} \frac{3}{2}(1 - x^2)\,dx = \frac{3}{2}\left[x - \frac{x^3}{3}\right]_{0.1}^{0.2}$

$$= \frac{3}{2}\left[\left\{0.2 - \frac{(0.2)^3}{3}\right\} - \left\{0.1 - \frac{(0.1)^3}{3}\right\}\right]$$

$$= \frac{3}{2}[0.0977] = 0.146$$

(ii) $P(X > 0.5) = \int_{0.5}^{1} \frac{3}{2}(1 - x^2)\,dx = \frac{3}{2}\left[x - \frac{x^3}{3}\right]_{0.5}^{1}$

$$= \frac{3}{2}\left[\left\{1 - \frac{1}{3}\right\} - \left\{0.5 - \frac{(0.5)^3}{3}\right\}\right]$$

$$= \frac{3}{2}[0.667 - 0.458] = 0.313$$

Example 2 : A continuous random variable X has the following probability law

$$f(x) = kx^2, \quad 0 \le x \le 2$$

Determine k and find the probabilities that (i) $0.2 \le X \le 0.5$, (ii) $X \ge 3/4$ given that $X \ge 1/2$. (M.U. 2005)

Sol. : Since the total probability i.e. the total area is unity

$$\int_a^b f(x)\,dx = \int_0^2 kx^2\,dx = 1$$

$$k\left[\frac{x^3}{3}\right]_0^2 = 1 \quad \therefore k \cdot \frac{8}{3} = 1 \quad \therefore k = \frac{3}{8}$$

(i) $P(0.2 \le X \le 0.5) = \frac{3}{8}\int_{0.2}^{0.5} x^2\,dx = \frac{3}{8}\left[\frac{x^3}{3}\right]_{0.2}^{0.5} = \frac{1}{8}\left[0.5^3 - 0.2^3\right] = 0.0123$

(ii) Let $A = (X \ge 1/2)$, $B = (X \ge 3/4)$

$$\therefore P(A) = P(X \ge 1/2) = \frac{3}{8}\int_{0.5}^{2} x^2\,dx = \frac{3}{8}\left[\frac{x^3}{3}\right]_{0.5}^{2}$$

$$= \frac{1}{8}\left[2^3 - 0.5^3\right] = 0.984$$

$$P(B) = P(X \ge 3/4) = \frac{3}{8}\int_{0.75}^{2} x^2\,dx = \frac{3}{8}\left[\frac{x^3}{3}\right]_{0.75}^{2}$$

$$= \frac{1}{8}\left[2^3 - 0.75^3\right] = 0.947$$

$$P(A \cap B) = P(B) = 0.947$$

$$\therefore P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{0.947}{0.984} = 0.96$$

Example 3 : Let X be a continuous random variable with probability distribution

$$p(x) = \begin{cases} \frac{x}{6} + k & \text{if } 0 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

Evaluate k and find $P(1 \le x \le 2)$. (M.U. 1995, 2004)

Sol. : Since the total probability is one

$$\int_{-\infty}^{\infty} p(x)\,dx = \int_0^3 \left(\frac{x}{6} + x\right)dx = \left[\frac{x^2}{12} + kx\right]_0^3 = \frac{3}{4} + 3k = 1$$

$$\therefore 3\left(\frac{1}{4} + k\right) = 1 \quad \therefore \frac{1}{4} + k = \frac{1}{3} \quad \therefore k = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$\therefore p(x) = \begin{cases} \frac{x}{6} + \frac{1}{12} & \text{if } 0 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

$$\therefore \quad P(1 \le x \le 2) = \int_1^2 \left(\frac{x}{6} + \frac{1}{12}\right) dx = \left[\frac{x^2}{12} + \frac{x}{12}\right]_1^2$$

$$= \frac{1}{12}[(4+2) - (1+1)] = \frac{1}{12}(4) = \frac{1}{3}.$$

Example 4 : Let X be a continuous random variable with p.d.f. $f(x) = kx(1-x)$, $0 \le x \le 1$. Find k and determine a number b such that $P(X \le b) = P(X \ge b)$. (M.U. 2003, 11, 15)

Sol. : Since $\int_{-\infty}^{\infty} f(x)\,dx = 1$, we have

$$k\int_0^1 (x - x^2)\,dx = 1 \quad \therefore \; k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1$$

$$\therefore \quad k\left[\frac{1}{2} - \frac{1}{3}\right] = 1 \quad \therefore \; k = 6.$$

Since, the total probability is 1 and $P(x \le b) = P(x \ge b)$, $P(x \le b) = 1/2$.

$$\therefore \quad \int_0^b f(x)\,dx = \frac{1}{2}$$

$$\therefore \quad 6\int_0^b (x - x^2)\,dx = \frac{1}{2} \quad \therefore \; \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^b = \frac{1}{12}$$

$$\therefore \quad \frac{b^2}{2} - \frac{b^3}{3} = \frac{1}{12} \quad \therefore \; 6b^2 - 4b^3 = 1 \quad \therefore \; 4b^3 - 6b^2 - 1 = 0$$

$$\therefore \quad 4b^3 - 2b^2 - 4b^2 - 2b + 2b - 1 = 0$$

$$\therefore \quad (2b-1)(2b^2 - 2b + 1) = 0 \quad \therefore \; b = 1/2.$$

Example 5 : The probability that a person will die in the time interval (t_1, t_2) is given by

$$P(t_1 \le t \le t_2) = \int_{t_1}^{t_2} f(t)\,dt$$

where, $$f(t) = \begin{cases} 3 \times 10^{-9}(100t - t^2)^2, & 0 < t < 100 \\ 0, & \text{elsewhere.} \end{cases}$$

Find (i) the probability that Mr. X will die between the ages 60 and 70. (ii) the probability that he will die between the ages 60 and 70, given that he has survived upto age 60. (M.U. 2005)

Sol. : (i) $$P(60 \le t \le 70) = \int_{60}^{70} 3 \times 10^{-9}(100t - t^2)^2\,dt$$

$$= 3 \times 10^{-9}\int_{60}^{70} (100^2 t^2 - 200t^3 + t^4)\,dt$$

$$= 3 \times 10^{-9}\left[100^2\frac{t^3}{3} - 200\frac{t^4}{4} + \frac{t^5}{5}\right]_{60}^{70}$$

$$= 3 \times 10^{-9}\left[27.89 \times 10^8 - 22.75 \times 10^8\right]$$

$$= 0.1542.$$

(ii) $$P\left(\frac{60 \le T \le 70}{T \ge 60}\right) = \frac{P(60 \le t \le 70 \cap t \ge 60)}{P(t \ge 60)} = \frac{P(60 \le t \le 70)}{P(60 \le t \le 100)}$$

$$= \frac{\int_{60}^{70} f(t)\,dt}{\int_{60}^{100} f(t)\,dt} = \frac{0.1542}{0.3174} = 0.4858.$$

EXERCISE - IV

1. A function is defined as

$$f(x) = \begin{cases} 0 & \text{for } x < 2 \\ \dfrac{2x+3}{18} & \text{for } 2 \le x \le 4 \\ 0 & \text{for } x > 4 \end{cases}$$

Show that $f(x)$ is a probability density function and find the probability that $2 < x < 3$. (M.U. 2005) [Ans. : 5 / 18]

2. A random variable X has the probability density function

$$f(x) = \begin{cases} 2e^{-2x} & \text{for } x \ge 0 \\ 0 & \text{for } x < 0 \end{cases}$$

Find $P(1 \le X \le 3)$, $P(X \ge 0.5)$. [Ans. : (i) 0.133, (ii) 0.368]

3. A continuous random variable X has the following probability density function

$$f(x) = \begin{cases} kx & 0 \le x \le 1 \\ k & 1 \le x \le 2 \end{cases}$$

Find (i) the value of k, (ii) $P(x \le 1.5)$. [Ans. : (i) $k = 2/3$, (ii) 2 / 3]

4. A continuous random variable has the following probability density function.

$$f(x) = \begin{cases} (x/4) + k & 0 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

Evaluate k and $P(1 \le X \le 2)$. [Ans. : $k = 1/4$, 5 / 8]

5. Find the value of k such that the following will be the probability density function. Find also $P(x \le 1.5)$.

$$f(x) = \begin{cases} kx & 0 \le x \le 1 \\ k & 1 \le x \le 2 \\ k(3-x) & 2 \le k \le 3 \end{cases}$$

(M.U. 2003) [Ans. : $k = \frac{1}{2}, \frac{1}{2}$]

13. Continuous Distribution Function

Probability distribution of X or the probability density function of X helps us to find the probability that X will be within a given interval $[a, b]$ i.e. $P(a \le X \le b) = \int_a^b f(x)\,dx$, other conditions being satisfied.

However, sometimes we need to know that probability that X will be less than a given value x. For a continuous random variable X, this probability is obtained by integrating $f(x)$ from $-\infty$ (or the lower limit of the interval) to x. The function so obtained is called **distribution function.**

Definition : If X is a continuous random variable X, having the probability density function $f(x)$ then the function

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\, dt, \quad -\infty < x < \infty$$

is called **distribution function** or **cumulative distribution function** of the random variable X.

Some Important Properties of Distribution Function $F(x)$ of a Continuous Random Variable

1. The function $F(x)$ is defined for every real number x.
2. Since $F(x)$ denotes probability and probability of X lies between 0 and 1,
   $0 \le F(x) \le 1$.
3. $F(x)$ is a non-decreasing function which means if $x_1 \le x_2$, then $F(x_1) \le F(x_2)$.
4. **The derivative of $F(x)$** *i.e.* $F'(x)$ exists at all points (except perhaps at a finite number of points) and **is equal to the probability density function $f(x)$.**
   $$F'(x) = \frac{d}{dx} F(x) = f(x) \ge 0 \text{ provided the derivative exists.}$$
5. If $F(x)$ is a distribution function of a continuous random variable then
   $P(a \le X \le b) = F(b) - F(a)$.

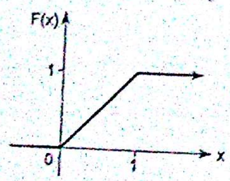
Example 1 : A continuous variable X has the following distribution function

$$F(x) = \begin{cases} 0, & x \le 0 \\ x, & 0 \le x \le 1 \\ 1, & 1 \le x \end{cases} \qquad \text{.................... (1)}$$

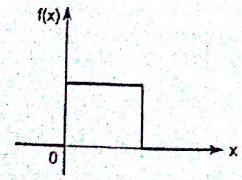
Find the probability density function and draw the graphs of both p.d.f. and c.d.f.

Sol. : The p.d.f. is

$$f(x) = F'(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 \le x \le 1 \\ 0 & 1 < x \end{cases}$$



Graph of c.d.f.



