Trigonometric Fourier Series

$x(t)$ = dc / avg. value of $x(t)$ + cosine terms + sine terms

$$\boxed{x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t}$$

Fourier coeff.

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt$$

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cdot \cos n\omega_0 t\,dt$$

$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\cdot \sin n\omega_0 t\,dt$$

Ex:- $x(t) = \underbrace{4}_{a_0} + 3\cos 2\omega_0 t + 5\sin 2\omega_0 t + 4\cos 3\omega_0 t + 2\sin 3\omega_0 t + \cdots$

2nd harmonics

3rd harmonics

$a_2 = 3$

$b_2 = 5$

$a_2 < b_2 \Rightarrow$ more sine

$a_3 = 4$, $b_3 = 2$ | $a_3 > b_3 \Rightarrow$ more cosine

Fourier coeff.

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt$$

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cdot \cos n\omega_0 t\,dt$$

$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\cdot \sin n\omega_0 t\,dt$$

a_0

$+ 4\cos 3\omega_0 t + 2\sin 3\omega_0 t + \cdots$

3rd harmonics

$a_2 = 3$

$b_2 = 5$

$a_2 < b_2 \Rightarrow$ more sine

$a_3 = 4$, $b_3 = 2$ | $a_3 > b_3 \Rightarrow$ more cosine

→ $x(t) \rightarrow$ Symm. about t-axis $\Rightarrow a_0 = 0$

→ $x(t) \rightarrow$ Even signal $\Rightarrow b_n = 0$

$$x(-t) = x(t)$$

→ $x(t) \rightarrow$ Odd signal $\Rightarrow a_n = 0$

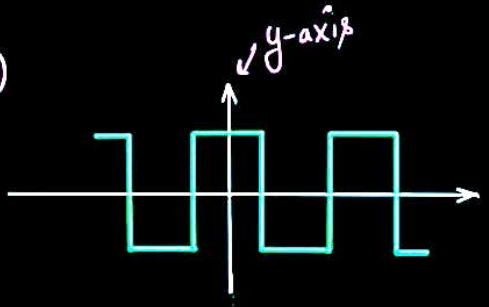
$$x(-t) = -x(t)$$

$x(-t) = x(t)$

$\rightarrow \quad x(t) \rightarrow$ odd signal $\Rightarrow a_n = 0$

$x(-t) = -x(t)$ ✓

i)

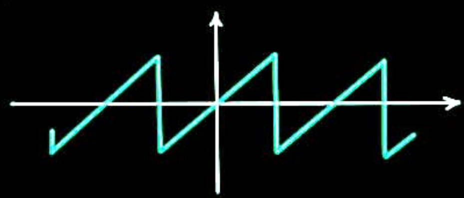


$a_0 = 0$

$b_n = 0$

$a_n \neq 0$

ii)

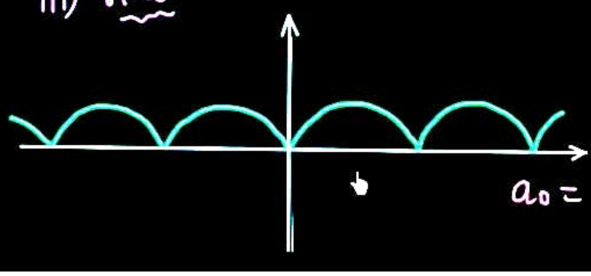


$a_0 = 0$

$a_n = 0$

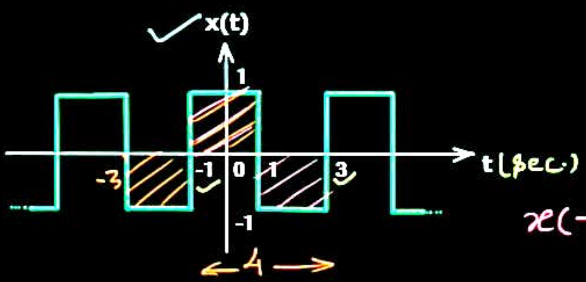
$b_n \neq 0$

iii) H.W.



$a_0 = ?$ $\quad a_n = ?$ $\quad b_n = ?$

Trigonometric Fourier Series (Example-1)



$$x(-t) = x(t)$$

$$x(t) = \cancelto{0}{a_0} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} \cancelto{0}{b_n} \sin n\omega_0 t$$

$$x(t) = \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$$

$$T_0 = 4 \text{ sec}$$

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cdot \cos n\omega_0 t \, dt$$

