CS6130: Paper Presentation Report

Yash Bhagwat and Krishna S.

IIT Madras: ep19b021@smail.iitm.ac.in, ee19b147@smail.iitm.ac.in

1 Introduction

- In our presentation, we explore the **lattice of stable matchings** and its **properties**. For this purpose, we discuss the properties of a general **distributive lattice**, followed by how stable matchings relate to distributive lattices. We also look at **rotations**, how to identify them and how they can be used to transform matchings. Finally, we also explore **stable edges** and how rotations can be used to detect such edges as well.
- The concepts explored in this paper are interesting as they describe the relations that exist between different stable matchings which can be exploited. They serve as the building blocks in providing algorithms for many useful problems, such as the robust matchings problem. The concept of rotations can be used to **identify all stable pairs**, and even identify the set of all stable matchings in $\mathrm{O}(n^2 + |M|n)$ time where $|M|$ is the number of stable matchings.

2 Intuition / Examples / Your Observations

- We have studied concepts like man-optimal and women-optimal stable matchings in class. We also know that there may be multiple stable matchings for a preference profile.
- Some of the matchings (say M) are better for every man than some other matchings (say M'). We showed that matching M is worse for every women than matching M'.
- Thus, we intuitively feel that these matchings can form a poset under the above relation which we called as the dominance relation.
- Similarly we can intuitively understand rotations as a cycle of men who swap partners cyclically and all end up with a worse preferred woman whereas all of the matched women partners of these men each get a better partner.
- Thus, it can be understood as moving away from man-optimality and towards woman-optimality in the above mentioned lattice of stable matchings. So, rotations are also a way to traverse the lattice of stable matchings. We showed this by showing that every matching which is not woman-optimal has a rotation. On eliminating this rotation, we get another stable matching which will be dominated by the current stable matching.
- In our presentation, we have used examples to illustrate the above ideas.

3 Results and Main Techniques

In our presentation, we proved a series of interesting lemmas to explain distributive lattice of stable matchings and rotations.

Regarding Distributive Lattices:

- To explain the tradeoff between two stable matchings, we show that for two stable matchings M and M', number of people who prefer M over M' is same as the number of people who prefer M' over M.
- We describe the dominance relation between matchings to build the distributive lattice of matchings.
- We do this by defining the meet, join for the lattice of stable matchings and finally prove the distributive property for the same.

Regarding Rotations:

- We define $s_M(m)$ as the first partner of m, after the matched partner of m, who herself prefers m over her own matched partner and $succ_M(m)$ as the matched partner of $s_M(m)$.
- Using the definition of successors, we define a rotation as a series of ordered pairs u_i, w_i in a matching M, such that these pairs themselves are a part of the matching M and succ(u_i)=w$_{i+1}$ and succ(u_{r-1})=w$_0$ for a set of r such pairs.
- We show how to eliminate a rotation in a stable matching M and how it is used to transform a matching. We prove that upon transformation, the new structure is indeed a matching which is stable.
- We show that any matching that is not woman optimal, must have at least one rotation exposed in it.
- We use the above, to also realize that any matching can be converted into the woman optimal stable matching via a series of elimination of rotations.
- We show how to detect a rotation in a matching, using a reduced preference profile in $O(n)$ time.
- We define stable edges as an edge which is present in some stable matching. Then, we show which edges can never be a part of any stable matching.
- Finally, we show that an edge can only be a stable edge if it is either a part of the woman optimal stable matching or it is a part of some rotation.
- We wrap up the presentation by acknowledging some other uses of rotations, like how it can be used to identify robust matchings.

Regarding reduced preference lists:

- For efficiently finding a rotation in $O(n)$ time, we mentioned that we could use reduced preference lists.
- A reduced preference list is a modification of the original preference list, where we remove edges which will certainly not appear in any matching dominated by a matching M.
- The key idea is that when a women gets a proposal by man m, she deletes all the men preferred below that man that are not already deleted from her preference list.
- Now we need to delete this woman from the man m's preference list also, to effectively maintain a reduced preference list.
- We would need to spend $O(n)$ time to trivially find the woman in the man's list and delete her.
- But if we use a doubly linked list data structure to store the preferences and then use two n by n matrices which contains pointers such that A[i][j] points to woman w_j in man m_i's preference list, we can have a quick deletion and maintain reduced preference lists in $O(n)$ time.

4 Key take aways from the paper

Here are some of the key take aways that we observed from the paper.

- For a given preference profile, there exists a partial ordering on all the stable matchings present, where M_0 (man-optimal) is the the minimal of these matchings, while M_z (woman-optimal) is the maximal of these matchings.
- This set of stable matchings and their partial ordering is used to define the distributive lattice of stable matchings.
- The concept of rotations and computation using the definition of successors and reduced preference lists.
- Transformation of matchings via elimination of rotations.
- Every matching that is not the M_z will have some rotation exposed, and so every matching can be eventually transformed into M_Z via a series of eliminations.
- The definition of stable edges, and what can and cannot be a stable edge.

5 Frame a question

(Question) A **fixed edge** is a stable edge, which appears in **every** stable matching. Can you, in $O(n^2)$ time, identify every fixed edge present with respect to some preference profile?
(Answer)

- We know that for an edge to be stable, it must appear in at least some rotation or M_z itself.
- As we are looking for fixed edges, it must be a part of M_z. It also cannot be a part of any rotation, as upon elimination of such a rotation, we would obtain a matching which does not contain the required fixed edge.
- Thus, we can realize that for all fixed edges, they must be a part of both the man and woman optimal stable matching, both of which can be computed in $O(n^2)$ time, and performing an intersection of these 2 matchings gives us the list of all fixed edges.