Graph of p.d.f.

Fig. 5.20

From the graph of $F(x)$ we see that $F(x)$ is continuous at all points including $x = 0$ and $x = 1$. $f(x)$ is obtained by differentiating $F(x)$.

Example 2 : For the distribution function given below, find p.d.f.

$$F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-x/4}, & x \ge 0 \end{cases}$$

Also find the probabilities : $P(X \le 4)$, $P(X \ge 8)$, $P(4 \le X \le 8)$.

Sol. : $F(x)$ satisfies all the conditions of a distribution function. If $f(x)$ is the corresponding probability density function

$$f(x) = F'(x) = \begin{cases} \frac{1}{4} e^{-x/4}, & x \ge 0 \\ 0, & x < 0 \end{cases}$$

Now, we have to verify that $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

$$\therefore \int_{-\infty}^{0} f(x)\, dx + \int_0^{\infty} \frac{1}{4} e^{-x/4}\, dx = 0 + \int_0^{\infty} \frac{1}{4} e^{-x/4}\, dx = \frac{1}{4}\left[\frac{e^{-x/4}}{-1/4}\right]_0^{\infty}$$

$$= -\left[e^{-x/4}\right]_0^{\infty} = -[0 - 1] = 1$$

Hence, $F(x)$ is a distribution function.

Now, $P(X \le 4) = F(4) = 1 - e^{-1} = 1 - \frac{1}{e} = \frac{e-1}{e}$

$$P(X \ge 8) = 1 - P(x \le 8) = 1 - F(8)$$

$$= 1 - \left[1 - e^{-2}\right] = e^{-2} = 1/e^2$$

$$P(4 \le X \le 8) = F(8) - F(4) = (1 - e^{-2}) - (1 - e^{-1})$$

$$= e^{-1} - e^{-2} = \frac{1}{e} - \frac{1}{e^2} = \frac{e-1}{e^2}$$

Example 3 : If $f(x) = \begin{cases} x e^{-x^2/2} & x \ge 0 \\ 0 & x < 0 \end{cases}$

(i) Show that $f(x)$ is a probability density function. (ii) Find its distribution function.

Sol. : If $f(x)$ is a p.d.f., we must have $f(x) \ge 0$ where $x \ge 0$ and $\int_0^{\infty} f(x)\, dx = 1$.

Clearly $f(x) = xe^{-x^2/2} \ge 0$ for $x \ge 0$.

Now, $\int_0^{\infty} x e^{-x^2/2}\, dx = \int_0^{\infty} e^{-t}\, dt \quad [t = x^2/2]$

$$= \left[-e^{-t}\right]_0^{\infty} = -[0 - 1] = 1$$

$\therefore$ $f(x)$ is a probability density function.

Now, its distribution function is given by

$$F(x) = \int_0^{x} f(x)\, dx = \int_0^{x} x e^{-x^2/2}\, dx = -\left[e^{-x^2/2}\right]_0^{x} \quad \text{[As above]}$$

$$= -\left[e^{-x^2/2} - 1\right] = 1 - e^{-x^2/2}, \quad x \ge 0$$

CHAPTER 6

Mathematical Expectation

1. Introduction

Suppose two coins are tossed twenty times. Let X be the number of heads obtained in a toss. X then, takes values 0, 1 and 2. Suppose further that no heads, one head and two heads were obtained 4, 10, 6 times respectively. Then, the average number of heads per toss

$$= \frac{4(0)+10(1)+6(2)}{6+10+4} = 1 \cdot 1$$

This is the average value and is not necessarily a possible outcome of the toss.

The ratios 4 / 20, 10 / 20, 6 / 20 of 0, 1, 2 heads to the total number of tosses are the relative frequencies of $X = 0, 1, 2$. If the experiment is repeated very large number of times, we know that, these relative frequencies tend to the probabilities 1 / 4, 1 / 2, 1 / 4 of 0, 1, 2 heads because in the toss of two coins we have the following.

Sample space	:	HH	$\underbrace{HT, TH}$	TT
Probability	:	1/4	1/2	1/4

The average calculated with probabilities in place of relative frequencies above is called **expected value or mathematical expectation** and is denoted by $E(X)$. Thus,

$$E(X) = \frac{1}{4}(0) + \frac{1}{2}(1) + \frac{1}{4}(2) = 1$$

$$E(X) = \text{sum of the products of the values and their probabilities}$$

$$= p_1 x_1 + p_2 x_2 + p_3 x_3 + \ldots..$$

This means, a person who throws two coins over and over again will get one head per toss on the average. This suggests us that the expected value of X can be obtained by multiplying the values of X by their respective probabilities and taking the sum. This leads us to the following definition of expectation of a discrete random variable X.

2. Expectation of a Random Variable

(a) Definition : If a discrete random variable X assumes values $x_1, x_2, \ldots.. x_n \ldots..$ with probabilities $p_1, p_2, \ldots.. p_n \ldots..$ respectively then the **mathematical expectation of X** denoted by $E(X)$ (if it exists) is defined by

$$\boxed{E(X) = p_1 x_1 + p_2 x_2 + \ldots.. + p_n x_n + \ldots..}$$

i.e. $\boxed{E(X) = \Sigma\, p_i x_i}$ where $\Sigma\, p_i = 1$.

if $\Sigma\, p_i x_i$ is absolutely convergent.

This value is also referred to as *mean value* of X. It is also denoted by μ_1'. $\therefore\ \mu_1' = E(X)$.

Notation : In this chapter we shall slightly deviate from our previous notation. Instead of denoting $P(X = x_i)$ by $p(x_i)$ we shall denote it simply by p_i. This will be found more convenient while dealing with expectations.

(b) Definition : Let X be a continuous random variable with probability density function $f(x)$. Then the **mathematical expectation of X**, denoted by $E(X)$ (if it exists), is defined by

$$\boxed{E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx} \quad \text{where, } \int_{-\infty}^{\infty} f(x)\, dx = 1$$

if the integral is absolutely convergent.

Notes ....

1. If X assumes only a finite number of values then $E(X) = \Sigma\, p_i x_i$ and can be considered as "weighted average" of the values $x_1, x_2, \ldots.. x_n$ with weights $p_1, p_2 \ldots.. p_n$.
2. If all values $x_1, x_2, \ldots.. x_n$ are equiprobable i.e. $p_1 = p_2 = \ldots.. = p_n = 1/n$ then $E(X) = (1/n) \Sigma\, x_i$ and can be seen to be simple arithmetic mean of the n values $x_1, x_2, \ldots.. x_n$.
3. One should guard oneself from being mislead by the term 'expectation'. $E(X)$ does *not* give us the value of X, we can expect in a single trial. In the Example 4 below $E(X) = 7/2$ is not even a possible value of X when a die is tossed.

   $E(X)$ denotes *mathematical* expectation of X in the sense that if we toss a die for a fairly large number of times, observe the frequencies of the outcomes 1, 2, 3, 4, 5, 6 then the average of these values will be closer to 7 / 2 the more often the die were tossed.
4. $E(X)$ is expressed in the *same units* as X.
5. **Expectation of a constant is constant**

   (i) $E(c) = \Sigma\, p_i c = c \Sigma\, p_i = c \quad [\because \Sigma\, p_i = 1]$

   (ii) $E(c) = \int_{-\infty}^{\infty} c\, f(x)\, dx = c \int_{-\infty}^{\infty} f(x)\, dx = c \quad \left[\because \int_{-\infty}^{\infty} f(x)\, dx = 1\right]$

Example 1 : A fair coin is tossed 3 times. A person received ₹ X^2 if he gets X heads. Find his expectation. **(M.U. 2004)**

Sol. : When a coin is tossed three times, the sample space is

$HHH, HHT, HTH, HTT, THH, THT, TTH, TTT$

The probability distribution of X is

X	:	0	1	2	3
$P(X = x)$	:	1/8	3/8	3/8	1/8

Now, X^2 takes the following values

X^2	:	0	1	2	3
$P(X^2)$	:	1/8	3/8	3/8	1/8

$$\therefore \quad E(X) = \Sigma\, p_i x_i = \frac{1}{8} \times 0 + \frac{3}{8} \times 1 + \frac{3}{8} \times 4 + \frac{1}{8} \times 9$$

$$= \frac{3+12+9}{8} = \frac{24}{8} = 3 ₹$$

Example 2 : There are 10 counters in a bag, 6 of which are worth 5 rupees each while the remaining 4 are of equal but unknown value. If the expectation of drawing a single counter at random is 4 rupees, find the unknown value. **(M.U. 2015)**

Sol. : Let x be the value of the remaining 4 counters.

$$P(\text{of counter worth of ₹ } 5) = \frac{6}{10}$$

$$P(\text{of counter of unknown value}) = \frac{4}{10}$$

$$E(X) = \Sigma p_i x_i \qquad \therefore \quad 4 = \frac{6}{10} \cdot 5 + \frac{4}{10} \cdot x$$

$$\therefore \quad 40 = 30 + 4x \qquad \therefore \quad 4x = 10 \qquad \therefore \quad x = ₹\ 2{\cdot}5.$$

Example 3 : A fair coin is tossed till a head appears. What is the expectation of the number of tosses required ? **(M.U. 1996, 2010)**

Sol. : Let X denote the order of the toss at which we get the first head. We have

Event	H	TH	TTH	$TTTH$	......
X	1	2	3	4	......
$P(X=x)$	$\frac{1}{2}$	$\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$	$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{8}$	$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{16}$	......

$$\therefore \quad E(X) = \Sigma p_i x_i = 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) + 3\left(\frac{1}{8}\right) + 4\left(\frac{1}{16}\right) + \ldots\ldots$$

Let
$$S = \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + 4 \cdot \frac{1}{16} + \ldots\ldots$$

$$\frac{1}{2} S = \frac{1}{4} + 2 \cdot \frac{1}{8} + 3 \cdot \frac{1}{16} + \ldots\ldots$$

$$\therefore \quad S - \frac{1}{2} S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots\ldots$$

$$\therefore \quad \frac{1}{2} S = \frac{1}{2} \cdot \frac{1}{1-(1/2)} = 1 \qquad \left[\text{G.P. } S_\infty = \frac{a}{1-r}\right]$$

$$\therefore \quad E(X) = 2.$$

Example 4 : Find the expectation of (i) the sum, (ii) the product of the number of points on the throw of n dice. **(M.U. 2004, 06)**

Sol. : Let X_i denote the number of points on the i th dice.

Then if S denotes the sum of the points of n dice then $S = \sum_{i=1}^{n} X_i$.

