Closure properties of regular language

1) Union
2) Concatnation
3) Kleen Star
4) Complement
5) Intersection
6) Difference
7) Reversal

Proof:

1) Union: Let $L_1, L_2 \in$ {set of regular language}

$L_1 \cup L_2 = L_3$

$r_1 + r_2 = r_3$.

r_1 = regular expression corresponding to L_1

r_2 = " " " " L_2

Since there exist regular expression for L_3 i.e. r_3. Hence $L_3 \in$ {set of regular language}

(2) Concatination:

$L_1 . L_2 = L_3$

$r_1 . r_2 = r_3$

Hence $L_3 \in$ {set of regular language}

(3) Kleen* star:

$(L_1)^* = L_2$

$(r_1)^* = r_2$

Hence $L_2 \in$ {set of regular language}

(4) Complement

$\overline{L_1} = L_2$

Design DFA for L_1.

Do complement of DFA by converting Final state to Non final and vice versa

$L_1 \exists$ DFA $\exists$ $\overline{\text{DFA}}$

Corresponding language for $\overline{\text{DFA}}$ is L_2

Since we can design DFA for L_2. Hence L_2 is also regular language

(5) $L_1 \cap L_2 = L_3$

$L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$

Since union and complement of regular language is also regular.

Hence $L_1 \cap L_2$ is also regular.

(6) $L_1 - L_2 = L_3$

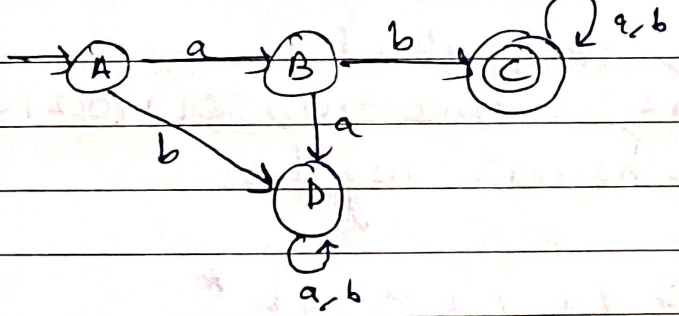
$L_1 - L_2 =$ ~~[illegible]~~ $L_1 \cap \overline{L_2}$

Since complement and intersection of regular language is also regular. Hence L_3 is regular.

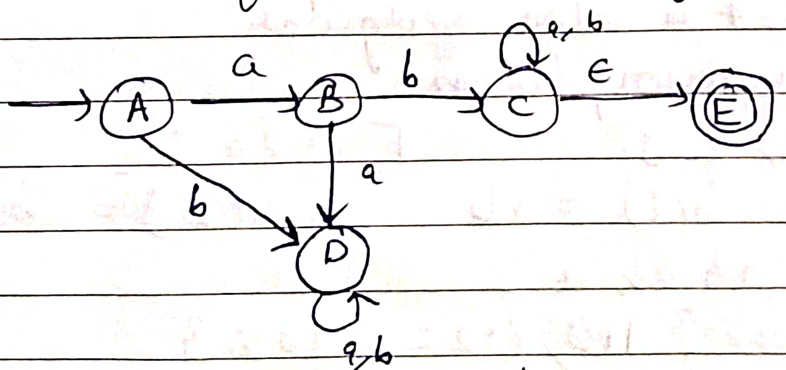
(7) Reversal:

Let L = starting with ab over $\Sigma\{a,b\}$.

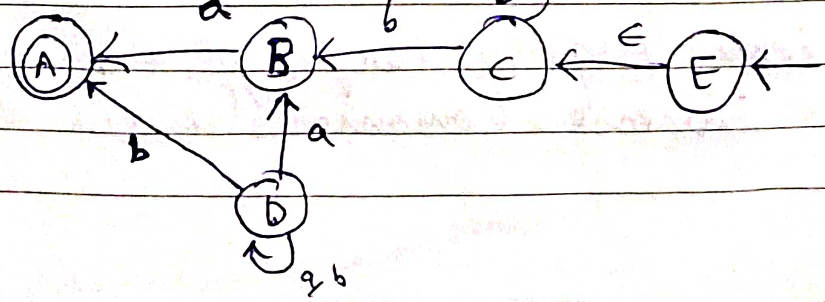
Finite Automata:



Convert final to non final with ϵ moves



Now final to initial and vice versa and change direction of arrows



Language corresponding to this finite automata is L^R

Hence regular language are closure under reversal.

