1) Causality Condition in Terms of Impulse Response
A discrete time LTI system is causal if and only if its impulse response sequence $\{h[n]\}$ is a causal sequence satisfying the condition

$$h[k] = 0 \quad \text{for} \quad k < 0$$

Discrete Fourier Transform (DFT)

The discrete fourier transform (DFT) of the length-N time-domain sequence $x[n]$ is defined by

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi kn/N} \qquad 0 \leq k \leq N-1$$

Put, $\boxed{W_N = e^{-j2\pi/N}}$ (Phase Factor or Twiddle Factor)

☆ $$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn} \qquad 0 \leq k \leq N-1$$

☆ Inverse Discrete Fourier Transform (IDFT) is given by

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, W_N^{-kn} \qquad 0 \leq n \leq N-1$$

Problems on DFT

$x(n) = a^n$

$$X[k] = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} = \sum_{n=0}^{N-1} a^n \left(e^{-j2\pi/N}\right)^{kn}$$

$$= \sum_{n=0}^{N-1} \left(a e^{-j2\pi k/N}\right)^n$$

$$= \frac{1 - \left[a e^{-j2\pi k/N}\right]^N}{1 - \left[a e^{-j2\pi k/N}\right]}$$

$$= \frac{1 - a^N e^{-j2\pi k}}{1 - a e^{-j2\pi k/N}} = \frac{1 - a^N}{1 - a e^{-j2\pi k/N}}$$

2) Find the DFT of the sequence $x[n] = 1$; $0 \le n \le 2$ for $N = 4$, 0 otherwise. Sketch the magnitude and phase spectrum.

Ans) $x[n] = \{1, 1, 1, 0\}$

$$X[k] = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} \quad ; \quad 0 \le k \le N-1$$

$$X[k] = \sum_{n=0}^{3} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

$$= x(0) + x(1)\, e^{-j\frac{2\pi k}{4}} + x(2)\, e^{-j\frac{2\pi k 2}{4}} + 0$$

$$= 1 + e^{-jk\frac{\pi}{2}} + e^{-jk\pi}$$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

$$e^{-jk\frac{\pi}{2}} = \cos\left(\frac{k\pi}{2}\right) - j\sin\left(\frac{k\pi}{2}\right)$$

$k = 0, 1, 2, 3$

For $k = 0$

$X[0] = 1 + 1 + 1 + 0 = 3$

$k = 1$

$X[1] = 1 + (-j) - 1 + 0 = -j$

$k = 2$

$X[2] = 1 - 1 + 1 + 0 = 1$

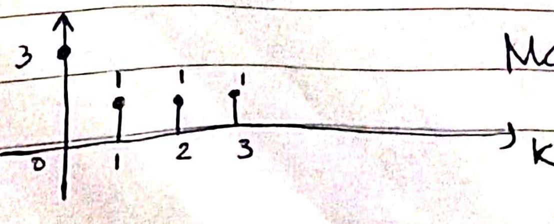
$k = 3$

$X[3] = 1 + j - 1 + 0 = j$

$X[k] = \{3, -j, 1, j\}$

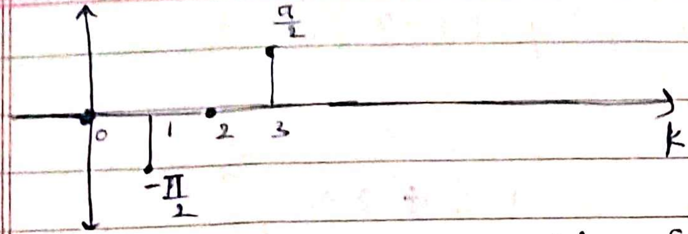
Magnitude : $\{3, 1, 1, 1\}$

Phase : $\{0, -\frac{\pi}{2}, 0, \frac{\pi}{2}\}$



Magnitude Spectrum

Phase Spectrum



5) Find IDFT of $X(k) = \{6, -2+2j, -2, -2-2j\}$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j\frac{2\pi}{N}kn} \qquad 0 \le n \le N-1$$

$$= \frac{1}{4} \sum_{k=0}^{3} X[k]\, e^{j\frac{2\pi}{4}kn}$$

$$= \frac{1}{4}\left[6 + (-2+2j)e^{j\frac{\pi}{2}n} + (-2)e^{j\frac{\pi}{2}2n} + (-2-2j)\, e^{j\frac{\pi}{2}3n}\right]$$

$= \cancel{\frac{1}{4}[6}$ $\quad e^{j\theta} = \cos\theta + j\sin\theta$

$$e^{j\frac{\pi}{2}n} = \cos\left(\frac{\pi}{2}n\right) + j\sin\left(\frac{\pi}{2}n\right)$$

$n=0 \Rightarrow 1+0 = 1$

$n=1 \Rightarrow 0+j = j$

$n=2 \Rightarrow -1+0 = -1$

$n=3 \Rightarrow 0+(-j) = -j$

$$e^{j\pi n} = \cos(\pi n) + j\sin(\pi n)$$

$n=0 \Rightarrow 1$

$n=1 \Rightarrow -1+0 = -1$

$n=2 \Rightarrow 1+0 = 1$

$n=3 \Rightarrow -1+0 = -1$

$$e^{j\frac{3\pi}{2}n} = \cos\left(\frac{3\pi}{2}n\right) + j\sin\left(\frac{3\pi}{2}n\right)$$