Now, $E(X_i) = \Sigma p_i x_i = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \ldots\ldots + \frac{1}{6} \cdot 6$

$$\therefore \quad E(X_i) = \frac{1}{6}(1+2+3+3+4+5+6) = \frac{21}{6} = \frac{7}{2}$$

$$\therefore \quad S = \sum_{i=1}^{n} X_i = n \cdot \frac{7}{2} = \frac{7n}{2}$$

If π denotes the product of the points

$$E(\pi) = E(X_1) \cdot E(X_2) \ldots\ldots E(X_n) = \left(\frac{7}{2}\right) \cdot \left(\frac{7}{2}\right) \cdots\cdots \left(\frac{7}{2}\right) = \left(\frac{7}{2}\right)^n.$$

Example 5 : A box contains n tickets numbered 1, 2, ..., n. If m tickets are drawn at random from the box. What is the expectation of the sum of the numbers on the tickets drawn ? **(M.U. 2001)**

Sol. : Let X_i denote the number on the i th ticket drawn.

Then $S = X_1 + X_2 + \ldots\ldots + X_m$

Now, $E(X_i) = \Sigma p_i x_i = \frac{1}{n} \cdot 1 + \frac{1}{n} \cdot 2 + \ldots\ldots + \frac{1}{n} \cdot n$

$$= \frac{1}{n}(1+2+\ldots\ldots+n) = \frac{1}{n} \cdot \frac{n}{2} \cdot (n+1) = \frac{n+1}{2}$$

$$\therefore \quad E(S) = \sum_{i=1}^{m} E(X_i) = \frac{m(n+1)}{2}.$$

Example 6 : Three urns contain respectively 3 green and 2 white balls, 5 green and 6 white balls, 2 green and 4 white balls. One ball is drawn from each urn. Find the expected number of white ball drawn. **(M.U. 2007, 09)**

Sol. : If X denotes the number of white balls drawn from an urn then the expectation of X is as follows.

$$E(X) = p_i x_i = p_1 x_1 + p_2 x_2$$

x takes two values. $X = 1$ if white ball is drawn and $X = 0$ if greenball is drawn.

$\therefore$ From first urn : $\quad E(X_1) = 1 \times \frac{2}{5} + 0 \times \frac{3}{5} = \frac{2}{5}$

From second urn : $\quad E(X_2) = 1 \cdot \frac{6}{11} + 0 \cdot \frac{5}{11} = \frac{6}{11}$

$$E(X_3) = 1 \cdot \frac{4}{6} + 0 \cdot \frac{2}{6} = \frac{2}{3}.$$

$\therefore$ The required expectation $= E(X_1) + E(X_2) + E(X_3)$

$$= \frac{2}{5} + \frac{6}{11} + \frac{2}{3} = \frac{266}{165} = 1{\cdot}61$$

Example 7 : A box contains 2^n tickets of which ${}^nC_r$ tickets bear the number r ($r = 0, 1, 2, \ldots\ldots, n$). A group of m tickets is drawn. What is the expectation of the sum of their numbers ?

Sol. : Let $X_1, X_2, \ldots\ldots, X_m$ be the variables denoting the number on the first, second, ..., m th ticket.

If S is the sum of the numbers on the tickets drawn then

$$S = \Sigma X_i \quad \text{and} \quad E(S) = \Sigma E(X_i)$$

Now, X_i is a random variable which can take any one of the value 0, 1, 2, ..., n with probabilities ${}^nC_0 / 2^n, {}^nC_1 / 2^n, \ldots\ldots, {}^nC_n / 2^n$.

$$\therefore \quad E(X_i) = \Sigma p_i x_i$$

$$= \frac{1}{2^n}\left[0 \cdot {}^nC_0 + 1 \cdot {}^nC_1 + 2 \cdot {}^nC_2 + 3 \cdot {}^nC_3 + \ldots\ldots + n \cdot {}^nC_n\right]$$

$$= \frac{1}{2^n}\left[1 \cdot n + 2 \cdot \frac{n(n-1)}{2!} + 3 \cdot \frac{n(n-1)(n-2)}{3!} + \ldots\ldots + n \cdot 1\right]$$

$$= \frac{n}{2^n}\left[1 + (n-1) + \frac{(n-1)(n-2)}{2!} + \ldots\ldots + 1\right]$$

$$\therefore \quad E(X_i) = \frac{n}{2^n}[1+1]^{n-1} = \frac{n}{2^n} \cdot 2^{n-1} = \frac{n}{2}.$$

$$\therefore \quad E(S) = \Sigma E(X_i) = m \cdot \frac{n}{2} = \frac{mn}{2}.$$

Cor. 1 : If two tickets are drawn then putting $m = 2$, we get

$$E(\text{Sum}) = 2 \cdot \frac{n}{2} = n.$$ (M.U. 2003)

Example 8 : A box contains 'a' white balls and 'b' black balls. 'c' balls are drawn from the box at random. Find the expected value of the number of white balls. (M.U. 2005)

Sol. : Let X_i be the variable denoting the result of the i th draw.

Let $X_i = 1$ if i th ball drawn is white and $X_i = 0$ if i th ball drawn is black.

Since, 'c' balls are drawn the sum of the white ball will be

$$S = X_1 + X_2 + \ldots\ldots + X_c = \sum_{i=1}^{c} X_i$$

Now, $P(X_i = 1) = P(\text{drawing a white ball}) = \dfrac{a}{a+b}$

$P(X_i = 0) = P(\text{drawing a black ball}) = \dfrac{b}{a+b}$

$$E(X_i) = 1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0)$$

$$= 1 \cdot \frac{a}{a+b} + 0 \cdot \frac{b}{a+b} = \frac{a}{a+b}$$

$$\therefore \quad E(S) = E(X_1) + E(X_2) + \ldots\ldots + E(X_c)$$

$$= c \cdot \frac{a}{a+b} = \frac{ac}{a+b}.$$

Example 9 : A die is thrown until a five is obtained, find the expectation of the number of throws.

Sol. : Probability of getting 5 in the first toss = 1 / 6,

Probability of getting 5 in the second = (5 / 6) · (1 / 6),

Probability of getting 5 in third = (5 / 6) (5 / 6) (1 / 6) and so on,

and X takes values 1, 2, 3, ...

$$E(X) = 1\,(1/6) + 2\,(5/6)\,(1/6) + 3\,(5/6)^2\,(1/6) + \ldots\ldots$$

$$= (1/6)\,[1 + 2x + 3x^2 + \ldots\ldots] \qquad \text{where, } x = 5/6$$

$$= (1/6)\,(1-x)^{-2} = (1/6)\,[1 - (5/6)]^{-2}$$

$$= 6$$

Example 10 : A and B throw a fair die for a stake of ₹ 44, which is won by the player who throws 6 first. If A starts first, find their expectations.

Sol. : A can win the game, in the first throw or in the third throw or in the fifth throw and so on.

$$P(\text{A winning}) = \frac{1}{6} + \left(\frac{5}{6}\right)\left(\frac{5}{6}\right) \cdot \frac{1}{6} + \left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right) \cdot \frac{1}{6} + \ldots\ldots$$

$$\therefore \quad P(\text{A winning}) = \frac{1}{6}\left[1 + \left(\frac{25}{36}\right) + \left(\frac{25}{36}\right)^2 + \ldots\ldots\right] = \frac{1}{6} \cdot \frac{1}{1-(25/36)} = \frac{6}{11}$$

$$P(\text{B winning}) = 1 - P(\text{A winning}) = \frac{5}{11}$$

$$\therefore \quad \text{Expectation of A} = p \cdot x = \frac{6}{11} \cdot 44 = ₹\,24$$

$$\text{Expectation of B} = p \cdot x = \frac{5}{11} \cdot 44 = ₹\,20.$$

Example 11 : A, B, C, D cut a pack of cards successfully in the order mentioned. The person who cuts a spade first wins ₹ 175. Find their expectations.

Sol. : Probability of cutting a spade $= \dfrac{13}{52} = \dfrac{1}{4}$.

Let A denote the success of A and $\bar{A}$ denote failure of A and so on.

Probability of A's success

$$= P(A) + P(\bar{A}\,\bar{B}\,\bar{C}\,\bar{D}\,A) + P(\bar{A}\,\bar{B}\,\bar{C}\,\bar{D} \cdot \bar{A}\,\bar{B}\,\bar{C}\,\bar{D}\,A) + \ldots\ldots$$

$$= \frac{1}{4} + \left(\frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{4}\right) + \left(\frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{4}\right) + \ldots\ldots$$

$$= \frac{1}{4} + \frac{81}{256} \cdot \frac{1}{4} + \left(\frac{81}{256}\right)^2 \cdot \frac{1}{4} + \ldots\ldots = \frac{1}{4} \cdot \frac{1}{1-(81/256)}$$

$$= \frac{1}{4} \cdot \frac{256}{256-81} = \frac{1}{4} \cdot \frac{256}{175} = \frac{64}{175}.$$

Probability of B's success

$$= P(\bar{A}\,B) + P(\bar{A}\,\bar{B}\,\bar{C}\,\bar{D} \cdot \bar{A}\,B) + \ldots\ldots$$

$$= \frac{3}{4} \cdot \frac{1}{4} + \left(\frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4}\right)\left(\frac{3}{4} \cdot \frac{1}{4}\right) + \ldots\ldots = \frac{3}{16}\left[1 + \frac{81}{256} + \ldots\ldots\right]$$

$$= \frac{3}{16}\left[\frac{1}{1-(81/256)}\right] = \frac{3}{16}\left[\frac{256}{175}\right] = \frac{48}{175}.$$

Probability of C's success

$$= \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} + \ldots\ldots = \frac{9}{64}\left[\frac{1}{1-(81/256)}\right] = \frac{9}{64} \cdot \frac{256}{256-81} = \frac{36}{175}.$$

Probability of D's success

$$= 1 - [P(A) + P(B) + P(C)] = 1 - \left[\frac{64}{175} + \frac{48}{175} + \frac{36}{175}\right] = \frac{27}{175}.$$

Now, the probabilities of A, B, C, D

$$\therefore \quad E(A) = px = \frac{64}{175} \times 175 = ₹\ 64, \qquad E(B) = px = \frac{48}{175} \times 175 = ₹\ 48,$$

$$E(C) = px = \frac{36}{175} \times 175 = ₹\ 36, \qquad E(D) = px = \frac{27}{175} \times 175 = ₹\ 27.$$

Example 12 : Find the expectation of number of failures preceeding the first success in an infinite series of independent trials with constant probabilities p and q of success and failure respectively. **(M.U. 1999, 2003, 17)**

Sol. : We have the following probability distribution

$$\begin{array}{llllll} X & : & 0 & 1 & 2 & 3 \quad \ldots \\ P(X=x) & : & p & qp & q^2p & q^3p \end{array}$$

