Matrix and their types and properties, Rank of a matrix, Consistency of system of linear equations, solution of simultaneous linear equation by elementary transformations, Eigen value and Eigen vectors, Cayley - Hamilton theorem and its application to find inverse, diagonalization of matrices.

Why do we study matrices?

(1) To solve linear equations

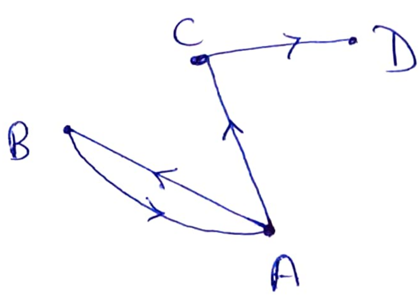
- Substitution method
- Elimination method
- Cross multiplication method
- Graphical method

(2) Coding (Java, C++)

(3) Gaming

(4) Computer | Mobile Phones

Suppose there are 4 players in a football ground namely A, B, C & D. A passes the ball to B and B passes the ball to A then A passes the ball to C then C passes the ball to D.



How ~~did the~~ computer understand this Commentary?

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Matrices are applied in the study of electrical circuits, quantum mechanics. We can calculate battery power outputs.

Matrix :- If m n numbers (Real or Complex) are arranged in the form of a rectangular array 'A' with m rows and n columns, is called a m×n matrix.

It is denoted by A ~ [ ] or ( )

An m×n matrix is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

Rows (horizontal)

Columns (vertical)

where a_{ij} is the element of A lying in the ith row and jth column. Thus $A = [a_{ij}]_{m \times n}$

NOTE:- (1) Matrix is an arrangement of numbers.

(2) Determinant is a value.

Types of matrix

(1) Real matrix :- A matrix is called real matrix if its each elements are real number

Ex: $\begin{bmatrix} \sqrt{2} & 0 & 1 \\ 3 & 5 & 6 \end{bmatrix}_{2 \times 3}$

(2) Square matrix :- A matrix in which number of rows is equal to is equal to number of columns then it is called square matrix.

eg:- $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2 \times 2}$, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{2 \times 2}$ → diagonal Principal diagonal Leading diagonal

For diagonal elements $i = j$
For non- diagonal elements $i \neq j$
For the elements lying above the diagonal, $i<j$
" " " " below " " , $i>j$

(3) Row matrix :- A matrix having one row and any number of columns, is called row matrix.

Ex :- $[1, 7 \quad 8]$, $[1 \quad 3 \quad 5 \quad 2]$

(4) Column matrix :- A matrix having one column and any number of rows, is called column matrix.

Ex :- $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 7 \end{bmatrix}$

(5) Null matrix :- A matrix in which all elements are zero, is called null or void or zero matrix.

Ex :- $O_{2\times2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $O_{2\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(6) Sub matrix :- A matrix obtained from any matrix A by deleting some rows or columns or both.

Ex :- Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times3}$

and submatrix $B = \begin{bmatrix} 0 & 6 \\ 7 & 9 \end{bmatrix}_{2\times2}$ is obtained by eleminating first row and second column.

(7) Diagonal matrix :- A square matrix in which all non-diagonal elements are zero, is called diagonal matrix.

Thus, for a diagonal matrix $A = [a_{ij}]$ we have $a_{ij} = 0$ for $i \neq j$.

Ex : $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3\times3}$, $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ where $a, b \neq 0$

(8) Scalar matrix :- A diagonal matrix in which all diagonal elements are equal to a scalar (k), is called scalar matrix.

Thus, for a scalar matrix we have $A = [a_{ij}]$ where $a_{ij} = \begin{cases} k & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

(9) Unit or Identity matrix :- A scalar matrix in which all diagonal elements are unity, is called unit matrix or identity matrix.

Thus, for a unit matrix we have $A = [a_{ij}]$ where $a_{ij} = \begin{cases} 1 ; & i = j \\ 0 ; & i \neq j \end{cases}$

Ex : $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$

(10) **Upper triangular matrix :-** A square matrix in which all the elements below the principal diagonal are zero, is called upper triangular matrix.

Thus, $A = [a_{ij}]$ is U.T.M. if $a_{ij} = 0$ for $i > j$

Ex :- $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}_{3\times 3}$

(11) **Lower triangular matrix :-** A square matrix in which all the elements above the principal diagonal are zero, is called lower triangular matrix.

Thus, $A = [a_{ij}]$ is L.T.M. if $a_{ij} = 0$ for $i < j$.

Ex :- $\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$

(12) **Triangular matrix :-** Triangular matrix is either L.T.M. or U.T.M.

(13) **Singular and non-singular matrix :-** A square matrix 'A' is called singular if $|A| = 0$ otherwise it is called non-singular matrix.

