Fourier Transforms

The main drawback of Fourier series is, it is only applicable to periodic signals. There are some naturally produced signals such as nonperiodic or aperiodic, which we cannot represent using Fourier series. To overcome this shortcoming, Fourier developed a mathematical model to transform signals between time (or spatial) domain to frequency domain & vice versa, which is called 'Fourier transform'.

Fourier transform has many applications in physics and engineering such as analysis of LTI systems, RADAR, astronomy, signal processing etc.

Deriving Fourier transform from Fourier series

Consider a periodic signal f(t) with period T. The complex Fourier series representation of f(t) is given as

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T_0}kt} \ldots\ldots (1)$$

Let $\frac{1}{T_0} = \Delta f$, then equation 1 becomes

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k\Delta ft} \ldots\ldots (2)$$

but you know that

$$a_k = \frac{1}{T_0} \int_{t_0}^{t_0+T} f(t)e^{-jk\omega_0 t} dt$$

Substitute in equation 2.

(2) $\Rightarrow f(t) = \Sigma_{k=-\infty}^{\infty} \frac{1}{T_0} \int_{t_0}^{t_0+T} f(t)e^{-jk\omega_0 t} dt\, e^{j2\pi k\Delta ft}$

Let $t_0 = \frac{T}{2}$

$$= \Sigma_{k=-\infty}^{\infty} [\int_{\frac{-T}{2}}^{\frac{T}{2}} f(t)e^{-j2\pi k\Delta ft} dt]\, e^{j2\pi k\Delta ft}. \Delta f$$

In the limit as $T \to \infty, \Delta f$ approaches differential $df, k\Delta f$ becomes a continuous variable f , and summation becomes integration

$$f(t) = lim_{T\to\infty} \left\{ \Sigma_{k=-\infty}^{\infty} [\int_{\frac{-T}{2}}^{\frac{T}{2}} f(t)e^{-j2\pi k\Delta ft} dt]\, e^{j2\pi k\Delta ft}. \Delta f \right\}$$

$$= \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} f(t) e^{-j2\pi ft} dt] e^{j2\pi ft} df$$

$$f(t) = \int_{-\infty}^{\infty} F[\omega] e^{j\omega t} d\omega$$

$$\text{Where } F[\omega] = [\int_{-\infty}^{\infty} f(t) e^{-j2\pi ft} dt]$$

Fourier transform of a signal

$$f(t) = F[\omega] = [\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt]$$

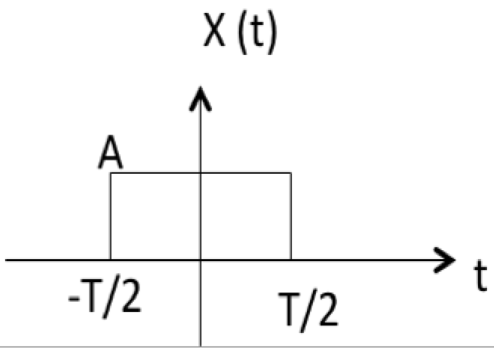
Inverse Fourier Transform is

$$f(t) = \int_{-\infty}^{\infty} F[\omega] e^{j\omega t} d\omega$$

Fourier Transform of Basic Functions

Let us go through Fourier Transform of basic functions:

FT of GATE Function



$$F[\omega] = ATSa(\frac{\omega T}{2})$$

FT of Impulse Function

$$FT[\omega(t)] = [\int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt]$$

$$= e^{-j\omega t} \mid t = 0$$

$$= e^{0} = 1$$

$$\therefore \delta(\omega) = 1$$

FT of Unit Step Function:

$$U(\omega) = \pi\delta(\omega) + 1/j\omega$$

FT of Exponentials

$$e^{-at}u(t) \xleftrightarrow{\text{F.T}} 1/(a + j\omega)$$

$$e^{-at}u(t) \xleftrightarrow{\text{F.T}} 1/(a + j\omega)$$

$$e^{-a\,|\,t\,|} \xleftrightarrow{\text{F.T}} \frac{2a}{a^2+\omega^2}$$

$$e^{j\omega_0 t} \xleftrightarrow{\text{F.T}} \delta(\omega - \omega_0)$$

FT of Signum Function

$$sgn(t) \xleftrightarrow{\text{F.T}} \frac{2}{j\omega}$$

Conditions for Existence of Fourier Transform

Any function f(t) can be represented by using Fourier transform only when the function satisfies Dirichlet's conditions. i.e.

- The function f(t) has finite number of maxima and minima.
- There must be finite number of discontinuities in the signal f(t),in the given interval of time.
- It must be absolutely integrable in the given interval of time i.e.

$$\int_{-\infty}^{\infty} |\, f(t)|\, dt < \infty$$

Discrete Time Fourier Transforms (DTFT)

The discrete-time Fourier transform (DTFT) or the Fourier transform of a discrete–time sequence x[n] is a representation of the sequence in terms of the complex exponential sequence $e^{j\omega n}$.

The DTFT sequence x[n] is given by

$$X(\omega) = \Sigma_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \ \ldots\ldots (1)$$

Here, X(ω) is a complex function of real frequency variable ω and it can be written as

$$X(\omega) = X_{re}(\omega) + jX_{img}(\omega)$$

Where $X_{re}(\omega)$, $X_{img}(\omega)$ are real and imaginary parts of X(ω) respectively.

$$X_{re}(\omega) = |\,X(\omega)|\cos\theta(\omega)$$

$$X_{img}(\omega) = |\, X(\omega)| \sin\theta(\omega)$$

$$|X(\omega)|^2 = |\, X_{re}(\omega)|^2 + |\, X_{im}(\omega)|^2$$

And X(ω) can also be represented as $X(\omega) = |\, X(\omega)| e^{j\theta(\omega)}$

Where $\theta(\omega) = arg X(\omega)$

$|\, X(\omega)|, \theta(\omega)$ are called magnitude and phase spectrums of X(ω).

Inverse Discrete-Time Fourier Transform

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega \ \ldots\ldots (2)$$

Convergence Condition:

The infinite series in equation 1 may be converges or may not. x(n) is absolutely summable.

$$\text{when } \sum_{n=-\infty}^{\infty} |\, x(n)| \, < \infty$$

An absolutely summable sequence has always a finite energy but a finite-energy sequence is not necessarily to be absolutely summable.

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