$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{4}$$

$$\omega_0 = \frac{\pi}{2} \text{ rad/sec}$$

$$a_n = \frac{1}{2}\int_{-1}^{3} x(t)\cdot \cos \frac{n\pi}{2} t \, dt$$

$$\omega_0 = \frac{\pi}{2} \text{ rad/sec}$$

$$a_n = \frac{1}{2}\int_{-1}^{3} x(t)\cdot \cos\frac{n\pi}{2}t\,dt$$

$$a_n = \frac{1}{2}\left[\int_{-1}^{1}(1)\cos\frac{n\pi}{2}t\,dt + \int_{1}^{3}(-1)\cdot\cos\frac{n\pi}{2}t\,dt\right]$$

$$a_n = \frac{1}{2}\left[\int_{-1}^{1}\cos\frac{n\pi}{2}t\,dt - \int_{1}^{3}\cos\frac{n\pi}{2}t\,dt\right]$$

$$a_n = \frac{1}{2}\left[\int_{-n\pi/2}^{n\pi/2}\cos\theta\,\frac{2}{n\pi}\,d\theta - \int_{n\pi/2}^{3n\pi/2}\cos\theta\,\frac{2}{n\pi}\,d\theta\right]$$

$$a_n = \frac{1}{\cancel{2}}\times\frac{\cancel{2}}{n\pi}\left[\int_{-n\pi/2}^{n\pi/2}\cos\theta\,d\theta - \int_{n\pi/2}^{3n\pi/2}\cos\theta\,d\theta\right]$$

$$a_n = \frac{1}{n\pi}\left[(\sin\theta)_{-n\pi/2}^{n\pi/2} - (\sin\theta)_{n\pi/2}^{3n\pi/2}\right]$$

$$\frac{n\pi}{2}t = \theta$$

$$\frac{n\pi}{2}dt = d\theta \Rightarrow dt = \frac{2}{n\pi}d\theta$$

$$t = -1 \Rightarrow \theta = -\frac{n\pi}{2}$$

$$t = 1 \Rightarrow \theta = \frac{n\pi}{2}$$

$$t = 3 \Rightarrow \theta = \frac{3n\pi}{2}$$

$$a_n = \frac{1}{n\pi}\left[(\sin\theta)_{-n\pi/2}^{n\pi/2} - (\sin\theta)_{n\pi/2}^{3n\pi/2} \right]$$

2π

$4\pi = 2 \times 360$

$\sin n\pi/2$

$n=1$

$\sin(\pi+\pi$

$-\sin\frac{n\pi}{2}$

$$a_n = \frac{1}{n\pi}\left[\sin\frac{n\pi}{2} - \sin(-n\pi/2) - \sin\frac{3n\pi}{2} + \sin\frac{n\pi}{2} \right]$$

$\sin(\pi-\theta)$

$+\sin\theta$

$\sin\left(0-\frac{n\pi}{2}\right)$

$-\sin\frac{n\pi}{2}$

Case I :- n = even

$$a_n = \frac{1}{n\pi}\left[\sin\frac{n\pi}{2} + \cancel{\sin\frac{n\pi}{2}} - \cancel{\sin\frac{n\pi}{2}} + \sin\frac{n\pi}{2} \right]$$

$$a_n = \frac{2}{n\pi}\sin\frac{n\pi}{2}$$

$n=2$

$$a_n = \frac{2}{2\pi}\sin\frac{2\pi}{2} \Rightarrow a_n = \frac{1}{\pi}\sin\pi \qquad \boxed{\sin n\pi = 0}$$

$$\boxed{a_n = 0}$$

Case II :- n = odd

Case a : $n = 1, 5, 9, 13, \ldots$

Case b : $n = 3, 7, 11, 15, \ldots$

Case II :- $n = \text{odd}$ $\qquad a_n = \frac{1}{n\pi}\left[\sin\frac{n\pi}{2} + \sin\frac{n\pi}{2} + \sin\frac{n\pi}{2} + \sin\frac{n\pi}{2}\right] \Rightarrow a_n = \frac{4}{n\pi}\sin\frac{n\pi}{2}$

case a: $n = 1, 5, 9, 13, \ldots$

$$a_n = \frac{4}{n\pi}$$

case b: $n = 3, 7, 11, 15, \ldots$

$$a_n = \frac{-4}{n\pi}$$

$$a_3 = \frac{-4}{3\pi}$$

$$x(t) = \sum_{n=1}^{\infty} a_n \cos n\frac{\pi}{2}t$$

$$= \frac{4}{\pi}\cos\frac{\pi}{2}t - \frac{4}{3\pi}\cos\frac{3\pi}{2}t + \frac{4}{5\pi}\cos\frac{5\pi}{2}t - \frac{4}{7\pi}\cos\frac{7\pi}{2}t + \cdots$$

$$x(t) = \frac{4}{\pi}\left[\cos\frac{\pi}{2}t - \frac{1}{3}\cos\frac{3\pi}{2}t + \frac{1}{5}\cos\frac{5\pi}{2}t - \frac{1}{7}\cos\frac{7\pi}{2}t + \cdots\right]$$

