# Radix 2 DIT FFT Algorithm

DIT → Decimation in Time

FFT → Fast Fourier Transform

Let $x(n)$ be a sequence of length N

$x(n) = \{x(0), x(1), x(2), \dots x(N-1)\}$

even indexed seq → $\{x(0), x(2), x(4), \dots x(N-2)\}$

odd " " → $\{x(1), x(3), x(5), \dots, x(N-1)\}$

We know that

$$X(K) = \sum_{n=0}^{N-1} x(n) W_N^{Kn} \quad ; \quad 0 \le K \le N-1 \quad \text{— (1)}$$

where $W_N = e^{-j\frac{2\pi}{N}}$

eqn (1) → Decimation → even & odd indexed seq.

$$X(K) = \sum_{\substack{n=0 \\ n \to \text{even}}}^{N-2} x(n) W_N^{Kn} + \sum_{\substack{n=0 \\ n \to \text{odd}}}^{N-1} x(n) W_N^{Kn} \quad \text{— (2)}$$

Let $n = 2r$ in 1st term and $n = 2r+1$ in 2nd term

$$X(K) = \sum_{r=0}^{\frac{N}{2}-1} x(2r) W_N^{K(2r)} + \sum_{r=0}^{\frac{N}{2}-1} x(2r+1) W_N^{K(2r+1)}$$

Let $\boxed{g(r) = x(2r)}$

and $\boxed{h(r) = x(2r+1)}$

$W_N = e^{-j\frac{2\pi}{N}} \Rightarrow W_N^2 = e^{-j\frac{2\pi \times 2}{N}} = e^{-j\frac{2\pi}{N/2}} = W_{N/2}$

$$\Rightarrow X(K) = \sum_{r=0}^{\frac{N}{2}-1} g(r) W_{N/2}^{Kr} + W_N^K \sum_{r=0}^{\frac{N}{2}-1} h(r) W_{N/2}^{Kr}$$

$$\Rightarrow \boxed{X(K) = G(K) + W_N^K H(K) \; ; \; 0 \le K \le \frac{N}{2} - 1}$$

$G(K)$ and $H(K)$ are $\frac{N}{2}$ point DFT of $g(r)$ and $h(r)$ respectively with period $\frac{N}{2}$

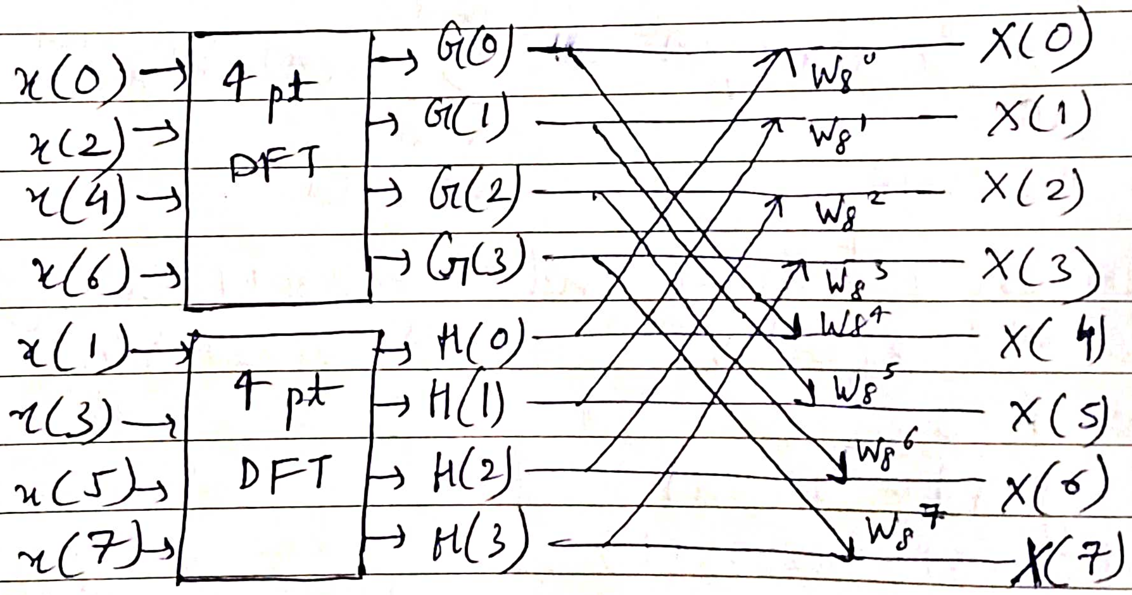
$$\boxed{X(K) = G\left(K - \frac{N}{2}\right) + W_N^K H\left(K - \frac{N}{2}\right) \; ; \; \frac{N}{2} \le K \le N-1}$$

If $N = 8$, then

$$X(0) = G(0) + W_8^0 H(0)$$
$$X(1) = G(1) + W_8^1 H(1)$$
$$X(2) = G(2) + W_8^2 H(2)$$
$$X(3) = G(3) + W_8^3 H(3)$$

$$X(4) = G(0) + W_8^4 H(0)$$
$$X(5) = G(1) + W_8^5 H(1)$$
$$X(6) = G(2) + W_8^6 H(2)$$
$$X(7) = G(3) + W_8^7 H(3)$$

Signal Flow Graph




★ $\frac{N}{2}$ point DFT to $\frac{N}{4}$ point DFT

$$G(k) = \sum_{r=0}^{\frac{N}{2}-1} g(r)\, W_{N/2}^{kr} \quad ; \quad 0 \le k \le \frac{N}{2} - 1 \qquad -②$$

$$g(r) = \{ g(0), g(1), g(2), \dots\ g(\tfrac{N}{2}-1) \}$$

eg eqn (2) → Decimation → even & odd indexed seq

$$G(K) = \sum_{r=0}^{\frac{N}{2}-2} g(r) W_{N/2}^{Kr} + \sum_{r=0}^{\frac{N}{2}-1} g(r) W_{N/2}^{Kr}$$

r is even r is odd

Put $r = 2l$ in 1st and $r = 2l+1$ in 2nd

$$G(K) = \sum_{l=0}^{\frac{N}{4}-1} g(2l) W_{N/2}^{K(2l)} + \sum_{l=0}^{\frac{N}{4}-1} g(2l+1) W_{N/2}^{K(2l+1)}$$

