Procedure to design Linear phase FIR Filters using windows.

EC Academy

$H_d(\omega) \rightarrow$ desired freq response

$H_d(\omega) \rightarrow$ F.T. of $h_d(n)$

$$\therefore H_d(\omega) = \sum_{n=0}^{\infty} h_d(n)\, e^{-j\omega n} \rightarrow (1)$$

$h_d(n) \rightarrow$ Inverse F.T. of $H_d(\omega)$

$$\therefore \underbrace{h_d(n)}_{\text{infinite duration}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega \rightarrow (2)$$

$h_d(n) \rightarrow$ finate $\rightarrow$ multiply $h_d(n)$ by window sequance 'M'

$\downarrow$ lenth M

Example: Rectangular window.

$$W_R(n) = \begin{cases} 1 \;; & n = 0, 1, 2, \ldots M-1 \\ 0 \;; & \text{otherwise.} \end{cases} \rightarrow (3)$$

$h_d(n) \rightarrow$ Sample response $\rightarrow$ Infinate

$$\therefore h(n) = h_d(n)\, W_R(n) \rightarrow (4)$$

$h_d(n) \rightarrow$ desired Sample response.

$$\therefore h(n) = \underbrace{\begin{cases} h_d(n) \;; & n = 0, 1, 2 \ldots M-1 \\ 0 \;; & \text{otherwise} \end{cases}}_{\text{Windowing}} \rightarrow (5)$$

Finate.

$(4) \Rightarrow h(n) = h_d(n) \cdot W_R(n)$

generally.

$h(n) = h_d(n) \cdot \underbrace{W(n)}_{\text{generalized window}} \rightarrow$ unit sample response FIR.

Freq response,

$$H(\omega) = F.T.\{h_d(n) \cdot W(n)\}$$

$$H(\omega) = H_d(\omega) * W(\omega) \rightarrow (6)$$

Design the Symmetric FIR lowpass Filter whose

$$H_d(w) = \begin{cases} e^{-jw\gamma} & ; \ |w| \le w_c \\ 0 & ; \ \text{otherwise} \end{cases}$$ with $M=7$ & $w_c = 1$ rad/sam.

use Rectangular window.

(i) obtain $h_d(n)$:

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(w)\, e^{jwn}\, dw \longrightarrow ①$$

$$H_d(w) = \begin{cases} e^{-jw\gamma} & ; \ -1 \le w \le 1 \\ 0 & ; \ \text{Otherwise.} \end{cases} \longrightarrow ②$$

$$\therefore h_d(n) = \frac{1}{2\pi}\int_{-1}^{1} e^{-jw\gamma} \cdot e^{jwn}\, dw = \frac{1}{2\pi}\int_{-1}^{1} e^{jw(n-\gamma)}\, dw$$

$$= \frac{1}{2\pi}\left[\frac{e^{jw(n-\gamma)}}{j(n-\gamma)}\right]_{-1}^{1} = \frac{1}{2\pi}\left[\frac{e^{j(n-\gamma)} - e^{-j(n-\gamma)}}{j(n-\gamma)}\right]$$

$$= \frac{1}{\pi(n-\gamma)}\left[\frac{e^{j(n-\gamma)} - e^{-j(n-\gamma)}}{2j}\right] \qquad \because \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$h_d(n) = \frac{\sin(n-\gamma)}{\pi(n-\gamma)} \qquad n \ne \gamma \longrightarrow ③$$

if $n = \gamma$

$$h_d(n) = \frac{1}{2\pi}\int_{-1}^{1} 1 \cdot dw = \frac{1}{2\pi}[2] = \frac{1}{\pi} \rightarrow ④$$

$$h_d(n) = \begin{cases} \dfrac{\sin(n-\gamma)}{\pi(n-\gamma)} & ; \ n \ne \gamma \\ \dfrac{1}{\pi} & ; \ n = \gamma \end{cases} \rightarrow ⑤$$

determine the Value of γ

$$h(n) = h(M-1-n)$$

$$\because h(n) = h_d(n) \cdot w(n)$$

$$h_d(n)w(n) = h_d(M-1-n)w(n)$$

$$h_d(n) = h_d(M-1-n)$$

$$\frac{\sin(n-\gamma)}{\pi(n-\gamma)} = \frac{\sin(M-1-n-\gamma)}{\pi(M-1-n-\gamma)}$$

$$\frac{-\sin(n-\tau)}{-\pi(n-\tau)} = \frac{\sin(M-1-n-\tau)}{\pi(M-1-n-\tau)} \qquad \because -\sin\theta = \sin(-\theta)$$