Ex :- (1) $\begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} = A \Rightarrow |A| = 0 \Rightarrow A$ is singular.

(2) $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

$\Rightarrow |A| = 6 \neq 0$

$\Rightarrow$ A is non-singular.

Transpose of a matrix :- Let A be any given matrix. Then a matrix obtained from A by interchanging rows into columns ~~into rows~~ and columns into rows is called transpose of A.

It is denoted by A' or A^T.

Ex:-(1) $A = \begin{bmatrix} a & h & f \\ h & b & g \\ f & g & c \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} a & h & f \\ h & b & g \\ f & g & c \end{bmatrix}$

$\Rightarrow A = A^T$.

(2) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then $A' = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

Properties :-

(1) $(A^T)^T = A$

(2) $(A+B)^T = A^T + B^T$

(3) $(AB)^T = B^T A^T$

Properties of determinant

(1) $|KA| = K^n |A|$, $n \to$ order

(2) $|A^T| = |A|$

(3) $|AB| = |A||B|$

(4) $|A^n| = |A|^n$

(14) Symmetric matrix :- A square matrix $A = [a_{ij}]$ is called symmetric matrix if $A^T = A$ i.e. $a_{ij} = a_{ji}$

(15) skew - symmetric matrix :- A square matrix $A = [a_{ij}]$ is called skew - symmetric matrix if $A^T = -A$ i.e. $a_{ij} = -a_{ji}$

NOTE :- For diagonal elements, put $i = j$

$$a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0$$

$$\Rightarrow a_{ii} = 0 \ \forall \ i$$

Hence, all diagonal elements of a skew - symmetric matrix are zero.

eg :- $A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 1 \\ 3 & -1 & 0 \end{bmatrix}$

$$A^T = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -1 \\ -3 & 1 & 0 \end{bmatrix} = -A$$

$\Rightarrow$ A is skew-symmetric matrix.

★
Every square matrix can be uniquely expressed as the sum of a symmetric and skew - symmetric matrix.

Proof :- Let A be any square matrix.

Then $A = \frac{1}{2}A + \frac{1}{2}A$

$$= \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

$$= B + C$$

where $B = \frac{1}{2}(A + A^T)$ and $C = \frac{1}{2}(A - A^T)$

Now $B = \frac{1}{2}(A + A^T)$

Taking transpose both sides

$$B^T = \frac{1}{2}(A + A^T)^T$$

$$= \frac{1}{2}(A^T + (A^T)^T)$$

$$= \frac{1}{2}(A^T + A)$$

$$= B \Rightarrow B = B^T$$

$\Rightarrow$ B is symmetric matrix.

and $C = \frac{1}{2}(A - A^T)$

$$\Rightarrow C^T = \frac{1}{2}(A - A^T)^T$$

$$= \frac{1}{2}(A^T - A)$$

$$= -\frac{1}{2}(A - A^T)$$

$$= -C \Rightarrow C = -C^T$$

$\Rightarrow$ C is skew-symmetric matrix.

Ques:- Express the following matrix as the sum of a symmetric matrix and skew-symmetric matrix where $A = \begin{bmatrix} 1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}$.

Solⁿ:- Given $A = \begin{bmatrix} 1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}$

(14 Then $A^T = \begin{bmatrix} 1 & 2 & 5 \\ 7 & 3 & 0 \\ 1 & 4 & 5 \end{bmatrix}$

if

We know that

(15

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = B + C$$

sk then $B = \frac{1}{2}(A + A^T)$

NO

$$= \begin{bmatrix} 1 & 9/2 & 3 \\ 9/2 & 3 & 2 \\ 3 & 2 & 5 \end{bmatrix} \rightarrow \text{symmetric matrix}$$

He

mc and $C = \frac{1}{2}(A - A^T)$

eg

$$= \begin{bmatrix} 0 & 5/2 & -2 \\ -5/2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix} \rightarrow \text{skew - symmetric matrix}$$

(16) **Orthogonal matrix** :- A square matrix 'A' is called orthogonal if $AA^T = I$.

Ex :- $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

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Then $A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Ps

17) Nilpotent matrix :- A square matrix '$A_{2\times 2}$' is called nilpotent if $A^2 = 0$.

or, If square matrix [A] is a nilpotent matrix of order n×n, then there must be $A^K = 0$ for all $K \geq n$.

Eg: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$A^2 = A \cdot A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow A^2 = 0$

$\Rightarrow$ A is Nilpotent matrix.

18) Idempotent matrix :- A square matrix A is called idempotent if $A^2 = A$.

Ex:- $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A$$

(19) Involutary matrix :- A square matrix 'A' is called involutary matrix if $A^2 = I$.