Ans

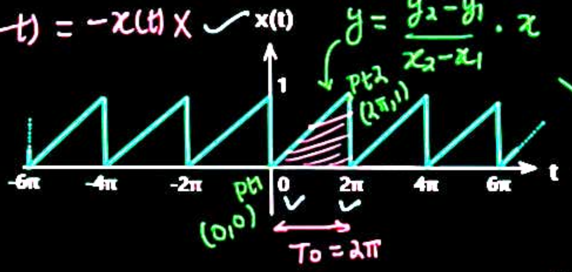
$x(-t) = x(t)$ ×

Trigonometric Fourier Series (Example-2)

IES

$x(-t) = -x(t)$ ×

$$y = \frac{y_2 - y_1}{x_2 - x_1} \cdot x$$



$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$y = \frac{1-0}{2\pi - 0} \cdot x \Rightarrow y = \frac{x}{2\pi} \Rightarrow x(t) = \frac{t}{2\pi}$$

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt \qquad \omega_0 = \frac{2\pi}{T_0}$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{t}{2\pi}\,dt \qquad \omega_0 = \frac{\cancel{2\pi}}{\cancel{2\pi}} = 1\ \frac{\text{rad}}{\text{sec}}$$

$$= \frac{1}{4\pi^2}\left[\frac{t^2}{2}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi^2}\left[\frac{4\pi^2}{2} - 0\right]$$

$$= \frac{1}{\cancel{4\pi^2}} \times \frac{\cancel{4\pi^2}}{2}$$

$$a_0 = 1/2$$

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos n\omega_0 t\,dt$$

$$a_n = \frac{\cancel{2}}{\cancel{2}\pi}\int_0^{2\pi} \frac{t}{2\pi}\cos nt\,dt$$

$$a_n = \frac{1}{2\pi^2}\int_0^{2\pi} \underbrace{t}_{A} \cdot \underbrace{\cos nt}_{T}\,dt$$

$$\int \underset{1}{u}\,\underset{2}{v}\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\int v\,dx\right]dx$$

ILATE

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1} b_n \sin n\omega_0 t$$

$$y = \frac{1-0}{2\pi - 0} \cdot x \Rightarrow y = \frac{x}{2\pi} \Rightarrow x(t) = \frac{t}{2\pi}$$

$$x(t) = 1/2 + \sum_{n=1}^{\infty} \frac{-1}{n\pi} \sin n\cancel{\omega_0} t$$

$$= \frac{1}{2} - \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin nt$$

$$\frac{2}{T_0} \int_{T_0} x(t) \cos n\omega_0 t \, dt$$

$$a_n = \frac{\cancel{2}}{\cancel{2}\pi} \int_0^{2\pi} \frac{t}{2\pi} \cos nt \, dt$$

$$a_n = \frac{1}{2\pi^2} \int_0^{2\pi} \underset{A}{t} \cdot \underset{T}{\cos nt} \, dt$$

1st

$$dx = u \int v dx - \int \left[\frac{du}{dx} \int v dx \right] dx$$

ILATE

$$\int t \cdot \cos nt \, dt = t \int \cos nt \, dt - \int \left[\frac{dt}{dt} \int \cos nt \, dt \right] dt$$

$$= \frac{t \cdot \sin nt}{n} - \int 1 \cdot \frac{\sin nt}{n} dt$$

$$= \frac{t \sin nt}{n} - \left[- \frac{\cos nt}{n^2} \right]$$

$$= \frac{t \sin nt}{n} + \frac{\cos nt}{n^2}$$

$$a_n = \frac{1}{2\pi^2} \left[\frac{t \sin nt}{n} + \frac{\cos nt}{n^2} \right]_0^{2\pi}$$

$$a_n = \frac{1}{2\pi^2} \left[\frac{2\pi \sin \overset{0}{\cancel{n 2\pi}}}{n} - \frac{(0) \sin \overset{0}{\cancel{n(0)}}}{n} + \frac{\overset{1}{\cancel{\cos n(2\pi)}}}{n^2} - \frac{\overset{1}{\cancel{\cos n(0)}}}{n^2} \right]$$

$$a_n = \frac{1}{2\pi^2} \left[0 - 0 + \cancel{\frac{1}{n^2}} - \cancel{\frac{1}{n^2}} \right]$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{-1}{n\pi}$$

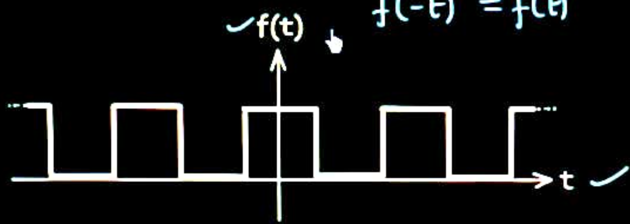
Trigonometric Fourier Series (Example-3)

Question: The Fourier expansion $f(t) = a_0 + \Sigma(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$ of the periodic signal shown below will contain the following nonzero terms

$\cos \omega_0 t$

$\cos 3\omega_0 t$

$f(-t) = f(t)$



(a) a_0 and b_n, $n = 1, 3, 5, \ldots\ldots \infty$

(b) a_0 and a_n, $n = 1, 2, 3, \ldots\ldots \infty$

(c) a_0, a_n and b_n, $n = 1, 2, 3, \ldots\ldots \infty$

(d) a_0 and a_n, $n = 1, 3, 5, \ldots\ldots \infty$

even + odd har.

only odd har.

