Lattice of Stable Matchings and their Properties

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Recap

- In class, we have dealt with Stable Matchings in detail.
- Many different stable matchings exist for a given preference profile.
- Today, we want to observe relations between these matchings
- In particular, that these matchings form a lattice with certain properties and that we can move from one matching to another using rotations.

Man and Woman Optimal Matchings (M_0 and M_z)

In class studying stable matchings, we showed

- If men are proposers the matching is Man Optimal (vice versa for women).

  In the Man-Optimal Stable Matching,

- Each man gets the best partner he can have in any stable matching.
- Each women gets the worst partner she can have in any stable matching.

Lattice Structure

- As seen in last slide, The Man-Optimal and similarly Woman Optimal matchings are extreme matchings.
- Also other stable matchings exists.

  We will now show,
- There exists a distributive lattice under a natural ordering relation for these matchings.
- Also, man-optimal and woman-optimal matchings are minimum and maximum elements of this lattice.

Example

Men's Preferences

$m_1 : (w_2, w_3, w_1, w_4)$
$m_2 : (w_3, w_4, w_2, w_1)$
$m_3 : (w_4, w_1, w_3, w_2)$
$m_4 : (w_1, w_2, w_4, w_3)$

Women's Preferences

$w_1 : (m_1, m_2, m_3, m_4)$
$w_2 : (m_2, m_3, m_4, m_1)$
$w_3 : (m_3, m_4, m_1, m_2)$
$w_4 : (m_4, m_1, m_2, m_3)$

- Using Gale Shapley, we can get Man and Woman Optimal Matchings
- What are they? (Just four proposals for both!)

Example

Men's Preferences

$m_1 : (w_2, w_3, w_1, w_4)$
$m_2 : (w_3, w_4, w_2, w_1)$
$m_3 : (w_4, w_1, w_3, w_2)$
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Women's Preferences

$w_1 : (m_1, m_2, m_3, m_4)$
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$w_3 : (m_3, m_4, m_1, m_2)$
$w_4 : (m_4, m_1, m_2, m_3)$

- Man optimal (M_0) : $\{(m_1, w_2), (m_2, w_3), (m_3, w_4), (m_4, w_1)\}$
- Woman optimal (M_0) :
  $\{(m_1, w_1), (m_2, w_2), (m_3, w_3), (m_4, w_4)\}$

Optimal Matchings for the example

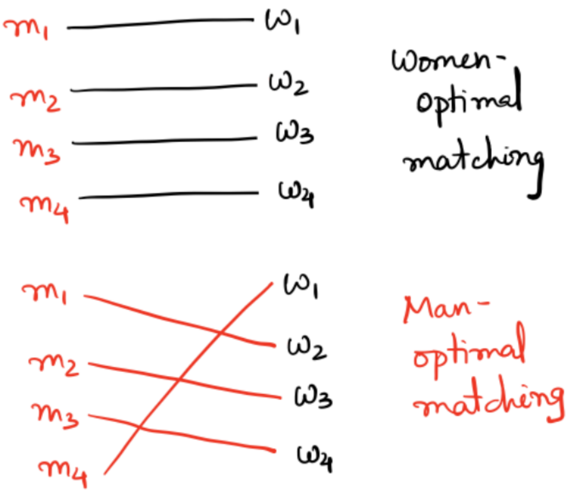

Figure 1: Man and Woman Optimal Matchings

Lattice Structure for matchings for the example

(w_i, w_j, w_k, w_l) represents the matched partners of (m_1, m_2, m_3, m_4). π_1 to π_5 are rotations we will explain later.