Ex: let $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

Then $$A^2 = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Ques:- Prove that $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is orthogonal matrix.

Soln:- Given $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ then

$$A^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Now $AA^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\cancel{\sin\theta} + \sin\theta\cancel{\cos\theta} \\ -\sin\theta\cancel{\cos\theta} + \cos\theta\cancel{\sin\theta} & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\Rightarrow AA^T = I$

$\Rightarrow$ A is orthogonal matrix.

Ques:- Find a, b, c if $A = \frac{1}{9}\begin{bmatrix} -8 & 4 & a \\ 1 & 4 & b \\ 4 & 7 & c \end{bmatrix}$ is orthogonal.

~~where $a>0, b>0, c>0$~~

Soln:- Given $A = \frac{1}{9}\begin{bmatrix} -8 & 4 & a \\ 1 & 4 & b \\ 4 & 7 & c \end{bmatrix}$

then $A^T = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ a & b & c \end{bmatrix}$

ogonal

Since A is orthogonal then $AA^T = I$

$$\Rightarrow \frac{1}{81}\begin{bmatrix} -8 & 4 & a \\ 1 & 4 & b \\ 4 & 7 & c \end{bmatrix}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ a & b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \frac{1}{81}\begin{bmatrix} 80+a^2 & 8+ab & -4+ac \\ 8+ab & 17+b^2 & 32+bc \\ -4+ac & 32+bc & 65+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

On comparing,

$$\frac{80+a^2}{81} = 1 \Rightarrow 80+a^2 = 81 \Rightarrow a^2 = 1$$
$$\Rightarrow a = 1$$

and $8+ab = 0 \Rightarrow 8+b = 0$ $(\because a=1)$
$\Rightarrow b = -8$

and $-4+ac = 0 \Rightarrow -4+c = 0$ $(\because a=1)$
$\Rightarrow c = 4$

$\therefore a = 1,\ b = -8,\ c = 4.$

NOTE:- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

then $A^{-1} = \frac{1}{|A|} \operatorname{adj} A = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

★ **Ques:-** If $A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$ then evaluate the value of the expression $A + 5I + 2A^{-1}$.

Soln:- Given $A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$

then $A^{-1} = \frac{1}{|A|} \text{adj} A$

$$= \frac{1}{2}\begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}$$

Now $A + 5I + 2A^{-1} = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2 \cdot \frac{1}{2}\begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}$

$$= \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} + \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Ans.

$A = \frac{1}{7}\begin{bmatrix} 3 & 2 & 6 \\ -6 & 3 & 2 \\ 2 & 6 & -3 \end{bmatrix}$ is orthogonal ?

Determine the value of a, b, c such that the given matrix $A = \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ is orthogonal.

$a = \pm\frac{1}{\sqrt{2}}$, $b = \pm\frac{1}{\sqrt{6}}$, $c = \pm\frac{1}{\sqrt{3}}$, $a^2+b^2+c^2 = 1$

Rank :— The rank of a matrix is an order of highest non-zero determinant obtained from the matrix.

Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

Then determinant obtained from matrix A are

$$D_1 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \quad D_4 = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$D_5 = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}, \quad D_6 = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}, \quad D_7 = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$D_8 = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad D_9 = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \quad D_{10} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$D_{11} = |a_{11}|, \; D_{12} = |a_{12}|, \; \dots, \; D_{19} = |a_{33}|.$$

Rank of a matrix A is denoted by $\rho(A)$ or Rank (A).

NOTE :— (1) $\rho(A) \leq$ Order (A)

(2) ρ(null matrix) $= 0$

Remark :— Non-zero row is that row which does not contain all the elements zero.

Ques (1) $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & -2 \\ 2 & -1 & 1 \end{bmatrix}$. Find rank of A.

then Soln:— $D_1 = \begin{vmatrix} 1 & 2 & -1 \\ -1 & 1 & -2 \\ 2 & -1 & 1 \end{vmatrix}$

Now

$$= 1\begin{vmatrix} 1 & -2 \\ -1 & 1 \end{vmatrix} - 2\begin{vmatrix} -1 & -2 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix}$$

$$= 1(1-2) - 2(-1+4) - 1(1-2)$$

$$= -1 - 6 + 1 = -6 \neq 0$$

$\therefore \rho(A) = \text{Order of } D_1 = 3.$

(2) Find rank of $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 5 \\ 1 & 4 & 4 \end{bmatrix}$.

#

Soln:— Given $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 5 \\ 1 & 4 & 4 \end{bmatrix}$

#

then $D_1 = 1(8-20) - 2(0-5) - 1(0-2)$

$$= -12 + 10 + 2 = 0$$

$D_2 = \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = 2 - 0 = 2 \neq 0$

$\Rightarrow \rho(A) = 2.$

(3) For what value of 'p' the rank of the matrix $\begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ p & 13 & 10 \end{bmatrix}$ is 2.