$a_0 \rightarrow$ avg. of $f(t) \neq 0$

Even signal $\Rightarrow b_n = 0$

[ESE-2011]

Trigonometric Fourier Series (Example-4)

Question: x(t) is a real valued function of a real variable with period T. Its trigonometric Fourier series expansion contains no terms of frequency $\omega = 2\pi(2k)/T$; k = 1,2,...... Also, no sine terms are present. Then x(t) satisfies the equation.

(a) $x(t) = -x(t-T)$
(b) $x(t) = x(T-t) = -x(-t)$
(c) $x(t) = x(T-t) = -x(t-T/2)$
(d) $x(t) = x(t-T) = x(t-T/2)$

[GATE-2006]

1. Time period $= T = T_0$

2. $\omega = \frac{2\pi}{T}(2k)$; $k = 1, 2, 3, \ldots$ ($\frac{2\pi}{T} = \omega_0$, $2k$ = int)

$\omega = (\text{int.})\,\omega_0$

$\omega_1 = 2\omega_0$
$\omega_2 = 4\omega_0$
} Even harmonics X $\Rightarrow$ Odd harmonics $\Rightarrow x(t) \leftarrow$ derived from HWS

$x(t) = -x(t \pm T/2)$

3. $b_n = 0 \Rightarrow$ Even signal $\Rightarrow x(-t) = x(t)$

$x(t) = x(t+T)$ ← left shifting

$t \to -t$

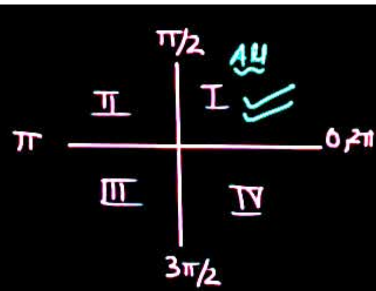
$x(-t) = x(-t+T)$

$x(t) = x(-t) = x(T-t)$

Trigonometric Fourier Series (Example-5)

Question: Prove $x(t)$ is periodic

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$



Condition for periodicity: $x(t) = x(t \pm \underset{\text{time period}}{T})$

$$x(t) = x(t + T_0)$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$\omega_0 = \frac{2\pi}{T_0}$$

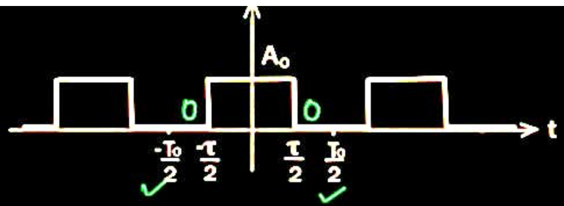
$$\omega_0 T_0 = 2\pi$$

$$x(t+T_0) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0(t+T_0) + \sum_{n=1}^{\infty} b_n \sin n\omega_0(t+T_0)$$

$$x(t+T_0) = a_0 + \sum_{n=1}^{\infty} a_n \cos n(\omega_0 t + \omega_0 T_0) + \sum_{n=1}^{\infty} b_n \sin n(\omega_0 t + \omega_0 T_0)$$

$$x(t+T_0) = a_0 + \sum_{n=1}^{\infty} a_n \cos n(\omega_0 t + 2\pi) + \sum_{n=1}^{\infty} b_n \sin n(\omega_0 t + 2\pi)$$

$$x(t+T_0) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t = x(t)$$



हल:- $C_n = \frac{1}{T_0} \int_{T_0} x(t) \cdot e^{-jn\omega_0 t} \, dt$

$$C_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \cdot e^{-jn\omega_0 t} \, dt$$

$$= \frac{1}{T_0} \left[\int_{-T_0/2}^{-\tau/2} \underbrace{0 \cdot e^{-jn\omega_0 t}}_{0} \, dt + \int_{-\tau/2}^{\tau/2} A_0 \cdot e^{-jn\omega_0 t} \, dt + \int_{\tau/2}^{T_0/2} \underbrace{0 \cdot e^{-jn\omega_0 t}}_{0} \, dt \right]$$