Let $a(l) = g(2l)$
and $b(l) = g(2l+1)$

$W_{N/2} = e^{-j\frac{2\pi}{N/2}}$ ∴ $W_{N/2}^{2} = e^{-j\frac{2\pi}{N/2} \cdot 2} = e^{-j\frac{2\pi}{N/4}} = W_{N/4}$

$$\Rightarrow G(k) = \sum_{l=0}^{\frac{N}{4}-1} a(l) W_{N/4}^{Kl} + W_{\frac{N}{2}}^{K} \sum_{l=0}^{\frac{N}{4}-1} b(l) W_{N/4}^{Kl}$$

$$\boxed{G(K) = A(K) + W_{N/2}^{K} B(K); \quad 0 \le K \le \frac{N}{4} - 1}$$

$A(K)$ and $B(K)$ are $\frac{N}{4}$ point DFT of $a(l)$ and $b(l)$ respectively with period $\frac{N}{4}$

$$\boxed{G(K) = A\left(K - \frac{N}{4}\right) + W_{N/2}^{K} B\left(K - \frac{N}{4}\right); \quad \frac{N}{4} \le K \le \frac{N}{2} - 1}$$

Similarity

$$H(K) = C(K) + W_{N/2}^{K} D(K); \quad 0 \le K \le \frac{N}{4} - 1$$

$$H(K) = C\left(K - \frac{N}{4}\right) + W_{N/2}^{K} D\left(K - \frac{N}{4}\right); \quad \frac{N}{4} \le K \le \frac{N}{2} - 1$$

If $N = 8$ then

$G(0) = A(0) + W_4^0 B(0)$

$G(1) = A(1) + W_4^1 B(1)$

$G(2) = A(0) + W_4^2 B(0)$

$G(3) = A(1) + W_4^3 B(1)$

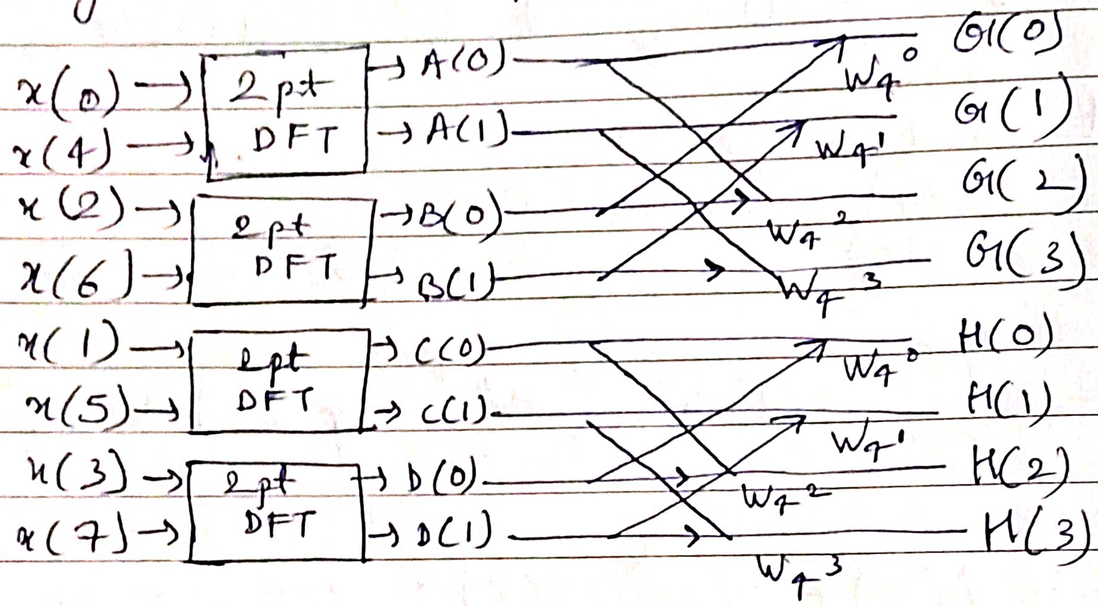
$H(0) = C(0) + W_4^0 \, D(0)$

$H(1) = C(1) + W_4^1 \, D(1)$

$H(2) = C(0) + W_4^2 \, D(0)$

$H(3) = C(1) + W_4^3 \, D(1)$

Signal Flow Graph




* $\frac{N}{4}$ point DFT to $\frac{N}{8}$ point DFT

$$A(K) = \sum_{n=0}^{N/4-1} a(n) \, W_{N/4}^{Kn} \quad ; \; 0 \le K \le \frac{N}{4} - 1$$

For $K = 8$

$$A(K) = \sum_{n=0}^{1} a(n) \, W_2^{Kn} \quad ; \; 0 \le K \le 1$$

$$A(0) = a(0) + W_2^0 \, a(1)$$

$$A(1) = a(0) + W_2^1 \, a(1)$$

Since

$a = \{ x(0), \; x(4) \}$

$b = \{ x(2), \; x(6) \}$

$c = \{ x(1), \; x(5) \}$

$d = \{ x(3), \; x(7) \}$

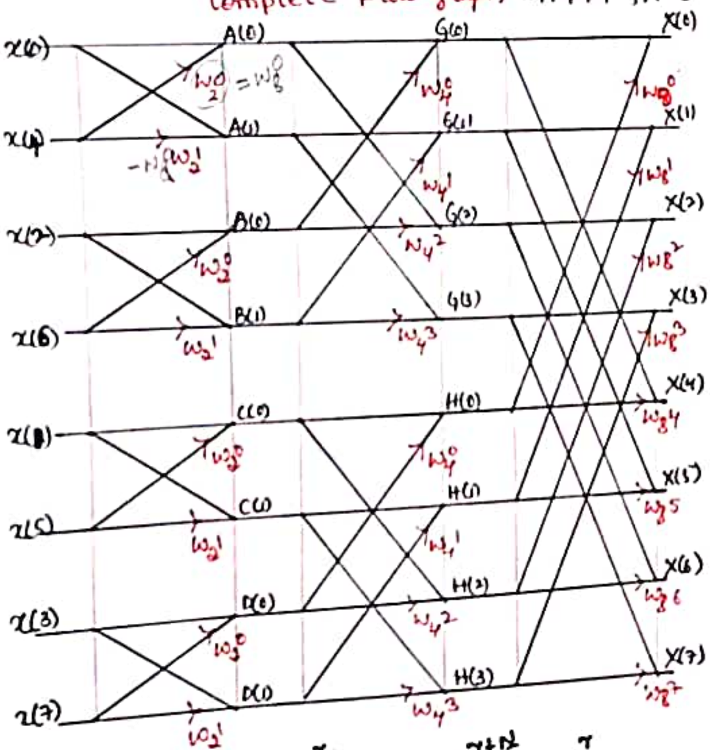
$$A(0) = x(0) + W_2^0\, x(4)$$
$$A(1) = x(0) + W_2^1\, x(4)$$

