Pumping Lemma (For Regular Languages)

>> Pumping Lemma is used to prove that a Language is NOT REGULAR

>> It cannot be used to prove that a Language is Regular

If A is a Regular Language, then A has a Pumping Length 'P' such that any string 'S' where $|S| \geqslant P$ may be divided into 3 parts $S = x\,y\,z$ such that the following conditions must be true:

(1) $x y^i z \in A$ for every $i \geqslant 0$

(2) $|y| > 0$

(3) $|xy| \leqslant P$

To prove that a language is not Regular using PUMPING LEMMA, follow the below steps:

(We prove using Contradiction)

-> Assume that A is Regular
-> It has to have a Pumping Length (say P)
-> All strings longer than P can be pumped $|S| \geqslant P$
-> Now find a string 'S' in A such that $|S| \geqslant P$
-> Divide S into x y z
-> Show that $x\,y^i z \notin A$ for some i
-> Then consider all ways that S can be divided into x y z
-> Show that none of these can satisfy all the 3 pumping conditions at the same time
-> S cannot be Pumped == CONTRADICTION

Pumping Lemma (For Regular Languages) - EXAMPLE (Part-1)

Using Pumping Lemma prove that the language $A = \{a^n b^n \mid n \geqslant 0\}$ is Not Regular

Proof:

Assume that A is Regular

Pumping length = p

$S = \underbrace{a^p b^p}_{x\ y\ z}$ $\Rightarrow S = aaaaaaabbbbbbb$

$p = 7$

Case 1: The Y is in the 'a' part

$\underbrace{aa}_{x}\underbrace{aaaa}_{y}\underbrace{a\,bbbbbbb}_{z}$

Case 2: The Y is in the 'b' part

$\underbrace{aaaaaaa\,bb}_{x}\underbrace{bbbb}_{y}\underbrace{b}_{z}$

Case 3: The Y is in the 'a' and 'b' part

$\underbrace{aaaaa}_{x}\underbrace{aabb}_{y}\underbrace{bbbbb}_{z}$

$xy^1z \Rightarrow xy^2z$ ✗

aa aaaa aaaa a bbbbbbb

$11 \neq 7$

$xy^1z \Rightarrow xy^2z$ ✗

aaaaaaa bb bbbb bbbb b

$7 \neq 11$

$xy^1z \Rightarrow xy^2z$ ✗

aaaaa aabbaabb bbbbb

$a^n b^n$

$|XY| \leq p \quad p=7$

Pumping Lemma (For Regular Languages) EXAMPLE (Part-2)

Using Pumping Lemma prove that the language $A = \{yy \mid y \in \{0,1\}^*\}$ is Not Regular

$\underbrace{01}\underbrace{01}$

Proof:

Assume that A is Regular

Then it must have a Pumping Length = P

$S = 0^P 1 0^P 1$

$x \quad y \quad z$

$P = 7$

$\underbrace{000}_{x}\underbrace{0000}_{y}\underbrace{100000001}_{z}$

$xy^iz \Rightarrow xy^2z$

00000000000100000001

$\notin A$

$|y| > 0$

$|xy| \leq P = 7$

A is not Regular

Pumping Lemma (For Context Free Languages)

Pumping Lemma (for CFL) is used to prove that a language is NOT Context Free

Context Free Language

In formal language theory, a Context Free Language is a language generated by some Context Free Grammar.

The set of all CFL is identical to the set of languages accepted by Pushdown Automata.