$$= \frac{A_0}{T_0} \int_{-\tau/2}^{\tau/2} e^{-jn\omega_0 t} \, dt$$

Let $-jn\omega_0 t = \theta$

$-jn\omega_0 dt = d\theta \Rightarrow dt = \frac{-1}{jn\omega_0} d\theta$

when, $t = -\tau/2$

$\theta = jn\omega_0 \tau/2$

$t = \tau/2$

$\theta = -jn\omega_0 \tau/2$

$$= \frac{A_0}{T_0}\int_{jnw_0\tau/2}^{-jnw_0\tau/2} e^{\theta}\left(\frac{-1}{jnw_0}\right)d\theta \Rightarrow C_n = \frac{-A_0}{jnw_0T_0}\int_{jnw_0\tau/2}^{-jnw_0\tau/2} e^{\theta}d\theta \Rightarrow C_n = \frac{-A_0}{jnw_0T_0}\left[e^{\theta}\right]_{jnw_0\tau/2}^{-jnw_0\tau/2}$$

$$C_n = \frac{-A_0}{jnw_0T_0}\left[e^{-j\underbrace{nw_0\tau/2}_{x}} - e^{j\underbrace{nw_0\tau/2}_{x}}\right]$$

$$e^{ix} = \cos x + i\sin x$$

$$C_n = \frac{-A_0}{jnw_0T_0}\left[e^{-jx} - e^{jx}\right] \Rightarrow C_n = \frac{-A_0}{jnw_0T_0}\left[\cos(-x) + j\sin(-x) - \{\cos x + j\sin x\}\right]$$

$$C_n = \frac{-A_0}{jnw_0T_0}\left[\cancel{\cos x} - j\sin x - \cancel{\cos x} - j\sin x\right]$$

$$C_n = \frac{-A_0}{jnw_0T_0}\left[-2j\sin n\frac{w_0\tau}{2}\right]$$

$$C_n = \frac{A_0}{jnw_0T_0}\left[2j\sin n\frac{w_0\tau}{2}\right]$$

$jnw_0\tau/2$ $jnw_0\tau/2$

$$C_n = \frac{-A_0}{jnw_0T_0}\left[e^{-j\underbrace{nw_0\tau/2}_{x}} - e^{j\underbrace{nw_0\tau/2}_{x}} \right]$$

$$e^{ix} = \cos x + i\sin x$$

$$C_n = \frac{-A_0}{jnw_0T_0}\left[e^{-jx} - e^{jx} \right] \Rightarrow C_n = \frac{-A_0}{jnw_0T_0}\left[\cos(-x) + j\sin(-x) - \{\cos x + j\sin x\} \right]$$

$$C_n = \frac{-A_0}{jnw_0T_0}\left[\cancel{\cos x} - j\sin x - \cancel{\cos x} - j\sin x \right]$$

$$Sa(t) = \frac{\sin t}{t}$$

$$C_n = \frac{-A_0}{jnw_0T_0}\left[-2j\sin \frac{nw_0\tau}{2} \right]$$

$$C_n = \frac{A_0}{\cancel{j}\cancel{nw_0}T_0} \frac{\left[\cancel{2j}\sin \frac{nw_0\tau}{2} \right]}{nw_0\tau/2} \times \cancel{nw_0}\tau/\cancel{2}$$

$$C_n = \frac{A_0\tau}{T_0} Sa\left(\frac{nw_0\tau}{2} \right)$$

Ans

Complex Exponential Fourier Series (Example-2)

Question: The Fourier series representation of an impulse train denoted by $s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$ is given by

(a) $\frac{1}{T_0}\sum_{n=-\infty}^{\infty} \exp -\frac{j2\pi nt}{T_0}$

(b) $\frac{1}{T_0}\sum_{n=-\infty}^{\infty} \exp -\frac{j\pi nt}{T_0}$

(c) $\frac{1}{T_0}\sum_{n=-\infty}^{\infty} \exp \frac{j\pi nt}{T_0}$

✓(d) $\frac{1}{T_0}\sum_{n=-\infty}^{\infty} \exp \frac{j2\pi nt}{T_0}$

$$s(t) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} \exp \frac{j2\pi nt}{T_0}$$

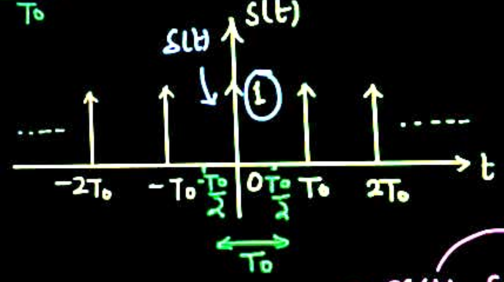
$$s(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$n = -2$ to $n = +2$ $\quad \omega_0 = \frac{2\pi}{T_0}$

$$s(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\frac{2\pi}{T_0}t}$$

$n = 1 \qquad n = -1$

$s(t) = \delta(t - T_0) \qquad s(t) = \delta(t + T_0)$



[GATE-1999]

$$\rightarrow x(t)\cdot\delta(t - t_1) = x(t_1)\cdot\delta(t - t_1)$$

$$C_n = \frac{1}{T_0}\int_{T_0} s(t)\cdot e^{-jn\omega_0 t}\,dt$$

$$C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)\cdot e^{-jn\omega_0 t}\,dt$$