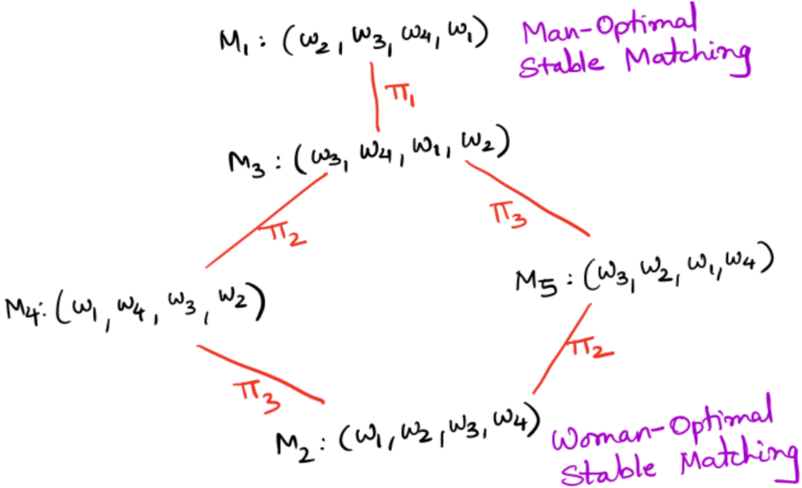

Figure 2: Lattice Structure of Stable Matchings

Preference Recap

- A person x prefers a matching M to a matching M' if x prefers his/her partner in matching M than matching M'
- Now we look at some interesting theorems related to preference.
- Let *SM* represent a Stable Matching.
- We also deal with n x n stable matchings only for simplicity.

Theorem 1

- In 2 *SM*'s M and M', if {u,w} are partners in M but not in M' or vice-versa
- Then, one of them prefers M over M' and other prefers M' over M.
- Also, the total number of people who prefer stable matching M over M' is the same those who prefer M' over M.

Proof 1, Part 1

- Let X and Y be sets of men and women who prefer M to M'.
- Also X' and Y' be sets of men and women who prefer M' to M.
- There can be no pair (m,w) in M such that $m \in X$ and $w \in Y$ since it would be a blocking pair for M'.
- So the partner of every man in X should be in Y'.
- Thus, $|X| \leq |Y'|$

Proof 1, Part 1

- Similarly, there can be no pair (m,w) in M such that $m \in X'$ and $w \in Y'$ since it would be a blocking pair for M.
- So the partner of every man in X' should be in Y.
- Thus, $|X'| \leq |Y|$

- But number of women whose partners changed must be equal to number of men whose partners changed.
- Thus, $|X| + |X'| = |Y| + |Y'|$

- From the above 3 results, $|X'| = |Y|$ and $|X| = |Y'|$
- Hence proved that if a pair is not matched in 1 of 2 stable matchings (M and M'), one partner prefers M whereas other partner prefers M'.

Proof 1, Part 2

- As $|X| + |X'| = |Y| + |Y'|$.
- Thus for two stable matchings M and M', the number of people who prefer M over M' are equal to the number of people who prefer M' over M.

Dominance Relation

- Let us go back to the idea of making a lattice of matchings.
- A stable matching M dominates stable matching M' if every man gets at least as good a partner in M than he has in M'.
- We write it as $M \preceq M'$
- Notice the preceding sign here.

- Since M_0 dominates every other matching, we later call it the minimum element in the lattice.
- Also M_z is dominated by every other matching, we later call it maximum element in the lattice.

Dominance Relation

- $\mathcal{M}$ represents the set of all stable matchings for this current stable marriage instance.
- This set $\mathcal{M}$ is partially ordered for by the dominance relation.
- Totally we call it $(\mathcal{M}, \preceq)$

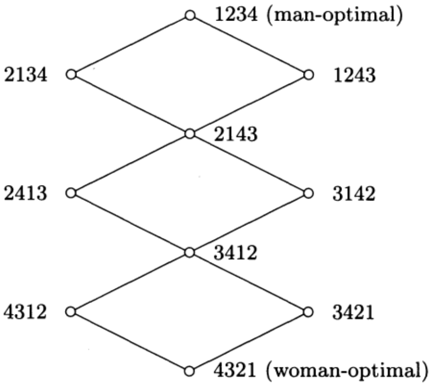

Figure 3: Lattice Structure of Stable Matchings

Distributive Lattice

- It is a partial order in which :
- Each pair (a,b) has a greatest lower bound (meet, denoted as $a \wedge b$) such that :
  - $i) a \wedge b \preceq a$,
  - $ii) a \wedge b \preceq b$,
  - $iii)$ There is no other c which satisfies (i) and (ii) such that $a \wedge b \prec c$,

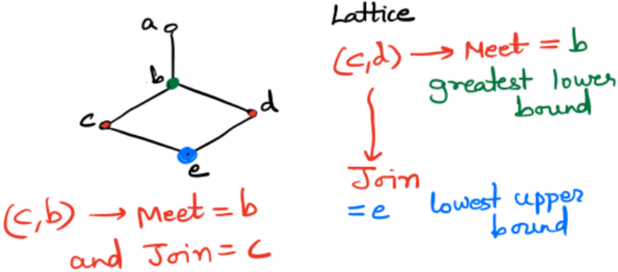

Figure 4: Join and Meet for a Lattice

Distributive Lattice

- Each pair (a,b) has a least upper bound (join, denoted as $a \vee b$) such that :
  - $i) a \preceq a \vee b$,
  - $ii) b \preceq a \vee b$,
  - $iii)$ There is no other c which satisfies (i) and (ii) such that $c \preceq a \vee b$,