Solⁿ:— Let $A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ p & 13 & 10 \end{bmatrix}$

Since $\rho(A) = 2$ (given)

$\Rightarrow D_1 = \begin{vmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ p & 13 & 10 \end{vmatrix} = 0$

$\Rightarrow 1\begin{vmatrix} 3 & 2 \\ 13 & 10 \end{vmatrix} - 5\begin{vmatrix} 0 & 2 \\ p & 10 \end{vmatrix} + 4\begin{vmatrix} 0 & 3 \\ p & 13 \end{vmatrix} = 0$

$\Rightarrow 4 + 5 \times 2p + 4\times(-3p) = 0$

$\Rightarrow 4 + 10p - 12p = 0$

$\Rightarrow 4 - 2p = 0$

$\Rightarrow 2p = 4$

$\Rightarrow p = 2.$ Ans.

Elementary Transformation:-

(1) Any two rows are interchangeble i.e. $R_i \leftrightarrow R_j$

(2) All the elements of any row can be multiplied by any non-zero number. i.e. $R_i \rightarrow kR_i$

(3) All the elements of a row can be added to corresponding elements of another row multiplied by any non-zero constant.

i.e. $R_i \longrightarrow R_i + kR_j$

Imp

Rank = No. of non-zero rows in upper triangular matrix.

Row-Echelon form or Echelon form :- Any matrix 'A' is said to be in Echelon form if

(1) The first non-zero element from the left of a non-zero row is 1 and is called leading entry or pivot element.

(2) If a column contains a leading entry then all entries below that leading entry are zero.

(3) Every zero row of a matrix occurs below a non-zero row.

(4) For each non-zero row, the leading entry in the lower row occurs to the right of the leading entry in the above row.

NOTE:- Only Row operations are allowed.

eg:- $\begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$ → Non-zero row ⇒ $\rho(A) = 3$ → Zero row

Rank (by Echelon form) :-

The number of non-zero rows in the echelon form of a matrix A is called the rank of matrix A. It is denoted by $\rho(A)$ or Rank (A).

eg:- $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

The given matrix is in echelon form and no. of non-zero rows are 4.

$\therefore \rho(A) = 4$

Ques:- Find the rank of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ by echelon form.

Soln:- Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

Using row elementry transformation,

By $R_2 \longrightarrow R_2 - 2R_1$

$R_3 \longrightarrow R_3 - 3R_1$

$$\sim \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

No. of non-zero rows = 1

$\Rightarrow \rho(A) = 1.$

Ans.

Ques:- $A = \begin{bmatrix} 2 & -1 & 3 & -1 \\ 1 & 2 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

Soln:- Given $A = \begin{bmatrix} 2 & -1 & 3 & -1 \\ 1 & 2 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

By $R_1 \longrightarrow R_1 - R_2$

$$\sim \begin{bmatrix} 1 & -3 & 6 & 0 \\ 1 & 2 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

By $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & -3 & 6 & 0 \\ 0 & 5 & -9 & -1 \\ 0 & 3 & -5 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

By $R_2 \leftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & -3 & 6 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 3 & -5 & 1 \\ 0 & 5 & -9 & -1 \end{bmatrix}$$

By $R_3 \to R_3 - 3R_2$, $R_4 \to R_4 - 5R_2$

$$\sim \begin{bmatrix} 1 & -3 & 6 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -8 & 4 \\ 0 & 0 & -14 & 4 \end{bmatrix}$$

By $R_3 \to \frac{R_3}{-8}$ & $R_4 \to R_4 + 14R_3$

$$\sim \begin{bmatrix} 1 & -3 & 6 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

By $R_4 \to \frac{R_4}{-3}$

$$\sim \begin{bmatrix} 1 & -3 & 6 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \text{Echelon form}$$

$\Rightarrow \rho(A) =$ no. of non-zero rows $= 4$. Ans.

NOTE:— The echelon form of a square matrix is an upper triangular matrix.

Tricks for Echelon form :-

For echelon form of matrix

(1) Columnwise zero elements may be decreasing by at least one zero.

(2) Row-wise zero elements may be increasing by at least one zero.

(3) Extra zero elements in rows and column does not effect echelon form.

Ques (1) Find the rank of A by echelon-form where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 9 \\ 3 & 6 & 10 & 13 \end{bmatrix}$$

Soln :— Given $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 9 \\ 3 & 6 & 10 & 13 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \begin{matrix} R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$\longrightarrow$ Echelon form

$\therefore \rho(A) =$ No. of non-zero rows $= 3$

Ans.