Since, we may get success in the first trial where the number of failures $X = 0$ and the probability is p; we may get success in the second trial when the number of failures $X = 1$ and the probability is $q\,p$ and so on.

$$\therefore \quad E(X) = \Sigma p_i x_i = p(0) + qp(1) + q^2p(2) + q^3p(3) + \ldots$$

$$= qp\left[1 + 2q + 3q^2 + \ldots\right] = qp(1-q)^{-2} = \frac{qp}{p^2} = \frac{q}{p}.$$

Example 13 : The daily consumption of electric power (in million kwh) is a random variable X with probability distribution function

$$f(x) = \begin{cases} kx\,e^{-x/3} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

Find the value of k, the expectation of x and the probability that on a given day the electric consumption is more than expected value. **(M.U. 2003, 04, 16)**

Sol. : We must have

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \quad i.e. \quad k\int_0^{\infty} x\,e^{-x/3}\,dx = 1$$

$$\therefore \quad k\left[(x)\left(\frac{e^{-x/3}}{-1/3}\right) - (1)\left(\frac{e^{-x/3}}{1/9}\right)\right]_0^{\infty} = 1$$

$$\therefore \quad k[0+9] = 1 \quad \therefore \quad 9k = 1 \quad \therefore \quad k = 1/9$$

$$E(X) = \int_{-\infty}^{\infty} x\,f(x)\,dx = \frac{1}{9}\int_0^{\infty} x^2 \cdot e^{-x/3} \cdot dx$$

$$= \frac{1}{9}\left[x^2\left(\frac{e^{-x/3}}{-1/3}\right) - (2x)\left(\frac{e^{-x/3}}{1/9}\right) + 2\left(\frac{e^{-x/3}}{-1/27}\right)\right]_0^{\infty}$$

$$= \frac{1}{9}[0 - 0 + 0 + 54] = 6$$

$$\therefore \quad P(X > 6) = \frac{1}{9}\int_6^{\infty} x \cdot e^{-x/3} \cdot dx$$

$$\therefore \quad P(X > 6) = \frac{1}{9}\left[(x)\left(\frac{e^{-x/3}}{-1/3}\right) - (1)\left(\frac{e^{-x/3}}{1/9}\right)\right]_6^{\infty}$$

$$= \frac{1}{9}\left[(0-0) - (-18\,e^{-2} - 9e^{-2})\right]$$

$$= 3\,e^{-2} = 0.406$$

Example 14 : Find k and then $E(X)$ if X has the p.d.f.

$$f(x) = \begin{cases} k\,x(2-x), & 0 \le x \le 2,\ k > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Sol. : Now $\displaystyle\int_0^2 kx(2-x)\,dx = k\int_0^2 2x - x^2\,dx = k\left[x^2 - \frac{1}{3}x^3\right]_0^2 = k \cdot \frac{4}{3}$

$$\therefore \quad k \cdot \frac{4}{3} = 1 \quad \therefore \quad k = \frac{3}{4}$$

By definition

$$E(X) = \int_0^2 x\,f(x)\,dx = \int_0^2 x \cdot \frac{3}{4}x(2-x)\,dx = \frac{3}{4}\int_0^2 (2x^2 - x^3)\,dx$$

$$\therefore \quad E(X) = \frac{3}{4}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2 = \frac{3}{4}\left[\frac{16}{3} - \frac{16}{4}\right] = \frac{3}{4} \cdot \frac{16}{12} = 1.$$

Example 15 : Find k and then $E(X)$ for the p.d.f.

$$f(x) = \begin{cases} k(x - x^2), & 0 \le x \le 1,\ k > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Sol. : Now $\displaystyle k\int_0^1 (x - x^2)\,dx = k \cdot \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = k\left[\frac{1}{2} - \frac{1}{3}\right] = k \cdot \frac{1}{6}.$

But $\quad k \cdot \dfrac{1}{6} = 1 \quad \therefore \quad k = 6$

By definition

$$E(X) = \int_0^1 x\,f(x)\,dx = \int_0^1 x \cdot 6(x - x^2)\,dx = 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{6}{12} = \frac{1}{2}.$$

EXERCISE - I

1. From an urn containing 3 red balls and 2 white balls, a man is to draw 2 balls at random without replacement. He gets ₹ 20 for each red ball and ₹ 10 for each white ball he draws. Find his expectation. **[Ans. : ₹ 32]**
2. Two urns contain respectively 5 white and 3 black balls; 2 white and 3 black balls. One ball is drawn from each urn. Find the expected number of white balls drawn. **[Ans. : 41 / 40]**
3. A and B toss a fair coin alternately. One who gets a head first wins ₹ 12. A starts. Find their mathematical expectations. **[Ans. : ₹ 8, 4]**

4. A, B, and C toss a fair coin. The first one to throw a head wins the game and gets ₹ 28. If A starts, find their mathematical expectations. [Ans. : ₹ 16, 8, 4]

5. A, B and C draw a card in that order from a well shuffled pack of 52 cards. The first to draw a diamond wins ₹ 740. If A starts, find their expectations. [Ans. : ₹ 320, 240, 180]

6. Two unbiased dice are thrown. Find the expectation of the sum. (M.U. 2007) [Ans. : 7]

7. A man with n keys in his pocket wants to open the door of his case by trying the keys independently and randomly one by one. Find the mean and the variance of the number of trials required to open the door if unsuccessful keys are kept aside. (M.U. 1998)

[Ans. : (i) $(n+1)/2$, (ii) $(n^2-1)/12$]

8. A player throwing an ordinary die is to receive ₹ $1/2^n$ where n denotes the number of throws required to get first 3. Find his expectation. (M.U. 2001, 04) [ans. : ₹ 1/7]

9. Three fair coins are tossed. Find the expectation and the variance of number of heads.

(M.U. 2004) [Ans. : $\overline{x} = 3/2$, Var $(X) = 3/4$]

10. From a box containing n tickets bearing numbers 1, 2, 3, ..., n a ticket is drawn. If X denotes the number on the ticket drawn, find the mean and variance of X.

[Ans. : $\overline{x} = (n+1)/2$, Var $(X) = (n^2-1)/2$]

11. In a game of chance a man is allowed to throw a fair coin indefinitely. He receives rupees 1, 2, 3, ... if he throws a head at the 1st, 2nd, 3rd, ... trial respectively. If the entry fee to participate in the game is ₹ 2, find the expected value of his net gain. [Ans. : Zero]

12. A person draws 3 balls from a bag containing 3 white, 4 red and 5 black balls. He is offered ₹ 10, ₹ 5 and ₹ 2 if he draws 3 balls of the same colour, 2 balls of the same colour and 1 ball of each colour respectively. Find his expectation. (M.U. 2004)

[Ans. : $p_1 = \frac{3}{44}$, $p_2 = \frac{29}{44}$, $p_3 = \frac{12}{44}$; ₹ 4·52]

13. A continuous random variable X has the density function $f(x) = k(1+x)$ where $2 \le x \le 5$. Find k, $P(x \le 4)$ and $E(X)$. (M.U. 2005) [Ans. : $k = \frac{2}{27}$; $\frac{16}{27}$; $\frac{11}{3}$]

3. Expectation of a Function of a Random Variable X

We can now extend the concept of expectation of a random variable to the function of a random variable.

1. Definition : Let X be a discrete random variable taking values $x_1, x_2, \ldots, x_n$ with probabilities $p_1, p_2, \ldots, p_n$ and $g(X)$ be a function of X then **mathematical expectation of $g(X)$** (if it exists) is defined by

$$\boxed{E[g(X)] = \Sigma\, p_i\, g(x_i)}$$

2. Definition : Let X be a continuous random variable with p.d.f. $f(x)$, let $g(X)$ be a function such that $g(X)$ is a random variable then $E[g(X)]$ (if it exists) is defined by,

$$\boxed{E[g(X)] = \int_{-\infty}^{\infty} g(x) \cdot f(x)\, dx}$$

Notes ....

1. If $g(X) = aX^n$, then $E[g(X)] = E[a X^n] = \Sigma\, p_i a x_i^n = a \Sigma\, p_i x_i^n = a E(X^n)$

And $E[g(X)] = E[a X^n] = \int_{-\infty}^{\infty} a x^n f(x)\, dx = a \int_{-\infty}^{\infty} x^n f(x)\, dx = a E(X^n)$

e.g. $E(aX) = a E(X)$

e.g. $E(aX^2) = a E(X^2)$, $E(aX^3) = a E(X^3)$

2. If $g(X) = aX + b$, then $E[g(X)] = E[aX + b] = a E(X) + b$.
3. If should be noted that, $E(X^2) \ne [E(X)]^2$ and $E(1/X) \ne 1/E(X)$
4. Putting $a = 1$, $E(X^n) = \Sigma\, p_i x_i^n$ and $E(X^n) = \int_{-\infty}^{\infty} x^n f(x)\, dx$

In particular $E(X^2) = \Sigma\, p_i x_i^2$ and $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$

$E(X^2)$ is denoted by μ_2'

$$\therefore \quad \mu_2' = \Sigma\, p_i x_i^2 \quad \text{or} \quad \mu_2' = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

4. Mean and Variance

If we know the probability density function, discrete or continuous, we can find the mean and variance of the random variable as follows.

$$\mu_1' = \text{Mean} = E(X) = \Sigma\, p_i x_i \quad \text{or} \quad \mu_1' = \int_{-\infty}^{\infty} x f(x)\, dx$$

We then find $\mu_2' = E(X^2) = \Sigma\, p_i x_i^2$ or $\mu_2' = \int_{-\infty}^{\infty} x^2 f(x)\, dx$

Now,

$$\begin{aligned} \text{Var}(X) &= E(X - \overline{X})^2 = E[X - E(X)]^2 \\ &= E\left[X^2 - 2X\,E(X) + \{E(X)\}^2\right] \\ &= E(X^2) - 2E(X) \cdot E(X) + [E(X)]^2 \end{aligned}$$

$$\boxed{\text{Var.}(X) = E(X^2) - [E(X)]^2}$$

But $E(X^2) = \mu_2'$ and $E(X) = \mu_1'$

$$\therefore \quad \boxed{\text{Var.}(X) = \mu_2' - \mu_1'^2}$$

Type I : Examples on Mean and Variance of a Discrete Probability Distribution

Example 1 : If X denotes the smaller of the two numbers that appear when a pair of dice is thrown, find the probability distribution of X, and also the mean and variance of X. **(M.U. 2004)**

Sol. : Refer to the table of Ex. 4 page 5-26.

We see that the number 1 appears as smaller (including equality) of the two numbers in 11 cases out of 36, the number 2 appears in 9 cases, 3 in 7 cases and so on.

The probability distribution of X is as given below.