$$\frac{\sin[-(n-\tau)]}{\pi(-(n-\tau))} = \frac{\sin(M-1-n-\tau)}{\pi(M-1-n-\tau)}$$

$$-(n-\tau) = M-1-n-\tau$$

$$-\cancel{n}+\tau = M-1-\cancel{n}-\tau$$

$$2\tau = M-1$$

$$\boxed{\tau = \frac{M-1}{2}}$$

(5) $\Rightarrow$
$$h_d(n) = \begin{cases} \dfrac{\sin\left(n-\frac{M-1}{2}\right)}{\pi\left(n-\frac{M-1}{2}\right)} & ; n \neq \frac{M-1}{2} \\ \dfrac{1}{\pi} & ; n = \frac{M-1}{3} \end{cases}$$

$\because M = 7$

$$h_d(n) = \begin{cases} \dfrac{\sin(n-3)}{\pi(n-3)} & ; n \neq 3 \\ \dfrac{1}{\pi} & ; n = 3 \end{cases}$$

Put $n = 0$ to 6

$n=0$; $h_d(0) = 0.01497$

$n=1$; $h_d(1) = 0.14472$

$n=2$; $h_d(2) = 0.26785$

$n=3$; $h_d(3) = \frac{1}{\pi}$

$n=4$; $h_d(4) = 0.26785$

$n=5$; $h_d(5) = 0.14472$

$n=6$; $h_d(6) = 0.01497$

$\because h(n) = h_d(n) \cdot w(n)$ $\quad w_n(n) = \begin{cases} 1 & ; 0 \le n \le 6 \\ 0 & ; \text{other} \end{cases}$

$$h(n) = \begin{cases} h_d(n) & ; 0 \le n \le 6 \\ 0 & ; \text{otherwise} \end{cases}$$

Co-efficients of FIR Filter

$\rightarrow h(0) = 0.01497$

$\rightarrow h(1) = 0.14472$

$\rightarrow h(2) = 0.26785$

$h(3) = \frac{1}{\pi}$

$\rightarrow h(4) = 0.26785$

$\rightarrow h(5) = 0.14472$

$\rightarrow h(6) = 0.01497$

Symmetric
$h(n) = h(6-n)$

Design the Symmetric FIR lowpass filter whose

$$H_d(w) = \begin{cases} e^{-jw\alpha} & ; |w| \le w_c \\ 0 & ; \text{Otherwise} \end{cases}$$ with $M=7$ & $w_c = 1$ rad/sam.

Use Hanning window.

$n=0$; $h_d(0) = 0.01497$
$n=1$; $h_d(1) = 0.14472$
$n=2$; $h_d(2) = 0.26785$
$n=3$; $h_d(3) = \frac{1}{\pi}$
$n=4$; $h_d(4) = 0.26785$
$n=5$; $h_d(5) = 0.14472$
$n=6$; $h_d(6) = 0.1497$

Hanning window $W(n) = 0.5\left[1-\cos\left(\frac{2\pi n}{M-1}\right)\right]$

$\because M=7$ $\quad W(n) = 0.5\left[1-\cos\left(\frac{2\pi n}{6}\right)\right]$

$\Rightarrow W(n) = 0.5\left[1-\cos\left(\frac{\pi n}{3}\right)\right]$ $\quad n=0$ to 6 $\quad M=7$

Rad $\quad 0.5\times(1-\cos(\pi A \div 3))$

$n=0$; $w(0) = 0$
$n=1$; $w(1) = 0.25$
$n=2$; $w(2) = 0.75$
$n=3$; $w(3) = 1$
$n=4$; $w(4) = 0.75$
$n=5$; $w(5) = 0.25$
$n=6$; $w(6) = 0$

$h(n) = h_d(n) \cdot w(n)$

$\therefore h(0) = 0$
$h(1) = 0.03618$
$h(2) = 0.20089$
$h(3) = \frac{1}{\pi}$
$h(4) = 0.20089$
$h(5) = 0.03618$
$h(6) = 0$

Determine the Filter coefficients $h_d(n)$ for desired frq. response of a low pass Filter given by,

$$H_d(e^{jw}) = \begin{cases} e^{-j2w} \; ; & -\frac{\pi}{4} \le w \le \frac{\pi}{4} \\ 0 \; ; & \frac{\pi}{4} \le |w| \le \pi \end{cases}$$

If we define new filter co-efficients by $h_d(n) \cdot w(n)$ where $w(n) = \begin{cases} 1 \; ; & 0 \le n \le 4 \\ 0 \; ; & \text{elsewhere} \end{cases}$

Determine $h(n)$ and also the frq. response $H(e^{jw})$ and compare with $H_d(e^{jw})$. Determine $H(e^{jw})$ using the Hamming window.