By R_2 ~ (2) $\begin{bmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = A$

$\rho(A) = 2.$

By R_2 ~ (3) $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\rho(A) = 2$

(4) By R_3 $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 2 & 1 & 3 & 13 \\ 5 & 2 & 1 & 12 \\ 2 & -3 & -2 & -10 \end{bmatrix}$

~ ~ $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & -3 & -4 & -18 \\ 0 & -5 & -4 & -22 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 5R_1$, $R_4 \to R_4 - 2R_1$

By R_3 ~ ~ $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -7 & -21 \\ 0 & 0 & -9 & -27 \end{bmatrix}$ $R_3 \to R_3 - 3R_2$, $R_4 \to R_4 - 5R_2$

By ~ $\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -7 & -21 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_4 \to 7R_4 - 9R_3$

$\therefore \rho(A) =$ No. of non-zero rows $= 3$

Ans.

$\Rightarrow$ ∫

Rank of matrix by Normal form :-
Canonical form
(1) Both row and column operations are allowed.

(2) Reduce to identity matrix.

(3) Normal form $\longrightarrow A \sim \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

(4) $\rho(A) = r =$ Order of identity matrix in Normal form

Determine the rank of matrix A by normal form where $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Soln :- Given $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

By $R_2 \longrightarrow R_2 - 2R_1$

$$A \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_2 \leftrightarrow R_3$

$$A \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Now we'll use column operations

By $C_3 \to C_3 - C_1$

$$A \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

By $C_2 \to C_2 - 2C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

By $C_2 \leftrightarrow C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

It is normal form $\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$

Hence $\rho(A) = 2$.

Ans.

Ques:- Reduce to normal form the following matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$$

Soln:- Given $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$

Using transformation $R_3 \to R_3 - 3R_1$,

$R_2 \to R_2 - 2R_1$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix}$$

Using $C_2 \rightarrow C_2 - 2C_1$, $C_3 \rightarrow C_3 - 3C_1$, $C_4 \rightarrow C_4 - 4C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix}$$

Using $R_2 \rightarrow -\frac{1}{3} R_2$, $R_3 \rightarrow -\frac{1}{6} R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{2}{3} & \frac{5}{3} \\ 0 & 1 & \frac{2}{3} & \frac{11}{3} \end{bmatrix}$$

Using $R_3 \rightarrow R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{2}{3} & \frac{5}{3} \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Using $C_3 \rightarrow C_3 - \frac{2}{3} C_2$, $C_4 \rightarrow C_4 - \frac{5}{3} C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Using $C_3 \leftrightarrow C_4$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

Using $C_3 \to \frac{C_3}{2}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\sim [I_3 \;\; 0]$ is normal form of A.

$\Rightarrow \rho(A) = 3$.

The rank of matrix is said to be r if
(i) It has at least one non-zero minor of order r
(ii) Every minor of A of order $(r+1)$/higher than r is zero.

The determinant of any square matrix A is a scalar, denoted by det (A).
Non-square matrices do not have determinants.

The rank of a matrix [A] is equal to the order of the largest non-singular submatrix of [A]. It follows that a non-singular square matrix of order $n \times n$ has a rank of n.

A minor of a (not necessarily square) matrix A is the determinant of a square matrix obtained by omitting some rows and/or some colums of A.

Solution of Simultaneous Equations

The matrix of the coefficient of x, y, z is reduced into Echelon form by elementary row transformation. At the end of the row transformation the value of z is calculated from the last equation and value of y and x are calculated by backward substitution. This is known as Gauss Jordan Method.

eg. Solve the following equations using ellimination Method

$$x-y+2z=3, \quad x+2y+3z=5, \quad 3x-4y-5z=-13$$

Soln:- In matrix form, the eqn are written as

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 3 \\ 3 & -4 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ -13 \end{bmatrix}$$

change this into Echelon form.

$R_2 \rightarrow R_2 - R_1, \quad R_3 \rightarrow R_3 - 3R_1$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \\ 0 & -1 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -22 \end{bmatrix}$$

$R_2 \rightarrow R_2/3$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1/3 \\ 0 & -1 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2/3 \\ -22 \end{bmatrix}$$

$R_3 \rightarrow R_3 + R_2$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1/3 \\ 0 & 0 & -32/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2/3 \\ -64/3 \end{bmatrix}$$