(a) $\frac{\;}{T_0}\sum_{n=-\infty} \exp - \frac{\;}{T_0}$

(b) $\frac{1}{T_0}\sum_{n=-\infty}^{\infty} \exp -\frac{j\pi n t}{T_0}$

(c) $\frac{1}{T_0}\sum_{n=-\infty}^{\infty} \exp \frac{j\pi n t}{T_0}$

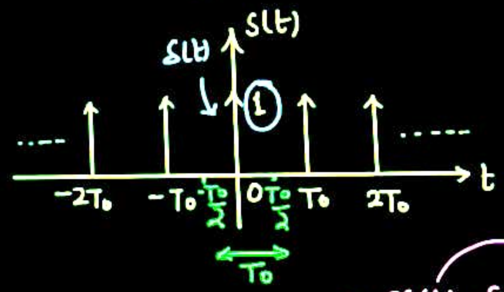
(d) $\frac{1}{T_0}\sum_{n=-\infty}^{\infty} \exp \frac{j2\pi n t}{T_0}$

$$S(t) = \sum_{n=-\infty} C_n e$$

$$S(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\frac{2\pi}{T_0}t}$$

$n = 1$ $\quad$ $n = -1$

$S(t) = \delta(t - T_0)$ $\quad$ $S(t) = \delta(t + T_0)$



[GATE-1999]

$\rightarrow x(t)\cdot\delta(t-t_1) = x(t_1)\cdot\delta(t-t_1)$

$$C_n = \frac{1}{T_0}\int_{T_0} S(t)\cdot e^{-jn\omega_0 t}\, dt$$

$$C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)\cdot e^{-jn\omega_0 t}\, dt$$

$$C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t-0)\cdot e^{-jn\omega_0 t}\, dt \Rightarrow C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)\cdot e^{0}\, dt \Rightarrow C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)\, dt$$

$$C_n = \frac{1}{T_0}\times 1$$

$$C_n = 1/T_0$$

$x(t) = \underbrace{3} + \underbrace{2\sin\omega_0 t} + \underbrace{\cos\omega_0 t} + \cos(\underbrace{2\omega_0 t + \pi/4}_{\alpha})$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$x(t) = \cdots + C_{-2} e^{-2j\omega_0 t} + C_{-1} e^{-j\omega_0 t} + C_0 + C_1 e^{j\omega_0 t} + C_2 e^{2j\omega_0 t} + \cdots\cdots$$

$$e^{i\alpha} = \cos\alpha + i\sin\alpha \text{ —— (I)}$$

$$e^{-i\alpha} = \cos\alpha - i\sin\alpha \text{ —— (II)}$$

(I) + (II) :—

$$e^{i\alpha} + e^{-i\alpha} = 2\cos\alpha \Rightarrow \cos\alpha = \frac{1}{2}\left[e^{i\alpha} + e^{-i\alpha}\right]$$

(I) - (II) :— $$e^{i\alpha} - e^{-i\alpha} = 2i\sin\alpha \Rightarrow \sin\alpha = \frac{1}{2i}\left[e^{i\alpha} - e^{-i\alpha}\right]$$

$$x(t) = 3 + 2\cdot\frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] + \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] + \frac{1}{2}\left[e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}\right]$$

$$x(t) = \dots + C_{-2}e^{-2j\omega_0 t} + C_{-1}e^{-j\omega_0 t} + C_0 + C_1 e^{j\omega_0 t} + C_2 e^{2j\omega_0 t} + \dots$$

$$e^{i\alpha} = \cos\alpha + i\sin\alpha \quad \text{—— (I)}$$

$$e^{-i\alpha} = \cos\alpha - i\sin\alpha \quad \text{—— (II)}$$

(I) + (II) :—

$$e^{i\alpha} + e^{-i\alpha} = 2\cos\alpha \Rightarrow \cos\alpha = \frac{1}{2}[e^{i\alpha} + e^{-i\alpha}]$$