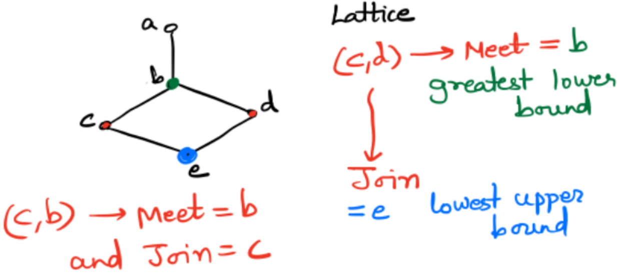

Figure 5: Join and Meet for Lattice

Distributive Lattice

- Distributive Property is satisfied :
  - $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,
  - $a \wedge (b \wedge c) = (a \vee b) \wedge (a \vee c)$,
- Now we will show $(\mathcal{M}, \preceq)$ is a distributive lattice by presenting lemmas to interpret meet and join.

Meet for Lattice of Matchings

Let M and M' be 2 *SM*'s. If each man is given the better of the two partners in M and M', the result is a *SM*.

- **Proof that it's a matching**:
- Let's say two men (m and m') receive the same partner w.
- Thus say in Matching M, w is matched to m, thus m prefers M over M'
- Also in Matching M', w is matched to m', thus m' prefers M' over M
- Earlier we gave a theorem that if there that if there is a pair (m, w) which exists in *SM* M but not *SM* M', then one of the partners prefers M and other prefers M'.

Meet for Lattice of Matchings

- Thus since m prefers M over M', w prefers M' over M.
- Thus w prefers being paired by m' rather than m.
- But on the flip side, same argument can be applied for the pair (m',w).
- From that we get that w prefers M over M' and this prefers m rather than m'.
- This is a contradiction.
- Thus this is indeed a matching.

Meet for Lattice of Matchings

- **Proof that this is a stable matching**
- Suppose its not and (m, w) is a blocking pair.
- Thus, m strictly prefers w over both $p_M(m)$ and $p_{M'}(m)$.
- Also w strictly prefers m to her partner in new matching.
- But woman w can only have two options for partner, either $p_M(w)$ or $p_{M'}(w)$
- If w and $p_M(w)$ are partners in this matching, then (m,w) should block matching M also.
- If w and $p_{M'}(w)$ are partners in this matching, then (m,w) should blocking matching M' also.
- Both of these are contradictions, hence M^* is a stable matching.

Meet for Lattice of Matchings

- Now since every man gets the better of his partners in M and M', this matching dominates both M and M'.
- i.e. This matching $M^* \preceq M$ and $M^* \preceq M'$.
- Thus, we anticipate (as we will show later) that this matching is greatest lower bound of M and M'.
- Thus we will call this matching as $M \wedge M'$.

Join for Lattice of Matchings

- Very similar to what we mentioned above, if each man is given the poorer of his partners in M and M', then the result is a stable matching.
- The proofs corresponding to this are very similar to the "better" partners case.
- We leave it as an exercise for the audience.
- We will represent this matching as the join of M and M' denoted as $M \vee M'$

Proof for Lattice

- Finally we are in possession of all tools required to show that $(\mathcal{M}, \preceq)$ is a distributive lattice.
- It is easy to show that $M \wedge M' \preceq M$ and $M \wedge M' \preceq M'$ since every man is getting better of the two partners.
- Also for every matching M^* satisfying $M^* \preceq M$ and $M^* \preceq M'$, each man must atleast have the better partner among M and M' for every man m.
- Thus, $M^* \preceq M \wedge M'$
- Thus, $M \wedge M'$ is the greatest lower bound(meet) for M and M' as we already notated.
- Similar proof is made for join.

Proof for Distributive Lattice

- Through defining meet and join for the $(\mathcal{M}, \preceq)$ we showed that it is a lattice.
- We can now show that the lattice is distributive.
- Let X, Y, Z be stable matchings.
- Let $U = X \wedge (Y \vee Z)$
- Let $V = (X \wedge Y) \vee (X \wedge Z)$
- Case 1) If $p_y(m) = p_z(m) = w$
- In both U and V we can easily observed that m will be partnered by either $p_x(m)$ or w based on which he prefers

Completing Proof

- Case 2) $p_y(m) \neq p_z(m)$
- In both U and V we can observe,
- if m prefers Y over Z over X, $p_z(m)$ will be partner
- Similarly, if m prefers Z over Y over X, $p_y(m)$ will be partner.
- Else, $p_x(m)$ will be partner.