X :	1	2	3	4	5	6
$P(X = x)$:	11/36	9/36	7/36	5/36	3/36	1/36

$$E(X) = \sum p_i x_i = \frac{11}{36}(1) + \frac{9}{36}(2) + \frac{7}{36}(3) + \frac{5}{36}(4) + \frac{3}{36}(5) + \frac{1}{36}(6) = 2.5278$$

$$E(X^2) = \frac{11}{36}(1^2) + \frac{9}{36}(2^2) + \frac{7}{36}(3^2) + \frac{5}{36}(4^2) + \frac{3}{36}(5^2) + \frac{1}{36}(6^2)$$

$$= \frac{301}{36} = 8.3611$$

$$\therefore \quad V(X) = E(X^2) - [E(X)]^2 = 8.3611 - (2.5278)^2 = 1.97$$

Example 2 : A discrete random variable has the probability density function given below.

X :	-2	-1	0	1	2	3
$P(X = x)$:	0.2	k	0.1	$2k$	0.1	$2k$

Find k, the mean and variance. **(M.U. 1997, 2001)**

Solution : We must have $\sum p_i = 1$.

$$\therefore \quad 5k + 0.4 = 1 \quad \therefore \; 5k = 0.6 \quad \therefore \; k = \frac{0.6}{5} = \frac{3}{25}$$

Hence, the probability distribution is

X :	-2	-1	0	1	2	3
$P(X = x)$:	$2/10$	$3/25$	$1/10$	$6/25$	$1/10$	$6/25$

Now, Mean $= E(X) = \sum p_i x_i = -\frac{4}{10} - \frac{3}{25} + 0 + \frac{6}{25} + \frac{2}{10} + \frac{18}{25} = \frac{60}{250} = \frac{6}{25}$

$$E(X^2) = \sum p_i x_i^2 = \frac{2}{10}(4) + \frac{3}{25}(1) + 0 + \frac{6}{25}(1) + \frac{1}{10}(4) + \frac{6}{25}(9) = \frac{73}{250}$$

$$\therefore \quad \text{Variance} = \sigma^2 = E(X^2) - [E(X)]^2 = \frac{73}{250} - \frac{36}{625} = \frac{293}{625}.$$

Example 3 : Find the value of k from the following data.

X :	0	10	15
$P(x)$:	$\frac{k-6}{5}$	$\frac{2}{k}$	$\frac{14}{5k}$

Also find the distribution function and the expectation of the distribution.

Sol. : Since $\sum p_i = 1$.

$$\frac{k-6}{5} + \frac{2}{k} + \frac{14}{5k} = 1 \quad \therefore \; k^2 - 11k + 14 = 0$$

$$\therefore \quad (k-8)(k-3) = 0 \quad \therefore \; k = 8 \text{ or } 3$$

But when $k = 3$ $P(x = 0) = \frac{3-6}{5} = -\frac{3}{5}$ which is impossible. $\therefore \; k = 8.$

$\therefore$ The p.d.f. and distribution function are

X :	0	10	15
$P(x)$:	$2/5$	$1/4$	$7/20$
$F(x)$:	$2/5$	$13/20$	1

$$\therefore \quad E(X) = \sum p_i x_i = \frac{2}{5}(0) + \frac{1}{4}(10) + \frac{7}{20}(15) \ldots \text{[illegible]}$$

Example 4 : If the mean of the following distribution is 16 find m, n and variance

X :	8	12	16	20	24
$P(X = x)$:	$1/8$	m	n	$1/4$	$1/12$

(M.U. 2006, 16)

Sol. : Since $\sum p_i = 1$,

$$\frac{1}{8} + m + n + \frac{1}{4} + \frac{1}{12} = 1 \quad \therefore \; m + n = \frac{13}{24} \quad \text{......... (1)}$$

Since mean $= 16$, $\sum p_i x_i = 16$

$$\therefore \quad 1 + 12m + 16n + 5 + 2 = 16$$

$$\therefore \quad 12m + 16n = 8 \qquad \therefore \; 3m + 4n = 2 \quad \text{......... (2)}$$

Multiply (1) by 3 and subtract from (2),

$$\therefore \quad 3m + 4n = 2 \;\; ; \;\; 3m + 3n = \frac{13}{8} \qquad \therefore \; n = \frac{3}{8}$$

$$\therefore \quad m + n = \frac{13}{24} \text{ gives } \quad m + \frac{3}{8} = \frac{13}{24} \quad \therefore \; m = \frac{4}{24} = \frac{1}{6}$$

To find variance, consider

$$E(X^2) = \sum p_i x_i^2 = \frac{1}{8}(64) + \frac{1}{6}(144) + \frac{3}{8}(256) + \frac{1}{4}(400) + \frac{1}{12}(576) = 276$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = 276 - 16^2 = 20.$$

Example 5 : A woman with n keys with her, wants to open the door of her house by trying keys independently and randomly one by one. Find the mean and the variance of the number of trials required to open the door, if unsuccessful keys are kept aside. **(M.U. 2016)**

Sol. : If unsuccessful keys are kept aside, she will get success in the first trial, or second trial or third trial and so on, the random variable X of the successful trial will take values 1, 2, 3, ..., n.

$$\therefore \quad P(\text{Success in the first trial}) = \frac{1}{n}$$

$$P(\text{Failure in the first trial}) = 1 - \frac{1}{n}$$

If there is failure in the first trial, the key is eliminated. There are now $(n-1)$ keys.

$$\therefore \quad P(\text{Success in the second trial}) = \frac{1}{n-1}$$

$\therefore$ P (Failure in the first trial and success in the second trial)

$$= \left(1 - \frac{1}{n}\right)\left(\frac{1}{n-1}\right) = \frac{n-1}{n} \cdot \frac{1}{n-1} = \frac{1}{n}$$

$\therefore$ P (Failure in the first trial, failure in the second trial and success in the third trial)

$$= \left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n-1}\right)\left(\frac{1}{n-2}\right)$$

$$= \frac{(n-1)}{n} \cdot \frac{(n-2)}{(n-1)} \cdot \frac{1}{(n-2)} = \frac{1}{n}$$

[illegible] the probability of success at any trial remains constant $= \frac{1}{n}$.

Thus, the probability distribution of X is

$$\begin{array}{lcccccc} X & : & 1 & 2 & 3 & 4 & \ldots & n \\ P(X=x) & : & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \ldots & \frac{1}{n} \end{array}$$

$$\therefore \quad E(X) = \Sigma\, x\, p(x) \cdot x = \frac{1}{n}\cdot 1 + \frac{1}{n}\cdot 2 + \frac{1}{n}\cdot 3 + \ldots + \frac{1}{n}\cdot n$$

$$= \frac{1}{n}(1+2+3+\ldots+n) = \frac{1}{n}\cdot\frac{n}{2}\cdot(n+1)$$

$$\therefore \quad E(X) = \frac{n+1}{2} \qquad \left[\because 1+2+\ldots+n = \frac{n+1}{2}\right]$$

$$E(X^2) = \Sigma\, p(x)\, x^2 = \frac{1}{n}\cdot 1^2 + \frac{1}{n}\cdot 2^2 + \frac{1}{n}\cdot 3^2 + \ldots + \frac{1}{n}\cdot n^2$$

$$= \frac{1}{n}(1^2+2^2+3^2+\ldots+n^2) = \frac{1}{n}\cdot\frac{n}{6}\cdot(n+1)(2n+1)$$

$$= \frac{(n+1)(2n+1)}{6} \qquad \left[\because 1^2+2^2+\ldots+n^2 = \frac{n}{6}(n+1)(2n+1)\right]$$

$$\therefore \quad V(X) = E(X^2) - [E(X)]^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$= \frac{2(2n^2+3n+1) - 3(n^2+2n+1)}{12} = \frac{n^2-1}{12}.$$

Type II : Examples on Mean and Variance of a Continuous Probability Distribution

Example 1 : A continuous random variable X has the p.d.f. defined by $f(x) = A + Bx$, $0 \le x \le 1$. If the mean of the distribution is 1/3, find A and B. **(M.U. 2004, 10)**

Sol. : Since $f(x)$ is a probability distribution

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

By data $\int_0^1 (A+Bx)\, dx = 1$

$$\therefore \quad \left[Ax + \frac{Bx^2}{2}\right]_0^1 = 1 \quad \therefore \quad A + \frac{B}{2} = 1 \qquad \ldots\ldots\ldots\text{(i)}$$

Since the mean is $\frac{1}{3}$, $\int_0^1 x\, f(x)\, dx = \frac{1}{3}$ $\quad\therefore \int_0^1 (A+Bx)\, x\, dx = \frac{1}{3}$

$$\therefore \quad \int_0^1 (Ax + Bx^2)\, dx = \frac{1}{3} \quad \therefore \quad \left[A\frac{x^2}{2} + \frac{Bx^3}{3}\right]_0^1 = \frac{1}{3}$$

$$\therefore \quad \frac{A}{2} + \frac{B}{3} = \frac{1}{3} \quad \therefore \quad 3A + 2B = 2 \qquad \ldots\ldots\ldots\text{(ii)}$$

Solving the equations (i) and (ii), we get $A = 2$, $B = -2$.