(8) Homomorphism :

Suppose Σ and Γ (tou) are alphabets. Then a function $h : \Sigma \rightarrow \Gamma^*$ is called homomorphism

1. $h(\epsilon) = \epsilon$
2. $h(xyz) = h(x) \cdot h(y) \cdot h(z)$

eg. $\Sigma = \{a, b\}$ $\quad \Gamma = \{0, 1, 2\}$

$h(a) = 010$

$h(b) = 102$

$\Rightarrow h(abba) = h(a) \cdot h(b) h(b) h(a)$

$= 010\ 102\ 102\ 010$

Let $L_1 = \{ab, aba, abb\}$

$\Rightarrow L_2 = \{010102, 010102010, 010102102\}$

is also finite, hence regular.

~~Hence~~ Let RE for $L_3 = ab^*$

then RE for $L_4 = 010(102)^*$

hence L_4 is also regular.

(9) Inverse Homomorphism :

$\Sigma = \{0, 1, 2\}$ $\quad \Gamma = \{a, b\}$

$h(0) = a$ $\quad h(1) = ab$ $\quad h(2) = ba$

(1) $L_1 = \{ab\, ab\, a\}$

$h^{-1}(L_1) = \{110, 022, 102\}$

$$\boxed{h(h^{-1}(L)) \subseteq L}$$

Hence regular language are closure under inverse homomorphism.

(9) Substitution: Every symbol of a language is replaced by other language.

$\Sigma = \{0, 1\}$ $\qquad\qquad \Gamma = \{a, b\}$

$f(0) = a^*$ $\qquad\qquad 0 \Rightarrow a^n \mid n \geq 0$

$f(1) = bb^*$ $\qquad\qquad 1 \rightarrow bb^n \mid n \geq 0$

(1) $L = 0 + 1^*$ $\qquad\qquad b^n \mid n \geq 1$

$f(L) = a^* + (bb^*)^*$

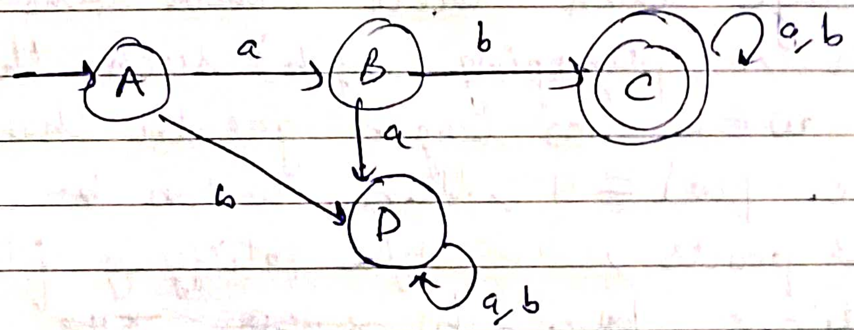
Hence we get regular expression after substitution So regular language are closure under substitution.

(10) INIT

eg. GATE

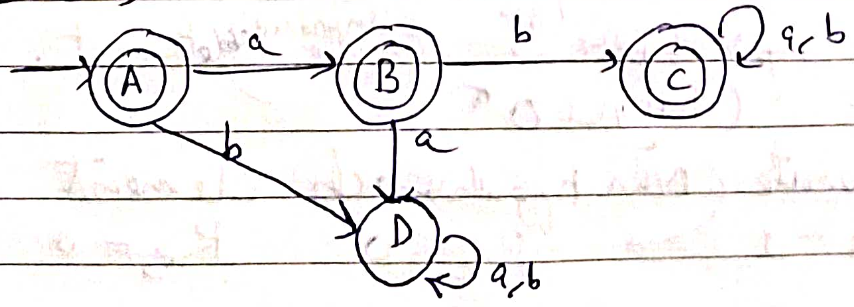
INIT will be $\{\epsilon, G, GA, GAT, GATE\}$

(1) L = set of all strings starting with ab



DFA for L

By converting all the non final states except dead state in DFA of L we get DFA of INIT(L).