Moving on (proof completion and next)

- Case 2) $p_y(m) \neq p_z(m)$
- In both U and V we can observe,
- if m prefers Y over Z over X, $p_z(m)$ will be partner
- Similarly, if m prefers Z over Y over X, $p_y(m)$ will be partner.
- Else, $p_x(m)$ will be partner.

- Now we move to the topic of rotations which help us move from one stable matching to another.

Successors

- In a matching M, for a man m, $s_M(m)$ is the first woman w on m's list who strictly prefers m over her current matched partner (denoted by $p_M(w)$).
- The current partner of $s_M(m)$ is called the $succ_M(m)$

Blue = M_1 = man optimal stable matching.

Men's Preferences	Women's Preferences
$m_1 : (\underline{w_2}, \underline{w_3}, w_1, w_4)$	$w_1 : (m_1, m_2, m_3, \underline{m_4})$
$m_2 : (\underline{w_3}, w_4, w_2, w_1)$	$w_2 : (m_2, m_3, m_4, \underline{m_1})$
$m_3 : (\underline{w_4}, w_1, w_3, w_2)$	$w_3 : (m_3, m_4, \underline{m_1}, \underline{m_2})$
$m_4 : (\underline{w_1}, w_2, w_4, w_3)$	$w_4 : (m_4, m_1, m_2, \underline{m_3})$

$S_M(m_1) = W_3$, $Succ(m_1) = m_2$

Reduced preference lists

- For a matching M, we compute the reduced preference list by doing the following:
- In the women's side of the preference list, we delete every man after her currently matched partner.
- In the men's side, we remove the same edges which were removed on the women's side.
- This will be useful to us later.

Reduced preference lists

$\Rightarrow M_2 =$

$\Rightarrow M/P = (m_1, w_3), (m_2, w_4), (m_3, w_1), (m_4, w_2)$

Reduced P.P

W.r.t M_2,

R.P.P

1 : –, 3, 1, 4
2 : –, 4, 2, 1
3 : –, 1, 3, 2
4 : –, 2, 4, 3

1: 1, 2, 3, –
2: 2, 3, 4, –
3: 3, 4, 1, –
4: 4, 1, 2, –

Rotations

- In a matching M, if an ordered sequence of such successor pairs can be obtained, such that a cycle is formed, then it is termed as a rotation. (Include pic below).
- $\rho = (m_0, w_0), (m_1, w_1)...(m_{r-1}, w_{r-1})$ such that succ(m_i)=m_{i+1} and succ$(m_{r-1}) = m_0$
- we say this rotation ρ is exposed in matching M.

Rotations

Blue = M_1 = man optimal stable matching.

Men's Preferences

$m_1 : (\underline{w_2}, \underline{w_3}, w_1, w_4)$
$m_2 : (\underline{w_3}, w_4, w_2, w_1)$
$m_3 : (\underline{w_4}, w_1, w_3, w_2)$
$m_4 : (\underline{w_1}, w_2, w_4, w_3)$

Women's Preferences

$w_1 : (m_1, m_2, m_3, \underline{m_4})$
$w_2 : (m_2, m_3, m_4, \underline{m_1})$
$w_3 : (m_3, m_4, \underline{m_1}, \underline{m_2})$
$w_4 : (m_4, m_1, m_2, \underline{m_3})$

$S_M(m_1) = W_3$, $Succ(m_1) = m_2$
$Succ(m_2) = m_3$
$Succ(m_3) = m_4$
$Succ(m_4) = m_1$

Thus, $P = (m_1, w_2), (m_2, w_3), (m_3, w_4), (m_4, w_1)$

Elimination of a rotation

- In a matching M, once we pick a rotation ρ, we perform the following transformation of the matching.
- If a
  man m or a woman w is not in ρ, then they retain their partner.

$\Rightarrow M_2 =$

$\Rightarrow M/\rho = (m_1, w_3), (m_2, w_4), (m_3, w_1), (m_4, w_2)$

- Else, the partner of man m_i becomes w_{i+1} and partner of m_{r-1} partner becomes w_0.
- This is termed as an elimination of a rotation, denoted by M/ρ.
- Question. Can you show why this elimination leads to a stable matching?