(I) − (II) :— $$e^{i\alpha} - e^{-i\alpha} = 2i\sin\alpha \Rightarrow \sin\alpha = \frac{1}{2i}[e^{i\alpha} - e^{-i\alpha}]$$

$$e^{a+b} = e^{a} \cdot e^{b}$$

$$e^{j\pi/4} = \cos\pi/4 + j\sin\pi/4 = 1/\sqrt{2} + j \cdot 1/\sqrt{2} = \frac{1+j}{\sqrt{2}}$$

$$e^{-j\pi/4} = \frac{1-j}{\sqrt{2}}$$

$$x(t) = 3 + \cancel{2}\cdot\frac{1}{\cancel{2}j}[e^{j\omega_0 t} - e^{-j\omega_0 t}] + \frac{1}{2}[e^{j\omega_0 t} + e^{-j\omega_0 t}] + \frac{1}{2}[e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}]$$

$$\frac{1}{j} \times \frac{j}{j} = \frac{j}{j^2} = \frac{j}{-1} = -j$$

$$= 3 + \frac{1}{j}e^{j\omega_0 t} - \frac{1}{j}e^{-j\omega_0 t} + \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t} + \frac{1}{2}e^{2j\omega_0 t} \cdot \underline{e^{j\pi/4}} + \frac{1}{2}e^{-2j\omega_0 t} \cdot \underline{e^{-j\pi/4}}$$

$$= 3 + \left(\frac{1}{2} - j\right)e^{j\omega_0 t} + \left(\frac{1}{2} + j\right)e^{-j\omega_0 t} + \frac{1+j}{2\sqrt{2}}e^{2j\omega_0 t} + \frac{1-j}{2\sqrt{2}}e^{-2j\omega_0 t}$$

$$\cdots + C_{-2} e^{-2j\omega_0 t} + C_{-1} e^{-j\omega_0 t} + C_0 + C_1 e^{j\omega_0 t} + C_2 e^{2j\omega_0 t} + \cdots$$

$$\cos\alpha + i\sin\alpha \quad \text{—— (I)}$$

$$\cos\alpha - i\sin\alpha \quad \text{—— (II)}$$

$$e^{a+b} = e^{a}\cdot e^{b}$$

$$e^{j\pi/4} = \cos\pi/4 + j\sin\pi/4$$

$$= 1/\sqrt{2} + j\cdot 1/\sqrt{2}$$

$$= \frac{1+j}{\sqrt{2}}$$

$$e^{-j\pi/4} = \frac{1-j}{\sqrt{2}}$$

$$i\alpha = 2\cos\alpha \Rightarrow \cos\alpha = \frac{1}{2}\left[e^{i\alpha} + e^{-i\alpha}\right]$$

$$- e^{-i\alpha} = 2i\sin\alpha \Rightarrow \sin\alpha = \frac{1}{2i}\left[e^{i\alpha} - e^{-i\alpha}\right]$$

$$C_{-2} = \frac{1-j}{2\sqrt{2}}$$

$$C_{-1} = \frac{1}{2} + j$$

$$C_0 = 3$$

$$C_1 = \frac{1}{2} - j$$

$$C_2 = \frac{1+j}{2\sqrt{2}}$$

$$3 + \cancel{2}\cdot\frac{1}{\cancel{2}j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] + \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] + \frac{1}{2}\left[e^{j(2\omega_0 t+\pi/4)} + e^{-j(2\omega_0 t+\pi/4)}\right]$$

$$3 + \frac{1}{j}e^{j\omega_0 t} - \frac{1}{j}e^{-j\omega_0 t} + \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t} + \frac{1}{2}e^{2j\omega_0 t}\cdot e^{j\pi/4} + \frac{1}{2}e^{-2j\omega_0 t}\cdot e^{-j\pi/4}$$

$$3 + \left(\frac{1}{2} - j\right)e^{j\omega_0 t} + \left(\frac{1}{2} + j\right)e^{-j\omega_0 t} + \frac{1+j}{2\sqrt{2}}e^{2j\omega_0 t} + \frac{1-j}{2\sqrt{2}}e^{-2j\omega_0 t}$$

Question: The signal $x(t)$ has period equal to 1 and the following Fourier coefficients

$$C_n = \left(\frac{-1}{3}\right)^n \;; \quad n \geq 0$$
$$\;\;\; = 0 \;; \quad n < 0$$

What is $x(t)$?

a) $x(t) = \dfrac{1}{1 - \frac{1}{3}e^{j2\pi t}}$

b) $x(t) = \dfrac{1}{1 + \frac{1}{3}e^{j2\pi t}}$

c) $x(t) = \dfrac{1}{1 + \frac{1}{3}e^{-j2\pi t}}$

d) $x(t) = \dfrac{1}{1 - \frac{1}{3}e^{-j2\pi t}}$

$$\omega_0 = \frac{2\pi}{1} = 2\pi$$

Sol:- $$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$
$$= \sum_{n=-\infty}^{\infty} C_n e^{jn2\pi t}$$
$$= \sum_{n=-\infty}^{0} (0) \cdot e^{jn2\pi t} + \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n \cdot e^{jn2\pi t}$$

Sol:- $x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$ $\qquad \omega_0 = \frac{2\pi}{1} = 2\pi$

$$= \sum_{n=-\infty}^{\infty} C_n e^{jn2\pi t}$$

$$= \sum_{n=-\infty}^{0} (0) \cdot e^{jn2\pi t} + \sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n \cdot e^{jn2\pi t}$$