(i) obtain $h_d(n)$:

$$h_d(n) = \frac{1}{2\pi} \int_{-2\pi}^{\pi} H_d(e^{jw}) e^{jwn} dw = \frac{1}{2\pi} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} e^{-j2w} \cdot e^{jwn} dw$$

$$= \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} e^{jw(n-2)} dw = \frac{1}{2\pi} \left[\frac{e^{jw(n-2)}}{j(n-2} \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\frac{\pi}{4}(n-2)} - e^{-j\frac{\pi}{4}(n-2)}}{j(n-2)} \right] = \frac{1}{\pi(n-2)} \left[\frac{e^{j\frac{\pi}{4}(n-2)} - e^{-j\frac{\pi}{4}(n-2)}}{2j} \right]$$

$$h_d(n) = \frac{\sin \frac{\pi}{4}(n-2)}{\pi(n-2)} \; ; \; n \neq 2$$

if $n = 2$

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} 1 \, dw = \frac{1}{2\pi} \left[\frac{2\pi}{4} \right] = \frac{1}{4}$$

$$\therefore h_d(n) = \begin{cases} \frac{\sin \frac{\pi}{4}(n-2)}{\pi(n-2)} \; ; & n \neq 2 \\ \frac{1}{4} \; ; & n = 2 \end{cases}$$

(ii) obtain $h(n)$:

$h(n) = h_d(n) \cdot w(n) \; ; \; 0 \le n \le 4$

$n=0$; $h(0) = h_d(0) = 0.159091$

$n=1$; $h(1) = h_d(1) = 0.224989$

$n=2$; $h(2) = h_d(2) = \frac{1}{4}$

$n=3$; $h(3) = h_d(3) = 0.224989$

$n=4$; $h(4) = h_d(4) = 0.159091$

(iii) Obtain $H(e^{jw})$:

$\because M=5$; $0 \le n \le 4$

$\hookrightarrow$ odd.

$$\therefore H(w) = e^{-jw\left(\frac{M-1}{2}\right)}\left\{h\left(\frac{M-1}{2}\right)+2\sum_{n=0}^{\frac{M-3}{2}} h(n)\cos w\left(n-\frac{M-1}{2}\right)\right\}$$

$$H(w) = e^{-j2w}\left[h(2)+2\sum_{n=0}^{1} h(n)\cos w(n-2)\right]$$

$$= e^{-j2w}\left[h(2)+2h(0)\cos w(-2)+2h(1)\cos w(-1)\right]$$

$$= e^{-j2w}\left[0.25+2\times 0.159091\cos 2w+2\times 0.224989\cos w\right]$$

$$= e^{-j2w}\left[0.25+0.318\cos 2w+0.45\cos w\right]$$

$$\boxed{|H(w)| = 0.25+0.318\cos 2w+0.45\cos w}$$

$$\angle H(w) = \begin{cases} -2w \ ; & |H(w)|>0 \\ -2w+\pi \ ; & |H(w)|<0 \end{cases}$$

$H(e^{jw})$ with $H_d(e^{jw}) \rightarrow$ different & will not same.

(iv) obtain $H(e^{j\omega})$ using Hamming window. $M=5$

Hamming window $w(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{M-1}\right)$; $0 \le n \le M-1$

$w(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{4}\right)$; $0 \le n \le 4$

$h(n) = h_d(n) \cdot w(n)$

$h_d(0) = 0.15901$
$h_d(1) = 0.224984$
$h_d(2) = 0.25$
$h_d(3) = 0.224984$
$h_d(4) = 0.159091$

$n=0$; $w(0) = 0.08$
$n=1$; $w(1) = 0.54$
$n=2$; $w(2) = 1$
$n=3$; $w(3) = 0.54$
$n=4$; $w(4) = 0.08$

$h(0) = 0.01273$
$h(1) = 0.12149$
$h(2) = 0.25$
$h(3) = 0.12149$
$h(4) = 0.01273$

$M=5$ → odd

$$H(\omega) = e^{-j2\omega}\left[h(2) + 2\sum_{n=0}^{1} h(n)\cos\omega(n-2)\right]$$

$$H(e^{j\omega}) = H(\omega) = e^{-j2\omega}\left[0.25 + 2\times 0.01273\cos 2\omega + 2\times 0.12149\cos\omega\right]$$

$$H(e^{j\omega}) = e^{-j2\omega}\left[0.25 + 0.02546\cos 2\omega + 0.243\cos\omega\right]$$