Similarly,

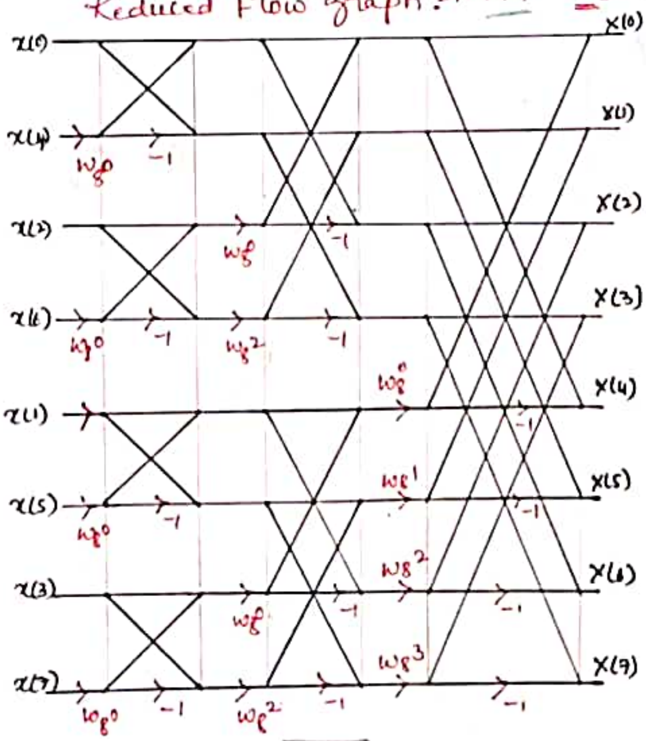
$$B(0) = x(2) + W_2^0\, x(6)$$
$$B(1) = x(2) + W_2^1\, x(6)$$
$$C(0) = x(1) + W_2^0\, x(5)$$
$$C(1) = x(1) + W_2^1\, x(5)$$
$$D(0) = x(3) + W_2^0\, x(7)$$
$$D(1) = x(3) + W_2^1\, x(7)$$

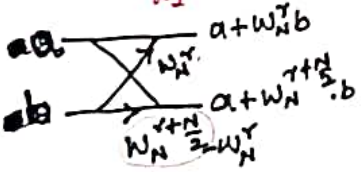
Property : $W_N^{k+\frac{N}{2}} = -W_N^k$

Complete Flow graph DIT-FFT ; N=8




Reduced Flow graph DIT-FFT N=8




$W_N^{r+\frac{N}{2}} = -W_N^r$

↳ Symmetry


**This is also known as BUTTERFLY DIAGRAM**

# Computational Efficiency of fft over dft:

Direct Computation of DFT

no of complex addion $= N(N-1)$

no of complex multiplication $= N^2$

Radix-2 FFT

no of complex addion $= N \log_2 N$

no of complex multipliaction $= \frac{N}{2} \log_2 N$

% Saving in Add. $= 100 - \frac{\text{no of addition FFT}}{\text{no of adition in DFT}} \times 100$

% saving in mul. $= 100 - \frac{\text{no of mul. FFT}}{\text{no of mul. DFT}} \times 100$

Ex:- $N = 1024$

Direct Computation of DFT

no of complex addions $= 1024(1024-1) = 1047552$

no of complex mul. $= (1024)^2 = 1048576$

Radix-2 FFT.

no complex add $= N \log_2 N = 1024 \log_2 1024$

$\log_2 x = \frac{\ln x}{\ln 2}$

$= 1024 \frac{\ln 1024}{\ln 2} = 10240$

no complex mul $= \frac{1024}{2} \log_2 1024$

$= 512 \frac{\ln 1024}{\ln 2} = 5120$

% Saving in Add $= 100 - \left[\frac{10240}{1047552}\right] \times 100$

$= 99\%$

% Savin in mul $= 100 - \left[\frac{5120}{1048576}\right] \times 100$

$= 99.5\%$

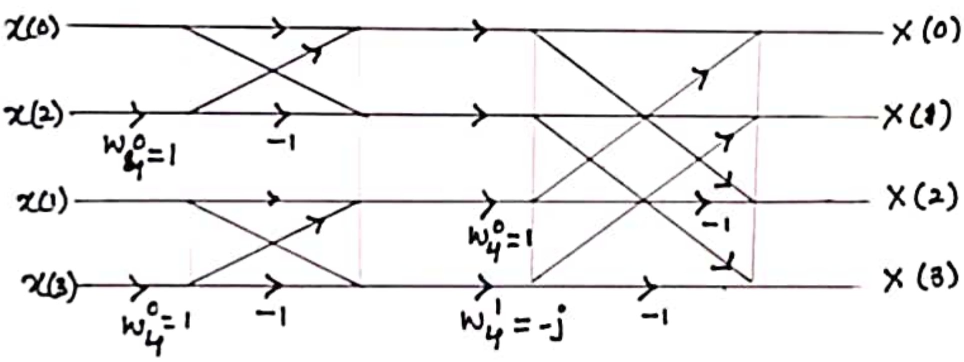
Given $x(n) = \{0, 1, 2, 3\}$, find $X(K)$ using DIT-FFT Algorithm.