Now back substitute

$$x - y + 2z = 3 \quad \text{——} (1)$$

$$y + \frac{1}{3}z = \frac{2}{3} \quad \text{——} (2)$$

$$-\frac{32}{3}z = -\frac{64}{3} \quad \text{——} (3)$$

$$z = \frac{64}{32} = 2$$

$$\boxed{z = 2}$$

Put in (2)

$$y + \frac{2}{3} = \frac{2}{3}$$

$$\boxed{y = 0}$$

Put in (1)

$$x - y + 2z = 3$$

$$x - 0 + 2(2) = 3$$

$$x = 3 - 4$$

$$\boxed{x = -1}$$

$\boxed{x = -1,\ y = 0,\ z = 2}$. Ans

TYPES OF LINEAR EQUATIONS

① **Consistent** :- A system of eqn is said to be consistent if they have one or more solution

$$\left.\begin{array}{l} x+2y=4 \\ 3x+2y=2 \end{array}\right]$$

unique soln

$$\left.\begin{array}{l} x+2y=4 \\ 3x+6y=12 \end{array}\right]$$

infinite soln,

② **Inconsistent** :- A system of equation has no solution, it is said to be inconsistent ie;

$$\begin{array}{l} x+2y=4 \\ 3x+6y=5 \end{array} \longrightarrow \left.\begin{array}{l} x+2y=4 \\ x+2y=5/3 \end{array}\right]$$

No soln

Inconsistent.

Consistency of a system of Linear Equations

$$a_{11}x_1+a_{12}x_2+ \text{———} +a_{1n}x_n=b_1$$

$$a_{21}x_1+a_{22}x_2+ \text{———} +a_{2n}x_n=b_2$$

$$\cdots\cdots\cdots\cdots$$

$$a_{m1}x_1+a_{m2}x_2+ \text{———} +a_{mn}x_n=b_m$$

$$\begin{bmatrix} a_{11} & a_{12} & \text{—} & a_{1n} \\ a_{21} & a_{22} & \text{—} & a_{2n} \\ | & & & \\ a_{m1} & a_{m2} & \text{—} & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ | \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ | \\ b_m \end{bmatrix}$$

$AX = B$

and $C = [A, B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & \vdots \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$

Where C is called Augmented matrix.

$AX = B$

- Consistent
  - unique sol^n
    $\rho[A] = \rho[A:b] = n$
    $|A| \neq 0$
  - Infinite sol^n
    $\rho[A] = \rho[A:b] < n$
    $|A| = 0$
- Inconsistent
  $\rho[A] \neq \rho[A:b]$
  (No sol^n)

Note :- In $AX = B$. if $B = 0$ then it is known as homogeneous equation.

if $B \neq 0$ the it is known as system of non-Homogeneous equation.

Ques Test the consistency and solve

$5x + 3y + 7z = 4$

$3x + 26y + 2z = 9$

$7x + 2y + 10z = 5.$

Sol^n:- Augmented Matrix $= [A:b] = C$

$$C = \left[\begin{array}{ccc|c} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{array}\right]$$

$R_1 \to R_1/5$

$$\left[\begin{array}{ccc|c} 1 & 3/5 & 7/5 & 4/5 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 7R_1$

$$\left[\begin{array}{ccc|c} 1 & 3/5 & 7/5 & 4/5 \\ 0 & 121/5 & -11/5 & 33/5 \\ 0 & -11/5 & 1/5 & -3/5 \end{array}\right]$$

$R_3 \to R_3 + \frac{1}{11} R_2$

$$\left[\begin{array}{ccc|c} 1 & 3/5 & 7/5 & 4/5 \\ 0 & 121/5 & -11/5 & 33/5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho[A] = \rho[A:b] = 2 < n$
< 3

Hence system is consistent and it has infinite no. of solution.

To more simplify $R_2 \to R_2\left(\frac{5}{11}\right)$

$$\left[\begin{array}{ccc|c} 1 & 3/5 & 7/5 & 4/5 \\ 0 & 11 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$R_1 \to R_1(5)$

$$\begin{bmatrix} 5 & 3 & 7 \\ 0 & 11 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$$

$$5x + 3y + 7z = 4 \quad \text{--- (1)}$$

$$11y - z = 3 \quad \text{--- (2)}$$

Let $z = k$, $\quad 11y - k = 3$

$$\boxed{y = \frac{3+k}{11}}$$

Put in (1)

$$5x + 3\left(\frac{3+k}{11}\right) + 7(k) = 4$$

$$5x = 4 - 7k - \frac{9}{11} - \frac{3k}{11}$$

$$5x = \frac{35}{11} - \frac{80k}{11}$$

$$x = \frac{35 - 80k}{11 \times 5} = \frac{1}{11}(7 - 16k)$$

$$\boxed{x = \frac{7 - 16k}{11}}$$

$k \in R$

infinite soln.