$n=0 \Rightarrow 1$

$n=1 \Rightarrow -j$

$n=2 \Rightarrow -1$

$n=3 \Rightarrow 0+j = j$

$$x(0) = \frac{1}{4}[6 + (-2+2j) - 2 - 2 - 2j]$$

$$x(0) = 0$$

$$x(1) = \frac{1}{4}\left[6 + (-2+2j)\,j + (-2)(-1) + (-2-2j)(-j)\right]$$

$$x(1) = \frac{1}{4}\left[6 - 2j - 2 + \cancel{2} \cancel{+} \cancel{2} + 2j\right]$$

$$\cancel{x(1) = \frac{1}{4}[4+4j] = (1+j)}$$

$$x(1) = \frac{1}{4}[4] = 1$$

$$x(2) = \frac{1}{4}\left[6 + (-2+2j)(-1) + (-2)(1) + (-2-2j)(-1)\right]$$

$$= \frac{1}{4}\left[6 + 2 - \cancel{2j} - \cancel{2} + \cancel{2} + \cancel{2j}\right]$$

$$x(2) = \frac{1}{4}[8] = 2$$

$$x(3) = \frac{1}{4}\left[6 + (-2+2j)(-j) + (-2)(-1) + (-2-2j)\,j\right]$$

$$x(3) = \frac{1}{4}\left[6 + \cancel{2j} + 2 + 2 - \cancel{2j} + 2\right]$$

$$x(3) = \frac{1}{4}[12] = 3$$

$$x(n) = \{0, 1, 2, 3\}$$

DSP as Linear Transformation (Matrix Method)

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{bmatrix}_{N\times 1} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^{2} & \cdots & W_N^{(N-1)} \\ 1 & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}_{N\times N} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}_{N\times 1}$$

$$\boxed{X_N = W_N \cdot x_N} \longrightarrow \text{DFT}$$

$$\boxed{x_N = \frac{1}{N} W_N^{*} X_N} \rightarrow \text{IDFT}$$

W_N^* means complex conjugate of W_N

eg1) Find 4 point DFT of $x(n) = \{1, 0, 0, 1\}$ using linear transform (Matrix Method)

Ans) $X_N = W_N \cdot x_N$

$$W_N = e^{-j\frac{2\pi}{N}} = e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) - j\sin\left(\frac{\pi}{2}\right)$$

$$= 0 - j(1) = -j$$

$$W_N = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & (-j) & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \qquad x_N = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$X_N = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$X_N = \begin{bmatrix} 1+0+0+1 \\ 1+j \\ 1-1 \\ 1+(-j) \end{bmatrix} = \begin{bmatrix} 2 \\ 1+j \\ 0 \\ 1-j \end{bmatrix}$$

$$X_k = \{2, 1+j, 0, 1-j\}$$

eg2) Find 6 point DFT of $x(n) = \{1, 1, 0, 0, 0, 2\}$ using matrix method [linear transformation]

$$W_N = e^{-j\frac{2\pi}{6}} = \cos\left(\frac{\pi}{3}\right) - j\sin\left(\frac{\pi}{3}\right)$$

$$W_N = \frac{1}{2} - j\frac{\sqrt{3}}{2} = \left(\frac{1 - j\sqrt{3}}{2}\right)$$

$$X_N = W_N \cdot x_N$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W_N^1 & W_N^2 & W_N^3 & W_N^4 & W_N^5 \\ 1 & W_N^2 & W_N^4 & W_N^6 & W_N^8 & W_N^{10} \\ 1 & W_N^3 & W_N^6 & W_N^9 & W_N^{12} & W_N^{15} \\ 1 & W_N^4 & W_N^8 & W_N^{12} & W_N^{16} & W_N^{20} \\ 1 & W_N^5 & W_N^{10} & W_N^{15} & W_N^{20} & W_N^{25} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

$$X_N = \begin{bmatrix} 4 \\ 2.5 + 0.866j \\ -0.5 + 0.866j \\ -2 \\ -0.5 - 0.866j \\ 0.5 - 0.866j \end{bmatrix}$$

eg 3) For $X[K] = \{2, 1+j, 0, 1-j\}$ find 4 point IDFT using matrix method

Ans) $$x_N = \frac{1}{N} X_N W_N^*$$

$$W_N = e^{-j\frac{2\pi}{4}} = \cos\left(\frac{\pi}{2}\right) - j\sin\left(\frac{\pi}{2}\right) = 0 - j(1)$$

$$= -j$$

$$W_N = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$W_N^* = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

$$X_N = \begin{bmatrix} 2 \\ 1+j \\ 0 \\ 1-j \end{bmatrix}$$

$$x_N = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 2 \\ 1+j \\ 0 \\ 1-j \end{bmatrix}$$

$$x_N = \frac{1}{4}\begin{bmatrix} 4 \\ \cancel{2} + \cancel{j} \cancel{-1} \cancel{-j} \cancel{-1} \\ \cancel{2} \cancel{-1} \cancel{-j} \cancel{-1} + \cancel{j} \\ 2 - \cancel{j} + 1 + \cancel{j} + 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ 0 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$x[N] = \{1, 0, 0, 1\}$

DFT Properties

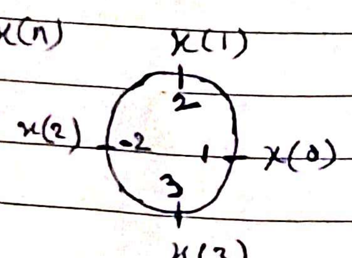
1) Linearity : If $DFT\{x_1(n)\} = X_1(K)$
and $DFT\{x_2(n)\} = X_2(K)$

then,
$$DFT\{a_1 x_1(n) + a_2 x_2(n)\} = a_1 X_1(K) + a_2 X_2(K)$$

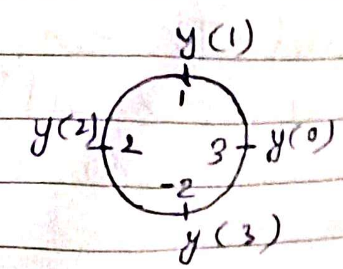
2) Periodicity : If $DFT\{x(n)\} = X(K)$
then, $x(n+N) = x(n)$ for all n
$X(K+N) = X(K)$ for all K

3) Circular Time Shift:
If $DFT\{x(n)\} = X(K)$
then $DFT\{x((n-n_0))_N\} = X(K)e^{-j\frac{2\pi}{N}Kn_0}$

eg. $x(n) = \{1, 2, -2, 3\}$
$y(n) = x((n-1))_4$



y(n)



−ve → Anticlockwise
+ve → clockwise

So we rotate, in anticlockwise direction
by one step

$y(n) = \{3, 1, 2, -2\}$

4) Circular Frequency Shift:

If $DFT\{x(n)\} = X(K)$

then, $DFT\{x(n)\, e^{j\frac{2\pi}{N}ln}\} = X((K-l))_N$

5) Time Reversal Property

If $DFT\{x(n)\} = X(k)$

then, $DFT\{x(N-n)\} = X(N-k)$

OR $DFT\{x((-n))_N = x(N-n)\} = X((-K))_N = X(N-K)$

6) Circular Convolution: If $DFT\{x(n)\} = X(k)$

then, $DFT\{x_1(n) \circledN x_2(n)\} = X_1(k) \cdot X_2(k)$

Ⓝ ~~denotes~~ represents circular convolution

7) Modulation or Multiplication Property.