Context Free Grammar is defined by 4 tuples as $G = \{ V, \Sigma, S, P \}$ where

V = Set of Variables or Non-Terminal Symbols

Σ = Set of Terminal Symbols

S = Start Symbol

P = Production Rule

Context Free Grammar has Production Rule of the form

$$A \rightarrow \alpha$$

where, $\alpha = \{V \cup \Sigma\}^*$ and $A \in V$

Pumping Lemma (For Context Free Languages)

Pumping Lemma (for CFL) is used to prove that a language is NOT Context Free

If A is a Context Free Language, then, A has a Pumping Length 'P' such that any string 'S', where $|S| \geqslant P$ may be divided into 5 pieces $S = uvxyz$ such that the following conditions must be true:

(1) $u v^i x y^i z$ is in A for every $i \geqslant 0$

(2) $|vy| > 0$

(3) $|vxy| \leqslant P$

To prove that a Language is Not Context Free using Pumping Lemma (for CFL) follow the steps given below: (We prove using CONTRADICTION)

-> Assume that A is Context Free
-> It has to have a Pumping Length (say P)
-> All strings longer than P can be pumped $|S| \geqslant P$
-> Now find a string 'S' in A such that $|S| \geqslant P$
-> Divide S into uvxyz
-> Show that $u v^i x y^i z \notin A$ for some i
-> Then consider the ways that S can be divided into uvxyz
-> Show that none of these can satisfy all the 3 pumping conditions at the same time
-> S cannot be pumped == CONTRADICTION

Pumping Lemma (for Context Free Languages) - Example (Part-1)

Show that $L = \{ a^N b^N c^N \mid N \geqslant 0 \}$ is Not Context Free

-> Assume that L is Context Free

-> L must have a pumping length (say P)

-> Now we take a string S such that $S = a^P b^P c^P$

-> We divide S into parts u v x y z

Eg. $P = 4$ So, $S = a^4 b^4 c^4$

Case I : v and y each contain only one type of symbol

$$\underbrace{a}_{u}\underbrace{aa}_{v}\underbrace{abbbbc}_{x}\underbrace{c}_{y}\underbrace{cc}_{z}$$

$u v^i x y^i z$ $(i = 2)$

$u v^2 x y^2 z$

a a a a a a b b b b c c c c c

$a^6 b^4 c^5 \notin L$

Pumping Lemma (for Context Free Languages) - Example (Part-2)

Show that $L = \{ ww \mid w \in \{0,1\}^* \}$ is NOT Context Free

-> Assume that L is Context Free
-> L must have a pumping length (say P)
-> Now we take a string S such that $S = 0^P 1^P 0^P 1^P$
-> We divide S into parts u v x y z

Eg. $P = 5$ So, $S = 0^5 1^5 0^5 1^5$

Case 1: vxy does not straddle a boundary

$$\underbrace{00000\,1}_{u}\,\underbrace{111}_{vxy}\,\underbrace{1\,00000\,11111}_{z}$$

$uv^i xy^i z$

$uv^2 xy^2 z$

0000011111110000011111

$\underbrace{0^5 1^7}_{} \neq \underbrace{0^5 1^5} \notin L$

Case 2a: vxy straddles the first boundary

$$\underbrace{000}_{u}\underbrace{00'111}_{v\ x\ y}\underbrace{11'00000'11111}_{z}$$

uv^1xy^1z

uv^2xy^2z

000 0000 1111111 00000 11111

$\underbrace{0^7\,1^7}_{\neq}\ \underbrace{0^5\,1^5}\ \notin L$

Case 2b: vxy straddles the third boundary

$$\underbrace{00000\,11111\,000}_{u}\;\underbrace{00\,111}_{v\ x\ y}\;\underbrace{11}_{z}$$

uv^2xy^2z

00000 11111 000 00001 11111 11

$\underbrace{0^5 1^5}\ \neq\ \underbrace{0^7 1^7}\ \notin L$

Case 3: vxy straddles the midpoint

$$\underbrace{0\,0\,0\,0\,0\,'\,1\,1}_{u}\;\underbrace{1\,1\,1\,'\,0\,0}_{v\;x\;y}\;\underbrace{0\,0\,0\,'\,1\,1\,1\,1\,1}_{z}$$

00000 11 11111 0000 000 11111

$$\underbrace{0^5\,1^7}\;\neq\;\underbrace{0^7\,1^5} \qquad \notin L$$