Introduction to Linear Algebra (MTH 113M)

Assignment 1, Jan 7, 2025

Only problems 2, 3, 5, 6, 7 will be discussed in the tutorial.

1. Given $A \in M_n(F)$, show that A can be written uniquely as

$$A = B + C$$

where B is a symmetric matrix and C is a skew-symmetric matrix.

2. Let $A \in M_{m,n}(F)$, $B \in M_{n,p}(F)$, $C \in M_{p,q}(F)$. Show that

- $(AB)C = A(BC)$,
- $(AB)^T = B^T A^T \quad \text{and} \quad (AB)^* = B^* A^*$.

3. Let $A, B \in M_{m,n}(\mathbb{R})$ such that $Ax = Bx$ for all $x \in \mathbb{R}^n$. Then prove that $A = B$.

4. Let A be an upper-triangular matrix. If $A^*A = AA^*$ then prove that A is a diagonal matrix. A similar result holds if A is a lower-triangular matrix.

5. A square matrix $A \in M_n(F)$ is said to be nilpotent if $A^k = 0$ for some natural number k.

- Give example of non-zero nilpotent matrices,
- Prove that if A is nilpotent then $A + \mathbb{I}_n$ is an invertible matrix, where $\mathbb{I}_n$ is the identity matrix of order n.

6. Let $A \in M_n(\mathbb{C})$ be a Hermitian matrix i.e. $A^* = A$.

- Then the diagonal entries of A are necessarily real numbers.
- For each $B \in M_n(\mathbb{C})$ prove that B^*AB is a Hermitian matrix.
- Further if $A^2 = 0$ then show that $A = 0$.
- Then x^*Ax is a real number, for any $x \in M_{n,1}(\mathbb{C})$.

7. Given a square matrix $A \in M_n(F)$, the *trace* of A denoted by $\mathrm{Tr}(A)$, is defined to be the sum of all diagonal entries, i.e.,

$$\mathrm{Tr}(A) := \sum_{i=1}^{n} a_{i,i}.$$

- Let $B \in M_n(F)$, then show that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$
- Show that if A is invertible then $\mathrm{Tr}(ABA^{-1}) = \mathrm{Tr}(B)$.