$\therefore$ The p.d.f. is $f(x) = 2 - 2x$, $0 \le x \le 1$

Example 2 : The distribution function of a r.v. X is given by $F_X(x) = 1 - (1+x)\, e^{-x}$, $x \ge 0$. Find the mean and variance. **(M.U. 2011)**

Sol. : We have

$$f_X(x) = \frac{d\, F_X(x)}{dx} = (1+x)\, e^{-x} - e^{-x} = x\, e^{-x}, \quad x \ge 0$$

$$\therefore \quad \text{Mean } \overline{X} = \int_0^{\infty} x \cdot f_X(x)\, dx = \int_0^{\infty} x^2 e^{-x}\, dx$$

$$= \left[x^2(-e^{-x}) - 2x(e^{-x}) + 2(1)\cdot(-e^{-x})\right]_0^{\infty} = 2$$

$$E(X^2) = \int_0^{\infty} x^2 f_X(x)\, dx = \int_0^{\infty} x^3 \cdot e^{-x}\, dx$$

$$E(X^2) = \left[x^3(-e^{x}) - 3x^2(e^{-x}) + 6x(-e^{-x}) - 6(e^{-x})\right]_0^{\infty} = 6$$

$$\therefore \quad V(X) = E(X^2) - [E(X)]^2 = 6 - 4 = 2.$$

Example 3 : A continuous random variable X has the p.d.f. $f(x) = kx^2 e^{-x}$, $x \ge 0$. Find k, mean and variance. **(M.U. 2004)**

Solution : We must have $\int_0^{\infty} k\, x^2 e^{-x} \cdot dx = 1$

$$\therefore \quad k\left[x^2(-e^{-x}) - \int -e^{-x} 2x\, dx\right]_0^{\infty} = 1$$

$$\therefore \quad k\left[-x^2 e^{-x} + 2x(-e^{-x}) - \int -2e^{-x} dx\right]_0^{\infty} = 1$$

[Integrating by parts]

$$k\left[-x^2 e^{-x} - 2x e^{-x} - 2e^{-x}\right]_0^{\infty} = 1$$

$$k[0 - (-0 - 0 - 2)] = 1 \quad \therefore \quad 2k = 1 \quad \therefore \quad k = \frac{1}{2}.$$

Now, mean $\overline{X} = \int_0^{\infty} x\, f(x)\, dx = \int_0^{\infty} \frac{1}{2} x^3 e^{-x} dx$

$$= \frac{1}{2}\left[x^3(-e^{-x}) - (3x^2)(e^{-x}) + (6x)(-e^{-x}) - (6)(e^{-x})\right]_0^{\infty}$$

(By the generalised rule of integration by parts.

$$\int uv\, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots$$

where dashes denote the derivatives and suffixes denote the integrals.)

$$\therefore \quad \overline{X} = \frac{1}{2}[0 - (-6)] = \frac{1}{2}\cdot 6 = 3$$

Now, $\mu_2' = \frac{1}{2}\int_0^{\infty} x^2 f(x)\, dx = \frac{1}{2}\int_0^{\infty} x^2 \cdot x^2 e^{-x}\, dx$

$$= \frac{1}{4}\int_0^{\infty} x^4 \cdot e^{-x} dx$$

$$= \frac{1}{2}\left[x^4(-e^{-x}) - (4x^3)(e^{-x}) + (12x^2)(-e^{-x}) - (24x)(e^{-x}) + 24(-e^{-x})\right]_0^{\infty}$$

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$$\therefore \quad \mu_2' = \frac{1}{2}[0-(-24)] = \frac{24}{2} = 12$$

$$\therefore \quad \text{Variance} = \mu_2' - \mu_1'^2 = 12 - 9 = 3.$$

Example 4 : If X is a continuous random variable with probability density function given by

$$f(x) = \begin{cases} k(x-x^3), & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find (i) k, (ii) mean, (iii) variance. (M.U. 2016)

Sol. : (i) We have $\int_0^1 k(x-x^3)\,dx = 1$

$$\therefore \quad k\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = 1 \quad \therefore \quad k\left[\frac{1}{2} - \frac{1}{4}\right] = 1 \quad \therefore \quad k \cdot \frac{1}{4} = 1 \quad \therefore \quad k = 4$$

(ii) Mean, $\bar{X} = \mu_1' = \int_0^1 x\,f(x)\,dx = \int_0^1 x \cdot 4(x-x^3)\,dx$

$$= 4\int_0^1 (x^2 - x^4)\,dx = 4\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = 4\left[\frac{1}{3} - \frac{1}{5}\right] = \frac{8}{15}$$

$$\mu_2' = \int_0^1 x^2 f(x)\,dx = \int_0^1 x^2 \cdot 4(x-x^3)\,dx = 4\int_0^1 (x^3 - x^5)\,dx$$

$$= 4 \cdot \left[\frac{x^4}{4} - \frac{x^6}{6}\right]_0^1 = 4 \cdot \left[\frac{1}{4} - \frac{1}{6}\right] = 4 \cdot \frac{1}{12} = \frac{1}{3}$$

(iii) Variance $(X) = \mu_2' - \mu_1'^2 = \frac{1}{3} - \frac{64}{225} = \frac{33}{225} = 0{\cdot}1467.$

Example 5 : Find the value of k, if the function

$$f(x) = kx^2(1-x^3), \quad 0 \le x \le 1$$
$$= 0 \quad \text{otherwise}$$

is a probability density function. Also find $P(0 \le x \le 1/2)$ and the mean and variance. (M.U. 2017)

Sol. : We have $\int_0^1 kx^2(1-x^3)\,dx = 1 \quad \therefore \quad \int_0^1 k(x^2 - x^5)\,dx = 1$

$$\therefore \quad k\left[\frac{x^3}{3} - \frac{x^6}{6}\right]_0^1 = 1 \quad \therefore \quad k\left[\frac{1}{3} - \frac{1}{6}\right] = 1 \quad \therefore \quad k \cdot \frac{1}{6} = 1 \quad \therefore \quad k = 6.$$

Now, $P(0 \le x \le 2) = 6\int_0^{1/2}(x^2 - x^5)\,dx = 6\left[\frac{x^3}{3} - \frac{x^6}{6}\right]_0^{1/2} = 6\left[\frac{1}{8 \cdot 3} - \frac{1}{64 \cdot 6}\right] = \frac{15}{64}.$

Mean $\bar{x} = \mu_1' = \int_0^1 x\,f(x)\,dx = 6\int_0^1 x \cdot x^2(1-x^3)\,dx = 6\int_0^1 (x^3 - x^6)\,dx$

$$= 6\left[\frac{x^4}{4} - \frac{x^7}{7}\right]_0^1 = 6\left[\frac{1}{4} - \frac{1}{7}\right] = \frac{18}{28} = \frac{9}{14}$$

$$\mu_2' = \int_0^1 x^2 f(x)\,dx = 6\int_0^1 x^2\left[x^2(1-x^3)\right]dx = 6\int_0^1 (x^4 - x^7)\,dx$$

$$= 6\left[\frac{x^5}{5} - \frac{x^8}{8}\right]_0^1 = 6\left[\frac{1}{5} - \frac{1}{8}\right] = \frac{18}{40} = \frac{9}{20}$$

$$\text{Variance} = \mu_2' - \mu_1'^2 = \frac{9}{20} - \frac{81}{196} = \frac{441-405}{980} = \frac{36}{980} = \frac{9}{245}.$$

Example 6 : A continuous random variable X has the following probability density function

$$f(x) = \begin{cases} kx & 0 \le x \le 2 \\ 2k & 2 \le x \le 4 \\ 6k - kx & 4 \le x \le 6 \end{cases}$$

Find k, $P(1 \le x \le 3)$ and the mean.

Sol. : For p.d.f. we must have $\int_{-\infty}^{\infty} f(x)\,dx = 1.$

$$\therefore \quad \int_0^2 kx\,dx + \int_2^4 2k\,dx + \int_4^6 (6k - kx)\,dx = 1$$

$$\therefore \quad k\left[\frac{x^2}{2}\right]_0^2 + 2k\left[x\right]_2^4 + k\left[6x - \frac{x^2}{2}\right]_4^6 = 1$$

$$\frac{k}{2}[4-0] + 2k[4-2] + k[(36-18)-(24-8)] = 1$$

$$2k + 4k + 6k = 1 \quad \therefore \quad 12k = 1 \quad \therefore \quad k = \frac{1}{2}$$

$$\therefore \quad P(1 \le x \le 3) = \int_1^2 \frac{x}{12}\,dx + \int_2^3 \frac{1}{6}\,dx = \frac{1}{12}\left[\frac{x^2}{2}\right]_1^2 + \frac{1}{6}\left[x\right]_2^3 = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

$$\text{Mean } \bar{x} = \mu_1' = \int_{-\infty}^{\infty} x\,f(x)\,dx$$

$$= \frac{1}{12}\int_0^2 x \cdot x\,dx + \frac{1}{6}\int_2^4 x\,dx + \frac{1}{12}\int_4^6 x(6-x)\,dx$$

$$\therefore \quad \text{Mean } \bar{x} = \frac{1}{12}\left[\frac{x^3}{3}\right]_0^2 + \frac{1}{6}\left[\frac{x^2}{2}\right]_2^4 + \frac{1}{12}\left[3x^2 - \frac{x^3}{3}\right]_4^6$$

$$= \frac{8}{36} + \frac{1}{12}[16-4] + \frac{1}{12}\left[\left(108 - \frac{216}{3}\right) - \left(48 - \frac{64}{3}\right)\right]$$

$$= \frac{2}{9} + 1 + \frac{1}{12} + \frac{28}{3} = \frac{383}{36}.$$

Example 7 : If the distribution function of a random variable is given by

$$F(x) = \begin{cases} 1 - 4/x^2, & x > 2 \\ 0, & x \le 2 \end{cases}$$

find (i) $P(x < 3)$, (ii) $P(4 < x < 5)$, (iii) mean and the variance.

Sol. : The probability density function $f(x)$ is given by

$$f(x) = F'(x) = \begin{cases} 8/x^3, & x > 2 \\ 0, & x \le 2 \end{cases}$$

(i) $P(x < 3) = \int_2^3 \frac{8}{x^3}\, dx = \left[-\frac{8}{2x^2}\right]_2^3 = \left[-\frac{4}{x^2}\right]_2^3 = -4\left[\frac{1}{9} - \frac{1}{4}\right] = \frac{5}{9}.$

(ii) $P(4 < x < 5) = \int_4^5 \frac{8}{x^3}\, dx = \left[-\frac{4}{x^2}\right]_4^5 = -4\left[\frac{1}{25} - \frac{1}{16}\right] = 0{\cdot}09$

(iii) $\mu_1' = \int_2^\infty x \cdot \frac{8}{x^3}\, dx = \int_2^\infty \frac{8}{x^2}\, dx = 8\left[-\frac{1}{x}\right]_0^\infty = -8\left[0 - \frac{1}{2}\right] = 4$

$\mu_2' = \int_2^\infty x^2 \cdot \frac{8}{x^3}\, dx = 8\int_2^\infty \frac{1}{x}\, dx = 8\left[\log x\right]_2^\infty = \infty$

$\therefore$ Var. $(x) = \mu_2' - \mu_1'^2$ $\quad\therefore$ Var. (x) does not exist.