Design a FIR linear phase Filter using Kaiser window to meet the following specifications

$$0.99 \le |H(e^{jw})| \le 1.01 \; ; \; 0 \le |w| \le 0.19\pi$$

$$|H(e^{jw})| \le 0.01 \; ; \; 0.21\pi \le |w| \le \pi$$

$$1-0.01 \le |H(e^{jw})| \le 1+0.01 \; ; \; 0 \le |w| \le \underbrace{0.19\pi}_{w_p}$$

$$|H(e^{jw})| \le 0.01 \; ; \; \underbrace{0.21\pi}_{w_s} \le |w| \le \pi$$

$\delta_1 = 0.01$ & $\delta_2 = 0.01$ $w_p = 0.19\pi$ (Passband edge freq) $w_s = 0.21\pi$ (Stopband edge freq)

$$\therefore \Delta w = w_s - w_p = 0.21\pi - 0.19\pi = 0.02\pi$$

δ = min of δ_1 & δ_2 $\Rightarrow \delta = 0.01$

Attenuation $A = -20\log_{10}\delta = -20\log_{10} 0.01 = 40$

(i) Cut off freq w_c:

$$w_c = \frac{w_p + w_s}{2} = \frac{0.19\pi + 0.21\pi}{2}$$

$$w_c = 0.2\pi$$

(ii) To obtain β & M:

$$\beta = \begin{cases} 0.1102(A-8.7) \; ; & A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21) \; ; & 21 \le A \le 50 \\ 0 \; ; & A < 21 \end{cases}$$

$$\therefore \beta = 0.5842(40-21)^{0.4} + 0.07886(40-21) = 3.395$$

$$M = \frac{A-8}{2.285\Delta w} = \frac{40-8}{2.285(0.02\pi)} = 223$$

(iii) eq.n for Kaiser window:

$$\alpha = \frac{M}{2} = \frac{223}{2} = 111.5$$

$$W_K(n) = \begin{cases} \dfrac{I_0\left\{\beta\left[1-\left[\frac{n-\alpha}{\alpha}\right]^2\right]^{1/2}\right\}}{I_0(\beta)} & ; 0 \le n \le M \\ 0 & ; \text{otherwise} \end{cases}$$

$$w(n) = \begin{cases} \dfrac{I_0\left\{3.395\left[1-\left[\frac{n-111.5}{111.5}\right]^2\right]^{1/2}\right\}}{I_0(3.395)} & ; 0 \le n \le 223 \\ 0 & ; \text{otherwise} \end{cases}$$

(iv) Obtain $h_d(n)$:

Ideal freq response

$$H_d(\omega) = \begin{cases} e^{-j\omega\left(\frac{M-1}{2}\right)} & ; -\omega_c \le \omega \le \omega_c \\ 0 & ; \text{otherwise} \end{cases}$$

I.F.T.

$$h_d(n) = \begin{cases} \dfrac{\sin\left[\omega_c\left[n-\frac{M-1}{2}\right]\right]}{\pi\left(n-\frac{M-1}{2}\right)} & ; n \ne \frac{M-1}{2} \\ \dfrac{\omega_c}{\pi} & ; n = \frac{M-1}{2} \end{cases}$$

$M = 223 \quad \therefore \underbrace{\frac{M}{2} = 111.5}_{h_d(n)} \rightarrow$

$$\therefore h_d(n) = \frac{\sin\left[0.2\pi\left[n-\frac{223}{2}\right]\right]}{\pi\left(n-\frac{223}{2}\right)}$$

$$h_d(n) = \frac{\sin\left[0.2\pi(n-111.5)\right]}{\pi(n-111.5)} \quad ; 0 \le n \le 223$$

(v) Obtain h(n):

$$h(n) = h_d(n) \cdot w(n)$$

$$\therefore h(n) = \begin{cases} \dfrac{\sin(0.2\pi(n-111.5))}{\pi(n-111.5)} \cdot \dfrac{I_0\left\{3.395\left[1-\left[\frac{n-111.5}{111.5}\right]^2\right]^{1/2}\right\}}{I_0(3.395)} & ; 0 \le n \le 223 \\ 0 & ; \text{otherwise} \end{cases}$$

Design of Linear Phase FIR filters using Frequency Sampling:

Desired frq response $\rightarrow H_d(\omega)$

This frq response is sampled at 'M' points $\rightarrow \boxed{\omega = \frac{2\pi}{M}K}$

$K = 0, 1, 2 \ldots M-1$

Discrete fourier tansform

$$H(K) = H_d(\omega) \quad ; K = 0, 1, 2 \ldots M-1$$

$$H(K) = H_d\left[\frac{2\pi}{M}K\right] ; K = 0, 1, 2 \ldots M-1$$

$H(K) \rightarrow$ M-Point DFT.