DFA for INIT(L)

Hence regular language are closure under INIT.

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(11) Quotient :

$L_1/L_2 = \{ y \mid xy \in L_1$ for some $x \in L_2 \}$ Left

$L_1/L_2 = \{ x \mid xy \in L_1$ for some $y \in L_2 \}$ Right

eg.

$L_1 = \{01, 001, 101, 0111, 1101\}$

$L_2 = \{01\}$

$L_1/L_2 = \{\epsilon, 11\}$ Left

$L_1/L_2 = \{\epsilon, 0, 1, 11\}$ Right

Regular language are closure under quotient.

(12) Infinite Union and Subset :

Regular language are not closure under infinite union and subset.

Closure Property of CFL

1) Union

$L_1 \cup L_2 = CFL$

Let $L_1 = a^n b^n \mid n \geq 1$

$\Rightarrow S_1 \rightarrow a S_1 b \mid \epsilon$

and $L_2 = c^n d^n \mid n \geq 1$

$\Rightarrow S_2 \longrightarrow c S_2 d \mid \epsilon$

$S \longrightarrow S_1 \mid S_2$

$S_1 \longrightarrow a S_1 b \mid \epsilon$

$S_2 \rightarrow c S_2 d \mid \epsilon$

Hence union is closure under CFL

2) Concatenation

$S \rightarrow S_1 S_2$

$S_1 \rightarrow a S_1 b \mid \epsilon$

$S_2 \rightarrow c S_2 d \mid \epsilon$

$L_1 = a^n b^n \mid n \geq 0$

$L_2 = c^n d^n \mid n \geq 0$

$L = a^n b^n c^m d^m \mid n, m \geq 0$

Hence intersection is closure under CFL

3) Kleene Closure

Let $L_1 = a^n b^n \mid n \geq 0$

$S_1 \rightarrow a S_1 b \mid \epsilon$

$S \rightarrow S_1 S \mid \epsilon$

Hence Kleene closure is closure under CFL

4) Intersection

Intersection is not closure under CFL

$L_1 = a^n b^n c^m \mid n, m \geq 0$

$L_2 = a^m b^n c^n \mid m, n \geq 0$

$L_1 \cap L_2 = a^n b^n c^n \mid n \geq 0$

$L_1 = \{ \epsilon, ab, abc, aabb, aabbc,$
$aabbcc, aaabbb, a^3b^3c, a^3b^3c^2,$
$a^3b^3c^3, \ldots$

$L_2 = \{ \epsilon, bc, abc, bbcc, abbcc, a^2b^2c^2,$
$b^3c^3, ab^3c^3, a^2b^3c^3, a^3b^3c^3, \ldots$

$L = L_1 \cap L_2 = \{ \epsilon, abc, a^2b^2c^2, a^3b^3c^3, \ldots$

$\Rightarrow L = a^n b^n c^n \mid n \geq 0$

L_1 and L_2 are CFL, but
L is not CFL

Hence, intersection is not closure under CFL

5) Complement

Let L_1 and L_2 be two CFL

$L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$

Let's assume that complement is closure under CFL, then

$\Rightarrow$ $\overline{L_1}$ and $\overline{L_2}$ are also ... CFL

We know that union is closure under CFL

so $\overline{L_1} \cup \overline{L_2}$ is also CFL

$\Rightarrow$ $\overline{\overline{L_1} \cup \overline{L_2}}$ is also CFL

$\Rightarrow$ $L_1 \cap L_2$ is CFL

Which is FALSE

Our assumption is WRONG

Hence, complement is not closure under CFL