Example 8 : A continuous random variable has probability density function

$$f(x) = k(x - x^2), \quad 0 \le x \le 1$$

Find (i) k, (ii) mean, (iii) variance. **(M.U. 1997, 2001, 03, 15, 16)**

Sol. : (i) We have

$$\int_0^1 k(x - x^2)\, dx = 1 \quad \therefore\ k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1$$

$$\therefore\ k\left[\frac{1}{2} - \frac{1}{3}\right] = 1 \quad \therefore\ k \cdot \frac{1}{6} = 1 \quad \therefore\ k = 6.$$

(ii) $\mu_1' = \int_0^1 x \cdot 6(x - x^2)\, dx = 6\int_0^1 (x^2 - x^3)\, dx$

$$= 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 6\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{2}$$

(iii) $\mu_2' = \int_0^1 x^2 \cdot 6(x - x^2)\, dx = 6\int_0^1 (x^3 - x^4)\, dx$

$$= 6\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = 6\left[\frac{1}{4} - \frac{1}{5}\right] = \frac{6}{20} = \frac{3}{10}$$

$$\therefore\ \text{Var. } (x) = \mu_2' - \mu_1'^2 = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}$$

EXERCISE - II

or $\mu_1 = \int_{-\infty}^{\infty} (x - \overline{X}) f(x)\, dx$

$$= \int_{-\infty}^{\infty} x f(x)\, dx - \overline{X} \int_{-\infty}^{\infty} f(x)\, dx$$

$$= \overline{X} - \overline{X} = 0$$

9. Moment Generating Function

(a) Definition (Discrete Random Variable) : The moment generating function (m.g.f.) of a discrete random variate X about a denoted by $M_a(t)$ is defined by

$$M_a(t) = E[e^{t(x-a)}] \qquad \therefore \qquad \boxed{M_a(t) = \Sigma\, p_i\, e^{t(x_i - a)}} \qquad \text{.................... (1)}$$

The m.g.f. is a function of the real parameter t. The subscript a shows the point about which the m.g.f. is taken.

Expanding the exponential in (1), we get

$$M_a(t) = \Sigma\, p_i \left[1 + \frac{t}{1!}(x_i - a) + \frac{t^2}{2!}(x_i - a)^2 + \frac{t^3}{3!}(x_i - a)^3 + ...\right]$$

$$= \Sigma\, p_i + \frac{t}{1!} \Sigma\, p_i (x_i - a) + \frac{t^2}{2!} \Sigma\, p_i (x_i - a)^2 + \frac{t^3}{3!} \Sigma\, p_i (x_i - a)^3 + ... \qquad \text{.......... (1a)}$$

But $\Sigma\, p_i (x_i - a)^r$ is the r-th moment of X about a *i.e.* μ'_r. Hence, we have

$$M_a(t) = 1 + \mu'_1 \cdot \frac{t}{1!} + \mu'_2 \cdot \frac{t^2}{2!} + \mu'_3 \cdot \frac{t^3}{3!} + ... + \mu'_r \cdot \frac{t^r}{r!} + ... \qquad \text{.................... (2)}$$

Thus, the coefficient of $(t^r / r!)$ is the r-th moment of X about a *i.e.* μ_r'. In this way $M_a(t)$ generates moments. This is the reason why the function $M_a(t)$ is called the **moment generating function.**

Thus, $\boxed{\mu_r' = \text{coefficient of } \frac{t^r}{r!}}$

(b) Definition (Continuous Random Variable) : The moment generating function (m.g.f.) of a continuous random variate X about a denoted $M_a(t)$ is defined by

$$M_a(t) = E[e^{t(x-a)}] \qquad \therefore \qquad \boxed{M_a(t) = \int_{-\infty}^{\infty} e^{t(x-a)} \cdot f(x)\, dx} \qquad \text{.................... (3)}$$

where $f(x)$ is the p.d.f. of X.

Expanding the exponential in (3), we get,

$$M_a(t) = \int_{-\infty}^{\infty} f(x) \cdot \left[1 + \frac{t}{1}(x - a) + \frac{t^2}{2!}(x - a)^2 + \frac{t^3}{3!}(x - a)^3 + ...\right] dx$$

$$M_a(t) = \int_{-\infty}^{\infty} f(x)\, dx + \frac{t}{1!} \int_{-\infty}^{\infty} (x - a) f(x)\, dx$$

$$+ \frac{t^2}{2!} \int_{-\infty}^{\infty} (x - a)^2 f(x)\, dx + \frac{t^3}{3!} \int_{-\infty}^{\infty} (x - a)^3 f(x)\, dx + ... \qquad \text{.......... (3a)}$$

But $\int_{-\infty}^{\infty} (x - a)^r f(x)\, dx$ is the r-th moment μ_r' of X about a. Hence,

$$M_a(t) = 1 + \mu_1' \cdot \frac{t}{1!} + \mu_2' \cdot \frac{t^2}{2!} + \mu_3' \cdot \frac{t^3}{3!} + \ldots + \mu_r' \cdot \frac{t^r}{r!} + \ldots \quad \ldots\ldots (4)$$

Thus, the coefficient of $(t^r / r!)$ is the r-th moment of X about a.

(c) To find moments of various orders from m.g.f. : It is clear from (2) and (4) that the moment of order r is the coefficient of $(t^r / r!)$ in the expansion of m.g.f. Hence, one way of obtaining various moments is to obtain expansion of the m.g.f. of X and find the coefficient of $(t^r / r!)$ in the expansion.

However, in practice many a time obtaining the expansion of m.g.f. is not convenient. In such cases we differentiate m.g.f. w.r.t. t for r times and equate it to zero to get μ_r.

Thus, differentiating $M_a(t)$ from (2) (or from 4), successively, we get,

$$\frac{d}{dt}[M_a(t)] = \mu_1' + \mu_2' t + \mu_3' \frac{t^2}{2!} + \ldots$$

Putting $t = 0$, $\quad \frac{d}{dt}[M_a(t)]_{t=0} = \mu_1'$

$$\frac{d^2}{dt^2}[M_a(t)] = \mu_2' + \mu_3' t + \mu_4' \frac{t^2}{2!} + \ldots$$

Putting $t = 0$, $\quad \frac{d^2}{dt^2}[M_a(t)]_{t=0} = \mu_2'$

In general, $\quad \boxed{\frac{d^r}{dt^r}[M_a(t)]_{t=0} = \mu_r'}$

(d) Moment generating function about origin : Putting $a = 0$ in (1), we get,

$$\boxed{M_0 = \Sigma\, p_i\, e^{t x_i}} \quad \ldots\ldots (5)$$

Putting $a = 0$ in (3), we get,

$$\boxed{M_0(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx} \quad \ldots\ldots (6)$$

Putting $a = 0$ in (1a), we get since $\Sigma p_i x_i^r = \mu_r'$ about the origin.

$$M_0(t) = 1 + \mu_1' t + \mu_2' \frac{t^2}{2!} + \ldots + \mu_r' \frac{t^2}{r!} + \ldots \qquad \therefore \quad M_0(t) = \Sigma \mu_r' \frac{t^r}{r!}$$

Note ....

It is assumed that the r.h.s. of (5) and (6) is absolutely convergent.

(e) If $L(t) = \log M(t)$ where $M(t)$ is the moment generating function of a random variable, prove that the mean = $L'(0)$ and variance = $L''(0)$. (M.U. 2006)

Proof : We have $\quad M(t) = 1 + \mu_1' t + \mu_2' \frac{t^2}{2} + \ldots$

$$\therefore \quad L(t) = \log M(t) = \log\left[1 + \left(\mu_1' t + \mu_2' \frac{t^2}{2!} + \ldots\right)\right]$$

$$= \left(\mu_1' t + \mu_2' \frac{t^2}{2} + \ldots\right) - \frac{1}{2}\left(\mu_1' t + \mu_2' \frac{t^2}{2!} + \ldots\right)^2 + \ldots$$

$$= \left(\mu_1' t + \mu_2' \frac{t^2}{2!} + \ldots\right) - \frac{1}{2}\left(\mu_1'^2 t^2 + \mu_1' \mu_2' t^3 + \ldots\right) + \ldots$$

$$\therefore \quad L'(t) = \mu_1' + \mu_2' t - \mu_1'^2 \cdot t - \frac{3}{2}\mu_1' \mu_2' t^2 + \text{terms in higher powers of } t$$

Putting $t = 0$, $\quad \therefore \quad L'(0) = \mu_1'$ $\quad \ldots\ldots$ (A)

Now, $\quad L''(t) = \mu_2' - \mu_1'^2 - 3\mu_1' \mu_1'^2\, t$ + terms in higher powers of t

Putting $t = 0$, $\quad L''(0) = \mu_2' - \mu_1'^2$

Hence, $\quad$ Mean, $\mu = L'(0)$, Variance, $\mu_2 = L''(0)$. $\quad \ldots\ldots$ (B)

(f) Moment generating function of the sum of two independent random variates : ***"The moment generating function of the sum of two independent random variates is equal to the product of the m.g.f.s of the two variates."***

Proof : Let X, Y be two independent random variates then the m.g.f. of their sum $X + Y$ about the origin is given by

$$M_{(X+Y)}(t) = E[e^{t(X+Y)}] = E(e^{tX} \cdot e^{tY})$$

$$= E(e^{tX}) \cdot E(e^{tY}) \qquad [\because X \text{ and } Y \text{ are independent}]$$

$$\therefore \quad \boxed{M_{X+Y}(t) = M_X(t) \cdot M_Y(t)}$$

Generalisation : If $X_1, X_2, \ldots, X_n$ are n independent random variates, then the m.g.f. of their sum is equal to the product of their m.g.f.s

$$M_{X_1 + X_2 + \ldots + X_n}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \ldots M_{X_n}(t)$$

(g) Uniqueness of Moment Generating Function : This is a very important property of m.g.f. It states that the m.g.f. of a distribution, if it exists, uniquely determines the distribution. In other words it means that for a given probability distribution there is one and only one m.g.f. and corresponding a given m.g.f. there is one and only one probability distribution. Thus, if m.g.f. of X and m.g.f. of Y are equal then X and Y must be identical.

Example 1 : Find the M.G.F. of the following distribution

X	:	-2	3	1
$P(X = x)$	:	1/3	1/2	1/6

Hence, find first four central moments. (M.U. 2010, 15)

Sol. : Since we want central moments we shall find m.g.f. about the mean.