$\therefore N = 4$

bit reversal

$4 = 2^{2}$

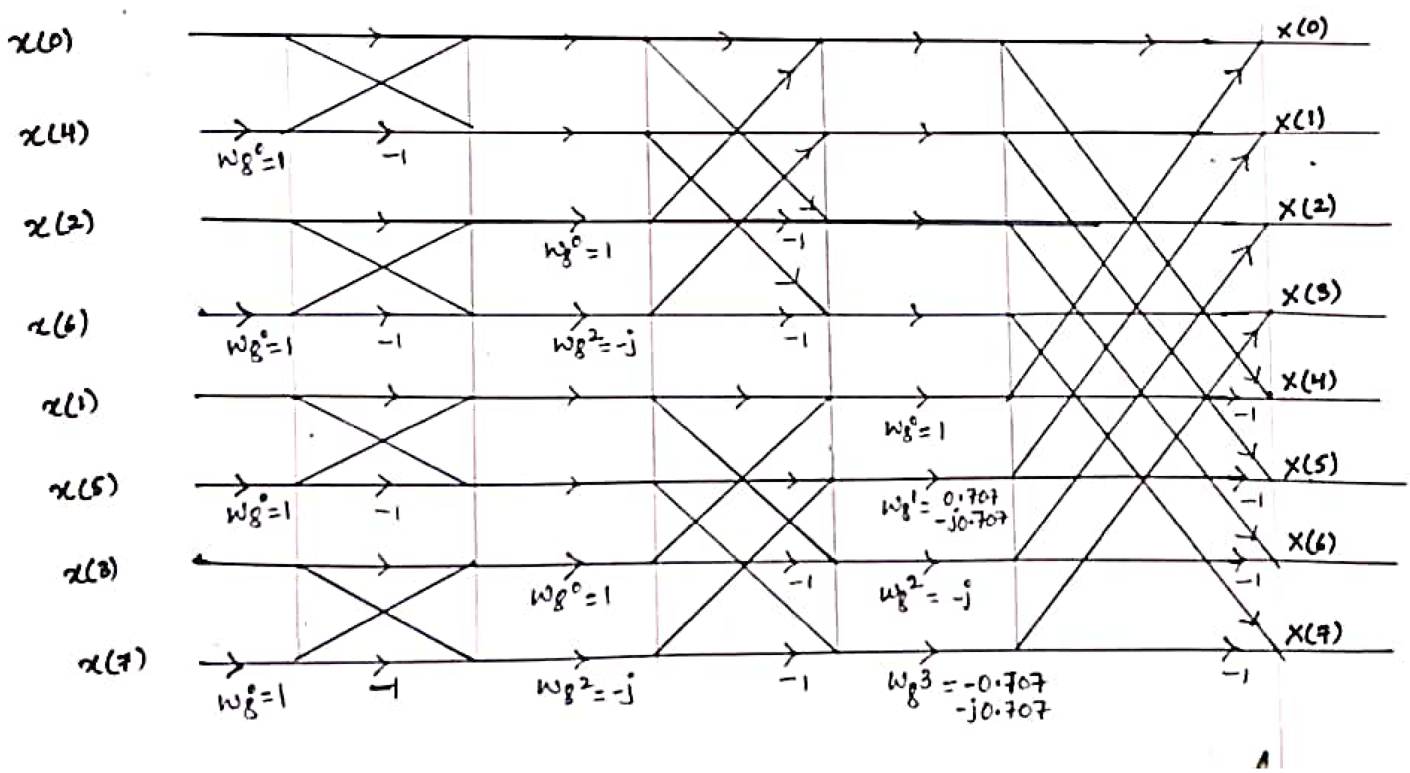
| | BR |
|---|---|
| $0 \rightarrow 00$ | $00 \rightarrow 0$ ✓ |
| $1 \rightarrow 01$ | $10 \rightarrow 2$ ✓ |
| $2 \rightarrow 10$ | $01 \rightarrow 1$ ✓ |
| $3 \rightarrow 11$ | $11 \rightarrow 3$ ✓ |




Flow-graph for DIT-FFT : N=4

Given $x(n) = \{1, 2, 4, 8, 16, 32, 64, 128\}$
Find $X(K)$ using DIT-FFT.

$\therefore N = 8$



# Radix 2 DIF FFT Algorithm

DIF → Decimation in Frequency

$$X(K) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \; ; 0 \le k \le N-1 \quad — (1)$$

$$X(K) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{n=\frac{N}{2}}^{N-1} x(n) W_N^{kn}$$

Put $n = n + \frac{n}{2}$ in second part

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) W_N^{k(n+\frac{N}{2})}$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + W_N^{k\frac{N}{2}} \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) W_N^{kn}$$

$$W_N^{\frac{KN}{2}} = e^{-j\frac{2\pi}{N}k\frac{N}{2}} = e^{-jk\pi} = \left(e^{-j\pi}\right)^K$$

~~$= \cos(k\pi) - j\sin(k\pi)$~~

~~$= \cos(k\pi)$~~

$$\left[\cos(\pi) - j\sin(\pi)\right]^K = (-1)^K$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + (-1)^K \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) W_N^{kn}$$

$$X(K) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^k x\left(n+\frac{N}{2}\right)\right] W_N^{kn}$$

Decompose $X(K)$ as even and odd indexed sequence

Put $K = 2r$

$$X(2r) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^{2r} x\left(n+\frac{N}{2}\right)\right] W_N^{2rn}$$

$$X(2r) = \sum_{n=0}^{\frac{N}{2}-1} g(n) W_{N/2}^{rn} \; ; \; 0 \le r \le \frac{N}{2}-1$$

where $g(n) = x(n) + x\left(n + \frac{N}{2}\right)$

Put $K = 2r+1$

$$X(2r+1) = \sum_{n=0}^{N/2-1}\left[x(n) + (-1)^{2r+1}\, x\left(n+\frac{N}{2}\right)\right] W_N^{(2r+1)n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1}\left[x(n) - x\left(n+\frac{N}{2}\right)\right] W_N^{n} \cdot W_N^{2rn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} h(n)\, W_N^{n}\, W_{N/2}^{rn}$$