If $DFT\{x_1(n)\} = X_1(K)$

& $DFT\{x_2(n)\} = X_2(K)$

then,

$$DFT\{x_1(n) \cdot x_2(n)\} = \frac{1}{N}\left[X_1(K) \, Ⓝ \, X_2(k)\right]$$

8) Parsval Theorem

If $DFT\{x_1(n)\} = X_1(k)$

& $DFT\{x_2(n)\} = X_2(k)$

then,

$$\sum_{n=0}^{N-1} x_1(n) \cdot x_2^*(n) = \frac{1}{N} \sum_{K=0}^{N-1} X_1(K) \cdot X_2^*(K)$$

$*$ represents complex conjugate

9) Circular Correlation Property

If $DFT\{x(n)\} = X(K)$

& $DFT\{y(n)\} = Y(K)$

then,

$DFT\{r_{xy}(l)\} = R_{xy}(K) = X(K) \cdot Y^*(K)$

10) Duality Property

If $x(n) \xrightarrow{DFT} X(K)$

then $DFT\{X(n)\} = N[x((-K))_N] = N x(N-K)$

11) Circular Symmetry Property

If $DFT\{x(n)\} = X(K)$

then, $DFT\{x_p(n)\} = X(K)$

$x_p(n) \Rightarrow$ periodic representation of $x(n)$

eg. $x(n) = \{4, 3, \underset{\uparrow}{2}, 1\}$

$x_p(n) = \{\underset{\uparrow}{2}, 1, 4, 3\}$

12) Symmetry Property

$x(n) \xrightarrow{DFT} X(K)$

$x(n) = x_R(n) + j x_I(n)$

$X(K) = X_R(K) + j X_I(K)$

$$X_R(K) = \sum_{n=0}^{N-1} \left[x_R(n) \cos \frac{2\pi}{N} kn + x_I(n) \sin \frac{2\pi}{N} kn \right]$$

$$X_I(K) = \sum_{n=0}^{N-1} \left[x_I(n) \cos \frac{2\pi}{N} kn - x_R(n) \sin \frac{2\pi}{N} kn \right]$$

$$x_R(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R(k)\cos\frac{2\pi}{N}kn - X_I(k)\sin\frac{2\pi}{N}kn\right]$$

$$x_I(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R(k)\sin\frac{2\pi}{N}kn + X_I(k)\cos\frac{2\pi}{N}kn\right]$$

1) Real & Even Sequences

$x_I(n) = 0$ & $x(n) = x_R(n)$

$$X(k) = \sum_{n=0}^{N-1} x(n)\cos\frac{2\pi}{N}kn$$

2) Real and odd sequences

$x_I(n) = 0$ & $x(n) = x_R(n)$

$$X(k) = \sum_{n=0}^{N-1} -j\,x(n)\sin\frac{2\pi}{N}kn$$

3) Purely imaginary sequence

$x_R(n) = 0$ & $x(n) = j\,x_I(n)$ or $x_I(n) = -jx(n)$

$$X_R(k) = \sum_{n=0}^{N-1} -j\,x(n)\sin\frac{2\pi}{N}kn$$

$$X_I(k) = \sum_{n=0}^{N-1} -j\,x(n)\cos\frac{2\pi}{N}kn$$

$$\boxed{X(N-k) = X^*(k)}$$ Symmetry Property

eg. The first five points of 8 point DFT $X(k)$ are $\{0.25,\ 0.125 - j0.3018,\ 0,\ 0.125 - j0.518,\ 0\}$. Determine the remaining three points. Estimate the value of $x(0)$

Ans) $X(5) = X(8-3) = X^*(3)$

$X(5) = 0.125 + j\,0.518$

$X(6) = X(8-2) = X^*(2)$

$X(6) = 0$

$X(7) = X(8-1) = X^*(1) = 0.125 + j\,0.3018$

By IDFT

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j2\pi kn/N}$$

$$x(0) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) = \frac{1}{8} \sum_{k=0}^{7} X(k)$$

$$x(0) = \frac{1}{8}\left(X(0) + X(1) + X(2) + X(3) + X(4) + X^*(3) + X^*(2) + X^*(1) \right]$$

$$= \frac{1}{8}\left[0.25 + 2\times0.125 + 0 + 2\times0.125 + 0\right]$$

$$= \frac{1}{8}\left[0.25 + 0.25 + 0.25\right] = \frac{0.75}{8} = 0.09375$$

eg. Find the circular convolution of
$x_1(n) = \{2, 1, 2, 1\}$ $x_2(n) = \{1, 2, 3, 4\}$

Ans) Matrix Method

$$\begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 14 \\ 16 \\ 14 \\ 16 \end{bmatrix}$$

Hence circular convolution $y(n) = \{14, 16, 14, 16\}$

eg) Find the linear convolution of
$x_1(n) = \{2, 1, 2, 1\}$ & $x_2(n) = \{1, 2, 3, 4\}$

$x_1(n)$				2	1	2	1
$x_2(n)$				1	2	3	4
				8	4	8	4
			6	3	6	3	
		4	2	4	2		
	2	1	2	1			
	2	5	10	16	12	11	4

$y(n) = \{2, 5, 10, 16, 12, 11, 4\}$
= Linear Convolution

Read Left to right for Linear Convolution

16	12	11 ~~9~~	4
	2	5	10
16	14	16	14

~~$y(n) = \{16, 14, 16, 14\}$ ← Circular Convolution~~

For circular convolution read from right to left

$y(n) = \{14, 16, 14, 16\}$ - Circular Convolution

eg) Compute the circular convolution of $x_1(n) = \{2, 1, 2, 1\}$ & $x_2(n) = \{1, 2, 3, 4\}$ using DFT and IDFT.