Now, $\quad \bar{X} = \Sigma p_i x_i = -\frac{2}{3} + \frac{3}{2} + \frac{1}{6} = \frac{-4+9+1}{6} = \frac{6}{6} = 1$

$\therefore$ M.G.F. about the mean

$$M_{\bar{x}}(t) = \Sigma p_i e^{t(x_i - \bar{x})} = \frac{1}{3} \cdot e^{t(-2-1)} + \frac{1}{2} e^{t(3-1)} + \frac{1}{6} e^{t(1-1)}$$

$$= \frac{1}{3}e^{-3t} + \frac{1}{2}e^{2t} + \frac{1}{6}$$

$$M_{\bar{x}}(t) = \frac{1}{3}\left[1 - 3t + \frac{9t^2}{2!} - \frac{27t^3}{3!} + \frac{81t^4}{4!} - \ldots\right] + \frac{1}{2}\left[1 + 2t + \frac{4t^2}{2!} + \frac{8t^3}{3!} + \frac{16t^4}{4!} + \ldots\right] + \frac{1}{6}$$

Now, μ_r = Coeff. of $\frac{t^r}{r!}$

$\therefore$ μ_0 = Coeff. of t^0 = constant term $= \frac{1}{3} + \frac{1}{2} + \frac{1}{6} = 1$

$\therefore$ μ_1 = Coeff. of $\frac{t}{1!} = \frac{1}{3}(-3) + \frac{1}{2}(2) = 0$

μ_2 = Coeff. of $\frac{t^2}{2!} = \frac{9}{3} + \frac{4}{2} = 5$

μ_3 = Coeff. of $\frac{t^3}{3!} = \frac{1}{3}(-27) + \frac{1}{2}(8) = -5$

μ_4 = Coeff. of $\frac{t^4}{4!} = \frac{1}{3}(81) + \frac{1}{2}(16) = 35.$

Example 2 : Find the m.g.f. of the random variable X about the origin whose p.m.f. is given above in Ex. 1. Also find the first two moments about the ogirion. **(M.U. 2004)**

Sol. : By definition

$$\mu_0(t) = E(e^{tX}) = \Sigma p_i e^{tx_i} = \frac{1}{3} \cdot e^{-2t} + \frac{1}{2}e^{3t} + \frac{1}{6}e^t$$

Now, $\mu_1' = \left[\frac{d}{dt}M_0(t)\right]_{t=0} = \left[-\frac{2}{3}e^{-2t} + \frac{3}{2}e^{3t} + \frac{1}{6}e^t\right]_{t=0}$

$$= -\frac{2}{3} + \frac{3}{2} + \frac{1}{6} = \frac{6}{6} = 1$$

$$\mu_2' = \left[\frac{d^2}{dt^2}M_0(t)\right]_{t=0} = \left[\frac{4}{3}e^{-2t} + \frac{9}{2}e^{3t} + \frac{1}{6}e^t\right]_{t=0}$$

$$= \frac{4}{3} + \frac{9}{2} + \frac{1}{6} = \frac{36}{6} = 6.$$

Example 3 : A random variable X has probability density function $1/(2^x)$, $x = 1, 2, 3 \ldots$ Find the m.g.f. and hence, find the mean and variance.

Sol. : Since, $P(X = x) = \frac{1}{2^x}$, $x = 1, 2, 3 \ldots$

$$M_0(t) = E(e^{tX}) = \Sigma p e^{tx}$$

$$= \Sigma \frac{1}{2^x} e^{tx} = \sum_{x=1} \left(\frac{e^t}{2}\right)^x = \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \ldots$$

$$= \frac{e^t}{2}\left[1 + \left(\frac{e^t}{2}\right) + \left(\frac{e^t}{2}\right)^2 + \ldots\right] = \frac{e^t}{2}\left[1 - \frac{e^t}{2}\right]^{-1}$$

$$\therefore \quad M_0(t) = \frac{e^t}{2} \cdot \frac{1}{1 - (e^t/2)} = \frac{e^t}{2} \cdot \frac{2}{2 - e^t} = \frac{e^t}{2 - e^t}$$

$$\therefore \quad \mu_1' = \left[\frac{d}{dt}M_0(t)\right]_{t=0} = \left[\frac{(2 - e^t)e^t - e^t(-e^t)}{(2 - e^t)^2}\right]_{t=0}$$

$$= 2\left[\frac{e^t}{(2 - e^t)^2}\right]_{t=0} = 2$$

$$\therefore \quad \mu_2' = \left[\frac{d^2}{dt^2}M_0(t)\right]_{t=0} = 2 \cdot \left[\frac{(2 - e^t)^2 \cdot e^t - e^t \cdot 2(2 - e^t) \cdot (-e^t)}{(2 - e^t)^4}\right]_{t=0}$$

$$= 2 \cdot \left[\frac{(2 - e^t) \cdot e^t + 2e^{2t}}{(2 - e^t)^3}\right]_{t=0} = \frac{2(1 + 2)}{1} = 6$$

$\therefore$ $\mu_2 = \mu_2' - \mu_1'^2 = 6 - 4 = 2$

$\therefore$ Mean = Variance = 2.

Example 4 : If X denotes the outcome when a fair die is tossed, find M.G.F. of X and hence, find the mean and variance of X. **(M.U. 2005, 09)**

Sol. : We have, here X taking values 1, 2, 3, 4, 5, 6 each with probability 1/6.

$$M_0(t) = E(e^{tx_i}) = \Sigma p_i e^{tx_i}$$

$$= \frac{1}{6}e^t + \frac{1}{6}e^{2t} + \frac{1}{6}e^{3t} + \ldots + \frac{1}{6}e^{6t}$$

$$= \frac{1}{6}(e^t + e^{2t} + \ldots + e^{6t})$$

$$\therefore \quad \mu_1' = \left[\frac{d}{dt}M_0(t)\right]_{t=0} = \frac{1}{6}\left[e^t + 2e^{2t} + \ldots + 6e^{6t}\right]_{t=0}$$

$$= \frac{1}{6}[1 + 2 + \ldots + 6] = \frac{21}{6} = \frac{7}{2}$$

$$\mu_2' = \left[\frac{d^2}{dt^2}M_0(t)\right]_{t=0} = \frac{1}{6}\left[e^t + 4e^{2t} + 9e^{3t} + \ldots + 36e^{6t}\right]_{t=0}$$

$$= \frac{1}{6}[1 + 4 + 9 + \ldots + 36] = \frac{91}{6}$$

$$\therefore \quad \mu_2 = \mu_2' - \mu_1'^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}.$$

$\therefore$ Mean $= \frac{7}{2}$ and Variance $= \frac{35}{12}$.

Example 5 : Find the m.g.f. of a random variable X if the r-th moment about the origin is given by $\mu_r' = r!$.

Sol. : Be definition, $\mu_r' = E(x^r) = r!$ ....(1)

$\therefore$ $E(x) = 1$, $E(x^2) = 2!$ $E(x^3) = 3!$ .....

Now, $M_0(t) = E(e^{tx})$

$$= E\left[1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \ldots\right]$$

$$= E(1) + t\,E(x) + \frac{t^2}{2!}E(x^2) + \frac{t^3}{3!}E(x^3) + \ldots$$

Putting the values of $E(X)$, $E(X)^2$, ... from (1)

$$M_0(t) = 1 + t + \frac{t^2}{2!}\cdot 2! + \frac{t^3}{3!}\cdot 3! + \ldots$$

$$= 1 + t + t^2 + t^3 + \ldots = (1-t)^{-1} = \frac{1}{1-t}$$

Example 6 : Find the m.g.f. of a random variable whose p.m.f. is

$$P(X = x) = \left(\frac{1}{2}\right)^x, \quad x = 1, 2, 3, \ldots$$
$$= 0 \quad \text{else where}$$

Hence, find the mean and variance of X.

Sol. : By definition

$$M_0(t) = E(e^{tx}) = \Sigma\, p_i\, e^{tx_i} = \Sigma \frac{1}{2^x} e^{tx} = \Sigma\left(\frac{e^t}{2}\right)^x$$

$$M_0(t) = \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \ldots = \frac{e^t}{2}\left[1 + \left(\frac{e^t}{2}\right) + \left(\frac{e^t}{2}\right)^2 + \ldots\right]$$

$$= \frac{e^t}{2}\cdot\frac{1}{1-(e^t/2)} = \frac{e^t}{2-e^t}$$

To find the mean and variance

$$\mu_1' = \left[\frac{d}{dt}M_0(t)\right]_{t=0} = \left[\frac{(2-e^t)\cdot e^t - e^t\cdot(-e^t)}{(2-e^t)^2}\right]_{t=0}$$

$$= \left[\frac{2e^t}{(2-e^t)^2}\right]_{t=0} = 2$$

$$\mu_2' = \left[\frac{d^2}{dt^2}M_0(t)\right]_{t=0} = \left[\frac{(2-e^t)^2\cdot e^t - e^t\cdot 2(2-e^t)(-e^t)}{(2-e^t)^4}\right]_{t=0}$$

$$= \frac{2[1+2]}{1} = 6$$

Variance $= \mu_2' - {\mu_1'}^2 = 6 - 4 = 2$.

Remark ...

Mean and variance of the above distribution can be obtained using $E(X)$ and $E(X^2)$. Find them.

Example 7 : A random variable X has the following probability density function

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Find m.g.f., μ_r', mean, and variance.

Sol. : We have

$$M_0(t) = E(e^{tx}) = \int_0^1 e^{tx}(1)\,dx = \left[\frac{e^{tx}}{t}\right]_0^1 = \frac{1}{t}\left[e^t - 1\right]$$

$$= \frac{1}{t}\left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots - 1\right] = \frac{1}{t}\left[t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\right]$$

$$= 1 + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots + \frac{t^r}{(r+1)!}$$

$\therefore$ μ_r' = Coefficient of $\frac{t^r}{r!} = \frac{1}{r+1}$, $r = 1, 2, \ldots$

Putting $r = 1, 2$

$\therefore$ Mean $= \mu_1' = \frac{1}{2}$; $\mu_2' = \frac{1}{3}$ $\quad\therefore$ Var. $(X) = \mu_2' - {\mu_1'}^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$.

Example 8 : A random variable X has the following probability density function

$$f(x) = \begin{cases} ke^{-kx}, & x > 0,\ k > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Find the m.g.f. and hence, the mean and variance.

Sol. : We have

$$M_0(t) = E(e^{tx}) = \int_0^\infty e^{tx}\cdot k\,e^{-kx}\,dx = k\int_0^\infty e^{(t-k)x}\,dx$$

$$= \frac{k}{t-k}\left[e^{(t-k)x}\right]_0^\infty = \frac{k}{t-k}[0-1] = \frac{k}{k-t} \qquad [t \neq k]$$

Now, $M_0(t) = \dfrac{k}{k[1-(t/k)]} = \left[1 - \dfrac{t}{k}\right]^{-1} = 1 + \dfrac{t}{k} + \dfrac{t^2}{k^2} + \dfrac{t^3}{k^3} + \ldots$

$\therefore$ μ_1' = Coefficient of $t = \frac{1}{k}$; $\quad \mu_2'$ = Coefficient of $\frac{t^2}{2!} = \frac{2}{k^2}$

$\therefore$ Mean $= \mu_1' = \frac{1}{k}$; Var. $(X) = \mu_2' - {\mu_1'}^2 = \frac{2}{k^2} - \frac{1}{k^2} = \frac{1}{k^2}$.

Example 9 : If a random variable has the moment generating function $M_t = \frac{3}{3-t}$, obtain the mean and the standard deviation. (M.U. 2010)

Sol. : We have

$$M_0(t) = \frac{3}{3-t} = \frac{3}{3[1-(t/3)]} = \left(1 - \frac{t}{3}\right)^{-1} = 1 + \frac{t}{3} + \frac{t^2}{9} + \frac{t^3}{27} + \ldots \infty.$$