Ans

- The only possible blocking pairs are those which are a part of the previous matching, but not a part of the current one, i.e they are a part of the rotation ρ.
- For the pairs which are a part of the rotations, by definition, w_i prefers the partner in M/ρ over its partner in M. Thus, there are no blocking pairs.
- We can also say that M/ρ is dominated by M, as all women have a partner they prefer at least as much as the partner found in M, and vice versa for the men.

Elimination of a rotation, continued - 1

- Lemma: If M is a any other stable matching that is not woman optimal, then some rotation ρ must be exposed in M.
- Proof - Suppose a man m has 2 different partners in M_z and M. Then, man m will prefer M to M_z. Thus, $s_M(m)$ exists. And as $s_M(m)$ exists, if $succ_M(m)$ exists too, and we call it m'. We can continue this, and say $succ_M(m')$ must also exist.
- If the above is untrue, then m' and $s_M(m)$ must be partners in the woman optimal stable matching (M_z). Thus, m prefers $s_M(m)$ to his partner in M_z, and $s_M(m)$ prefers m over her partner in M_z, which is a contradiction.
- This leads us to an interesting idea that every matching, via a series of rotations, can lead us to the woman optimal stable matching.

How to detect a rotation?

- After computing a reduced preference profile, we can compute a rotation by doing the following.
- The previous slide gives us an interesting idea on the same. For each man m who is not matched to his partner in M_z, create the following graph: we draw a directed edge from m to $s_M(m)$

How to detect a rotation?

For our P.P,

M_2 = green

Men's Preferences

$m_1 : (w_2, w_3, w_1, w_4)$
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Women's Preferences

$w_1 : (m_1, m_2, m_3, m_4)$
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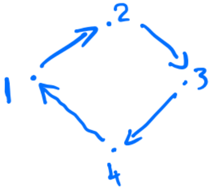

We have a rotation!

Stable edge

- For a pair (m,w), if it is a part of some stable matching, then we call it a stable edge.
- Before we ask what is a stable edge, let us show that some edges can never be a stable edge.
- In some rotation ρ, for some m_j, if a woman w is between w_i and w_{i+1} in his preference list, then m_i and w can never be a part of any stable matching.
- Proof: Let this (m_j,w) be a part of some matching M'. By definition, w prefers her partner over m_j in M, and m prefers his partner in M over m_j. Thus, (m_j,w) can never be a part of any stable matching.

Stable edges and rotations

- Lemma: For an edge to be stable, it must be either a part of the Woman optimal stable matching, or be present in some rotation.
- Proof: First, check if the edge is a part of the woman optimal matching. If not, check if it a part of any rotations. If it is not a part of any rotation, then it cannot be a part of any stable matching.

Robust matchings

- A d-robust matching is a matching which is still stable, even upon making upto d swaps of adjacent members of a preference profile.
- It is a very interesting topic, and knowledge of stable edges help us here.
- Using rotations, and $O(N^4)$ pre-processing time, we can compute if any 2 stable edges are present in some stable matching in $O(1)$ time.

Robust matchings

For our P.P,

M_2 = green

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Women's Preferences

$w_1 : (m_1, m_2, m_3, m_4)$
$w_2 : (m_2, m_3, m_4, m_1)$
$w_3 : (m_3, m_4, m_1, m_2)$
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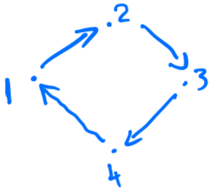

We have a rotation!

Stable quadruples

- a stable quadruple satisfies the following conditions.
- U*,W* are acceptable to each other.
- (U*,W) and (U,W*) exist in some stable matching.
- If a stable quadruple exists, we can perform the following swaps - Pull W* just ahead of W in U* preference list, and vice versa. Then, U* and W* will be a blocking pair of the aforementioned matching.
- This is interesting, as if a stable quadruple exists whose swap distance is less than d with respect to the matching, then the matching is not d-robust.
- Trivially, we can check all O(N^{2d}) quadruples, but with the help of rotations, we can check only a subset of them, and time complexity reduces to O(N^4)

History

- Seminal Paper - Gale and Shapley - 1962 - All basic stable matching results
- The distributive lattice nature and related theorems - Knuth - 1976
- Representation of Rotations - Irving and Leather - 1986
- Book from which did the presentation - Gusfield and Irving - 1989