Ans) We know that

$$DFT\{x_1(n) \circledN x_2(n)\} = X_1(K) \cdot X_2(K)$$

$$x_1(n) \circledN x_2(n) = IDFT\{X_1(K) \cdot X_2(K)\}$$

$$\begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

$$W_N = e^{-j\frac{2\pi}{4}} = \cos\left(\frac{\pi}{2}\right) - j\sin\left(\frac{\pi}{2}\right) = -j$$

$$X_1(K) = \begin{bmatrix} 6 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

$$X_1(K) \cdot X_2(K) = \begin{bmatrix} 6 \\ 0 \\ 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix} = \begin{bmatrix} 60 \\ 0 \\ -4 \\ 0 \end{bmatrix}$$

IDFT of $X_1(K) \cdot X_2(K)$

$$x_N = \frac{1}{N}[W_N^* \cdot X_N]$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 60 \\ 0 \\ -4 \\ 0 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 56 \\ 64 \\ 56 \\ 64 \end{bmatrix} = \begin{bmatrix} 14 \\ 16 \\ 14 \\ 16 \end{bmatrix}$$

$y(n) = \{14, 16, 14, 16\}$

Linear Convolution using DFT

2g) $x_1(n) = \{1, 2\}$ $\quad l_{x_1} = 2$

$x_2(n) = \{1, 2, 1\}$ $\quad l_{x_2} = 3$.

length of $y(n) = l_{x_1} + l_{x_2} - 1 = 2 + 3 - 1 = 4$

We add 2 zeroes to $x_1(n)$ and one zero to $x_2(n)$

$x_1(n) = \{1, 2, 0, 0\}$

$x_2(n) = \{1, 2, 1, 0\}$

$x_1(n) \circledast x_2(n) = \text{IDFT}\{X_1(K) \cdot X_2(K)\}$

$$\begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1-2j \\ -1 \\ 1+2j \end{bmatrix}$$

$$W_N = e^{-j\frac{2\pi}{N}} = e^{-j\frac{2\pi}{4}} = \cos\left(\frac{\pi}{2}\right) - j\sin\left(\frac{\pi}{2}\right) = -j$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -2j \\ 0 \\ 2j \end{bmatrix}$$

$$X_1(K) \cdot X_2(K) = \begin{bmatrix} 3 \\ 1-2j \\ -1 \\ 1+2j \end{bmatrix}\begin{bmatrix} 4 \\ -2j \\ 0 \\ 2j \end{bmatrix} = \begin{bmatrix} 12 \\ -4-2j \\ 0 \\ -4+2j \end{bmatrix}$$

$$y(n) = \frac{1}{N} W_N^{*} X_N$$

$$= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 12 \\ -4-2j \\ 0 \\ -4+2j \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 4 \\ 16 \\ 20 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 5 \\ 2 \end{bmatrix}$$

$x_1(n) \circledast x_2(n) = \{1, 4, 5, 2\}$

Overlap Add Method

For $h(n) = \{3, 2, 1, 1\}$ & $x(n) = \{1, 2, 3, 3, 2, 1, -1, -2, -3, 5, 6, -1, 2, 0, 2, 1\}$. Find the o/p using overlap add method assume block length 7

Step 1 $N = 7$

$N = M + L - 1 \Rightarrow 7 = 4 + L - 1 \Rightarrow L = 4$

Step 2 Append $L-1$ 0's to $h(n)$

$h(n) = \{3, 2, 1, 1, 0, 0, 0\}$

Step 3 We divide $x(n)$ into multiple sequences by selecting L samples at a time and append $M-1$ 0's to them

$x_1(n) = \{1, 2, 3, 3, 0, 0, 0\}$

$x_2(n) = \{2, 1, -1, -2, 0, 0, 0\}$

$x_3(n) = \{-3, 5, 6, -1, 0, 0, 0\}$

$x_4(n) = \{2, 0, 2, 1, 0, 0, 0\}$

Step 4 Find Convolution

$y_1(n) = x_1(n) \text{ (N) } h(n)$

$y_2(n) = x_2(n) \text{ (N) } h(n)$

$y_3(n) = x_3(n) \text{ (N) } h(n)$

$y_4(n) = x_4(n) \text{ (N) } h(n)$

$$y_1(n) = \begin{bmatrix} 1 & 0 & 0 & 0 & 3 & 3 & 2 \\ 2 & 1 & 0 & 0 & 0 & 3 & 3 \\ 3 & 2 & 1 & 0 & 0 & 0 & 3 \\ 3 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 3 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$y_1(n) = \{3, 8, 14, 18, 11, 6, 3\}$ Similarily

$y_2(n) = \{6, 7, 1, -5, -4, -3, -2\}$

$y_3(n) = \{-9, 9, 25, 11, 9, 5, -1\}$

$y_4(n) = \{6, 4, 8, 9, 4, 3, 1\}$

Step5 Last M-1 samples of previous convolution would be added to first M-1 samples of next convolution

3	8	14	18	11	6	3												
				6	7	1	-5	-4	-3	-2								
								-9	9	25	11	9	5	-1				
													6	4	8..			

$y(n) = \{3, 8, 14, 18, 17, 13, 4, -5, -13, 6, 23, 11, 15, 9, 7, 9, 4, 3, 1\}$

Overlap Save Method

Que) Find $y(n)$ for $h(n) = \{1, 1, 1\}$ and $x(n) = \{3, -1, 0, 1, 3, 2, 0, 1, 2, 1\}$ using Overlap Save Method

Ans) $h(n) = \{1, 1, 1\}$ $M = 3$

Step 1: $N = 2^M = 2^3 = 8$ $\boxed{N=8}$

$N = M + L - 1 \Rightarrow 8 = 3 + L - 1 \Rightarrow \boxed{L = 6}$

Step 2: Append L-1 0s to $h(n)$

$h(n) = \{1, 1, 1, 0, 0, 0, 0, 0\}$

Step 3: Select L samples from $x(n)$ and insert M-1 0s at their starting to get $x_1(n)$

$x_1(n) = \{0, 0, 3, -1, 0, 1, 3, 2\}$

Step 4: Select remaining samples from $x(n)$, select M-1 samples from last from $x_1(n)$ and insert them in beginning of $x_2(n)$

Insert 0s at end of $x_2(n)$ such that its length is N.