Q2. Imp Determine for what values of λ and μ the following equations have
(1) No soln (2) unique soln (3) infinite no. of sol^n

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu.$$

Soln

$$C = [A:b] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$R_2 \longrightarrow R_2 - R_1$, $R_3 \longrightarrow R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right]$$

$R_3 \longrightarrow R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

Infinite
(1) ~~No solution~~

$\rho[A] = \rho[A:b] < n < 3$

Let $\rho[A] = \rho[A:b] = 2$

if $\rho[A] = 2$ then $\lambda - 3 = 0$

$\boxed{\lambda = 3}$

if $\rho[A:b] = 2$

$\Rightarrow \lambda - 3 = 0$ and $\mu - 10 = 0$

$\boxed{\lambda = 3 \text{ and } \mu = 10}$

(2) Unique solution.

$\rho[A] = \rho[A:b] = n = 3.$

For $\rho[A] = 3 \Rightarrow \lambda - 3 \neq 0$

$\lambda \neq 3.$

but for $\rho[A:b] = 3$

$\lambda - 3 \neq 0$ and μ can be anything

$\Rightarrow$ ies for unique sol^n $\lambda \neq 3$ and μ can be any real NO.

(3) ~~of~~ NO. of solution

~~For~~ sol^n $\rho[A] \neq \rho[A:b]$

Let $\rho[A] = 2$ then $\lambda - 3 = 0$

$\lambda = 3$

Let $\rho[A:b] = 3$, $\Rightarrow \mu - 10 \neq 0$

For NO solution, $\lambda = 3$, $\mu - 10 \neq 0$

$\boxed{\lambda = 3, \ \mu \neq 10}$

Q. For what values of λ the equation

$$x+y+z=1$$
$$x+2y+4z=\lambda$$
$$x+4y+10z=\lambda^2$$

have solution and solve them completely in each case.

Solⁿ: $C = [A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & \lambda \\ 1 & 4 & 10 & \lambda^2 \end{array}\right]$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda-1 \\ 0 & 3 & 9 & \lambda^2-1 \end{array}\right]$$

$R_3 \to R_3 - 3R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & \lambda-1 \\ 0 & 0 & 0 & \lambda^2-3\lambda+2 \end{array}\right]$$

$\lambda^2-1-3(\lambda-1)$

$\lambda^2-3\lambda+2$

① Unique solution

For unique solⁿ $\rho[A] = \rho[A:B] = n = 3$

But $\rho[A] = 2$

So unique solⁿ doesnot exist.

② Infinite No. of solⁿ

$\rho[A] = \rho[A:B] < n = 2$

$\rho[A] = 2$, But for $\rho[A:B] = 2$, $\lambda^2 - 3\lambda + 2 = 0$

$$\lambda^2 - 3\lambda + 2 = 0$$
$$\lambda^2 - 2\lambda - \lambda + 2 = 0$$
$$\lambda(\lambda-2) - 1(\lambda-2) = 0$$

$(\lambda-2)(\lambda-1)=0$

$\boxed{\lambda=1,2}$

For $\lambda=1,2$ system has infinite no. of soln.

For $\lambda=1$,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda-1 \\ \lambda^2-3\lambda+2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$x+y+z=1$$

$$y+3z=0$$

$$z=-\frac{y}{3}$$

Let $\boxed{y=k}$, $\boxed{z=-\frac{k}{3}}$.

$$x+k-\frac{k}{3}=1$$

$$\boxed{x=1-\frac{2k}{3}}$$

For $\lambda=2$,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$x+y+z=1$$

$$y+3z=1$$

Let $\boxed{y=k,}$ $3z=1-y=1-k$

$$\boxed{z=\frac{1-k}{3}}$$

$x+y+z=1 \Rightarrow x+k+\frac{1-k}{3}=1 \Rightarrow x=1-\frac{1+2k}{3}$

$$\boxed{x=\frac{2(1+k)}{3}}$$

Homogeneous Equations $AX=0$

$A_{n \times n}$

Let $A_{m \times n}$ be the coefficient matrix

$\boxed{AX=0}$ → Homogeneous Equation.

$|A| \neq 0$
consistent with unique solⁿ

OR

$\rho[A] = n$
trivial solⁿ
OR
zero solution.

$|A| = 0$
consistent with infinite solution.

OR

$\rho[A] < n$
Infinite solⁿ
Non-trivial solⁿ

Que 1: Find all solution of the system of equation

$$x+2y-z=0$$
$$2x+y+z=0$$
$$x-4y+5z=0$$

Solⁿ

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$AX=0$ → Homog. eqn.