where $h(n) = x(n) - x\left(n+\frac{N}{2}\right)$

$$\Rightarrow \quad g(n) = x(n) + x\left(n+\frac{N}{2}\right)$$

$$h(n) = x(n) - x\left(n+\frac{N}{2}\right)$$

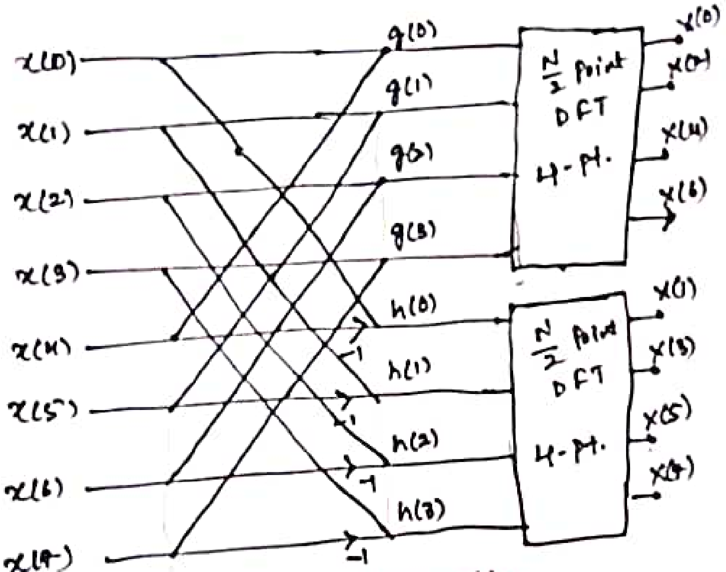
For $N=8$

$g(0) = x(0) + x(4)$ $\qquad$ $h(0) = x(0) - x(4)$

$g(1) = x(1) + x(5)$ $\qquad$ $h(1) = x(1) - x(5)$

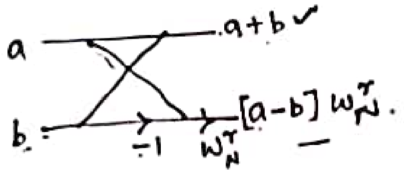
$g(2) = x(2) + x(6)$ $\qquad$ $h(2) = x(2) - x(6)$

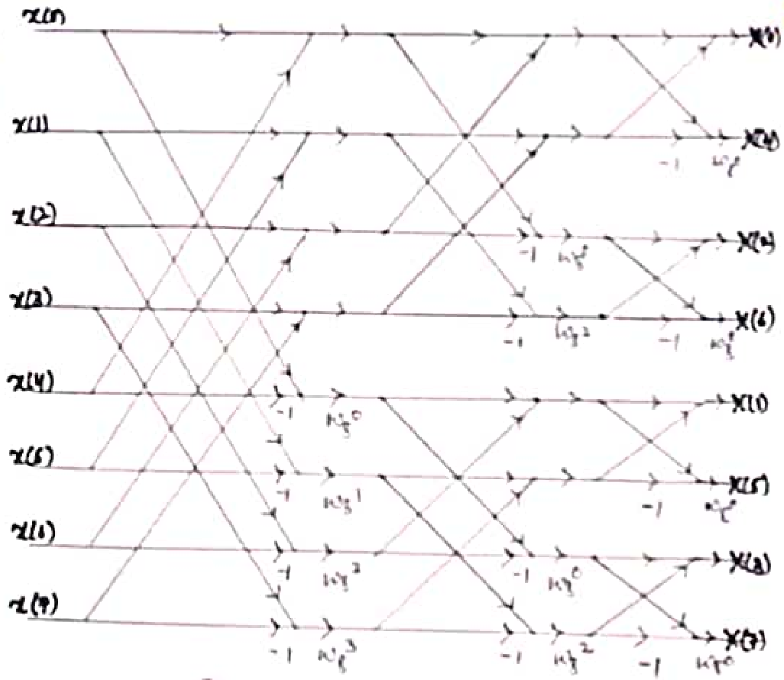
$g(3) = x(3) + x(7)$ $\qquad$ $h(3) = x(3) - x(7)$




$1^{st}$ stage of decimation.