$x_2(n) = \{3, 2, 0, 1, 2, 1, 0, 0\}$

Step 5) $y_1(n) = x_1(n) \, (N) \, h(n)$

$y_2(n) = x_2(n) \, (N) \, h(n)$

$$\begin{bmatrix} 0 & 2 & 3 & 1 & 0 & -1 & 3 & 0 \\ 0 & 0 & 2 & 3 & 1 & 0 & -1 & 3 \\ 3 & 0 & 0 & 2 & 3 & 1 & 0 & -1 \\ -1 & 3 & 0 & 0 & 2 & 3 & 1 & 0 \\ 0 & -1 & 3 & 0 & 0 & 2 & 3 & 1 \\ 1 & 0 & -1 & 3 & 0 & 0 & 2 & 3 \\ 3 & 1 & 0 & -1 & 3 & 0 & 0 & 2 \\ 2 & 3 & 1 & 0 & -1 & 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$y_1(n) = \{5, 2, 3, 2, 2, 0, 4, 6\}$

Similarity

$y_2(n) = \{3, 5, 5, 3, 3, 4, 3, 1\}$

Step 6) Discard M-1 samples from beginning of $y_1(n)$ and $y_2(n)$ and join the two sequences to get $y(n)$

$y(n) = \{3, 2, 2, 0, 4, 6, 5, 3, 3, 4, 3, 1\}$

Computation of DFT of Real sequences

1. N Point DFT of two real sequences using single N point DFT

Let $g(n)$ and $h(n)$ be two real sequences of length N, with $G(k)$ and $H(k)$ denoting their respective N point DFTs

$x(n) = y(n) + h(n)$

$G[k] = \frac{1}{2}\{X[k] + X^*[((-k))_N]\}$

$H[k] =$

Computation of DFT of Real Sequences

1. N Point DFTs of two real sequences using single N Point DFT

Let $g[n]$ and $h[n]$ be two real sequences of length N each, with $G[k]$ and $H[k]$ denoting their respective N Point DFTs.

$$x(n) = g(n) + jh(n)$$

$$G(k) = \frac{1}{2}\{X(k) + X^*((-k)_N)\}$$

$$H(k) = \frac{1}{2j}\{X(k) - X^*((-k)_N)\}$$

Note that, $X^*((-k)_N) = X^*((N-k)_N)$

eg. Compute the 4 point DFT of two real sequences $g(n) = \{1, 2, 0, 1\}$ $h(n) = \{2, 2, 1, 1\}$ $0 \le n \le 3$

Ans) $x(n) = g(n) + j\,h(n)$

$x(n) = \{1+2j \quad 2+2j \quad j \quad 1+j\}$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_N^1 & W_N^2 & W_N^3 \\ 1 & W_N^2 & W_N^4 & W_N^6 \\ 1 & W_N^3 & W_N^6 & W_N^9 \end{bmatrix} \begin{bmatrix} 1+2j \\ 2+2j \\ j \\ 1+j \end{bmatrix}$$

$$W_N = e^{-j\frac{2\pi}{4}} = \cos\left(\frac{\pi}{2}\right) - j\sin\left(\frac{\pi}{2}\right)$$

$$W_N = 0 - j(1) = -j$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1+2j \\ 2+2j \\ j \\ 1+j \end{bmatrix}$$

$$X(K) = \begin{bmatrix} 4+6j \\ 2 \\ -2 \\ 2j \end{bmatrix} \qquad X(K) = \{4+6j,\ 2,\ -2,\ 2j\}$$

$$X^*(K) = \{4-6j,\ 2,\ -2,\ -2j\}$$

$$X^*(4-K) = \{4-6j,\ -2j,\ -2,\ 2\}$$

$$G(K) = \frac{1}{2}\{8,\ 2-2j,\ -4,\ 2+2j\}$$

$$G(K) = \{4,\ 1-j,\ -2,\ 1+j\}$$

$$H(K) = \frac{1}{2j}\{12j,\ 2+2j,\ 0,\ 2j-2\}$$

$$H(K) = \{6,\ 1-j,\ 0,\ 1+j\}$$

2. 2N point DFT of real sequence using a single N-Point DFT

Let $v[n]$ be a real sequence of length $2N$ with $V[K]$ denoting its $2N$ point DFT. Define two real sequences $g[n]$ and $h[n]$ of length N each as

$$g[n] = v[2n], \quad h[n] = v[2n+1] \qquad 0 \le n \le N-1$$

with $G[K]$ and $H[K]$ denoting their N point DFTs

$$x[n] = g[n] + j\,h[n]$$

$$G(K) = \frac{1}{2}\{X(K) + X^*((-K)_N)\}$$

$$H(K) = \frac{1}{2j}\{X(K) - X^*((-K)_N)\}$$

$V[K] = G((K)_N) + W_{2N}^{K} H((K)_N) \qquad 0 \le K \le 2N-1$

eg. Determine 8 point DFT $V[K]$ of length-8 real sequence $v[n]$ given below:

$v[n] = \{1, 2, 2, 2, 0, 1, 1, 1\} \qquad 0 \le n \le 7$

Ans) $g[n] = v[2n]$

$g[n] = \{1, 2, 0, 1\} \qquad 0 \le n \le 3$

$h[n] = v[2n+1]$

$h[n] = \{2, 2, 1, 1\} \qquad 0 \le n \le 3$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1+2j \\ 2+2j \\ j \\ 1+j \end{bmatrix} = \begin{bmatrix} 4+6j \\ 2 \\ -2 \\ 2j \end{bmatrix}$$

As computed in previous example

$G(K) = \{4, 1-j, -2, 1+j\}$

$H(K) = \{6, 1-j, 0, 1+j\}$

$V(K) = G((K)_N) + W_{2N}^{K} H((K)_N)$

$W_8 = e^{-j\frac{2\pi}{8}} = \cos\left(\frac{\pi}{4}\right) - j\sin\left(\frac{\pi}{4}\right) = \left(\frac{1-j}{\sqrt{2}}\right)$