$\downarrow 0$

$$= \sum_{n=0}^{\infty} \left(-\frac{1}{3} \cdot e^{j2\pi t}\right)^n$$

$$x(t) = \underset{a}{1} + \left(-\frac{1}{3} \cdot e^{j2\pi t}\right) + \left(-\frac{1}{3} \cdot e^{j2\pi t}\right)^2 + \cdots\cdots$$

$\cos 2\pi t + j\sin 2\pi t$

$$r = -\frac{1}{3} \cdot e^{j2\pi t}$$

$|r| < 1$

GP

$$S_\infty = \frac{a}{1-r} \quad -1 < r < 1$$

$$S_n = \frac{a(1-r^n)}{1-r} \quad -1 < r < 1$$

0.25

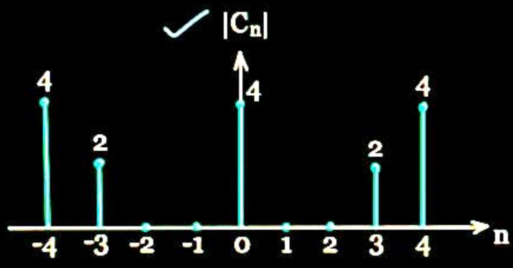
etc

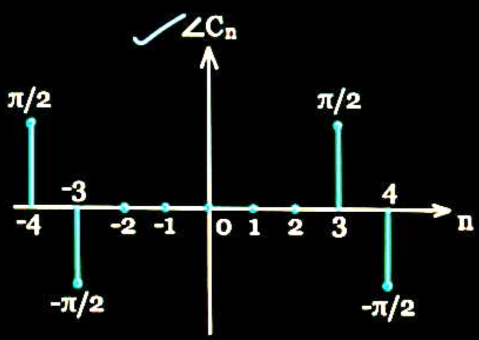
$$S_\infty = x(t) = \frac{1}{1 - \left(-\frac{1}{3} \cdot e^{j2\pi t}\right)}$$

$$x(t) = \frac{1}{1 + \frac{1}{3} e^{j2\pi t}}$$

Complex Exponential Fourier Series (Example-5)

Question: Find $x(t)$





Sol. :-

$$C_n = |C_n| e^{j\angle C_n}$$

$$= |C_n| \cdot (\cos\angle C_n + j \sin \angle C_n)$$

$C_0 = 4 \cdot e^{j0}$ $\quad C_1 = 0 \cdot e^{j0}$ $\quad C_2 = 0 \cdot e^{j0}$ $\quad C_3 = 2 \cdot e^{j\pi/2}$ $\quad C_4 = 4 \cdot e^{-j\pi/2}$

$C_0 = 4$ $\quad C_1 = 0$ $\quad C_2 = 0$

$C_{-1} = 0$ $\quad C_{-2} = 0$ $\quad C_{-3} = 2\, e^{-j\pi/2}$ $\quad C_{-4} = 4 \cdot e^{j\pi/2}$

$C_5 = C_{-5} \cdots$

$C_5 = C_{-5} \cdots = 0$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$= C_0 + C_3 e^{j3\omega_0 t} + C_4 e^{j4\omega_0 t} + C_{-3} e^{-j3\omega_0 t} + C_{-4} e^{-j4\omega_0 t}$$

$$\frac{e^{jx} + e^{-jx}}{2} = \cos x$$

$$= 4 + 2e^{j\frac{\pi}{2}} \cdot e^{j3\omega_0 t} + 4e^{-j\frac{\pi}{2}} \cdot e^{j4\omega_0 t} + 2e^{-j\frac{\pi}{2}} \cdot e^{-j3\omega_0 t} + 4e^{j\frac{\pi}{2}} \cdot e^{-j4\omega_0 t}$$

$$e^{jx} + e^{-jx} = 2\cos x$$

$$= 4 + \underline{2} e^{j(3\omega_0 t + \pi/2)} + \underline{4} e^{j(4\omega_0 t - \pi/2)} + \underline{2} e^{-j(3\omega_0 t + \pi/2)} + \underline{4} e^{-j(4\omega_0 t - \pi/2)}$$

$$= 4 + 2\left[e^{j\underbrace{(3\omega_0 t + \pi/2)}_{x_1}} + e^{-j\underbrace{(3\omega_0 t + \pi/2)}_{x_1}}\right] + 4\left[e^{j\underbrace{(4\omega_0 t - \pi/2)}_{x_2}} + e^{-j\underbrace{(4\omega_0 t - \pi/2)}_{x_2}}\right]$$

$$= 4 + 2\left[\underbrace{e^{jx_1} + e^{-jx_1}}\right] + 4\left[\underbrace{e^{jx_2} + e^{-jx_2}}\right]$$

$$= 4 + 2 \cdot 2\cos x_1 + 4 \cdot 2\cos x_2$$