$\frac{N}{2} \rightarrow \frac{N}{4}$ point, 2-point DFT $\rightarrow$ N-point

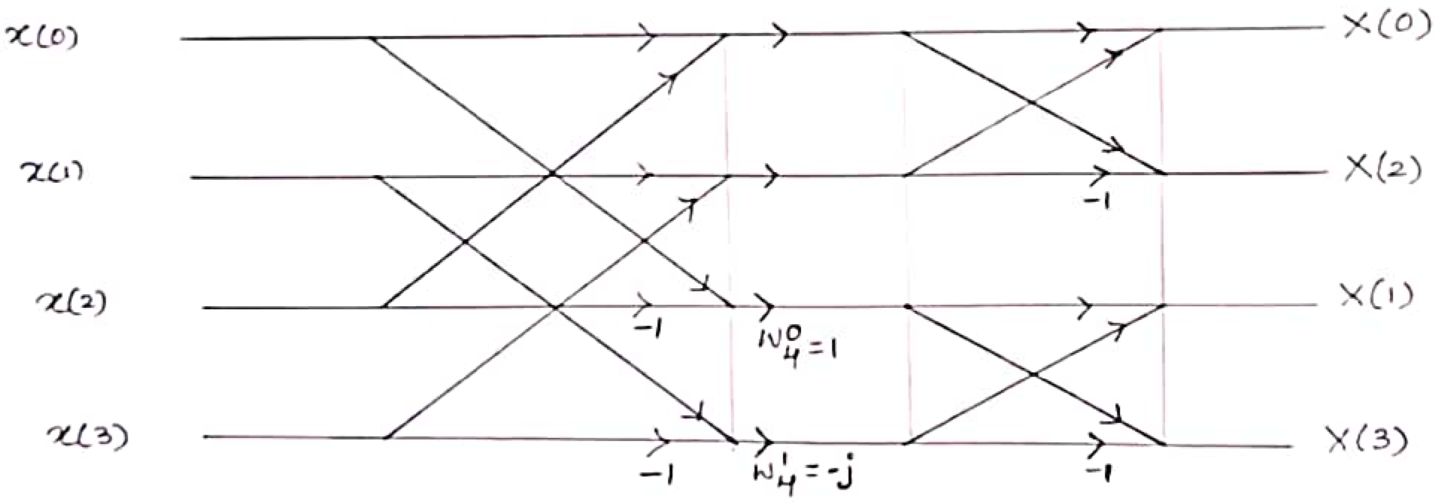



Reduced flowgraph for N=8 DIF-FFT algorithm

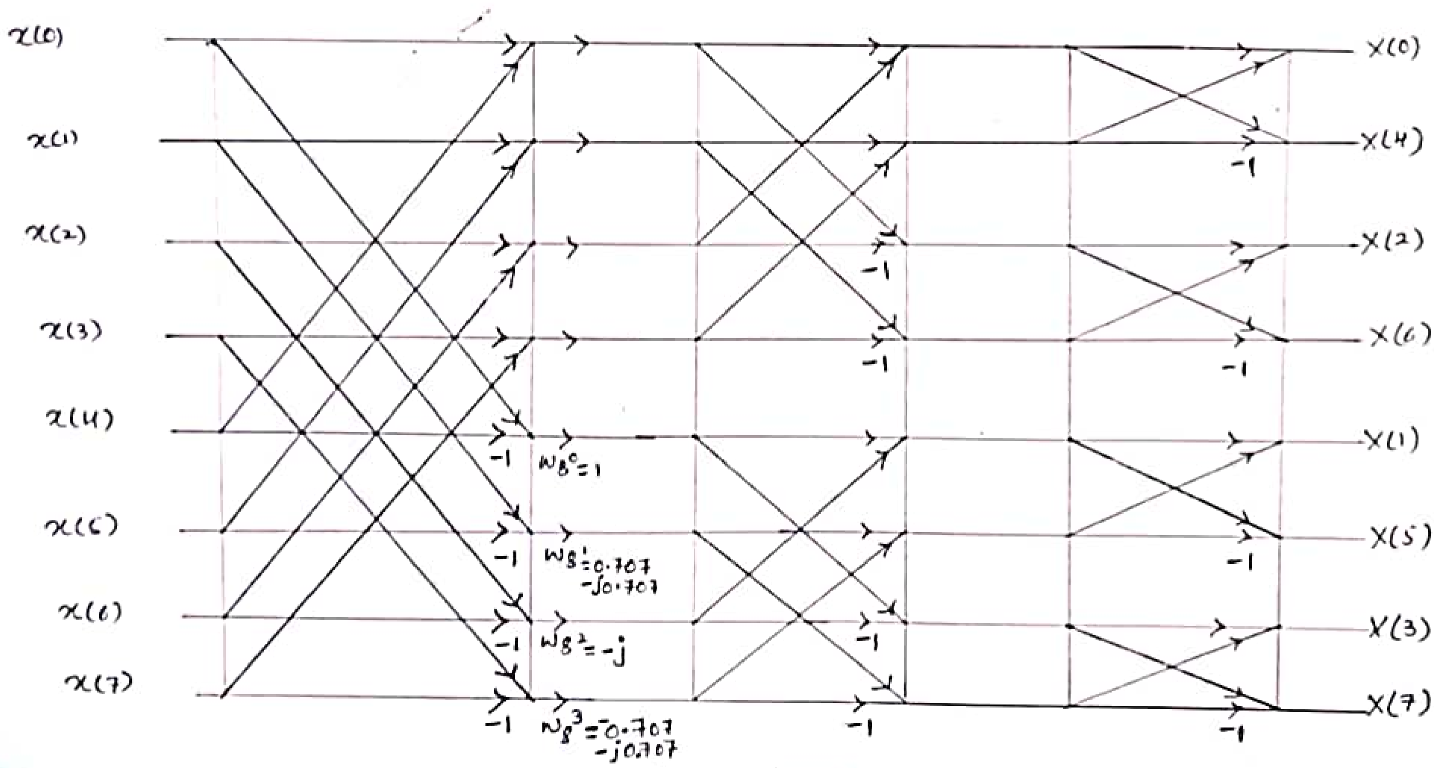
Compute the DFT of $x(n) = \cos\frac{n\pi}{2}$ where $N=4$ using DIF-FFT.

$x(n) = \cos\frac{n\pi}{2}$




Flow graph for 4-Point DIF FFT

Given $x(n) = n+1$ for $0 \le n \le 7$ find $X(K)$ using DIF-FFT Algorithm. $x(n) = \{1, 2, 3, 4, 5, 6, 7, 8\}$

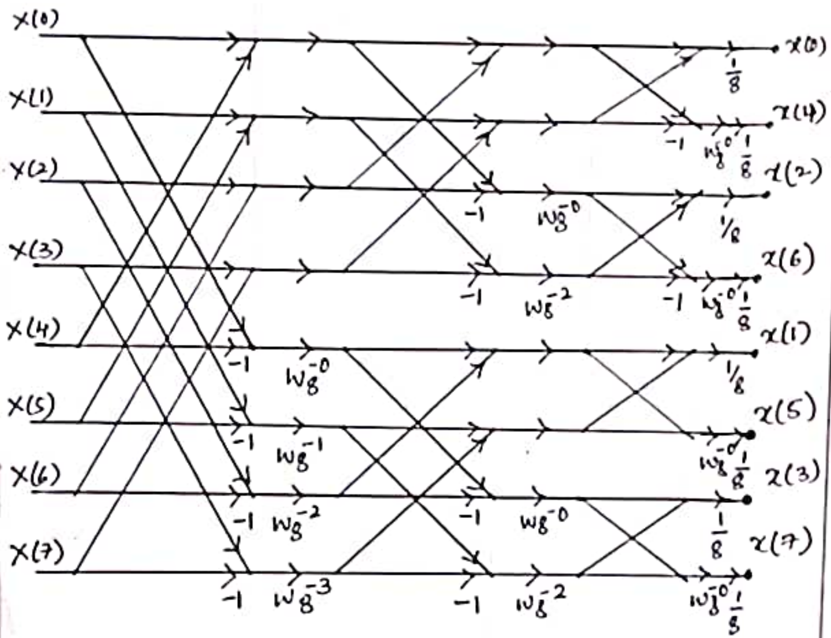



8-Point DIF FFT.