$V(0) = G(0) + W_8^0 H(0)$

$V(0) = 4 + 6 = 10$

$V(1) = G(1) + \left(\frac{1-j}{\sqrt{2}}\right) H(1) = (1-j) + \frac{(1-j)(1-j)}{\sqrt{2}}$

$V(1) = 1 - j - \sqrt{2}j = 1 - 2.414j$

$V(2) = G(2) + W_8^2 H(2) = -2$

$V(3) = G(3) + W_8^3 H(3) = 1 - 0.414j$

$V(4) = G(0) + W_8^4 H(0) = -2$

$V(5) = G(1) + W_8^5 H(1) = 1 + 0.414j$

$V(6) = G(2) + W_8^6 H(2) = -2$

$V(7) = G(3) + W_8^7 H(3) = 1 + 2.414j$

$V(K) = \{10, 1-2.414j, -2, 1-0.414j, -2, 1+0.414j, -2, 1+2.414j\}$

Z Transform

Z Transform of a discrete time signal $x(n)$ is represented by $x(n) \xrightarrow{Z.T} X(z)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Inverse Fourier Transform is given by

$$x(n) = \frac{1}{2\pi j} \int X(z) z^{n-1} dz$$

eg. Finite sequence $x(n)$ is defined as $x(n) = \{5, 3, -3, 0, 4, -2\}$. Find $X(z)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = x(0) z^0 + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + x(4) z^{-4} + x(5) z^{-5}$$

$$= 5 + 3z^{-1} + (-3) z^{-2} + 0 + 4z^{-4} + (-2) z^{-5}$$

$$= 5 + 3z^{-1} - 3z^{-2} + 4z^{-4} - 2z^{-5}$$

eg. A finite duration sequence $x(n) = \{5, 3, 0, \underset{\uparrow}{1}, 2, 4\}$ Find it's z transform

Ans) $X(z) = x(-3) z^3 + x(-2) z^2 + x(-1) z^1 + x(0) z^0 + x(1) z^{-1} + x(2) z^{-2}$

$$X(z) = 5z^3 + 3z^2 + 0 + 1 + 2z^{-1} + 4z^{-2}$$

$$X(z) = 5z^3 + 3z^2 + 1 + 2z^{-1} + 4z^{-2}$$

eg. Find the signal $x(n)$ for which the z transform is $X(z) = 4z^4 - z^3 - 3z + 4z^{-1} + 3z^{-2}$

Ans) $x(n) = \frac{1}{2\pi j} \int X(z) z^{n-1} dz$

simply

$x(n) = \{4, -1, 0, -3, \underset{\uparrow}{0}, 4, 3\}$

eg. Find z transform of the sequence $x(n) = a^{-n} u(-n-1)$

Ans) $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} a^{-n} u(-n-1) z^{-n}$

$= \sum_{n=-\infty}^{-1} a^{-n} z^{-n} = \sum_{n=-\infty}^{-1} (az)^{-n}$

$= \sum_{n=1}^{\infty} (az)^{n} = \frac{az}{1-az}$

eg. Find the z transform of the sequence $x(n) = -a^{n} u(-n-1)$

Ans) $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} -a^{n} u(-n-1) z^{-n}$

$= \sum_{n=-\infty}^{-1} -a^{n} z^{-n} = -\sum_{n=1}^{\infty} a^{-n} z^{n}$

$= -\sum_{n=1}^{\infty} (a^{-1} z)^{n} = -\left[\frac{a^{-1}z}{1-a^{-1}z}\right] = -\left[\frac{z}{a-z}\right]$

$= \frac{z}{z-a}$

eg) Find the z transform of the signals $a^{n} u(n)$ & $a^{-n} u(n)$

Ans) $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} a^{n} u(n) z^{-n}$

$= \sum_{n=0}^{\infty} a^{n} z^{-n} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$

$X(z) = \sum_{n=-\infty}^{\infty} a^{-n} u(n) z^{-n} = \sum_{n=0}^{\infty} (a^{-1} z^{-1})^{n}$

$= \frac{1}{1-a^{-1}z^{-1}} = \frac{az}{az-1}$

eg. Find the z transform of $u(n)$

$X(z) = \sum_{n=-\infty}^{\infty} u(n) z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$

eg. Find the z transform of impulse function

Ans) $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = 1(1) = 1$

eg. Find z transform of $\cos(n\omega)u(n)$

Ans) $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} \cos(n\omega)\, u(n)\, z^{-n}$

$$= \sum_{n=0}^{\infty} \cos(n\omega) z^{-n}$$

$$= \sum_{n=0}^{\infty} \left[\frac{e^{jn\omega} + e^{-jn\omega}}{2}\right] z^{-n}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} (e^{j\omega} z^{-1})^n + \frac{1}{2}\sum_{n=0}^{\infty} (e^{-j\omega} z^{-1})^n$$

$$= \frac{1}{2}\left(\frac{1}{1-e^{j\omega}z^{-1}}\right) + \frac{1}{2}\left(\frac{1}{1-e^{-j\omega}z^{-1}}\right)$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{j\omega}} + \frac{z}{z-e^{-j\omega}}\right]$$

$$= \frac{z}{2}\left[\frac{1}{z-e^{j\omega}} + \frac{1}{z-e^{-j\omega}}\right]$$

$$= \frac{z}{2}\left[\frac{2z - (2\cos\omega)}{z^2 - z(2\cos\omega) + 1}\right]$$

$$= z\left[\frac{z - \cos\omega}{z^2 - 2z\cos\omega + 1}\right]$$

$$= \left(\frac{z^2 - z\cos\omega}{z^2 - 2z\cos\omega + 1}\right)$$

Initial & Final Value Theorem in Z Transform

$$x(0) = \lim_{z\to\infty} X(z)$$

$$x(\infty) = \lim_{z\to 1} (z-1)X(z)$$

eg) $X(z) = \left(\frac{z}{z-1}\right)$

$$x(0) = \lim_{z \to \infty} \left(\frac{z}{z-1}\right) = \lim_{z \to \infty} \left(\frac{1}{1-\frac{1}{z}}\right) = 1$$