Method I :- Find Rank of A using echelon Form

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -4 & 5 \end{bmatrix}$$

$R_2 \longrightarrow R_2 - 2R_1$, $R_3 \longrightarrow R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 3 \\ 0 & -6 & 6 \end{bmatrix}$$

$R_2 \longrightarrow R_2/(-3)$, $R_3 \longrightarrow R_3/(-6)$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$R_3 \longrightarrow R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

↳ This is echelon form

No. of non zero rows $= 2 =$ Rank

$$\rho[A] = 2 < 3$$

System of eqn have infinite soln.

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + 2y - z = 0$$

$$y - z = 0$$

$$\boxed{y = z}$$

Let $\boxed{y = k}$, where $k \in R$.

$y = z \Rightarrow \boxed{z = k}$

$x + 2y - z = 0$

$x + 2k - k = 0$

$x + k = 0$

$\boxed{x = -k}$

$\boxed{x = k, \quad y = k, \quad z = k}$

Q. Determine the value of λ so that equation

$2x + y + 2z = 0$
$x + y + 3z = 0$
$4x + 3y + \lambda z = 0$ have non-zero solution.

Sol^n For non-zero solution,

$\rho[A] < n < 3$ or $|A| = 0$

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & \lambda \end{bmatrix}$$

$|A| = 2(\lambda - 9) - 1(\lambda - 12) + 2(3 - 4)$

$= \cancel{4\lambda - 18 - \lambda + 12 - 10}$

$= \cancel{3\lambda - 4}$

$= 2\lambda - 18 - \lambda + 12 - 2$

$|A| = \lambda - 8$

$$|A| = 0$$
$$\lambda - 8 = 0$$
$$\boxed{\lambda = 8}.$$

To solve this, reduce into echelon form.

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda = 8$,
$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \longleftrightarrow R_1$
$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 2 \\ 4 & 3 & 8 \end{bmatrix}$$

$R_4 \longrightarrow R_4 - 4R_1$

$R_2 \longrightarrow R_2 - 2R_1$
$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & -1 & -4 \end{bmatrix}$$

$R_3 \longrightarrow R_3 - R_2$
$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + y + z = 0$$
$$-y - 4z = 0$$

$y = -4z$

Let $\boxed{y = k}$, $z = -\frac{y}{4}$

$$\boxed{z = -\frac{k}{4}}$$

$x + y + 3z = 0$

$x + k - 3\left(\frac{k}{4}\right) = 0$

$x + \frac{k}{4} = 0$

$$\boxed{x = -\frac{k}{4}}$$

Que 3. Find the value of λ st the following equations have unique soln

$\lambda x + 2y - 2z = 1 \Rightarrow \lambda x + 2y - 2z - 1 = 0$

$4x + 2\lambda y - z - 2 = 0$

$6x + 6y + \lambda z - 3 = 0$

Soln

$\lambda x + 2y - 2z = 1$

$4x + 2\lambda y - z = 2$

$6x + 6y + \lambda z = 3$

This is non-Homogeneous system of eqn

$$\begin{bmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

This system will have unique solution

if $\rho[A] = \rho[A:B] = n = 3$

when $\rho[A] = 3 \Rightarrow |A| \neq 0$.

<u>Note:-</u> If $A_{n\times n}$ matrix. If A is non-singular matrix ie; $|A| \neq 0$ then Rank $A = n$.

so, for unique soln $|A| \neq 0$

$$\begin{vmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{vmatrix} \neq 0$$

$$\lambda[2\lambda^2+6] - 2[4\lambda+6] - 2[24-12\lambda] \neq 0$$

$$2\lambda^3+6\lambda - 8\lambda - 12 - 48 + 24\lambda \neq 0$$

$$2\lambda^3+22\lambda - 60 \neq 0$$

$$\lambda^3+11\lambda - 30 \neq 0 \quad \text{---} (1)$$

Let $\lambda = 2$, $\quad 2^3 + 11(2) - 30 = 0$

so $(\lambda-2)$ is root of (1)

$$\lambda^3+11\lambda-30 \neq 0$$

$$\lambda^2(\lambda-2) + 2\lambda(\lambda-2) + 15(\lambda-2) \neq 0$$

$$(\lambda-2)(\lambda^2+2\lambda+15) \neq 0.$$

for $\lambda^2+2\lambda+15 = 0$

$$\lambda = \frac{-2 \pm \sqrt{(2)^2-4(15)}}{2}$$

$$\lambda = \frac{-2 \pm \sqrt{-56}}{2}$$

$$\lambda = \frac{-2 \pm \sqrt{-2\times 2\times 2\times 7}}{2}$$

$$= \frac{-2 \pm 2\sqrt{14}\,i}{2}$$

$$\lambda = -1 \pm \sqrt{14}\,i$$

So,

$$\lambda^3 + 2\lambda + 15 \neq 0$$

For $\lambda \neq 2, -1 \pm \sqrt{14}\,i$, system of equation have unique solⁿ.