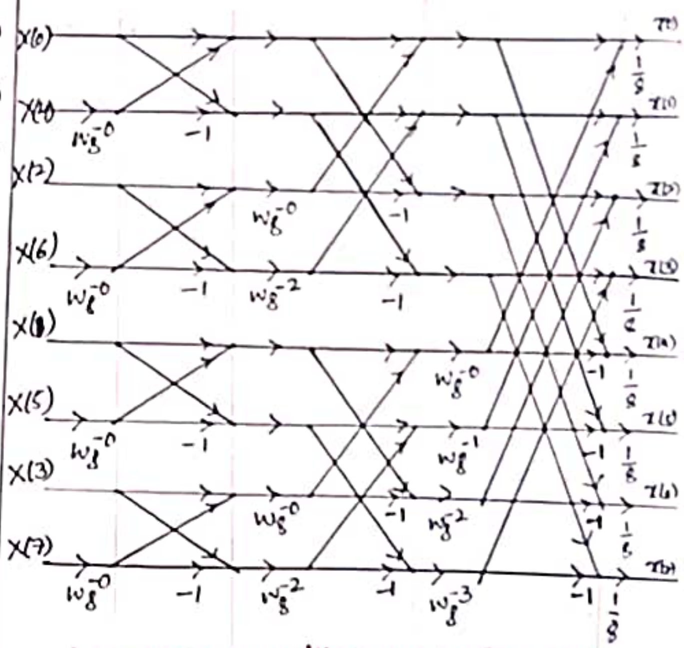
## IDFT using FFT

IDFT ; $x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, W_N^{-Kn}$ ; $0 \le n \le N-1$

DFT ; $X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{Kn}$ ; $0 \le K \le N-1$