$$x(\infty) = \lim_{z \to 1} (z-1) \frac{z}{(z-1)} = 1$$

Basic Z Transform Pairs

$x(n)$	$X(z)$
$\delta(n)$	1
$u(n)$	$\frac{z}{z-1}$
$-u(-n-1)$	$\frac{z}{z-1}$
$\delta(n-m)$	z^{-m}
$a^n u(n)$	$\frac{z}{z-a}$
$-a^n u(-n-1)$	$\frac{z}{z-a}$
$n a^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$
$a^n \cos(\omega n) u(n)$	$\frac{z^2 - az\cos\omega}{z^2 - 2az\cos\omega + a^2}$
$a^n \sin(\omega n) u(n)$	$\frac{az\sin\omega}{z^2 - 2az\cos\omega + a^2}$

Region of Convergence

1) The ROC in z transform is indicated as circle contains a unit circle
2) If $x(n)$ is a right sided sequence & if $|z| = r$ in the ROC, then all finite values of z for which $|z| > r$ will also be in ROC
3) If $x(n)$ is a left sided sequence & if $|z| = r$ in the ROC, then all finite values of z for which $|z| < r$ will also be in ROC.
4) If $x(n)$ is a two sided signal & if $|z| = r$ circle is

in ROC, then the ROC will contain a ring in z plane that includes $|z| = r$

5) If the z transform of $x(n)$ is rational, then its ROC is bounded by poles or extended to ∞

6) If the z transform of $x(n)$ is rational and right sided, then ROC is z plane outside the outermost pole

7) If the z transform of $x(n)$ is rational & left sided then ROC is the region in z plane inside the innermost pole.

Properties of Z Transform

1. **Linearity**: If $x_1(n) \xrightarrow{Z.T.} X_1(z)$ &
$x_2(n) \xrightarrow{Z.T.} X_2(z)$

the linearity property states that

$$a_1 x_1(n) + a_2 x_2(n) \longleftrightarrow a_1 X_1(z) + a_2 X_2(z)$$

2. **Time shifting**: If $x(n) \xleftrightarrow{Z.T} X(z)$

then, $x(n-m) \xrightarrow{Z.T.} z^{-m} X(z)$

and, $x(n+m) \xrightarrow{Z.T.} z^{m} X(z)$

3. **Time Reversal Property**:

If $x(n) \xrightarrow{Z.T} X(z)$

then, $x(-n) \xrightarrow{Z.T.} X(z^{-1})$

4. **Scaling Property**: If $x(n) \xrightarrow{Z.T.} X(z)$

then, $a^n x(n) \xrightarrow{Z.T.} X\left(\frac{z}{a}\right)$

5. **Differentiation in Z Domain**:

If $x(n) \xrightarrow{Z.T.} X(z)$

$n x(n) \xrightarrow{Z.T.} -z \frac{d}{dz} X(z)$

6. **Multiplication and Convolution Property**

If $x_1(n) \xrightarrow{Z.T.} X_1(z)$

$x_2(n) \xrightarrow{Z.T.} X_2(z)$

Multiplication property states that

$x_1(n) \cdot x_2(n) \xrightarrow{ZT} X_1(z) \circledast X_2(z)$

Convolution property states that

$x_1(n) \circledast x_2(n) \xrightarrow{ZT} X_1(z) \cdot X_2(z)$

eg. Determine the system function and unit sample response of the system described by differential equation

$$y(n) = \frac{1}{2} y(n-1) + 2x(n)$$

Ans) Taking z transform on both sides

$$Y(z) = \frac{1}{2} z^{-1} Y(z) + 2X(z)$$

$$Y(z)\left[1 - \frac{1}{2} z^{-1}\right] = 2X(z)$$

$$\frac{Y(z)}{X(z)}\left[\frac{2 - z^{-1}}{2}\right] = 2$$

$$\frac{Y(z)}{X(z)} = \frac{4}{2 - z^{-1}} = \frac{2}{1 - \frac{z^{-1}}{2}} = \frac{2z}{z - \frac{1}{2}} = 2\left(\frac{z}{z - \frac{1}{2}}\right)$$

We know that

$$\boxed{H(z) = \frac{Y(z)}{X(z)}}$$

$H(z) = \frac{2z}{z - \frac{1}{2}}$ = System Function

$$h(n) = IZT\left[2\left(\frac{z}{z - \frac{1}{2}}\right)\right] = 2\left(\frac{1}{2}\right)^n u(n)$$

$h(n) = 2^{1-n} u(n)$ = Unit Sample Response

Causality and stability according to ROC of Z Transform

* A discrete time LTI system is causal if and only if
1. The ROC is the exterior of the circle outside the outermost pole
2. If $H(z)$ is expressed as ratio of polynomial in z,

the order of numerator can not be greater than order of denominator.

★ An LTI system is stable if

1. It's system function $H(z)$ includes the unit circle
2. A causal LTI system is stable if and only if all the poles of $H(z)$ lies inside the unit circle.

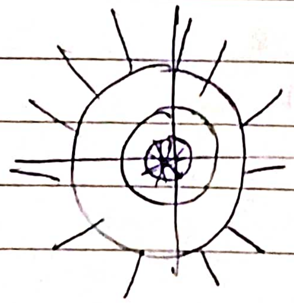
eg. Find the ROC of $x(n) = 3^{|n|}$

Ans) $x(n) = 3^{|n|} = 3^{n} u(n) + 3^{-n} u(-n-1)$

$$X(z) = \frac{z}{z-3} + \frac{z}{z-\frac{1}{3}}$$

Right sided signal — $|z| > 3$

left sided signal — $|z| < \frac{1}{3}$

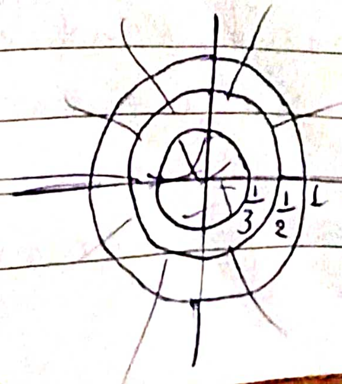


As there is no common region of convergence. So region of convergence can't be determined.