Take IDFT of $H(K)$ to get $h(n)$

$h(n) \rightarrow$ unit sample response of FIR filter.

If $M \Rightarrow$ odd:

$$h(n) = \frac{1}{M}\left[H(0) + 2\sum_{K=1}^{\frac{M-1}{2}} Re\left\{H(K)\, e^{j\frac{2\pi}{N}Kn}\right\}\right]$$

If $M \Rightarrow$ even:

$$h(n) = \frac{1}{M}\left[H(0) + 2\sum_{K=1}^{\frac{M}{2}-1} Re\left\{H(K)\, e^{j\frac{2\pi}{N}Kn}\right\}\right]$$

Window

$$H_d(\omega) \xrightarrow{IDFT} h_d(n) \xrightarrow{finite} \overbrace{h_d(n) \cdot w(n)}^{h(n)}$$

$h_d(n)$ ↑ infinite

Determine the impulse response $h(n)$ of a filter having desired frq. response

$$H_d(e^{j\omega}) = \begin{cases} e^{-j(N-1)\omega/2} & ; 0 \le |\omega| \le \frac{\pi}{2} \\ 0 & ; \frac{\pi}{2} \le |\omega| \le \pi \end{cases}$$

$M = N = 7$, use frq. sampling approach.

(i) Desired frq. response:

$$\underline{N=7} \Rightarrow H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega} & ; 0 \le |\omega| \le \frac{\pi}{2} \\ 0 & ; \frac{\pi}{2} \le |\omega| \le \pi \end{cases}$$

(ii) Sample $H_d(e^{j\omega})$:

$$e^{-j3 \cdot \frac{2\pi k}{7}} = e^{-j\frac{6\pi k}{7}}$$

Put $\omega = \frac{2\pi k}{N}$; $k = 0, 1, 2 \dots N-1$

For $N=7 \Rightarrow \omega = \frac{2\pi k}{7}$; $k = 0, 1, 2 \dots 6$

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\frac{6\pi k}{7}} & ; 0 \le \frac{2\pi k}{7} \le \frac{\pi}{2} \\ 0 & ; \frac{\pi}{2} \le \frac{2\pi k}{7} \le \pi \end{cases}$$

$$\frac{2\pi k}{7} = \frac{\pi}{2} \Rightarrow \boxed{k = \frac{7}{4}}$$

$$\frac{2\pi k}{7} = \frac{\pi}{2} \Rightarrow \boxed{k = \frac{7}{4}}$$

$$\frac{2\pi k}{7} = \pi \Rightarrow \boxed{k = \frac{7}{2}}$$

$$\cancel{H_d(e^{j\omega})} = H_d(k) = \begin{cases} e^{-j\frac{6\pi k}{7}} & ; 0 \le k \le \frac{7}{4} \\ 0 & ; \frac{7}{4} \le k \le \frac{7}{2} \end{cases}$$

(iii) To obtain $h(n)$:

$M = N = 7 \rightarrow$ odd

$$h(n) = \frac{1}{M}\left[H(0) + 2\sum_{k=1}^{\frac{M-1}{2}} Re\left\{H(k) e^{j\frac{2\pi}{N}kn}\right\}\right]$$

$$\frac{M-1}{2} \Rightarrow \frac{7-1}{2} \Rightarrow 3 \quad ; H(0) = 1$$

$$h(n) = \frac{1}{7}\left[1 + 2 \cdot \sum_{k=1}^{3} Re\left\{e^{-j\frac{6\pi k}{7}} \cdot e^{j2\pi kn/7}\right\}\right]$$

$$h(n) = \frac{1}{7}\left[1 + 2\sum_{k=1}^{3} Re\left\{e^{-j2\pi k(3-n)/7}\right\}\right]$$

$$e^{-j\theta} = \underbrace{\cos\theta}_{Re} - j\sin\theta \Rightarrow Re\left[e^{-j\theta}\right] = \cos\theta$$

$$h(n) = \frac{1}{7}\left[1 + 2\sum_{k=1}^{3}\cos\left[\frac{2\pi k(3-n)}{7}\right]\right] \quad ; n = 0, 1, 2 \dots 6$$