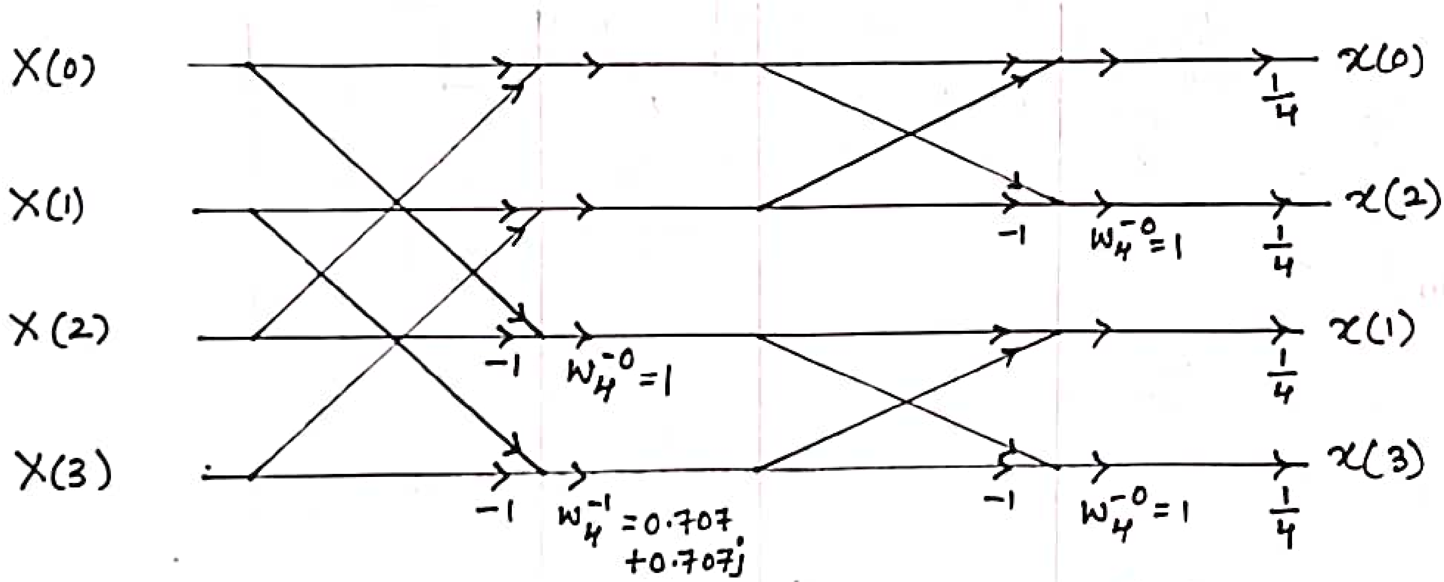



DIT-FFT algorithm for IDFT ; N=8



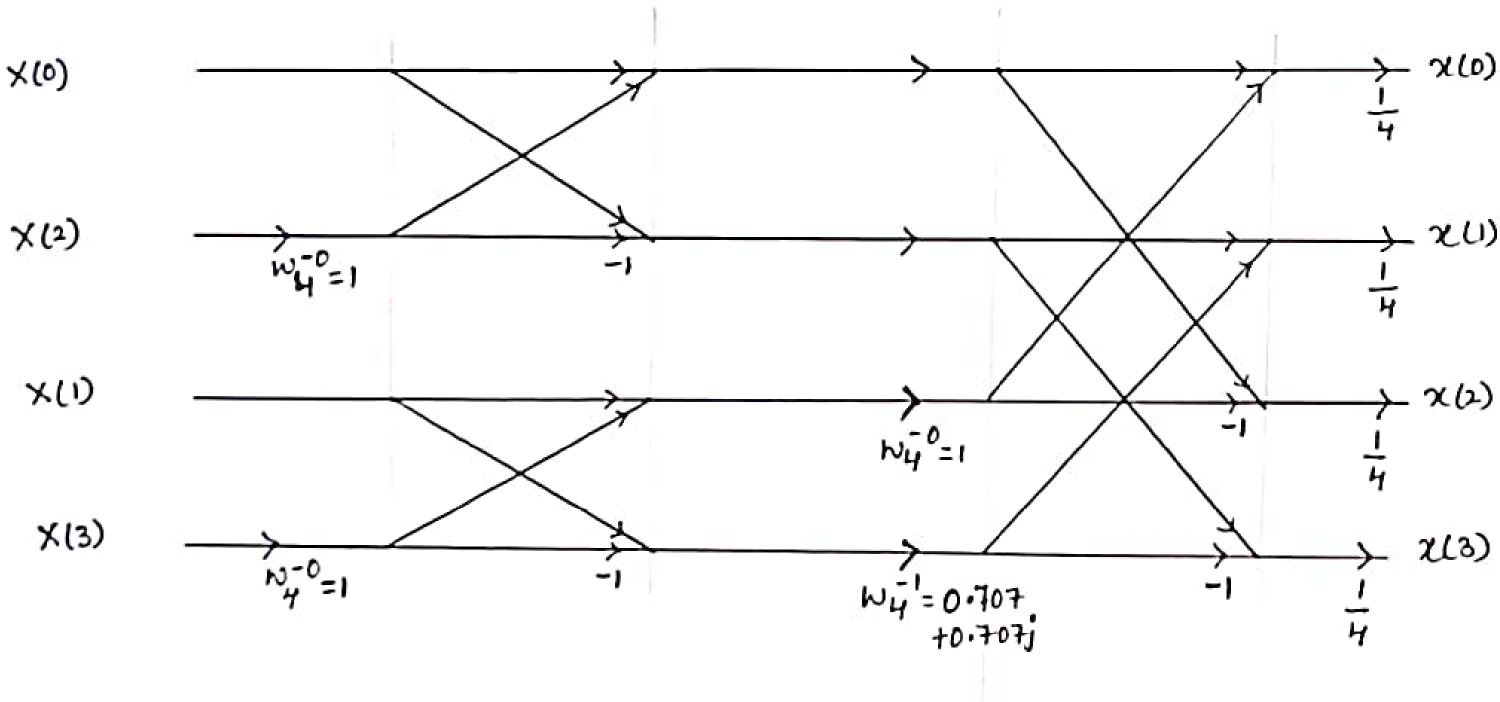

DIF-FFT algorithm for IDFT ; N=8

For $X(k) = \{32, -4, 0, -4\}$ compute IDFT using DIT-FFT.

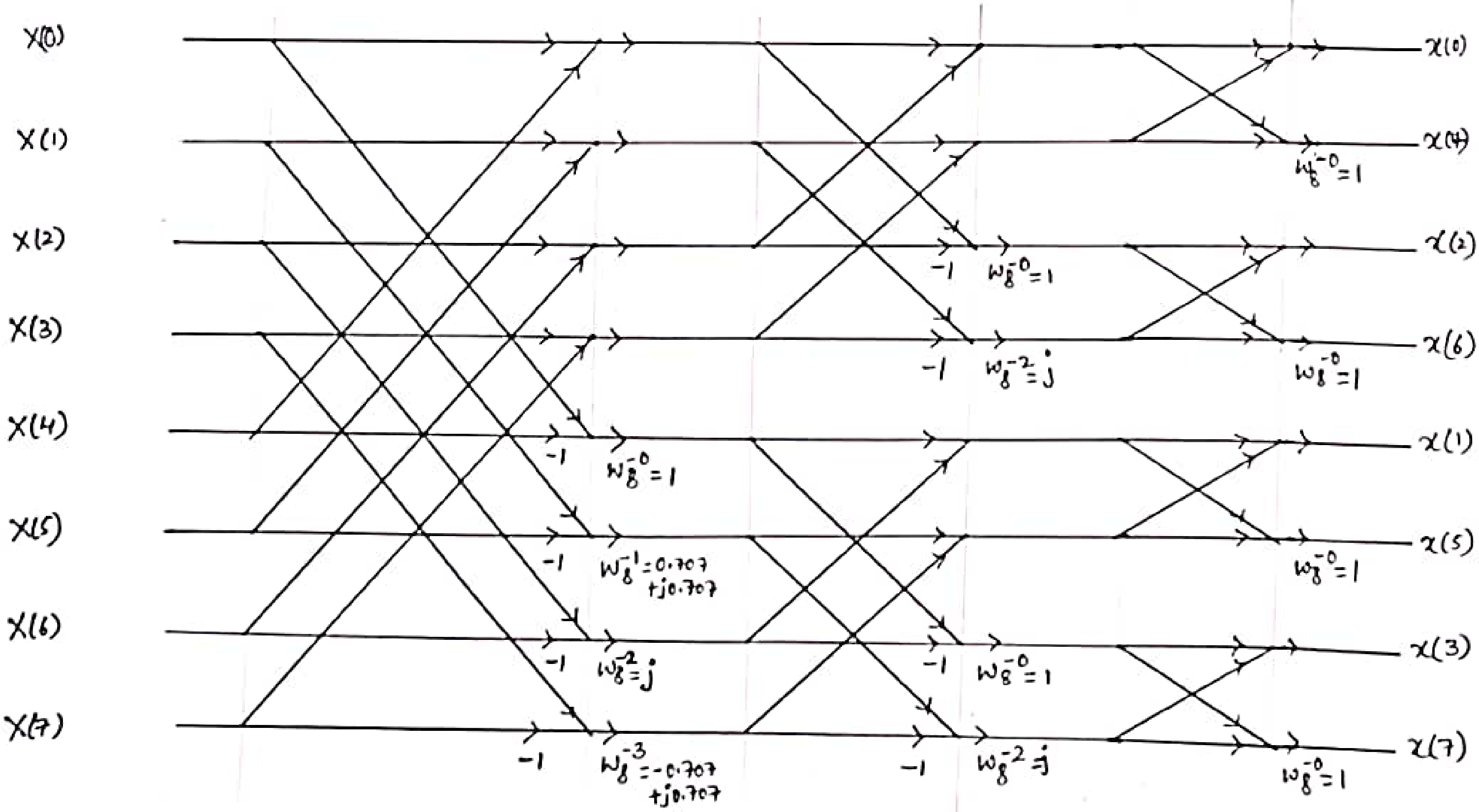



Flow graph for IDFT using DIT FFT
N=4

Compute $x(n)$ for $X(K) = \{32, -4, 0, -4\}$ using DIF-FFT



Find the seq. $x(n)$ corresponding to 8-point DFT using DIT-FFT for $X(K) = \{4,\ 1-j2.414,\ 0,\ 1-j0.414,\ 0,\ 1+j0.414,\ 0,\ 1+j2.414\}$



Find the seq. x(n) Corresponding to 8-point IDFT using DIF-FFT for X(K) = {4, 1-j2.414, 0, 1-j0.414, 0, 1+j0.414, 0, 1+j2.414}