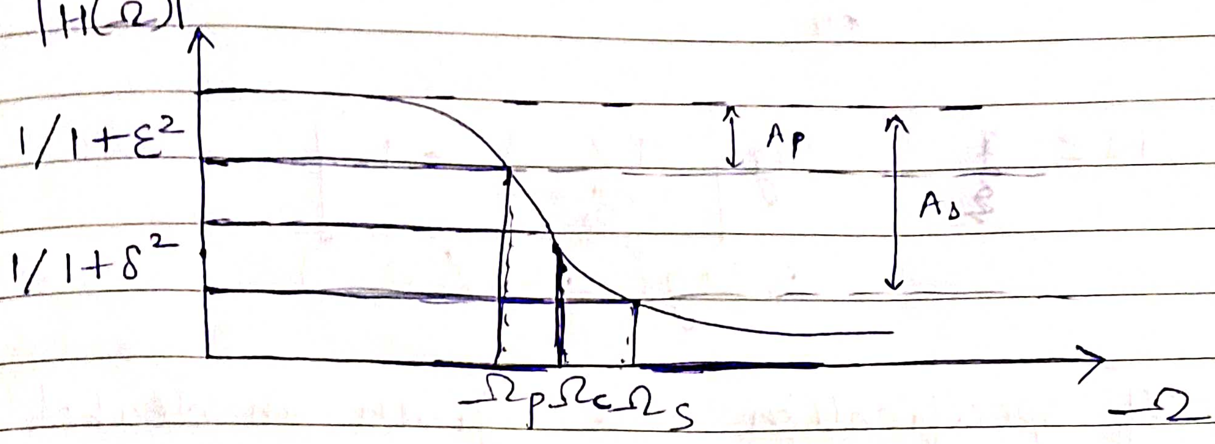
Butterworth Filter



$$|H(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

Ω_p = passband edge frequency
Ω_c = cutoff frequency
ε = Ripple related parameter in passband
δ = " " " " stopband

Steps to design Butterworth Filter

1) Obtain equivalent analog filter

Impulse Invariant

$$\Omega_p = \frac{\omega_p}{T_s}$$

$$\Omega_s = \frac{\omega_s}{T_s}$$

Bilinear

$$\Omega_p = \frac{2}{T_s} \tan\left(\frac{\omega_p}{2}\right)$$

$$\Omega_s = \frac{2}{T_s} \tan\left(\frac{\omega_s}{2}\right)$$

2) Calculate the Order of Filter

$$N \geq \frac{1}{2} \left\{ \frac{\log\left[\left(\frac{1}{A_s^2} - 1\right) / \left(\frac{1}{A_p^2} - 1\right)\right]}{\log\left(\frac{\Omega_s}{\Omega_p}\right)} \right\}$$

A_p = passband attenuation
A_s = stop band attenuation

or

$$N \geq \frac{\log\left[\left(\frac{1}{\delta_s^2}\right) - 1\right]}{2\log(\Omega_s / \Omega_c)}$$

If specification are given in decibel

$$N \geq \frac{1}{2} \frac{\log\left[\frac{10^{0.1 A_s} - 1}{10^{0.1 A_p} - 1}\right]}{\log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

3) Now Calculate cutoff frequency Ω_c

Impulse Invariance

$$\Omega_c = \frac{\omega_c}{T_s}$$

If ω_c is not given

$$\Omega_c = \frac{\Omega_p}{\left(\frac{1}{A_p^2} - 1\right)^{1/2N}}$$

Bilinear

$$\Omega_c = \frac{2}{T_s} \tan\left(\frac{\omega_c}{2}\right)$$

If Specification is in dB

$$\Omega_c = \frac{\Omega_p}{10^{0.1 A_p} - 1}$$

4) Calculate Poles

$$P_K = \pm \Omega_c e^{j(N+2k+1)\pi / 2N}$$

$k = 0, 1, 2, 3, \ldots N-1$

we select poles where $Re(P_K) < 0$

4) $$H(s) = \frac{\Omega_c^N}{(s-P_1)(s-P_2)\ldots}$$

5) Convert to $H(z)$

Impulse Invariance

$$H(s) = \sum_{K=1}^{N} \frac{C_K}{s - P_K}$$

$$H(z) = \sum_{K=1}^{N} \frac{C_K}{1 - e^{P_K T} z^{-1}}$$

Bilinear Transformation

$$s = \frac{2}{T_s}\left[\frac{z-1}{z+1}\right]$$

eg.