eg Check the causality and stability of system with $x(n) = \left(\frac{1}{2}\right)^{n} u(n) + 3^{-n} u(-n-1)$

$$\frac{z}{z-\frac{1}{2}} - \frac{z}{z-\frac{1}{3}}$$

$|z| > \frac{1}{2}$ $\quad$ $|z| < \frac{1}{3}$



There is no common ROC so we can't check causality and stability

Long Division Method to get Inverse Z Transform

$$X(z) = \frac{z}{z-\frac{1}{2}}$$

$$
\begin{array}{r|l|l}
z-\frac{1}{2} & z & 1+0.5z^{-1}+0.25z^{-2}+\dots \\
 & \pm z \mp 0.5 & \\
\hline
 & 0.5 & \\
 & \pm 0.5 \mp 0.25z^{-1} & \\
\hline
 & 0.25z^{-1} & \\
 & \pm 0.25z^{-1} \mp 0.125z^{-2} & \\
\hline
 & 0.125z^{-2} &
\end{array}
$$

$x(n) = \{1, 0.5, 0.25, 0.125, \dots\}$

$x(n)$ is positive side signal

$x(n) = \left(\frac{1}{2}\right)^n u(n)$

Note : (1) If $|z| > 1$, then we arrange numerator and denominator in descending power of z

eg. $X(z) = \frac{z+z^2}{3+z+z^3}$ $\quad |z|>1$

$= \frac{z^2+z}{z^3+z+3}$ [Descending power of z]

(2) If $|z| < 1$, then we arrange numerator and denominator in ascending power of z

eg. $X(z) = \frac{z^2+z}{z^3+z+3}$ $\quad |z|<1$

$= \frac{z+z^2}{3+z+z^3}$ [Ascending power of z]

Inverse Z Transform by Direct Computation / Partial Fraction

eg. Given $H(z) = \frac{z^2+2z}{z^2-3z+2}$. Find impulse response $h(n)$

Ans) Step 1: $\frac{H(z)}{z}$

Step 2: Partial Fraction
Step 3: Send z to right side
Step 4: Take I.Z.T.

We use this method when power of numerator $\geq$ power of denominator

$$\frac{H(z)}{z} = \frac{z+2}{z^2-3z+2} = \frac{(z+2)}{(z-1)(z-2)}$$

$$= \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

$$= \frac{(A+B)z + (-2A-B)}{(z-1)(z-2)}$$

$A+B = 1$ $\qquad$ $2A+B = -2$

$A+B = 1$

$\boxed{B = 4}$ $\qquad$ $\boxed{A = -3}$

$$\frac{H(z)}{z} = \frac{-3}{(z-1)} + \frac{4}{(z-2)}$$

$$H(z) = -3\left(\frac{z}{z-1}\right) + 4\left(\frac{z}{z-2}\right)$$

$$h(n) = -3\,u(n) + 4 \cdot 2^n\,u(n)$$

$$h(n) = -3\,u(n) + 2^{n+2}\,u(n)$$

Residue Method to Calculate Inverse Z Transform

$x(n) = \Sigma$ Residue of $X(z) \cdot X(z)\, z^{n-1}\Big|$ at the pole of residue

eg. $$X(z) = \frac{z^{18}}{\left(z-\frac{1}{2}\right)(z-1)(z-4)}$$

$X(z)$ converges for $|z| = 1$. Find $x(-16)$

Ans) $$x(n) = \cancel{\left(z-\frac{1}{2}\right)} \frac{z^{18}\; z^{n-1}}{\cancel{\left(z-\frac{1}{2}\right)}(z-1)(z-4)}\Bigg|_{\text{at } z=\frac{1}{2}} +$$

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$$\left.\frac{\cancel{(z-1)}\, z^{18}\, z^{n-1}}{\left(z-\frac{1}{2}\right)\cancel{(z-1)}(z-4)}\right|_{at\ z=1} + \left.\frac{\cancel{(z-4)}\, z^{18}\, z^{n-1}}{\left(z-\frac{1}{2}\right)(z-1)\cancel{(z-4)}}\right|_{at\ z=4}$$

$$= \cancel{z^{n-1}\left[\frac{z^{18}}{\left(z-\frac{1}{2}\right)(z-4)}\right|_{z=1}}$$

$$= \cancel{z^{18} z^{n-1}\left[\frac{1}{\left(-\frac{1}{2}\right)\left(-\frac{7}{2}\right)} + \frac{1}{\left(\frac{1}{2}\right)(-3)} + \frac{1}{\left(\frac{7}{2}\right)(3)}\right]}$$

$$= \left.\frac{z^{18} z^{n-1}}{(z-1)(z-4)}\right|_{at\ z=\frac{1}{2}} + \left.\frac{z^{18} z^{n-1}}{\left(z-\frac{1}{2}\right)(z-4)}\right|_{at\ z=1} + \left.\frac{z^{18} z^{n-1}}{\left(z-\frac{1}{2}\right)(z-1)}\right|_{at\ z=4}$$

As $X(z)$ converges for $|z|=1$ So we consider only

$$\left.\frac{z^{18} z^{n-1}}{\left(z-\frac{1}{2}\right)(z-4)}\right|_{at\ z=1}$$

$$x(n) = \frac{1}{\left(\frac{1}{2}\right)(-3)} = \frac{1}{\left(\frac{-3}{2}\right)} = \frac{-2}{3}$$

$$x(-16) = \frac{-2}{3}$$

ROC of discrete time sequence in Z Transform

1. If sequence is purely right sided or causal
   ROC: entire z plane except at $z=0$
   eg. $\{1, 2, 4, 0\}$
2. For left sequence and anticausal sequence
   ROC: entire z plane except at $z=\infty$
   eg. $\{1, 2, 4, \underset{\uparrow}{0}\}$

3) For two sided sequence ROC is entire z plane except at $z = 0$ and $z = \infty$

eg. $\{1, 2, \underset{\uparrow}{4}, 0\}$